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Y.-F. S. PÉTERMANN

About a theorem of Paolo Codecà's and omega estimates for arithmetical convolutions. Second part

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About a Theorem of Paolo Codecà's and Omega Estimates for Arithmetical Convolutions

Second Part

Y.-F. S. PÉTERMANN

1. - Introduction

Consider the real valued functions h defined on $[1, \infty)$

$$h(x) = \sum_{n \leq z} \alpha(n) n^a f\left(\frac{x}{n}\right), \tag{1.1}$$

where $\alpha(n)$ is a sequence of real numbers satisfying

$$\sum_{n \leq x} |\alpha(n)| = O(x), \tag{1.2}$$

$-1 \leq a < 0$, f is a periodic function of period 1, of bounded variation on $[0, 1]$ and such that

$$\int_0^1 f(u) \mathrm{d}u = 0, \tag{1.3}$$

and $z = z(x) \leq x$ is a positive, strictly increasing, continuous and unbounded function (z will always be assumed to satisfy these properties in the sequel).

We say that h is $C_z(a, \alpha, f)$.

In the first part of this work [8], inspired by an article of Codecà's [1], I considered functions g nearly $C_x(-1, \alpha, f)$ (i.e. short of a $o(1)$), where α possesses an asymptotic mean K, and such that

$$g - K \int_1^\infty u^{-1} f(u) \mathrm{d}u$$

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is nearly $C_z(-1,\alpha,f)$ for some $z = z(x) = o(x)$, as $x \to \infty$.

In the case where z may be taken small enough, I obtained a general expression for the mean of g on an arithmetical progression $An + B$, $n \leq x$, (Theorem 1), from which I could then, for particular functions g, prove omega estimates by suitably choosing the parameters A, B, and x.

Among the functions for which I found this process to be successful are the classical error terms H and E related to the Euler function ϕ and to the sum-of-divisors function σ, an error term of Landau (see e.g. [14]) related to the function $n/\phi(n)$, and the "Chowla-Walum functions"

$$G_{a,k}(x) := \sum_{n \leq \sqrt{x}} n^a \psi_k\left(\frac{x}{n}\right) \tag{1.4}$$

(where $\psi_k(y) = B_k(\{y\})$ denotes the k-th Bernoulli polynomial of argument the fractional part $\{y\}$ of y) when $a \leq -1$. This brings us to the main purpose of this sequel.

The $G_{a,k}$ are related to various divisor problems (see e.g. [5], [6], [9], [10], [11]). Conjectures were proposed as to the "best" O and Ω estimates satisfied by the functions $G_{a,k}$, originating with the Piltz-Hardy-Landau conjecture on the famous Dirichlet divisor problem, generalized in 1963 by Chowla and Walum [2]. If we gather the conjectures that appear reasonable so far in view of the various investigations made by a number of authors, we can state them in the compact form described below.

Let $\alpha_k(a)$ and $\beta_k^*(a)$, where $*$ is allowed to denote $+$, $-$, $\pm$, or nothing at all (i.e. not even a blank), be the smallest α, respectively the largest β, for which $G_{a,k}(x) = O(x^{\alpha+\varepsilon})$, resp. $G_{a,k}(x) = \Omega_*(x^{\beta-\varepsilon})$, for every $\varepsilon > 0$. Set

$$g(a) = \begin{cases} \frac{a}{2} + \frac{1}{4} & \text{if } a \geq -\frac{1}{2}, \\ 0 & \text{if } a \leq -\frac{1}{2}. \end{cases} \tag{1.5}$$

CONJECTURE. *For every real number a, every positive integer k, and for $*$ denoting $+$ and $-$, we have*

$$\alpha_k(a) \leq g(a) \tag{$O.k.a.$}$$

and

$$\beta_k^*(a) \geq g(a). \tag{$\Omega_*.k.a$}$$

REMARKS.

(i) The Piltz-Hardy-Landau conjecture is in this notation $(O.1.0)$; the Chowla-Walum conjecture is: $(O.k.a)$ for all $a \geq 0$ and all positive integers k.

(ii) The assertions "$(O.k.a)$ and $(\Omega.k.a)$" and "$\alpha_k(a) = g(a)$" are equivalent.

(iii) For a brief review of the results known to date towards these conjectures see [11].

In the first part of this paper [8] (see the Addendum), the truth of $(\Omega_{\pm}.k.a)$, $k = 1, 2, \ldots$, is proved for $a \leq -1$. Here, through an extension of the main result of [8] to $C_z(a, \alpha, f)$, $-1 < a < 0$, for suitable z and α (Theorem 1 in Section 2 below), we obtain Ω-estimates for the $G_{a,k}(x)$ (Theorem 2 just below), and as a corollary the truth of $\Omega_{\pm}.k.a)$, $k = 1, 2, \ldots$, for $a \leq -\frac{1}{2}$.

THEOREM 2. *For $-1 < a < 0$ and every positive integer k, we have*

$$G_{a,k}(x) = \Omega_{\pm}\left(\exp\left\{(1 + o(1))\xi_{a,k}\, \frac{(-a/2)^{1+a}}{1+a}\, \frac{(\log\, x)^{1+a}}{\log\log\, x}\right\}\right), \tag{1.6}$$

where

$$\xi_{a,k} = \begin{cases} 1 & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2^a & \text{if } k > 1 \text{ is odd.} \end{cases}$$

As another corollary of Theorem 2, we obtain in Section 3:

THEOREM 3. *Let $E_a(x)$ be the error term*

$$E_a(x) := \sum_{n \leq x} \sigma_a(n) \left(\frac{\zeta(1+a)}{1+a} x^{1+a} + \zeta(1+a)x - \frac{\zeta(-a)}{2}\right), \tag{1.7}$$

$$a \neq -1, 0.$$

Then, for $\frac{25}{38} < |a| < 1$,

$$E_a(x) = \Omega_{\pm}\left(x^{\frac{a+|a|}{2}} \exp\left\{(1 + o(1))\, \frac{(|a|/2)^{1-|a|}}{1 - |a|}\, \frac{(\log\, x)^{1-|a|}}{\log\log\, x}\right\}\right). \tag{1.8}$$

It appears that no nontrivial Ω-estimate for $E_a(x)$, with $-1 < a < -\frac{1}{2}$, was known so far; and, when a is positive, (1.7) improves in the indicated range both results

$$E_a(x) = \Omega_{\pm}\left((x\, \log\, x)^{\frac{a}{2} + \frac{1}{4}}\right) \tag{1.9}$$

and

$$E_a(x) = \Omega(x^a) \tag{1.10}$$

of Hafner's [3], and should be compared, on the one hand with MacLeod's [7]

$$\overline{\lim_{x \to \infty}}\, \frac{E_a(x)}{x^a} = \pm\, \frac{\zeta(a)}{2}, \qquad a > 1, \tag{1.11}$$

and our [12]

$$\overline{\underline{\lim}}_{x\to\infty} \frac{E_1(x)}{x \log\log x} \gtreqless \pm\frac{e^\gamma}{2}, \tag{1.12}$$

and, on the other hand, with the following consequence of conjecture $(O.1.a)$

$$E_a(x) = O(x^{a+\varepsilon}), \qquad a \geq \frac{1}{2}, \tag{1.H}$$

(see (3.10)).

Finally in Section 4 we give another application of Theorem 1. We define the functions

$$P_a(x) = \sum_{n\leq\sqrt{x}} n^a \cos\left(\frac{x}{n}\right) \tag{1.13}$$

and

$$Q_a(x) = \sum_{n\leq\sqrt{x}} n^a \sin\left(\frac{x}{n}\right). \tag{1.14}$$

In [8] we prove (see [4])

$$P_{-1}(x) = \Omega_\pm(\log\log x) \tag{1.15}$$

and

$$Q_{-1}(x) = \Omega_\pm\left((\log\log x)^{\frac{1}{2}}\right). \tag{1.16}$$

Here we obtain

THEOREM 4. *For $-1 < a < 0$, we have*

$$P_a(x) = \Omega_\pm\left(\exp\left\{(1+o(1))\frac{(-a/2)^{1+a}}{1+a}\frac{(\log x)^{1+a}}{\log\log x}\right\}\right) \tag{1.17}$$

and

$$Q_a(x) = \Omega_\pm\left(\exp\left\{(1+o(1))2^a\frac{(-a/2)^{1+a}}{1+a}\frac{(\log x)^{1+a}}{\log\log x}\right\}\right). \tag{1.18}$$

2. - The main result

Let the notation be that of Section 1 and consider a function h being some $C_z(a, \alpha, f)$, where $-1 < a < 0$ and, in addition to (1.2), the arithmetical function α satisfies the submultiplicative property

$$|\alpha(nm)| = O(|\alpha(n)\alpha(m)|) \tag{2.1}$$

for all positive integers n and m. Let $A = A(x) > 0$ be an integer valued function, and $B = B(x) \geq 0$ (we do not require that B be an integer: see [8, Addendum]). Then we have

THEOREM 1. *Set*

$$u = u(x) := z(Ax + B),$$

$$v_i := z\left(\frac{Ax}{2^i} + B\right), \qquad i \geq 0,$$

$$\alpha_b(A) := \sum_{d|A} |\alpha(d)| d^b,$$

and suppose that there exists a function $\eta = \eta(x)$ *decreasing to* 0 *as* $x \to \infty$ *and such that*

$$A \leq v_N, \tag{2.2}$$

where $N := \left[-\frac{\log \eta}{\log 2}\right]$. *Then*

$$\begin{aligned} &\frac{1}{x}\sum_{n\leq x} h(An + B) \\ &= \sum_{k\leq u} \alpha(k) k^a \left(\frac{1}{k^*}\sum_{n\leq k^*} f\left(\frac{n}{k^*} + \frac{B}{k}\right)\right) \\ &+ O\left(\frac{u^{a+2}}{x} + v_N^a \alpha_0(A) + \eta\alpha_a(A)\right), \end{aligned} \tag{2.3}$$

where k^* *denotes* $\frac{k}{(A,k)}$.

PROOF. The proof goes along the same line as that of Theorem 1 in [8]. We let $w(k)$ be the inverse function of $v(y) := z(Ay+B)$, $1 \leq y \leq x$ if $k \geq u(1)$, and $w(k) = 1$ otherwise, and we obtain as in [8]

$$\frac{1}{x}\sum_{n\leq x} h(An + B) = \beta + \delta + \varepsilon, \tag{2.4}$$

where

$$\beta := \sum_{k\leq u} k^a \alpha(k) \left(\frac{1}{k^*}\sum_{n\leq k^*} f\left(\frac{n}{k^*} + \frac{B}{k}\right)\right), \tag{2.5}$$

where by (1.2),

$$\varepsilon = O\left(\frac{1}{x}\sum_{k\le u} k^a k^* |\alpha(k)|\right) = O\left(\frac{u^{a+2}}{x}\right), \tag{2.6}$$

and where

$$\delta = O\left(\frac{1}{x}\sum_{k\le u} k^{a-1}(A,k)|\alpha(k)|w(k)\right). \tag{2.7}$$

In order to estimate δ we define, as in [8],

$$R_i := \begin{cases} \{k\in\mathbb{N}\,|\,\max(v(1),v_i) < k \le v_{i-1}\} & \text{if } i = 1,2,\dots,M := \left[\frac{\log\ x}{\log\ 2}\right]+1,\\ \{k\in\mathbb{N}\,|\,k\le v := v(1)\} & \text{if } i = M+1,\end{cases}$$

and may thus rewrite the sum in (2.7) as

$$\sum_{i=1}^{M+1}\sum_{k\in R_i}(A,k)k^{a-1}|\alpha(k)|w(k) \le \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \tag{2.8}$$

say, where $\sum_3 := \sum\limits_{k\le v} k^a|\alpha(k)|$, and

$$\sum\nolimits_{1,2} := \sum_i \frac{x}{2^{i-1}}\sum_{k\in R_i}(A,k)k^{a-1}|\alpha(k)|, \tag{2.9}$$

the ranges of summation in (2.9) being respectively

$$1\le i\le N \text{ and } N+1\le i\le M$$

($\sum_3$ corresponding to $i = M+1$). By (1.2)

$$\sum\nolimits_3 = O(v^{a+1}) = O(u^{a+2}). \tag{2.10}$$

By (2.1) the inside sum on the right side of (2.9) is a O of

$$\sum_{d|A} d^a|\alpha(d)| \sum_{d^*\ge\max(1,\frac{v_i}{d})} d^{*a-1}\ |\alpha(d^*)|, \tag{2.11}$$

which in turn is, by (1.2), a O of

$$\sum_{d|A} d^a\ |\alpha(d)| = \alpha_a(A), \tag{2.12}$$

and, in the case of $\sum_1$, is by (2.2) a O of

$$\sum_{d|A} |\alpha(d)| v_i^a \leq v_N^a \alpha_0(A). \tag{2.13}$$

Thus

$$\sum\nolimits_2 = O\left(x\eta\alpha_a(A)\right) \tag{2.14}$$

and

$$\sum\nolimits_1 = O\left(xv_N^a \alpha_0(A)\right). \tag{2.15}$$

The theorem now follows from (2.6), (2.7), (2.8), (2.10), (2.14) and (2.15). □

3. - Proofs of Theorems 2 and 3

In this section we let h be a $G_{a,\ell}$ as in (1.4). Thus $z = \sqrt{x}$, $f = \psi_\ell$ and $\alpha(n) = 1$ for all n. A well known identify for Bernoulli polynomials [13, (6.1)] implies that

$$\frac{1}{k^*}\sum_{n\leq k^*} \psi_\ell\left(\frac{n}{k^*} + \frac{B}{k}\right) = \frac{1}{k^{*\ell}}\psi_\ell\left(\frac{B}{(A,k)}\right), \tag{3.1}$$

whence, if $B = O(A)$ and $A = o(x)$, an application of Theorem 1 yields

$$\begin{aligned} &\frac{1}{x}\sum_{n\leq x} G_{a,\ell}(An+B) \\ &= \sum_{k\leq u} (A,k)^\ell k^{a-\ell}\psi_\ell\left(\frac{B}{(A,k)}\right) + O\left[A(Ax)^{\frac{a}{2}} + A^a\sigma_0(A)\right] + o\left(\sigma_a(A)\right) \end{aligned} \tag{3.2}$$

and, with a special choice of the parameters A, B and x, the

LEMMA 1. *If* $-1 < a < 0$ *and*

$$A = \prod_{\substack{p\leq y \\ p\in P_\ell}} p = x^{-\frac{a}{2+a}},$$

where $P_\ell = \{p \equiv 1(2), p \text{ prime}\}$ *if* ℓ *is either* 1 *or even, and* $P_\ell = \{p \equiv 1(3), p \text{ prime}\}$ *if* $\ell > 1$ *is odd, then there are non negative numbers* $B_i < A$, $i =$

1, 2, *such that as* $y \to \infty$,

$$\frac{1}{x}\sum_{n\le x} G_{a,\ell}(An+B) = \begin{cases} \Omega_+(\sigma_a(A)), & \text{if } B = B_1 \\ \Omega_-(\sigma_a(A)), & \text{if } B = B_2 \end{cases} \tag{3.3}$$

PROOF. We shall make use of the following elementary properties of the Bernoulli polynomials (see [13], Chapter I).

$$\text{If } \ell > 1, \quad \begin{cases} \psi_\ell(0)\psi_\ell\left(\frac{1}{2}\right) < 0 & \text{when } \ell \text{ is even,} \\ \psi_\ell\left(\frac{1}{3}\right)\psi_\ell\left(\frac{2}{3}\right) < 0 & \text{when } \ell \text{ is odd.} \end{cases} \tag{3.4}$$

(i) With the choice of A and with $B = 0$ the right side of (3.2) becomes, for a certain set D of integers containing 1,

$$\psi_\ell(0)\sum_{d|A} d^a \sum_{\substack{d^*\le \frac{y}{d} \\ d^*\in D}} d^{*a-\ell} + o(\sigma_a(A)), \tag{3.5}$$

and we thus obtain the Ω_--estimate for $\ell = 1$, and one of the Ω-estimates for ℓ even.

(ii) Let $B = A - 1$ and $\ell = 1$. The right side of (3.2) becomes

$$\frac{1}{2}\sum_{k\le u}(A,k)k^{a-1} - \varsigma(1-a) + O(u^a) + o(\sigma_a(A)), \tag{3.6}$$

since

$$\psi_1\left(\frac{A-1}{(A,k)}\right) = \frac{1}{2} - \frac{1}{(A,k)}.$$

Whence the Ω_+-estimate in the case where $\ell = 1$.

(iii) With $B = A/2$ we obtain, by virtue of (3.4), the other Ω-estimate for ℓ even.

(iv) Finally, when $\ell > 1$ is odd, each one of the choices $B = A/3$, $B = 2A/3$ yields one of the Ω-estimates (again we use (3.4)). □

Now we need an estimate for $\sigma_a(A)$.

LEMMA 2. *For* $-1 < a < 0$ *and*

$$A = \prod_{\substack{p\le y \\ p\equiv k(n)}} p,$$

where $(k,n) = 1$, *we have*

$$\sigma_a(A) = \exp\left((1+o(1))\left\{\frac{\phi(n)^a}{a+1}\,\frac{(\log A)^{a+1}}{\log\log A}\right\}\right). \tag{3.7}$$

PROOF. For this choice of A we have

$$\log \sigma_a(A) = \sum_{\substack{p\le y\\ p\equiv k(n)}} \log(1+p^a)$$

$$= \sum_{\substack{p\le y\\ p\equiv k(n)}} p^a + \begin{cases} O(1), & -1 < a < -\frac{1}{2},\\ O(\log\log y), & a = -\frac{1}{2},\\ O(y^{2a+1}), & -\frac{1}{2} < a < 0.\end{cases} \tag{3.8}$$

Now the Euler summation formula and the prime number theorem for arithmetical progressions yield

$$\sum_{p\le y} p^a = (1+o(1))\, \frac{\phi(n)^a}{a+1}\, \frac{y^{a+1}}{\log y}, \tag{3.9}$$

and the lemma follows, after another application of the prime number theorem. □

Theorem 2 is now a direct consequence of Lemmata 1 and 2. □

As for theorem 3, it easily follows from Theorem 2, from [9, (1.3)]

$$E_a(x) = -x^a G_{-a,1}(x) - G_{a,1}(x) + O(x^{\frac{a}{2}}), \qquad |a| < 2, \tag{3.10}$$

and from [10, (5.4)]

$$\alpha_1(b) \le \frac{b}{2} + \frac{25}{76}, \qquad b > 0. \tag{3.11}$$ □

4. - Proof of Theorem 4

We shall need

LEMMA 3. *With the notation of Theorem 1, we have*

$$\sum_{n\le k^*} {\sin \atop \cos} \left(2\pi\left(\frac{n}{k^*} + \frac{B}{k}\right)\right) = \begin{cases} {\sin \atop \cos} \left(\frac{2\pi B}{k}\right) & \text{if } k^* = 1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

The proof of which is quite straightforward.

To prove (1.17) we set

$$A = \prod_{2<p\le y} p = x^{-\frac{a}{a+2}}, \quad B = 0, \quad z = \sqrt{x}, \tag{4.2}$$

and we obtain, with Theorem 1 and Lemma 3,

$$\frac{1}{x}\sum_{n\leq x} P_a(2\pi(An+B)) = \sum_{k|A} k^a + o(\sigma_a(A)) = \sigma_a(A)(1+o(1)) \tag{4.3}$$

which, with Lemma 2, implies the Ω_+-estimate; as for the Ω_--estimate, it is obtained similarly with $B = A/2$ instead of $B = 0$ in (4.2).

To prove the Ω_+-estimate in (1.18), we set

$$A = 4B = x^{-\frac{a}{a+2}}, \quad B = \prod_{\substack{p\leq y\\ p\equiv 1(4)}} p, \quad z = \sqrt{x}. \tag{4.4}$$

Theorem 1 and Lemma 3 yield this time

$$\frac{1}{x}\sum_{n\leq x} Q_a(2\pi B(4n+1)) = 4^a\sum_{k|B} k^a + o(\sigma_a(A)) = 4^a\sigma_a(B)(1+o(1)), \tag{4.5}$$

and we conclude again with Lemma 2. The Ω_--estimate is similarly obtained with

$$A = \frac{4B}{3} = x^{-\frac{a}{a+2}} = 4D, \qquad D = \prod_{\substack{p\leq y\\ p\equiv 1(4)}} p, \qquad z = \sqrt{x}. \qquad \square \tag{4.6}$$

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Université de Genève
Section de Mathématiques
2-4, rue du Lièvre
C.P. 240,
CH-1211 Genève 24