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1. - Introduction

According to recent results in [2] and [10], the family of proper holomorphic maps from $D \subset\subset \mathbb{C}^n$ to $G \subset\subset \mathbb{C}^n$ of multiplicity bounded from above by some $m \in \mathbb{N}$ is normal. That is any sequence f_j of such maps either has a convergent subsequence or is compactly divergent.

If D and G are in addition C^∞ smoothly bounded and pseudoconvex of finite type [6], then by [5], [7] the maps f_j are known to extend smoothly up to the boundary of D. Here we study the behaviour of this extension as $j \to \infty$. We denote by $\mathrm{Prop}(D, G, m)$ the set of proper holomorphic maps from D to G of multiplicity m.

THEOREM 1. *Let $D, G, \subset\subset \mathbb{C}^n$ be C^∞-smoothly bounded pseudoconvex domains of finite type, and $f_j \in \mathrm{Prop}(D, G, m)$, $f_j \to f : D \to \overline{G}$. Then, if $f \in \mathrm{Prop}(D, G, m_0)$, one has*

i) *if $m_0 = m$, then $f_j \to f$ in $C^\infty(\overline{D})$*

ii) *if $m_0 < m$, then $\exists j'$, $\{p_i\}_1^{m-m_0} \subset \partial D$ with $f_{j'} \to f$ in $C^\infty(\overline{D} - \{p_i\})$.*

Otherwise, f is a constant map to some $q \in \partial G$, and ii) holds with $m_0 = 0$ and C^∞ replaced by C^0.

In [1], Bell gave an analogous result for biholomorphic maps. The points $\{p_i\}$ in ii) are limits of $f_{j1}^{-1}(w)$ for $w \in G$. The example of a sequence of m-fold Blanschke products as maps from the unit disc in $\mathbb{C}$ to itself shows that one cannot expect smooth convergence at these points, see also [1], [2] and [10]. The main ingredients of the proof are: the transformation rule for the Bergman kernel function under proper maps [3]; a Proposition of Bell [1] on the density of span $K_2(\cdot, w)$, $w \in G$, in $A^\infty(G) = A(G) \cap C^\infty(\overline{G})$; $C^\infty(\overline{G} \times \overline{G} - \Delta)$-regularity of the Bergman kernel for pseudoconvex domains of finite type, see [4], [9]. Here, Δ is the boundary diagonal of $G \times G$, $A(G)$ the holomorphic functions and

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K_2 the Bergman kernel function of G. Finally a division Theorem in $A^\infty(D)$, which is of independent interest; we write $|\cdot|_{\ell,D}$ for the C^ℓ-sup norm on D.

THEOREM 2: *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Assume that*

a) $u_j \in A^\infty(D)$ *converge in* $C^\infty(\overline{D})$ *to* $u \in A^\infty(D)$.

b) *the order of vanishing of the* u_j, u *in* $\overline{D}$ *is of uniformly bounded order.*

c) $h_j \in A(D)$ *are uniformly bounded:* $|h_j(z)| \leq M$ *for all* $z \in D$, $j \in \mathbb{N}$.

d) *for all* $m \geq 0$, $\{u_j \cdot h_j^m\}_{j\geq 1}$ *is bounded in* $A^\infty(D)$, *that is:* $\forall \ell \geq 1$, $m \geq 0$ $\exists c_1(\ell, m) \forall j :$ $|u_j \cdot h_j^m|_{\ell,D} \leq c_1(\ell, m)$.

Then $\{h_j\}_{j\geq 1}$ *is bounded in* $A^\infty(D)$: $\forall \ell \exists c_2(\ell, u,$ *finitely many* $c_1)$: $|h_j|_{\ell,D} \leq c_2$.

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2. - Proof of Theorem 1

If $f \in \text{Prop}(D, G, m)$, then f^{-1} is an m-valued holomorphic map or correspondence from G to D or a holomorphic map $f^{-1} : G \to D^m_{\text{sym}}$, the m-fold symmetric product of D, see [12].

PROPOSITION 3. *Assume* $D, G \subset\subset \mathbb{C}^n$ *and* $f_j \in \text{Prop}(D, G, m)$, $f_j \to f : D \to \overline{G}$. *Then, if* $f \in \text{Prop}(D, G, m_0)$, *one has*

i) *if* $m_0 = m$, *then* $f_j^{-1}(w) \to f^{-1}(w)$ *in* D^m_{sym}.

ii) *if* $1 \leq m_0 < m$, *then there is a subsequence* j' *and an* $(m - m_0)$*-valued holomorphic map* $h : G \to \partial D$ *with* $f_{j'}^{-1} \to (f^{-1} \cup h) : G \to (\overline{D})^m_{\text{sym}}$.

Otherwise, f *is into* ∂G, *and* $\exists j'$, $h : G \to (\partial D)^m_{\text{sym}}$ *with* $f_{j'}^{-1}(w) \to h$. *If in addition* D *and* G *are pseudoconvex of finite type, then the maps* h *above are constant:* $h : G \to \{p_i\}_1^{m-m_0} \subset \partial D$, *and in case* $f(D) \subset \partial G$, f *is constant and* $m_0 = 0$.

PROOF. By [2], [10], either $f \in \text{Prop}(D, G, m_0)$ for some $1 \leq m_0 \leq m$ or f maps D into ∂G. We may pass to a subsequence j' such that $f_{j'}^{-1}$ converges to an m-valued map $F : G \to \overline{D}$. If $f(D) \subset \partial G$, then $F(G) \subset \partial D$, and $h = F$. Otherwise, given $K_1 \subset\subset D$ there exists $K_2 \subset\subset G$ such that $f_j(K_1) \subset K_2$ for all j. Therefore $f_{j'}^{-1} \circ f_{j'} \to F \circ f$ as $j' \to \infty$. Note that we may write $f_j^{-1} \circ f_j = id \cup g_j$, where g_j is an $(m-1)$- valued map from D to D, therefore $F \circ f = id \cup g$. This implies that $F = f^{-1} \cup h$ for some $(m - m_0)$-valued map h. If $m_0 = m$, then f^{-1} is m-valued, so $h = \emptyset$, and $F = f^{-1}$. We see that every subsequence of f_j^{-1} has a subsequence that converges to f^{-1}. This proves i). In case ii) we need to show that $h(G) \subset \partial D$. In this case, if ∂D is pseudoconvex of finite type [6], it does not contain any complex varieties, and h must be constant: $h = \{p_i\}_1^{m-m_0}$ for some $p_i \in \partial D$. Since $f : D \to G$ is proper, given $K_2 \subset\subset G$ there exist $K_1 \subset\subset D$ such that $f^{-1}(K_2) \subset K_1$. Claim: $K_1 \cap F(w) \subset f^{-1}(w)$ for $w \in K_2$. It follows that $h(G) \subset \partial D$. Proof of claim: Let $z_{j'} \in K_1 \cap f_{j'}^{-1}(w)$ and $z_{j'} \to z$. Then $f_{j'}(z_{j'}) = w$, and we may pass to $j' \to \infty$: $f(z) = w$. □

PROPOSITION 4. [1, Fact 1]. *Let $G \subset\subset \mathbb{C}^n$ be a smooth pseudoconvex domain of finite type. Then $\forall r \in \mathbb{N}\ \exists \ell \in \mathbb{N}$, $\{w_k\}_1^\ell \subset G$, $c > 0\ \forall h \in A^\infty(G)$, $p \in G \exists \{c_k\}_1^\ell \in \mathbb{C}$:*

i) $\displaystyle\sum_{k=1}^{\ell} c_k K_2(z, w_k) = h(z) + 0(|z-p|^{r+1})$,

ii) $|c_k| \le c|h|_{r,G}$

Next consider the transformation formula of the Bergman kernel function under proper maps [3]:

$$u_j(z) K_2(f_j(z), w) = \sum_{i=1}^{m} K_1(z, F_j^{(i)}(w)) \overline{U_j^{(i)}(w)}. \tag{1}$$

Here, $\{F_j^{(i)}(w)\}_{i=1}^m = f_j^{-1}(w)$ are the branches of the multi-valued inverses, and $U_j^{(i)} = \det(F_j^{(i)})'$. We follow Bell [1]. Now let $h \in A^\infty(D)$, $q \in D$, $r \in \mathbb{N}$. By Proposition 4 where p is replaced by $f_j(q)$ and by (1) there exist $w_k \in G$, $c_k \in \mathbb{C}$ depending on j with

$$u_j(z) h \circ f_j(z) = \sum_{k=1}^{\ell} \sum_{i=1}^{m} c_k K_1(z, F_j^{(i)}(w_k)) U_j^{(i)}(w_k) + 0(|z-q|^{r+1}).$$

In case i) of the Theorem, F_j and U_j converge uniformly on $\{w_k\}$ as $j \to \infty$ by Proposition 3. Then, since $K_1 \in C^\infty(\overline{D} \times \overline{D} \backslash \Delta)$ (see [4]), and since the c_k are bounded independently of q and j, we conclude that $\{u_j \cdot h \circ f_j\}$ is bounded in $C^\infty(\overline{D})$. Letting $h = 1$, we conclude that $\{u_j\}$ is bounded in $C^\infty(\overline{D})$ and therefore converges in $C^\infty(\overline{D})$ to $u = \det f'$. By [5], u and u_j vanish at most of order $m \cdot n$ at any point in $\overline{D}$. Letting $h(w) = w_i^m$, $h_j = h \circ f_j$ for $i = 1, \dots, n$, $m \ge 0$, we finally verify the assumptions c) and d) of Theorem 2. We may then conclude that $\{f_j\}$ is bounded in $C^\infty(\overline{D})$. This proves part i).

In case ii) by Proposition 3 we may pass to a subsequence j' such that $\{u_{j'} \cdot h \circ f_{j'}\}$ is bounded in $C^\infty(\overline{D} \backslash \{p_i\})$. Here again the regularity of K_1 is used. A local version of Theorem 2 allows to conclude that convergence of f_j takes place in $C^\infty(\overline{D} \backslash \{p_i\})$. As to the case of f being a constant map, the same reasoning as above shows that for some subsequence j', $u_{j'}$ converges to $u \equiv 0$ in $C^\infty(\overline{D} \backslash \{p_i\}_1^m)$. Now the proof of Theorem 1, part B in [1] yields the conclusion that convergence takes place in $C^0(\overline{D} \backslash \{p_i\})$.

3. - A Division Theorem with Estimates

Assuming that $h \in A(D)$ is bounded and that $uh^m \in A^\infty(D)$ for all $m \geq 0$, we wish to show that h is in $A^\infty(D)$ and give estimates for h. Certainly, this cannot hold if u vanishes to infinite order at some point in $\overline{D}$. One is reduced to studying the question in the neighbourhood of a point p in ∂D at which u vanishes of finite order k. We restrict the considered functions u, h to a complex line L_p at p with this property. The division of uh^m by u will be carried out on such lines L_x for $x \in \partial D \cap U$, U a neighbourhood of p, and we will prove that for every ℓ, the function $h|_{\overline{D} \cap L_x \cap U}$ is in $C^\ell(D \cap L_x \cap U)$ with uniform estimates in x. The point is to keep track of the C^ℓ-sup norm estimate of h during the division process which proceeds by dividing the zeroes of u out of uh^m one at a time. To facilitate this procedure we introduce a normalizing transformation of $D \cap U$ which preserves analyticity on the complex lines $D \cap L_x \cap U$. We may choose holomorphic coordinates $(z_1, \dots, z_n)$ such that L_p is the z_1-axis and p is the origin. Let $x = (z_2, \dots, z_n) \in \mathbb{R}^{2n-2}$ and G be a smooth domain in D with $\partial D \cap U = \partial G \cap U$ for a neighbourhood U of the origin such that for some $r > 0$ and $|x| < r$ the slices $G_x = \{z \in \mathbb{C} : (z, x) \in G\} \subset \mathbb{C}$ are simply connected. Let $a \in \mathbb{C}$ be a fixed point which lies in all G_x and let Φ_x be the Riemann mapping function from G_x onto the unit disc Δ, with $\Phi_x(a) = 0$ and $\phi'_x(a) > 0$. Note that $0 \in \partial G_0$. Next let Ψ denote a conformal map of the unit disc onto $\Delta_- = \Delta \cap \{\operatorname{Im} z < 0\}$ which takes $\Phi_0(0) \in \partial A$ to $0 \in \partial \Delta_-$. The coordinate change given by $(z, x) \to (\Psi \circ \Psi_x(z), x/r)$ transforms $\bigcup_{|x|<r} G_x \subset D$ to $\Delta_- \times V \subset \mathbb{C} \times \mathbb{R}^{2n-2}$, where $V = \{|x| < 1\}$. One knows from the classical theory of conformal mappings that this change is C^∞-smooth up to $\partial G \cap U_1$ and maps this set onto $\{(-1, +1) \times V\}$ (by normalization) for some neighbourhood U_1 of the origin in $\mathbb{C}^n$. The function $u \in A(G)$ is transformed to a smooth function $u(z, x)$ on $\Delta_- \times V$ which is holomorphic in z for fixed x. For smooth functions u on $\Delta_- \times V$ we define the norm

$$|u(\cdot, x)|_{\ell, \Delta_-} := \sup_{\substack{z \in \Delta_- \\ i+j \leq \ell}} \left| \frac{\partial^{i+j}}{\partial z^i \partial \overline{z}^j} u(z, x) \right| .$$

Next we define the class of functions we will work in.

DEFINITION: a) $\Gamma^-(\ell, c)$ is the set of complex valued functions u on $\Delta_- \times V$ with $u(\cdot, x) \in A(\Delta_-)) \cap C^\ell(\overline{\Delta}_-)$, and $|u(\cdot, x)|_{\ell, \Delta_-} \leq c$ for each $x \in V$.

b) $\Gamma(\ell, c)$ are the functions on $\Delta \times V$ with $u(\cdot, x) \in A(A_-) \cap C^\ell(\overline{\Delta})$ and $|u(\cdot, x)|_{\ell, \Delta} \leq c$ for each $x \in V$.

Let $u^{(i)}(z, x) = \dfrac{1}{i!} \dfrac{\partial^i}{\partial z^i} u(z, x)$ and $u^{(i,j)} = \dfrac{1}{i!j!} \dfrac{\partial^{i+j}}{\partial z^i \partial \overline{z}^j} u$. The objective of this section is to prove the following.

THEOREM 5. *Let $h(\cdot, x) \in A(\Delta_-)$ and $|h(\cdot, x)| \leq c_1$ for $x \in V$, and assume that for all $m \geq 0$ there exists $c_1(m)$ with $u \cdot h^m \in \Gamma^-(\ell_1, c_1(m))$. Assume* *furthermore that* u *vanishes of order* k *at* $(0,0)$ *and that for some* $c_2 > 0$*:*

i) $|u^{(k,0)}(0,0)| \geq c_2^{-1}$

ii) $u(z,\cdot) \in C^1(V)$ *and* $\left|\frac{d}{dx}u(z,\cdot)\right| \leq c_2$ *on* $\Delta_- \times V$.

Then $h \in \Gamma^-(\ell, c)$*, where* $\ell = \left(\frac{1}{2}\right)^k \ell_1 - 2k - 2$*, and* c *depends only on* ℓ_1*,* k*, finitely many* c_1*, and* c_2*.*

Theorem 5 implies Theorem 2: Note that the assumption i) of Theorem 5 is verified uniformly for all u_i since they converge in $C^\infty(\overline{D})$ to u which we assume to vanish of at most finite order in $\overline{D}$. Assumption ii) follows from $|u_i \cdot h_i^0|_{1,D} = |u_i|_{1,D} \leq c_1(0,1)$. Now the conclusion of Theorem 9 gives for all ℓ uniform C^ℓ-estimates for $h_i|_{L\cap D\cap U}$ for complex lines L transversal to the boundary of D and some neighbourhood U of any boundary point of D.

Note that by Cauchy estimates, the uniform boundedness of h_i in D gives uniform boundedness of $h_i|_K$ in $C^\infty(K)$ for compact subsets K of D. Pick any point p in $D\backslash K$. Any ℓ-th order derivarive of u at p can be expressed as a finite linear combination of derivatives of u in the direction of complex lines L transversal to the boundary. Since we can choose the L from an open cone of directions at each boundary point, we conclude that the sequence h_i is bounded in $C^\infty(\overline{D})$.

The proof of Theorem 5 proceeds by four propositions. We closely follow Diederich-Fornaess [7]. Here is a well-known fact on bounded extension [8, p. 277].

PROPOSITION 6. *Let* $u \in \Gamma^-(\ell, c)$*. Then there exists a* $v \in \Gamma(\ell, c)$ *with* $v(\cdot, x)|\Delta_- = u(\cdot, x)$*.*

LEMMA 7. *Let* $u \in \Gamma(\ell, c_1)$ *and* $\varsigma_1, V \to \Delta$ *be any map. Then there exists* $\tilde{u} \in \Gamma(\ell, c), c$ *depending only on* ℓ *and* c_1*, with*

a) $u(\cdot, x) = \tilde{u}(\cdot, x)$ *in* Δ_-

b) $\tilde{u}(z,x) = \sum_0^{\ell-1} u^{(i)}(\varsigma_1(x), x) \cdot (z - \varsigma_1(x))^i + \sigma_\ell(z,x)$*, where* σ_ℓ *vanishes of order* ℓ *at* $z = \varsigma_1(x)$ *for all* $x \in V$*.*

PROOF. Conclusion b) says that the anti-holomorphic derivatives of $\tilde{u}$ up to order $\ell - 1$ vanish at ς_1.

The Taylor expansion for u at ς_1 is given by

$$u(z,x) = \sum_{i+j=0}^{\ell-1} u^{(i,j)}(\varsigma_1, x)(z - \varsigma_1)^i(\overline{z} - \overline{\varsigma}_1)^j$$

$$+\frac{1}{(\ell-1)!}(z-\varsigma_1)^{-\ell} \sum_{i+j=\ell} \int_{\varsigma_1}^{z} (z-w)^{\ell-1} u^{(i,j)}(w,x)dw \cdot (z-\varsigma_1)^i(\overline{z}-\overline{\varsigma}_1)^j$$

Clearly b) holds for $\tilde{u} \equiv u$ if $\varsigma_1(x) \in \overline{\Delta}_-$, and if $\varsigma_1(x) \notin \overline{\Delta}_-$, we set

$$\tilde{u} = u - \sum_{\substack{j \geq 1 \\ i+j \leq \ell-1}} \varphi\left(\frac{z-\varsigma_1}{\operatorname{Im}\varsigma_1}\right) \cdot u^{(i,j)}(\varsigma_1, x) \cdot (z-\varsigma_1)^i (\overline{z}-\overline{\varsigma}_1)^j.$$

Here, $\varphi \in C_0^\infty\left(\frac{1}{2}\Delta\right)$, $\varphi \equiv 1$ for $|z| < \frac{1}{4}$. We see that a) and b) hold. Note that since $u \in A(\Delta_-) \cap C^\ell(\overline{\Delta}_-)$, $u^{(0,1)}$ vanishes of order $\ell - 1$ on Im $z = 0$. We may estimate

$$|u^{(i,j)}(z,x)| \leq c_2 \cdot |u|_\ell \cdot |\operatorname{Im}\ z|^{\ell-i-j}, \quad i+j \leq \ell, \ j > 1.$$

Here c_2 depends only on ℓ, c_1. Denote by A_{ij} the entries of the above sum. For $z \in \operatorname{supp} \varphi\left(\frac{z-\varsigma_1}{\operatorname{Im}\ \varsigma_1}\right)$, we have $|z - \varsigma_1| \leq |\operatorname{Im}\ z|$, and for $z \notin$ supp, A_{ij} and all its derivatives vanish. Therefore

$$|A_{ij}(z,x)| \leq c_3 |u|_\ell \cdot |\operatorname{Im}\ z|^\ell.$$

Now every derivative up to order ℓ of A_{ij} with respect to z or $\overline{z}$ will take away one power of $|\operatorname{Im}\ z|$ in this estimate and change the constant c_3, making it dependent on the first ℓ derivatives of φ.

We conclude that $|\tilde{u}|_\ell \leq c_4 \cdot |u|_\ell \leq c_4 c_1$. □

LEMMA 8. *Let $u \in \Gamma(\ell, c_1)$, $\varsigma_1 : V \to \Delta$ satisfy the conclusion b) of Lemma 7 and $u(\varsigma_1, x) = 0$. Then there exists $u_1 \in \Gamma(\ell - 1, c)$, c depending only on ℓ and c_1, with*

$$u = (z - \varsigma_1) \cdot u_1 \qquad \text{on } \Delta \times V.$$

PROOF. Let σ_ℓ denote the ℓ-th order Taylor remainder term in the development of $u(z,x)$ around $(\varsigma_1(x), x)$. Define

$$u_1 = \sum_1^{\ell-1} u^{(i)}(\varsigma_1, x) \cdot (z - \varsigma_1)^{i-1} + \frac{\sigma_\ell(z,x)}{z - \varsigma_1}.$$

Then

$$\left|\sum_1^{\ell-1} u^{(i)} \cdot (z-\varsigma_1)^{i-1}\right|_{\ell-1} \leq c_2 \cdot |u|_\ell \text{ in } \Delta \times V.$$

The expression $\left(\frac{\sigma_\ell}{z-\varsigma_1}\right)^{(i,j)}$ for $i+j \leq \ell - 1$ is a sum of terms of the form $\sigma_\ell^{(p,q)} \cdot (z-\varsigma_1)^{-r}$ with $p+q+r \leq \ell, r \geq 1$. From the integral formula for σ_ℓ it follows that $|\sigma_\ell^{(p,q)}(z,x)| \leq c_3 |u|_\ell \cdot |z - \varsigma_1|^{\ell-p-q}$. This implies $|u_1|_{\ell-1} \leq c_4 \cdot |u_\ell| \leq c_4 c_1$. □

PROPOSITION 9. *Let $u(\cdot, x) \in \Gamma(\ell, c_1)$ vanish of order $k \le \ell - 1$ at 0, and*

i) $|u^{(k,0)}(0,0)| \ge c_1^{-1}$

ii) $u(z, \cdot) \in C^1(V)$ *and* $\left|\frac{d}{dx} u(z, \cdot)\right| \le c_1$ *on* $\Delta \times V$.

Then, after shrinking Δ, V to Δ_ϵ, V_r, where ϵ, r depend only on k, ℓ, c_1, there exist $u_k \in \Gamma(\ell - k, c)$, c depending only on k, ℓ, c_1 and maps $\varsigma_j : V \to \Delta$, $j = 1, \ldots, k$ with

a) $$u = u_k \prod_1^k (z - \varsigma_j) \text{ on } \Delta_- \times V$$

b) $$|u_k(z, x)| \ge 2^{-k-3} c_1^{-1} \text{ on } \Delta \times V.$$

PROOF. By the k-th order vanishing of u at 0,

$$u(z, 0) = \sum_{i+j=k} u^{(i,j)}(0,0) z^i \bar{z}^j + \sigma_{k+1}(z, 0).$$

Since $u \in A(\Delta_-) \cap C^\ell(\overline{\Delta}_-)$, we have $u^{(i,j)}(0) = 0$ for $j \ge 1$. Therefore

$$\begin{aligned} u(z, 0) &= u^{(k,0)}(0) z^k + \sigma_{k+1}(z) \\ &= z^k \cdot \left(u^{(k,0)}(0) + \frac{\sigma_{k+1}}{z^k}\right) \\ &= z^k v(z). \end{aligned}$$

Since $|\sigma_{k+1}(z, 0)| \le c_2 |u|_{k+1} \cdot |z|^{k+1} \le c_2 c_1 |z|^{k+1}$, we see that $|v(z)| \ge \frac{1}{2} c_1^{-1}$ for $|z| < \frac{1}{2} \frac{1}{c_1^2 c_2} =: \epsilon_0$. Therefore $|u(z, 0)| \ge \frac{\epsilon^k}{2} \cdot c_1^{-1}$ for $\epsilon \le |z| < \epsilon_0$. By assumption ii), there exists $r_0(\epsilon, c_1)$ with

(2) $$|u(z, x)| \ge \frac{\epsilon^k}{4} c_1^{-1} \text{ for } \epsilon \le |z| < \epsilon_0, \ |x| < r_0.$$

We now see that

(3) $\log u(\cdot, x)$ increases its value by $2\pi i k$ around $|z| = \epsilon_0$ for $|x| \le r_0$.

Therefore, there exists a map $\varsigma_1 : V_{r_1} \to \Delta_{\epsilon_0/2}$ with $u(\varsigma_1(x), x) = 0$. Applying Lemma 7 to u, ς_1 gives a $\tilde{u} \in \Gamma(\ell, c_3)$ with properties a) and b). Next apply Lemma 8 to $\tilde{u}, \varsigma_1$ and get $u_1 \in \Gamma(\ell - 1, c_4)$ with

(4) $$\tilde{u} = u_1 \cdot (z - \varsigma_1) \text{ on } \Delta_{\epsilon_0} \times V_{r_1}.$$

Since $\tilde{u} = u$ on $|z| = \epsilon_0$, we conclude from (4) that $u_1 \ne 0$ on $|z| = \epsilon_0$ and (3) holds for u_1 and $k - 1$.

We may repeat this argument k times and conclude that

$$\tilde{u}_j = u_{j+1} \cdot (z - \varsigma_{j+1}) \text{ on } \Delta_{\epsilon_0} \times V_{r_1} \tag{5}$$

with $u_{j+1} \epsilon \Gamma(\ell - k, c_5)$, $j = 1, \dots, k-1$. Therefore

$$\tilde{\tilde{u}} = u_k \cdot \prod_1^k (z - \varsigma_1) \text{ on } \Delta_{\epsilon_0} \times V_{r_1}.$$

Since $u = \tilde{u} = \tilde{\tilde{u}}$ on $\Delta_{\epsilon_0,-} \times V_{r_1}$, this proves a). To prove b), we make the following claim:

Given $\epsilon > 0$, $\exists r(\epsilon, c_1, k)$ such that $|\varsigma_j(x)| < \epsilon$ for $|x| < r$, $j = 1, \dots, k$.

PROOF. Let $\epsilon < \epsilon_0$ be given. By (4), $\tilde{u}(\varsigma_1(x), x) = 0$ for $|x| < r_1$. Since $\tilde{u}(\varsigma_1, x) = 0 = u(\varsigma_1, x)$, (2) implies that $|\varsigma_1(x)| \le \epsilon$. Continuing inductively, assume that $\varsigma_1, \dots, \varsigma_j$ have modulus smaller that ϵ. By (5), we have $\tilde{u}_j(\varsigma_{j+1}, x) = 0$ for $|x| < r_1$. Now $\tilde{\tilde{u}} = u_j \cdot \prod_1^j (z - \varsigma_i)$. Outside Δ_-, this $\tilde{\tilde{u}}$ does not have to coincide with the $\tilde{\tilde{u}}$ above. Since in Lemma 7 we have $\tilde{u}_j(\varsigma_{j+1}) = u_j(\varsigma_{j+1})$, we conclude that $\tilde{\tilde{u}}(\varsigma_{j+1}) = 0$. If now also $u(\varsigma_{j+1}) = 0$, then (2) implies that $|\varsigma_{j+1}(x)| < \epsilon$. Otherwise one has $\tilde{\tilde{u}}(\varsigma_{j+1}) \ne u(\varsigma_{j+1})$, and since by construction $\tilde{\tilde{u}}$ differs from u only in ϵ-neighbourhoods of $\varsigma_1, \dots, \varsigma_j$, we have for some q

$$|\varsigma_{j+1}(x)| \le |\varsigma_{j+1} - \varsigma_q| + |\varsigma_q| < 2\epsilon.$$

This proves the claim.

To conclude the proof, note that given $\epsilon > 0$, we have by (2) for $|z| = \epsilon$

$$|u_k(z, x)| = \frac{|u(z, x)|}{\prod_1^k |z - \varsigma_j|} \ge \frac{\epsilon^k}{4} \frac{c_1^{-1}}{(2\epsilon)^k} = 2^{-k-2} c_1^{-1}$$

for $|x| < r$, r chosen as in the claim. Since $u_k \epsilon \Gamma(\ell - k, c_5)$, we may choose ϵ small enough, depending on k, c_1, c_5 such that this implies $|u_k(z, x)| \ge 2^{-k-3} c_1^{-1}$ for $|z| < \epsilon$. □

Proof of Theorem 5: First we apply Proposition 9 to u, which gives $u = u_k \prod_1^k (z - \varsigma_j)$ on $\Delta_- \times V$.

We will successively divide the $(z - \varsigma_j)$ out of uh. To retain estimates on the way, we need to take into account those for $uh^m \in \Gamma^-(\ell_1, c_1(m))$ which by Proposition 6 we may assume to lie in $\Gamma(\ell_1, c_1(m))$.

a) Note that $\frac{(uh)^2}{z - \varsigma_1} = (uh^2) \cdot u_k \prod_2^k (z - \varsigma_j)$ on $\Delta_- \times V$. By the assumption concerning uh^2, and Proposition 9 concerning u_k, we may conclude that

$$\frac{(uh)^2}{z - \varsigma_1} \in \Gamma^-(\ell_1 - k, c_2). \tag{6}$$

We wish to show that $\frac{uh}{z-\varsigma_1} \in \Gamma^-(\ell, c)$ for some ℓ, c. If $\varsigma_1 \in \overline{\Delta}_1$, then (6) implies that $(hu)(\varsigma_1, x) = 0$, and we are done by Lemma 8 with $\ell = \ell_1 - 1$.

If $\varsigma_1 \notin \overline{\Delta}_-$, we proceed as follows. Since $uh \in \Gamma(\ell_1, c_1)$, we may apply Lemma 7 to uh, ς_1 :

$$(\widetilde{uh}) = \sum_0^{\ell-1} (uh)^{(i)}(\varsigma_1, x)(z-\varsigma_1)^i + \sigma_\ell(z, x).$$

Now

$$\frac{(\widetilde{uh})^2}{z-\varsigma_1} = \frac{(uh^{(0)})^2}{z-\varsigma_1} + 2uh^{(0)} \cdot \left(\sum_1^{\ell-1} uh^{(i)}(z-\varsigma_1)^{i-1} + \frac{\sigma_\ell}{z-\varsigma_1} \right)$$
$$+ \frac{1}{z-\varsigma_1} \left(\sum_1^{\ell-1} uh^{(i)}(z-\varsigma_1)^i + \sigma_\ell \right)^2 .$$

Since $\widetilde{uh} = uh$ on $\Delta_- \times V$, the left hand side is in $\Gamma^-(\ell_1 - k, c_2)$. As in the proof of Lemma 8 we may estimate the second and third terms on the right hand side to see that they are in $\Gamma(\ell_1 - 1, c_3)$. Therefore $\frac{(uh^{(0)})^2}{z-\varsigma_1} \in \Gamma^-(\ell_1 - k, c_4)$. By differentiating, this implies $\frac{(uh^{(0)})^2}{(z-\varsigma_1)^{\ell_1-k+1}} \in \Gamma^-(0, c_5)$, and $\frac{uh^{(0)}}{(z-\varsigma_1)^p} \in \Gamma^-(0, c_6)$. Here, $p = \left[\frac{\ell_1 - k + 1}{2}\right]$. Assume $p \geq 2$, and consider the integral

$$\int_{\frac{-i}{2}}^{z} \frac{uh^{(0)}}{(w-\varsigma_1)^p} dw = \frac{1}{1-p} \left(\frac{uh^{(0)}}{(z-\varsigma_1)^{p-1}} - \frac{uh^{(0)}}{\left(\frac{-i}{2} - \varsigma_1\right)^{p-1}} \right),$$

where we integrate along a straight line. The left hand side is in $\Gamma^-(1, c_7)$, and since $\varsigma_1 \notin \overline{\Delta}_-$, the second term on the right hand side, independent of z, is bounded by $\frac{1}{p-1}2^{p-1}c_1$. We conclude that $\frac{uh^{(0)}}{(z-\varsigma_1)^{p-1}} \in \Gamma^-(1, c_8)$. Repeating this gives $\frac{uh^{(0)}}{z-\varsigma_1} \in \Gamma^-(p-1, c_9)$. Now we have on $\Delta_- \times V$:

$$\frac{uh}{z-\varsigma_1} = \frac{uh^{(0)}}{z-\varsigma_1} + \sum_1^{\ell-1} uh^{(i)} \cdot (z-\varsigma_1)^{i-1} + \frac{\sigma_\ell}{z-\varsigma_1} \in \Gamma^-(p-1, c_{10}).$$

b) We prove by induction the following statement:

(7i) $$u_k \prod_i^k (z-\varsigma_j) h^m \in \Gamma^-(\ell, c) \quad \forall m \geq 0.$$

($i = 1$) is the assumption of the Theorem. ($i = 2$) was proved in part a) for $m = 1$. We will show that (7i) implies (7i+1). Let $g_m = u_k \prod_i^k (z-\varsigma_j) h^m \in \Gamma^-(\ell, c)$, and note that

$$\frac{g_m^2}{z - \varsigma_i} = g_{2m} \cdot u_k \cdot \prod_{i+1}^{k} (z - \varsigma_i) \text{ on } \Delta_- \times V.$$

The right hand side is in $\Gamma^-(\ell - k, c_1)$. We may now conclude as in a) that this implies

$$\frac{g_m}{z - \varsigma_1} \in \Gamma^-(\ell_1, c_2) \text{ where } \ell_1 = \left[\frac{\ell - k + 1}{2}\right] - 1.$$

c) Letting $m = 1$ in $(i = k)$, we get $u_k \cdot h \in \Gamma^-(\ell, c)$. Now the conclusions of Proposition 9 allow us to infer that $h = \frac{(u_k \cdot h)}{u_k} \in \Gamma^-(\ell - k, c_1)$. The expression given for ℓ_1 in Theorem 5 now follows from the one in part b) above. □

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