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Geometric Regularity Versus Analytic Regularity Higher Codimensional Case

E. AMAR - L. LEMPERT*

Introduction

In this work we deal with the following problem:
let Ω be a bounded pseudo-convex domain of $\mathbb{C}^n$, with smooth $\mathcal{C}^\infty$ boundary; let Y be a $\mathcal{C}^\infty$ smooth manifold defined in a neighbourhood of $\overline{\Omega}$; and suppose, finally that $Y \cap \Omega$ is an analytic submanifold of Ω; as can be seen all hypothesis are $\mathcal{C}^\infty$ smooth and the question is:

can $Y \cap \Omega$ be defined by functions holomorphic in Ω and smooth up to $\partial\Omega$?

Actually this question was asked by F. Forstneric.

In the (complex) codimension one case, a fairly complete answer was found by the first author [2]; but the case of higher codimension is much more involved because it does not seem possible to apply the same method as for the previous case; let us see why:

Using the implicit function theorem, we are led, locally, to the following situation: Y is defined by $u := (u_1, \dots, u_s) = 0$ where $u_i \in \mathcal{C}^\infty$ in a neighbourhood of $\overline{\Omega}$ and verify that:

i) $\overline{\partial} u_i$ is flat to infinite order on $Y \cap \overline{\Omega}$, $i = 1, \dots, s$

ii) $\partial u_1 \wedge \dots \wedge \partial u_s \not\equiv 0$ on $Y \cap \overline{\Omega}$.

In order to apply the codimension one method, we search for a matrix H such that:

iii) $v = H \cdot u$ verifies : $\overline{\partial} v = \overline{\partial} H \cdot u + H \cdot \overline{\partial} u \equiv 0$ in Ω

iv) H is invertible

in order to introduce no extra zeroes for v.

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We can rewrite the equation i) as: $(*)\overline{\partial}u = -B\cdot u$ and $(**)H^{-1}\overline{\partial}H = B$.

There are a lot of matrices B verifying $(*)$, but in order to verify $(**)$, B must satisfy the following compatibility condition:

$$(***) \qquad \overline{\partial}B = \overline{\partial}(H^{-1}\overline{\partial}H) = -H^{-1}\overline{\partial}H\ H^{-1}\wedge\overline{\partial}H = -B\wedge B.$$

Unfortunately the "easy" B associated to the problem does not satisfy $(***)$. It may be that the non commutative cohomology theory of A. Connes could be used to correct this B but we were not able to apply successfully this idea here.

So we have to use an other method:

a) find η_i such that η_i is flat at least up to second order on $Y\cap\Omega$ and $\overline{\partial}\eta_i = \overline{\partial}u_i$ in Ω, then $v_i = u_i - \eta_i$ is then holomorphic in Ω and we still have:

v) $v := (v_1,\dots,v_s) = 0$ on $Y\cap\Omega$

vi) $\partial v_i\wedge\ldots\wedge\partial v_s \neq 0$ on $Y\cap\overline{\Omega}$

because of vi), the v_i's have no extra zeroes than $Y\cap\Omega$ in a neighbourhood of $Y\cap\overline{\Omega}$; this is done in §2, but with a η_i only in $\mathcal{C}^k(\overline{\Omega})$ so v_i is also only in $\mathcal{C}^k$ and we want a $\mathcal{C}^\infty$ solution; but when we let k go to infinity, then the "good" neighbourhood where the v_i's have no extra zeroes may shrink to Y and we are left with nothing at the end. This is why we must use an approximation theorem done in §4 which relies on a division theorem done in §3. The end of the local result is performed via a Mittag-Leffler series, and this is done in §5.

Finally we get the following theorems:

THEOREM (semi-local): *Let Ω be a strictly pseudo-convex domain, Y be such that Y and $\partial\Omega$ are transversal, and verify* i) *and* ii) *above, then there are functions $f_1,\dots,f_s$, in $\mathcal{C}^\infty(\overline{\Omega})$, holomorphic in Ω such that:*

$$f_i = 0 \text{ on } Y\cap\Omega;\ i=1,\dots,s$$

$$\partial f_1\wedge\ldots\wedge\partial f_s \neq 0 \text{ on } Y\cap\overline{\Omega}.$$

THEOREM (global): *Let $\Omega\subset\mathbb{C}^n$ be a strictly pseudo-convex smoothly ($\mathcal{C}^\infty$) bounded domain, Y a smooth submanifold of a neighbourhood of $\overline{\Omega}$,* codim $Y = 2s$. *Assume that Y and $\partial\Omega$ are transversal and that $\Omega\cap Y$ is a complex submanifold X of Ω. Then there are functions $v_1,\dots,v_t\in A^\infty(\Omega)$ such that:*

$$\overline{X} = Y\cap\overline{\Omega} = \{z\in\overline{\Omega} : v_1(z) = \ldots = v_t(z) = 0\}$$

and for every point $z\in\overline{X}$ there are $v_{j_1},\dots v_{j_s}$ such that $\partial v_{j_1}\wedge\ldots\wedge\partial v_{j_s}(z)\neq 0$. One can always do with $t = n+1$.

Here $A^\infty(\Omega)$ is the set of holomorphic functions in $\Omega, \mathcal{C}^\infty$ smooth up to $\partial\Omega$.

The paper is organized this way:

In §1 we recall the notion of regular situation (in the Lojasiewicz sense), and we define a plurisubharmonic function in Ω.

In §2 we use the plurisubharmonic function defined in §1 to solve a $\bar{\partial}$ problem with bounds both by Hörmander's method and by Kohn's method in order to get the η_i's.

In §3 we state a new division theorem with $\mathcal{C}^k$ bounds we need in order to get both the local and the global theorem. This result may be of some interest in itself.

In §4 we prove a new approximation result also needed for our theorems.

In §5 we give the semi-global theorem and in §6, using a theorem by Grauert, the global one.

In the appendix a very simple proof of the fact that

$$H^1(F) = 0$$

is given, where F is an A^∞ vector bundle over a Stein manifold $\overline{X}$ with strictly pseudo-convex boundary. This proof can also be used in the $\mathcal{C}^k$ case where the other methods of sheaf theory break down.

The authors thank B. Berndtsson for interesting talks on this subject.

1. - Regular situation and flatness

DEFINITION 1. Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$ and A and B two closed sets in $\mathcal{U}$. We say that A and B are regularly separated in $\mathcal{U}$ of exponent $\alpha > 0$, for short A and B are $R.S.(\alpha)$, if:

$$\exists C > 0 \text{ s.t. } \forall x \in \mathcal{U},\ \ d(x, A) + d(x, B) \geq C d(x, A \cap B)^\alpha$$

DEFINITION 2. A and B are transversal if they are $RS(1)$.

If A is a closed set in $\mathbb{R}^n$, we denote by $\mathcal{E}^\ell(A)$ the set of all ℓ-differentiable functions on A in Whitney's sense [11].

LEMMA 1.1. *Let $f \in \mathcal{E}^\ell(A)$, $g \in \mathcal{E}^\ell(B)$ and $f - g = 0$ in $\mathcal{E}^\ell(A \cap B)$ (in the Whitney sense); if A and B are $R.S.(\alpha)$, then there is $h \in \mathcal{E}^k(A \cap B)$, with $k = [\ell/\alpha]$, such that: $h = f$ on A, $h = g$ on B.*

Here $[\ell/\alpha]$ is the integral part of ℓ/α.

PROOF. In [11] [p. 82] there is a function φ such that:

$$(1.1)\qquad \begin{cases} \varphi \in \mathcal{E}^\infty(\mathcal{U} \backslash A \cap B) \\ \varphi = 0 \text{ in a neighbourhood of } A \backslash A \cap B \\ \varphi = 1 \text{ in a neighbourhood of } B \backslash A \cap B \\ \forall a \in \mathbb{N}^n,\ \forall x \in \mathcal{U},\ |D^a \varphi(x)| \leq C d(x,\ A \cap B)^{-\alpha|a|} \end{cases}$$

So we put:

$$h' = f' + \varphi(g' - f') \tag{1.2}$$

Where f' and g' are Whitney extensions of f and g in $\mathcal{E}^{\ell}(\mathcal{U})$.
Because of the ℓ-flatness of $f - g$ we get $h' \in \mathcal{E}^{k}(\mathcal{U})$ with $k = [\ell/\alpha]$.
It remains to put:

$$h = h'|_{A \cup B} \text{ in the Whitney sense.} \tag{1.3}$$

Now let Ω' be a bounded domain of $\mathbb{C}^n$, then we have:

COROLLARY 1.1. *Let the manifold Y defined by $\{u_1 = \ldots = u_s = 0\}$ in a neighbourhood $\mathcal{U}$ of $\overline{\Omega}'$ and $\overline{\Omega}'$ be R.S.(α). If $\omega \in \mathcal{C}^{\ell}(\overline{\Omega}')$ with ω ℓ-flat on $Y \cap \Omega'$ then: $\omega/|u|^k$ is continuous on $\overline{\Omega}'$ with $k = [\ell/\alpha]$.*

PROOF. Let $A = \overline{\Omega}'$, $B = Y$, $f = \omega$ on A, $g = 0$ in $\mathcal{E}^{\ell}(B)$ then by the lemma 1.1 we get $h \in \mathcal{E}^{k}(A \cup B)$ such that $h = f$ on A and $h = g$ on B i.e. h is k-flat on Y with $k = [\ell/\alpha]$.

But Y is defined, as a manifold, by $u = (u_1, \ldots, u_s)$ so we have:

$$d(z, Y) \simeq |u(z)| := \left(\sum_{i=1}^{s} |u_i(z)|^2\right)^{1/2} \tag{1.4}$$

and

$$h'/|u|^k \text{ is continuous in } \mathcal{U} \tag{1.5}$$

Where h' is a Whitney extension of h in $\mathcal{U}$.

So by restriction to $\overline{\Omega}'$ we get the corollary.

Now we have:

LEMMA 1.2. *Let $u = (u_1, \ldots, u_s)$ with $u_i \in \mathcal{C}^{\ell+1}(\overline{\Omega}')$, $i = 1, 2, \ldots, s$ and $\bar{\partial} u_i$ ℓ-flat on $Y \cap \Omega'$. Let: $\varphi = \Lambda|z|^2 + \log|u|^2$. Then, if Y and $\overline{\Omega}'$ are R.S.(α), $\ell \geq 5\alpha$ and Λ big enough, φ is strictly pluri-subharmonic in a neighbourhood of $\overline{\Omega}'$.*

PROOF. Because of corollary 1.1 we get:

$$\text{with } k = [\ell/\alpha],\ \bar{\partial} u_i/|u|^k \in \mathcal{C}(\overline{\Omega}'). \tag{1.6}$$

By a simple calculus we get:

$$\partial\bar{\partial} \log|u|^2 = \sum_{i=1}^{s} \frac{\partial u_i \wedge \bar{\partial} u_i}{|u|^2} - \frac{(\sum u_i \partial u_i) \wedge (\sum u_i \bar{\partial} u_i)}{|u|^4} + E, \tag{1.7}$$

where E is $(k-4)$ flat on $Y \cap \Omega'$, because there is always a $\bar{\partial} u_i$ involved.

Clearly the sum of the 2 first terms in (1.7) is positive (it corresponds to the holomorphic case) so we have:

$$\partial\overline{\partial}\log|u|^2 = P + E \tag{1.8}$$

with P a positive (1,1) form and E a $(k-4)$ flat (1,1) form on $Y \cap \Omega'$.

So, with $\beta = \sum dz_i \wedge d\overline{z}_i$ we get:

$$\partial\overline{\partial}\varphi = \Lambda\beta + P + E \tag{1.9}$$

and, if Λ is big enough for absorbing the negativity of E we get the lemma.

2. - A solution of $\overline{\partial}\eta = \omega$ with bounds

In this paragraph, Ω and Ω', $\Omega' \supset \Omega$, will be pseudo-convex domains, bounded, with smooth $\mathcal{C}^\infty$ boundary;

$$\Omega = \{z \in \mathbb{C}^n,\ \ \rho(z) < 0\}$$

$$\Omega' = \{z \in \mathbb{C}^n,\ \ \rho'(z) < 0\}$$

with $\rho \in \mathcal{C}^\infty(\mathbb{C}^n)$, $\partial\rho \neq 0$ on $\partial\Omega$.

Y will be a smooth submanifold of a neighbourhood $\mathcal{U}$ of $\overline{\Omega}'$ in $\mathbb{C}^n$;

$$Y = \{u_1 = \ldots = u_s = 0\} \text{ with}$$

$$\begin{cases} u_i \in \mathcal{C}^\infty(\mathcal{U});\ \ i = 1, 2, \ldots, s. \\ \partial u_1 \wedge \ldots \wedge \partial u_s \neq 0 \text{ on } Y \\ \overline{\partial}u_i \text{ flat on } Y \cap \Omega \end{cases}$$

and such that

$$\begin{cases} Y \cap \Omega = Y \cap \Omega' \\ \forall z \in \Omega,\ \ d(z, \Omega'^c) \succeq d(z, Y)^m (\Rightarrow \Omega' \supset \overline{\Omega}\backslash Y). \end{cases}$$

The notation $a \preceq b$ means that a/b is bounded. The last assumption says that $\partial\Omega'$ is of finite order over $\partial\Omega$ along $Y \cap \Omega$.

Let Ω be a $(0,1)$ form in $\mathcal{C}^\ell_{(0,1)}(\overline{\Omega}')$, $\overline{\partial}\omega = 0$ in Ω' and such that:

$$\omega \text{ is } \ell - \text{flat on } Y \cap \overline{\Omega}' \tag{2.1}$$

If ℓ is big enough, $\ell > k\alpha$, the corollary 1.1 gives:

$$\omega/|u|^k \in \mathcal{C}(\overline{\Omega}') \subset L^2(\Omega',\ e^{-k\Lambda|z|^2}),\quad \text{i.e. } \omega \in L^2(\Omega',\ e^{-k\varphi}) \tag{2.2}$$

where φ is as in Lemma 1.2.

So we can use Hörmander's solution of $\overline{\partial}\eta = \omega$, [7]:

$$\exists \eta \in L^2(\Omega',\ e^{-k\varphi}) \text{ s.t. } \overline{\partial}\eta = \omega, \text{ and } \int_{\Omega'} |\eta|^2 e^{-k\varphi} \leq C N_k^2(\omega) \tag{2.3}$$

where C is independent of ω and:

$$N_k^2(\omega) := \int_{\Omega'} |\omega|^2 e^{-k\varphi} d\lambda, \text{ with } d\lambda \text{ Lebesgue's measure in } \mathbb{C}^n \tag{2.4}$$

because, by lemma 1.2, $k\varphi$ is strictly plurisubharmonic in Ω'.

On the other hand, by J. Kohn [8] we also have a solution v:

$$\exists v \in H^\ell(\Omega') \text{ s.t. } \overline{\partial}v = \omega, \text{ and } \|v\|_{H_\ell} \leq C\|\omega\|_{H^\ell} \tag{2.5}$$

Where C is independent of ω and H^ℓ is the Sobolev ℓ space of Ω'.

By the Sobolev imbedding theorem we have:

$$v \in \mathcal{C}^{\ell-n}(\overline{\Omega'}) \text{ and } \|v\|_{\ell-n} \leq C\|\omega\|_{H^\ell} \tag{2.6}$$

So we have:

$$h := v - \eta \text{ is holomorphic in } \Omega' \tag{2.7}$$

Now let $z \in \overline{\Omega}\backslash Y$; because $\partial\Omega'$ is of finite order over $\partial\Omega$ along $Y \cap \partial\Omega$, we have:

$$\exists r(z) \text{ s.t. } r(z) > d(z, Y)^m \text{ and } B(z, r(z)) \subset \Omega' \tag{2.8}$$

where $B = B(z, r)$ is the ball centered at z and of radius r.

So we have, h being holomorphic in Ω':

$$h(z) = \frac{1}{|B|}\int_B h(\varsigma)d\lambda(\varsigma) = \frac{1}{|B|}\int_B v(\varsigma)d\lambda - \frac{1}{|B|}\int_B \eta(\varsigma)d\lambda \tag{2.9}$$

where $|B|$ is the volume of $B = B(z, r(z))$.

From (2.9) we get:

$$\eta(z) = \frac{1}{|B|}\int_B \eta d\lambda + v(z) - \frac{1}{|B|}\int_B v d\lambda \tag{2.10}$$

but, by (2.8):

$$|B| \simeq r(z)^{2n} \succeq d(z, Y)^{2nm} \tag{2.11}$$

Where $a \simeq b$ means that both a/b and b/a are bounded.
So:

$$|\eta(z)| \preceq \int_B \frac{|\eta(\varsigma)|}{d(z,Y)^{2nm}} d\lambda + \left| v(z) - \frac{1}{|B|} \int_B v(\varsigma) d\lambda \right| \tag{2.12}$$

By Schwarz and (2.3):

$$\int_B \frac{|\eta|}{d(z,Y)^{2nm}} \preceq |B|^{1/2} \left[\int_{\Omega'} \frac{|\eta|^2}{|u|^{4nm}} d\lambda \right]^{1/2} \preceq |B|^{1/2} N_k(\omega) \tag{2.13}$$

as soon as $k \geq 2nm$, because:

$$d(z,Y) \simeq |u(z)| (\partial u_1 \wedge \ldots \wedge \partial u_s \neq 0 \text{ on } Y!)$$

and we have:

$$d(\varsigma, Y) \leq d(z,Y) + |\varsigma - z| \preceq d(z,Y) + d(z,Y)^m \preceq d(z,Y) \text{ if } \varsigma \in B(z, r(z));$$

hence: $|u(z)|^{-1} \preceq |u(\varsigma)|^{-1}$ and (2.13).
So we get:

$$|\eta(z)| \leq C_k d(z,Y)^{nm} N_k(\omega) + \Gamma(v,z) \tag{2.14}$$

Where:

$$\Gamma(v,z) := \left| v(z) - \frac{1}{|B|} \int_B v(\varsigma) d\lambda \right| \tag{2.15}$$

But if $v \in C^1(\overline{\Omega}')$, using Taylor expansion we easily get:

$$\Gamma(v,z) \preceq r(z) \|v\|_{C^1} \simeq d(z,Y)^m \|v\|_{C^1} \tag{2.16}$$

So, finally:

$$|\eta(z)| \leq C_k d(z,Y)^m \left[N_k(\omega) + \|\omega\|_{H^\ell(\Omega)} \right] \tag{2.17}$$

for $k \geq 2nm$ i.e. $\ell \geq 2nm$.
Exactly the same way we get:

$$\frac{\partial h}{\partial z_i}(z) = \frac{1}{|B|} \int_B \frac{\varsigma_i - z_i}{r^2} v d\lambda - \frac{1}{|B|} \int_B \frac{\varsigma_i - z_i}{r^2} \eta d\lambda = \frac{\partial v}{\partial z_i}(z) - \frac{\partial \eta}{\partial z_i}(z) \tag{2.18}$$

So expanding v up to second order we get again:

$$\left|\frac{\partial\eta}{\partial z_i}(z)\right| \leq C_k d(z,Y)^m \left[N_k(\omega) + \|\omega\|_{H^\ell(\Omega')}\right]. \tag{2.19}$$

We do the same for all holomorphic derivatives of η. For antiholomorphic (and mixed) derivatives we have, of course:

$$\overline{\partial}\eta = \omega \Rightarrow \overline{\partial}\eta \in \mathcal{C}^\ell(\overline{\Omega'}) \text{ and } \overline{\partial}\eta \text{ is } \ell-\text{flat on } Y \cap \overline{\Omega}. \tag{2.20}$$

So we have proved:

THEOREM 2.1. *Let Ω, Ω' and Y as above. Let $\omega \in \mathcal{C}^\ell_{(0,1)}(\overline{\Omega'})$, $\overline{\partial}\omega = 0$ in Ω' and ω ℓ-flat on $Y \cap \Omega$. Then there is a function η in $\mathcal{C}^k(\overline{\Omega})$, $\overline{\partial}\eta = \omega$ in Ω and:*

$$\forall a \in \mathbb{N}^n,\ |a| \leq k,\ |D^a\eta(z)| \preceq d(z,Y)^m[N_{\ell'}(\omega) + \|\omega\|_{H^\iota(\Omega')}]$$

Where: $\ell' = [\ell/\alpha]$ and $k = [\ell/2m\alpha] - n$.
In particular, η is k-flat on $Y \cap \Omega$.

REMARK 2.0. We get, more precisely:

$$\|\eta\|_{C^r(\Omega)} \preceq \left[N_{\ell'}(\omega) + \|\omega|_{H^t(\Omega')}\right]$$

with: $r = [t/2m\alpha] - n$, $\ell' = [t/\alpha]$ and $t \leq \ell$.

REMARK 2.1. We can replace $\partial u_1 \wedge \ldots \wedge \partial u_s \neq 0$ on $Y \cap \overline{\Omega}$ (which implies $d(z,Y) \simeq |u(z)|$) by: $|u|^2 \preceq d(z,Y)^\nu$ on $\overline{\Omega}$; the theorem will be still true but with a ℓ multiplied by a constant factor.

REMARK 2.2. If we know, a priori, that ω has a solution v in $\mathcal{C}^{\ell-n}(\overline{\Omega'})$ then Ω' needs not to have a smooth boundary and we have to replace $\|v\|_{H^l}$ by $\|v\|_{C^m}$ with $m = \ell - n$.
We can easily get a result on interpolation:

COROLLARY 2.1. *Let $f \in \mathcal{C}^{\ell+1}(\overline{\Omega'})$ with $\overline{\partial}f$ ℓ-flat on $Y \cap \Omega$, then there is a $F \in A^k(\Omega)$ such that the k-jet of F equals the k-jet of f on $Y \cap \Omega$.*

PROOF. $\exists\eta \in \mathcal{C}^k(\overline{\Omega})$ s.t. $\overline{\partial}\eta = \overline{\partial}f$ and η is k-flat on $Y \cap \overline{\Omega}$, so:

$$F := f - \eta \text{ is a solution.} \tag{2.21}$$

Let $A^k(\Omega)$ denote the set of holomorphic functions in Ω and in $\mathcal{C}^k(\Omega)$, as usual. Then we have:

THEOREM 2.2. *Let Ω, Ω', Y as above, then:*

for every $k \in \mathbb{N}$, for every $i \in \{1,2,\ldots,s\}$, there exists $v_i \in A^k(\Omega)$ s.t.:

i) $v_i = 0$ *on* $Y \cap \overline{\Omega}$

ii) $\partial v_i \wedge \ldots \wedge \partial v_s \neq 0$ *on* $Y \cap \overline{\Omega}$.

In particular there is a tubular neighbourhood $\mathcal{U}_k$ *of* $Y \cap \Omega$ *such that* $v = (v_1, \ldots, v_s)$ *defines* $Y \cap \Omega$ *as a holomorphic submanifold of* $\mathcal{U}_k$ *in* $A^k(\Omega \cap \mathcal{U}_k)$.

PROOF. For $i = 1, 2, \ldots, s$, let $\omega_i = \overline{\partial} u_i$, then for any $\ell \in \mathbb{N}$, ω_i is ℓ-flat so, for any $k \in \mathbb{N}$, we choose ℓ big enough in order to apply theorem 2.1; we get a $\eta_i \in \mathcal{C}^k(\Omega)$, k-flat on $Y \cap \Omega$ and we put:

$$v_i = u_i - \eta_i \tag{2.22}$$

to get a solution.

3. - A division theorem

In this paragraph, Ω will denote a strictly pseudo-convex (s.p.c.) bounded domain in $\mathbb{C}^n$ with smooth boundary; Y will be a $\mathcal{C}^\ell$ submanifold of a neighbourhood $\mathcal{U}$ of Ω such that:

$$\begin{cases} Y = \{v := (v_1, \ldots, v_s) = 0\} \text{ with } v_i \in \mathcal{C}^\ell(\mathcal{U}) \\ \partial v_1 \wedge \ldots \wedge \partial v_s \neq 0 \text{ on } Y \cap \Omega \\ v_i \text{ is holomorphic in } \Omega,\ i = 1, 2, \ldots, s. \end{cases} \tag{3.1}$$

Moreover we make the assumption:

$$Y \text{ and } \Omega \text{ are transversal.} \tag{3.2}$$

Now we can state, with $\mathcal{V}$ a neighbourhood of a point p in $Y \cap \partial\Omega$:

PROPOSITION 3.1. *Let* Ω *and* Y *as above, with* $v_i \in \mathcal{C}^\ell(\mathcal{U})$, *and* f *in* $\mathcal{C}^{m+1}(\Omega \cap \mathcal{V})$, *with* $\ell \geq m+s+5$, $\overline{\partial} f$ *m-flat on* $Y \cap \Omega \cap \mathcal{V}$, *then there are* $\mathcal{V}' \subset \mathcal{V}$, *a neighbourhood of* p, *and* $F \in A^k(\Omega \cap \mathcal{V}')$ *with* $k = m - (s-1)$, *such that* $F = f$ *on* $Y \cap \overline{\Omega}$.

Let us suppose first that $\operatorname{codim}_{\mathbb{C}} X = 1$, with $X := Y \cap \Omega$. Then we can put:

$$\omega := \overline{\partial} f / v \tag{3.3}$$

because $\{v = 0\}$ is transversal to $\partial\Omega$ then, by corollary 1.1:

$$\omega \in \mathcal{C}^m_{(0,1)}(\overline{\Omega} \cap \mathcal{V}), \text{ and of course, } \overline{\partial}\omega = 0. \tag{3.4}$$

By the existence of admissible neighbourhoods [1], we can find a open set $\mathcal{V}' \subset \mathcal{V}$, $\mathcal{V}'$ containing p, such that

(3.5) $\mathcal{V}' \cap \Omega$ is a s.p.c. domain with smooth $\mathcal{C}^\infty$ boundary.

Using the estimates of Lieb and Range [9] we get:

$$\exists g \in \mathcal{C}^{m+1/2}(\overline{\Omega} \cap \mathcal{V}') \text{ s.t. } \overline{\partial} g = \omega. \tag{3.6}$$

Now let

$$F = f - gv \in \mathcal{C}^m(\overline{\Omega} \cap \mathcal{V}') \tag{3.7}$$

then F fullfills all the requirements of proposition 3.1.

Now if $\operatorname{codim}_{\mathbb{C}} X \geq 2$, by a linear change in $(v_1, \dots, v_s)$ near p we can manage to have that:

(3.8) $\Omega_j := \{v_1 = \dots = v_j = 0\}$ is transversal to $\partial\Omega$, $j = 1, \dots, s$.

So we get that:

(3.9) Ω_j is a s.p.c. manifold near p with $\mathcal{C}^\ell$ boundary, $j = 1, \dots, s$.

Then we can apply recursively the $\operatorname{codim}_{\mathbb{C}}$ one solution:

$$\operatorname{codim}_{\mathbb{C}} X = 1 \text{ in } \Omega_{s-1}. \tag{3.10}$$

So we can extend f in neighborhood of p in Ω_{s-1}:

$$\exists F_1 \in A^m(\Omega_{s-1} \cap \mathcal{V}_1) \text{ s.t. } F_1 = f \text{ on } Y \cap \Omega \cap \mathcal{V}_1 \tag{3.11}$$

provided that $\ell \geq m + 5$, because we have to solve in a $\mathcal{C}^\ell$ domain instead of a $\mathcal{C}^\infty$ one [1].

So going on:

$$\exists F_2 \in A^{m-1}(\Omega_{s-2} \cap \mathcal{V}_2) \text{ s.t. } F_2 = F_1 \text{ on } \Omega_{s-1} \cap \mathcal{V}_2 \tag{3.12}$$

if $\ell \geq m + 6$.

So finally:

$$\exists F \in A^{m-(s-1)}(\Omega \cap \mathcal{V}_s) \text{ s.t. } F = f \text{ on } Y \cap \Omega \cap \mathcal{V}_s. \tag{3.13}$$

if $\ell \geq m + s + 5$.

And the proof of proposition 3.1.

Now we will need a division theorem in A^m:

THEOREM 3.1. *Under the same hypotheses on* Ω, Y, l *and* m, *if* h *is in* $A^m(\Omega)$ *and* $h = 0$ *on* $Y \cap \Omega$, *then* h *can be factorized in* $A^k(\Omega)$ *i.e.:*

$$\exists h_i \in A^k(\Omega) \text{ s.t. } h = \sum_{i=1}^{s} h_i v_i$$

with $k = m - s$.

PROOF OF THEOREM 3.1. Let

$$I := \{\text{germs of } f \in A^m(\Omega) \text{ s.t. } f = 0 \text{ on } X\} \tag{3.14}$$

and

$$J := \left\{\text{gems of } f \in A^m(\Omega) \text{ s.t. } \exists f_i \in A^{m-s}(\Omega) \text{ with } f = \sum_{i=1}^{s} f_i v_i\right\} \tag{3.15}$$

These are two sheaves on Ω and we have:

$$I = J \tag{3.16}$$

exactly as in [1], because the proof is based on the local extension (Proposition 3.1).

Again using the same method as in [1] we can write a Koszul complex as a presentation of J because $(v_1, \dots, v_s)$ is a regular sequence:

$$\dots \longrightarrow [A^{m-k_r}]^{s_r} \longrightarrow \dots \longrightarrow [A^{m-k_1}]^{s_1} \longrightarrow [A^{m-s}]^s \longrightarrow J \longrightarrow 0 \tag{3.17}$$

is exact.

Note that J is *not* coherent over A^m but is not too far from it.

So by cutting (3.17) into short exact sequences we get

$$\begin{cases} 0 \longrightarrow \mathcal{R}_1 \longrightarrow [A^{m-s}]^s \longrightarrow J \longrightarrow 0 \\ \dots\dots\dots \\ 0 \longrightarrow \mathcal{R}_r \longrightarrow [A^{m-k_r}]^s \longrightarrow \mathcal{R}_{r-1} \longrightarrow 0 \\ \dots\dots\dots \end{cases} \tag{3.18}$$

$$H^p(\Omega, J) \simeq H^{p+1}(\Omega, \mathcal{R}_1) \simeq \dots \simeq H^{p+r}(\Omega, \mathcal{R}_r) \simeq 0 \tag{3.19}$$

because if r is big enough $H^{p+r}(\Omega, \mathcal{F}) = 0$ by dimension theory and because:

$$H^q(\Omega, A^\ell) \simeq 0, \ \forall q \geq 1, \ \forall \ell \geq 1$$

by [6] (or [1] with the results of [8] for solving the $\bar\partial$ equation in $\mathcal{C}^m(\Omega)$ instead of [7] for solving it in $\mathcal{C}^\infty(\Omega)$).

In particular we have: $H^1(\Omega, J) = 0$ so:

$$\Gamma(\Omega, [A^{m-s}]^s) \longrightarrow \Gamma(\Omega, J) \longrightarrow 0 \text{ is exact} \tag{3.20}$$

and any section of J over Ω comes from a section of $[A^{m-s}]^s$ over Ω: this is exactly the theorem 3.2.

REMARK 3.1. As already noticed we cannot apply results on coherent sheaves of A^s-modules as in [6] because our sheaves are *not* coherent. For instance, in the unit ball of $\mathbb{C}^2$:

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2, \ |z| < 1\}$$

the function $f = z_1^4/(1 - z_2)$ is in $A^1(\mathbb{B})$ and 0 on $\{z_1 = 0\}$ but cannot be written $f = z_1 g$ with $f \in A^1(B)$!

More subtle related pathologies are true in $\mathbb{C}^n$, $n \geq 3$ [3]. For instance the intersection of two finitely generated ideals in $A^m(\mathbb{B})$ is not finitely generated.

4. - Approximation

We have the classical

PROPOSITION 4.1. *Let Ω a s.p.c. bounded domain in $\mathbb{C}^n$ with smooth boundary, then, $A^\infty(\Omega)$ is dense in $A^\ell(\Omega)$, $\forall \ell \in \mathbb{N}$.*

PROOF. Let $\epsilon > 0$ and let $\{\mathcal{U}_i, \ i = 1, \ldots, N\}$ be a finite covering of $\partial\Omega$ such that with z_i a fixed point in $\partial\Omega \cap \mathcal{U}_i$ and ν_i the exterior normal at z_i:

$$\forall z \in \mathcal{U}_i, \ d(z, \partial\Omega) < \epsilon, \ z - \epsilon\nu_i \in \mathcal{U}_i \cap \Omega \tag{4.1}$$

Then, with $h \in A^\ell(\Omega)$:

$$h_i(z) := h(z - \epsilon\nu_i) \in \mathcal{H}(\mathcal{U}_i) \tag{4.2}$$

Now let $\mathcal{U}_0$ be an open set, relatively compact in Ω and such that:

$$\mathcal{U}_0 \cup \{\mathcal{U}_i\} \supset \overline{\Omega}; \ h_0 := h \text{ on } \mathcal{U}_0;$$

and $\{\chi_i, \ i = 0, 1, \ldots\}$ a partition of unity relative to $\mathcal{U}_0, \mathcal{U}_1, \ldots$ with $\chi_i \in \mathcal{C}_0^\infty(\mathcal{U}_i)$, then:

$$h' := \sum_{i=0}^{N} \chi_i h_i \in \mathcal{C}^\infty(\overline{\Omega}'),$$

where Ω' is a s.p.c. domain such that:

$$\overline{\Omega} \subset \Omega' \subset\subset \bigcup_{i=0}^{N} \mathcal{U}_i, \quad \text{and } d(\partial\Omega', \Omega) < \epsilon. \tag{4.4}$$

Then:

$$\omega := \overline{\partial} h' = \sum_{i=0}^{N} h_i \overline{\partial} \chi_i \text{ is a global } (0,1) - \text{ form in } \Omega' \tag{4.5}$$

but we have:

$$\omega = \sum_{i=0}^{N} \overline{\partial} \chi_i (h_i - h) \tag{4.6}$$

In this formula h is extended to a $\mathcal{C}^\ell$ function in $\mathbb{C}^n$.
So $\omega \in \mathbb{C}^\infty(\Omega')$ but we also have:

$$||\omega||_{\mathcal{C}^\ell(\overline{\Omega}')} \preceq \delta, \text{ if } \epsilon \text{ is small enough} \tag{4.7}$$

because: with a small $\epsilon > 0$:

$$||h_i - h||_{\mathcal{C}^\ell(\overline{\Omega}' \cap \mathcal{U}_i)} < \frac{\delta}{N} \tag{4.8}$$

and N and χ_i are independent of ϵ! (only Ω' depends on ϵ).
Now we can solve $\overline{\partial}$ equation in Ω' with small bounds in $\mathcal{C}^{\ell+1/2}$:

$$\exists g \in \mathcal{C}^{\ell+1/2}(\Omega') \text{ s.t. } \overline{\partial} g = \omega \text{ and } ||g||_{\mathcal{C}^{\ell+1/2}(\overline{\Omega}')} \preceq \delta \tag{4.9}$$

But, because of the ellipticity of $\overline{\partial}$ on $(0,1)$ form we still get:

$$g \in \mathcal{C}^\infty(\Omega'), \tag{4.10}$$

so finally:

$$H := h' - g \in \mathcal{C}^\infty(\Omega') \tag{4.11}$$

verifies:

$$||H - h||_\ell \leq ||h - h'||_\ell + ||g||_\ell \tag{4.12}$$

$$h' - h = \sum \chi_i (h_i - h) \Rightarrow ||h' - h|| \leq \sup_i ||h_i - h|| \leq \delta \tag{4.13}$$

and the proposition 4.1.

Now we want to prove a stronger approximation result. Let Ω a s.p.c. domain with smooth C^∞ boundary and:

(4.14) $\Omega = \{\rho < 0\}$ with ρ strictly plurisubharmonic near $\partial\Omega$, $\partial\rho \neq 0$ on $\partial\Omega$.

Let X be a holomorphic submanifold of Ω, and $v = (v_1, \dots, v_s)$ be such that:

$$v_i \in A^\ell(\Omega),\ v = 0 \text{ on } X,\ \partial v_1 \wedge \dots \wedge \partial v_s \neq 0 \text{ on } X \cap \overline{\Omega} \tag{4.15}$$

and let us suppose that:

$$v_i \in A^\ell(\Omega'), \tag{4.16}$$

where Ω' is defined now by:

$$\Omega' = \left\{ \rho - \delta \left(\sum_{i=1}^{s} |v_i|^2 \right)^2 < 0 \right\} \tag{4.17}$$

for $\delta > 0$ small enough. This is not the Ω' as before!.

Here we have *not* that X is defined by v in Ω' nor in Ω; nevertheless we have:

THEOREM 4.1. *Let $h \in A^m(\Omega')$ with $h = 0$ on X, $v \in C^\ell(\Omega')$, and $\ell \geq 8(m+n+1)$. Then for any $\epsilon > 0$ there is a H in $A^{m+1}(\Omega)$ s.t.*

i) *$H = 0$ on X*

ii) *$\|H - h\|_{A^k(\Omega)} < \epsilon$ with $k = \left[\frac{m-s}{8}\right] - n$*

PROOF. Because of (4.15) there is a neighborhood $\mathcal{T}$ of $X \cap \overline{\Omega}'$ in $\overline{\Omega}'$ such that X is defined in $\mathcal{T}$ by v; moreover we can choose $\mathcal{T}$ to be strictly pseudo-convex with smooth boundary and we can apply theorem 3.1 to h in $\mathcal{T}$:

$$h = \sum_{i=1}^{s} h_i v_i,\ h_i \in A^{m-s}(\mathcal{T}). \tag{4.18}$$

Now we can approximate h_i in $\mathcal{T} \subset \Omega'$ by Proposition 4.1:

$$\forall \eta > 0,\ \exists h_i' \in A^\infty(\mathcal{T}),\ \exists h' \in A^\infty(\Omega'),\ \text{s.t. } \|h_i' - h_i\|_{C^{m-s}(\overline{\mathcal{T}})} < \eta \tag{4.19}$$

and:

$$\|h' - h\|_{C^m(\overline{\Omega}')} < \eta.$$

Now let:

$$g' = \sum_{i=1}^{s} v_i h'_i \in A^\ell(\mathcal{T}), \text{ because } v_i \in A^\ell(\Omega') \tag{4.20}$$

and the idea is to patch: g' near X with h' away from X; so let:

$$\chi \in \mathcal{C}^\infty(\mathbb{C}^n) \text{ s.t. } \chi \equiv 1 \text{ near } X \text{ and } \equiv 0 \text{ outside } \mathcal{T}. \tag{4.21}$$

(Remark: χ depends only on $\mathcal{T}$, not on the approximation.)
and put

$$g = \chi g' + (1-\chi)h' \in \mathcal{C}^\ell(\overline{\Omega}'). \tag{4.22}$$

We have again:

$$\|g-h\|_{\mathcal{C}^{m-s}(\overline{\Omega}')} \preceq \|g'-h\|_{\mathcal{C}^{m-s}(\overline{\mathcal{T}})} + \|h'-h\|_{\mathcal{C}^{m-s}(\overline{\Omega}')} \tag{4.23}$$

because: χ is independent of the approximation.
So, choosing well the $\eta > 0$ we get:

$$\|g-h\|_{\mathcal{C}^{m-s}(\overline{\Omega}')} < \epsilon. \tag{4.24}$$

Moreover we have:

$$\omega := \overline{\partial} g = (g'-h')\overline{\partial}\chi \tag{4.25}$$

and, clearly, ω is in $\mathcal{C}^\ell(\overline{\Omega}')$ but is small in $\mathcal{C}^{m-s}(\overline{\Omega}')$ and is ℓ-flat on X because here $\overline{\partial}\chi \equiv 0$.

So we can apply theorem 2.1: (the m of theorem 2.1 is 4 and $\alpha = 1$):

There is a $\eta' \in \mathcal{C}^{k'}(\overline{\Omega})$, with $k' = [\ell/8] - n$, $\overline{\partial}\eta' = \omega$ and η' is k'-flat on $X \cap \Omega$.

Moreover, from the remark 2.0, and the fact that

$$N_\ell(\omega) \preceq \|\omega\|_{L^\infty} \preceq \|\omega\|_{\mathcal{C}^{m-s}} (\overline{\partial}\chi \equiv 0 \text{ near } u - 0)$$

we have:

$$\|\eta'\|_{\mathcal{C}^k(\overline{\Omega})} < \|\omega\|_{\mathcal{C}^{m-s}(\overline{\Omega}')} \text{ with } k = \left[\frac{m-s}{8}\right] - n. \tag{4.26}$$

The point here being that k' depends only on ℓ but not on m.
So finally we have:

$$H = g - h \in A^{k'}(\Omega),\ H = 0 \text{ on } X \text{ because } \eta \text{ is } k'- \text{ flat on } X \tag{4.27}$$

and

$$\|H-h\|_k \leq \|g-h\|_k + \|\eta\|_k \text{ with } k=[(m-s)/8]-n. \tag{4.28}$$

So the theorem provided that $[\ell/8]-n \geq m+1$.

5. - The semi-global result

Let Ω be a strictly pseudo-convex domain in $\mathbb{C}^n$, bounded with $\mathcal{C}^\infty$ smooth boundary, $\Omega = \{\rho < 0\}$ with ρ a strictly plurisubharmonic defining function for Ω.

Let Y be a $\mathcal{C}^\infty$ submanifold of a neighbourhood $\mathcal{U}$ of $\overline{\Omega}$ such that:

$$\begin{cases} Y = \{u := (u_1, \dots, u_s) = 0\} \text{ with } u_i \in \mathcal{C}^\infty(\mathcal{U}),\ i = 1, \dots, s \\ \bar{\partial} u_i \text{ flat on } Y \cap \Omega \text{ and } \partial u_1 \wedge \dots \wedge \partial u_s \neq 0 \text{ on } Y \\ Y \text{ and } \partial\Omega \text{ transversal} \end{cases} \tag{5.1}$$

Let $\delta_k > 0\ \forall k$ and $\delta_k \searrow 0$ when $k \longrightarrow \infty$ and also:

$$\Omega_k := \left\{ \rho - \delta_k \left[\sum_{i=1}^{s} |u_i|^2 \right]^2 < 0 \right\} \tag{5.2}$$

Because ρ is strictly plurisubharmonic in $\mathcal{U}$, if δ_1 is small enough, Ω_k, for $k \geq 1$ is also strictly pseudo-convex with $\mathcal{C}^\infty$ smooth boundary, and Y and $\partial\Omega_k$ are transversal.

Now we have that for any $\ell \in \mathbb{N}$, there are $v_1, \dots, v_s$ in $A^\ell(\Omega_2)$ such that:

$$\begin{cases} \text{i)} & v := (v_1, \dots, v_s) = 0 & \text{on } Y \cap \Omega \\ \text{ii)} & \partial v_1 \wedge \dots \wedge \partial v_s \neq 0 & \text{on } Y \cap \overline{\Omega} \end{cases} \tag{5.3}$$

by Theorem 2.2.

REMARK 5.1. We have not $Y \cap \Omega_2 = \{v = 0\}$! This is only true in a tubular neighbourhood of $Y \cap \Omega$ because of ii).

Now let $\alpha(\ell) := [(\ell - s)/8] - n$ and choose ℓ such that $\alpha(\ell) \geq 2$ and let:

$$w^\ell := (w_1, \dots, w_s) \in \mathcal{C}^\ell, \text{ with } w_i \text{ given by (5.3)} \tag{5.4}$$

we have, of course:

$$w^\ell \in [A^\ell(\Omega_\ell)]^s;\ w^\ell = 0 \text{ on } Y \cap \Omega;\ \partial w_1^\ell \wedge \dots \wedge \partial w_s^\ell \neq 0 \text{ on } Y \cap \overline{\Omega}. \tag{5.5}$$

Now we apply theorem 4.1 to $h := w_i^\ell$, $i = 1, \dots, s$, with ϵ replaced by $\epsilon 2^{-\ell-1}$, and new functions:

$$v := (v_1, \dots, v_s) \in A^{\ell+n+1}(\Omega_\ell)$$

again given by theorem 2.2, so we get:

$$(5.6) \qquad \begin{cases} \exists w^{\ell+1} \in [A^{\ell+1}(\Omega_{\ell+1})]^s; \ w^{\ell+1} = 0 \text{ on } Y \cap \Omega, \text{ and:} \\ \|w^{\ell+1} - w^\ell\|_{C^{\alpha(\ell)}(\overline{\Omega}_{\ell+1})} < \epsilon 2^{-\ell-1} \end{cases}$$

Going on this way we get:

$$(5.7) \qquad \begin{cases} \exists w^{\ell+k+1} \in [A^{\ell+k+1}(\Omega_{\ell+k+1})]^s; \ w^{\ell+k+1} = 0 \text{ on } Y \cap \overline{\Omega}; \\ \|w^{\ell+k+1} - w^{\ell+k}\|_{C^{\alpha(\ell+k)}(\overline{\Omega}_{\ell+k+1})} < 2^{-(\ell+k+1)}\epsilon \end{cases}$$

Now we consider the Mittag-Leffler series:

$$(5.8) \qquad w = w^\ell + \sum_{k=\ell}^{+\infty} (w^{k+1} - w^k) = w^m + \sum_{k=m}^{+\infty} (w^{k+1} - w^k).$$

We have:

$$(5.9) \qquad w \in [A^\infty(\Omega)]^s; \ w = 0 \text{ on } Y \cap \Omega; \ \partial w_1 \wedge \dots \wedge \partial w_s \neq 0 \text{ on } Y \cap \overline{\Omega}$$

this latter assumption certainly holds if we choose ϵ small enough, for the $C^1(\overline{\Omega})$ norm of:

$$\left\| \sum_{\ell}^{\infty} (w^{k+1} - w^k) \right\|_{C^1(\overline{\Omega})} \leq \sum_{\ell}^{\infty} \|w^{k+1} - w^k\|_{C^{\alpha(k)}(\overline{\Omega})} \leq \epsilon$$

to be sure not to destroy (5.5): $\partial w_1^\ell \wedge \dots \wedge \partial w_s^\ell \neq 0$ on $Y \cap \overline{\Omega}$.

So we have proved:

THEOREM 5.1. *Let Ω a strictly pseudo-convex domain in $\mathbb{C}^n$, bounded, Y a smooth submanifold of a neighbourhood $\mathcal{U}$ of $\overline{\Omega}$, with $\partial\Omega$ and Y transversal, $Y = \{u_1 = \dots = u_s = 0\}$ with $u_i \in C^\ell(\mathcal{U})$, $\overline{\partial} u_i$ flat on $Y \cap \Omega$. Then there are $v_1, \dots, v_s$ in $A^\infty(\Omega)$ such that*

i) $v_i = 0$ *on* $Y \cap \overline{\Omega}$

ii) $\partial v_1 \wedge \dots \wedge \partial v_s \neq 0$ *on* $Y \cap \overline{\Omega}$.

COROLLARY 5.1. *(the local result). Let Ω be a strictly pseudo-convex domain in $\mathbb{C}^n$, bounded with C^∞ smooth boundary; let Y be C^∞ submanifold of $\mathbb{C}^n$, transversal to $\partial\Omega$ and such that $Y \cap \Omega$ is a complex analytic submanifold of Ω of pure codimension $s \geq 1$, then for any $p \in \partial\Omega \cap Y$ there is a neighbourhood $\mathcal{U}$ of p in $\mathbb{C}^n$ and $v_1, \dots, v_s$ in $A^\infty(\Omega \cap \mathcal{U})$ such that:*

i) $Y \cap \Omega \cap \mathcal{U} = \{v_1 = \dots = v_s = 0\}$

ii) $\partial v_1 \wedge \ldots \wedge \partial v_s \neq 0$ *on* $Y \cap \overline{\Omega} \cap \mathcal{U}$.

PROOF. The fact that $Y \cap \Omega$ is a holomorphic submanifold of Ω is equivalent to the fact that the tangent splace to Y at $z \in \Omega \cap Y$ is $\mathbb{C}$-linear; by continuity this is also true if $z \in Y \cap \partial\Omega$. So let $p \in Y \cap \partial\Omega$, by affine change of variables we can suppose that:

$$p = O \in \mathbb{C}^n, \ T_0 Y = \{z_1 = \ldots = z_s = 0\} \tag{5.10}$$

The implicit function theorem says that there is a neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}^n$ and $f_1, \ldots, f_s$ in $\mathbb{C}^\infty(\mathcal{U})$, such that:

$$Y \cap \mathcal{U} = \{z \in \mathcal{U} \text{ s.t. } z_i = f_i(z''), \ i = 1, \ldots, s\} \tag{5.11}$$

with $z'' = (z_{s+1}, \ldots, z_n)$.

Let us write the tangent space at $x \in Y \cap \mathcal{U}$ to Y:

$$z_j - x_j = \sum_{s+1}^{n} \frac{\partial f_j}{\partial z_k}(x'') \cdot (z_k x_k) + \sum_{s+1}^{n} \frac{\partial f_j}{\partial \overline{z}_k}(x'') \cdot (\overline{z}_k - \overline{x}_k), \ j = 1, \ldots, s. \tag{5.12}$$

But $T_x Y$ being $\mathbb{C}$ linear, if $(z - x) \in T_x Y \Rightarrow i(z - x) \in T_x Y$ this implies:

$$i(z_j - x_j) = i \sum \frac{\partial f_j}{\partial z_k}(x'')(z_k - x_k) - i \sum \frac{\partial f_j}{\partial \overline{z}_k}(x'')(\overline{z}_k - \overline{x}_k), \ j = 1, \ldots, s. \tag{5.13}$$

So adding (5.12) and (5.13) (after dividing by i):

$$\forall (z_k - x_k) \in \mathbb{C}, \ k = s+1, \ldots, n, \ \sum_{s+1}^{n} \frac{\partial f_j}{\partial \overline{z}_k} \cdot (z_k - x_k) = 0, \ \forall j = 1, \ldots, s. \tag{5.14}$$

So finally:

$$\forall k = s+1, \ldots, n, \ \forall j = 1, \ldots, s, \ \partial f_j / \partial \overline{z}_k(x'') = 0, \ \forall (x', x'') \in Y \cap \Omega \cap \mathcal{U}. \tag{5.15}$$

Let π the canonical projection: $\mathbb{C}^n \longrightarrow \mathbb{C}^{n-s}$ such that:

$$\pi(x) = x'' \tag{5.16}$$

then we have

$$f_j o \pi \in \mathcal{C}^\infty(\mathcal{U}) \text{ and is holomorphic in } \pi^{-1}(Y \cap \Omega \cap \mathcal{U}). \tag{5.17}$$

So let:

$$u_j = z_j - f_j, \ j = 1, \ldots, s \tag{5.18}$$

we get that u_j are in $\mathcal{C}^\infty(\mathcal{U})$, and holomorphic in $\pi^{-1}(Y \cap \Omega \cap \mathcal{U})$, but this set contains a neighbourhood, in Ω, of $Y \cap \Omega$ so:

$$\bar{\partial} u_j \text{ are flat on } Y \cap \Omega \cap \mathcal{U}, \quad j = 1, \ldots, s. \tag{5.19}$$

Moreover we have, by the existence of admissible neighbourhoods of 0 in $\mathbb{C}^n$ such that $\Omega' = \mathcal{V} \cap \Omega$ is strictly pseudo-convex with smooth boundary and now we can apply theorem 5.1 with Ω' and u_j to get the corollary.

6. - The global result

Our purpose in this paragraph is to prove:

THEOREM 6.1. *Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudo-convex smoothly ($\mathcal{C}^\infty$) bounded domain, Y a smooth submanifold of a neighbourhood of $\overline{\Omega}$,* $\operatorname{codim} Y =$ *$2t$. Assume that Y and $\partial\Omega$ are transverse and that $\Omega \cap Y$ is a complex submanifold X of Ω. Then there are functions $v_1, \ldots, v_s \in A^\infty(\overline{\Omega})$ such that:*

$$\overline{X} = Y \cap \overline{\Omega} = \{z \in \overline{\Omega} : v_1(z) = \ldots = v_s(z) = 0\}$$

and for every point $z \in \overline{X}$ there are $v_{j_1}, \ldots, v_{j_t}$ such that $\partial v_{j_1} \wedge \ldots \wedge \partial v_{j_t}(z) \neq 0$. One can always do with $s = n+1$.

REMARK 6.1. The theorem is true when, more generally, Ω is a subdomain of a Stein manifold M^n rather that $\mathbb{C}^n$. This general version can be reduced to Theorem 6.1. by embedding M in $\mathbb{C}^m$ with m sufficiently large, and constructing a s.p.c. domain $\Omega' \subset \mathbb{C}^m$ so that $M \cap \Omega' = \Omega$, and M and $\partial\Omega'$ are tranverse. Then the functions that define $\overline{X}$ in $\overline{\Omega}'$ restrict to functions on $\overline{\Omega}$ that define $\overline{X}$. By taking generic linear combinations of the defining functions one then shows that $n+1$ functions suffice even in the general case.

The proof of theorem 6.1. follows the proof of the local result. The most delicate point there seems to be the division theorem, where it is important to known that the normal bundle of the submanifold is trivial (see 3.1). This is clearly not true in general; to avoid the complications arising from this circumstance, we shall embed $\overline{\Omega}$ into a higher dimensional manifold E so that the normal bundle of $\overline{X}$ will be trivial.

Let us start with the construction of E.

Let TY denote the tangent bundle of Y, $TX = TY|_X$, $T\overline{X} = TY|_{\overline{X}}$. Introduce the normal bundles:

$$N_Y := N_{\mathbb{C}^n} Y := T\mathbb{C}^n_{|Y}/TY; \quad N_X := N_{\mathbb{C}^n} Y_{|X}; \quad N_{\overline{X}} := N_{\mathbb{C}^n}\overline{X} := N_{\mathbb{C}^n} Y_{|\overline{X}}$$

Then N_X is holomorphic and $N_{\overline{X}}$ is a complex vector bundle. In fact, the complex structure of $N_{\overline{X}}$ can be continued to N_Y, after an eventual shrinking of Y. In what follows, we shall treat N_Y as a complex vector bundle.

Choose now a neibourhood $\mathcal{U} \subset \mathbb{C}^n$ of $\overline{X}$ which is a domain of holomorphy and which can be (topologically) retracted to $\overline{X}$. Pull back the tangent bundle $T\overline{X}$ by this retraction, to obtain a topological (hence smooth) complex vector bundle $\pi : E \longrightarrow \mathcal{U}$. By Grauert's theorem [5], E carries a holomorphic structure. We can think of $\mathcal{U}$ (and $\Omega \cap \mathcal{U}$, and $\overline{X}$, Y) as submanifolds of E, embedded by the zero section.

PROPOSITION 6.1. *The normal bundle* $N_E\overline{X} = TE|_{\overline{X}}/T\overline{X}$ *is* A^∞*-trivial.*

PROOF: As smooth bundles we have:

$$N_E X \simeq N_E\mathcal{U}|_{\overline{X}} \oplus N_{\mathbb{C}^n}\overline{X} \simeq E|_{\overline{X}} \oplus N_{\mathbb{C}^n}\overline{X} \simeq T\overline{X} \oplus N_{\mathbb{C}^n}\overline{X} \simeq T\mathbb{C}^n|_{\overline{X}}. \tag{6.1}$$

Hence $N_E\overline{X}$ is $\mathcal{C}^\infty$-trivial. By a result of A. Sebbar this implies that it is A^∞ trivial too. (Sebbar in [10] proves his result for A^∞-bundles over strictly pseudo-convex smooth domain in $\mathbb{C}^n$; our base here is not a domain but a Stein manifold with strictly pseudo-convex boundary, nevertheless the same theorem holds in this situation as well with the same proof).

PROPOSITION 6.2. *There is a diffeomorphism* Φ *of class* $\mathcal{C}^\infty$ *from a neighbourhood of* $\overline{X}$ *in* N_EY *to a neighbourhood of* $\overline{X}$ *in* E *which is holomorphic on* N_EX.

PROOF. (These type of results are well known.)
From the exact sequence of sheaves (of A^∞-modules):

$$0 \longrightarrow T\overline{X} \longrightarrow TE|_{\overline{X}} \longrightarrow N_E\overline{X} \longrightarrow 0 \tag{6.2}$$

we infer:

$$\ldots \longrightarrow H^0(TE|_{\overline{X}}) \longrightarrow H^0(N_E\overline{X}) \longrightarrow H^1(T\overline{X}) \longrightarrow \ldots \tag{6.3}$$

Now $H^1(T\overline{X}) = 0$ (see the appendix) hence every A^∞ section of $N_E\overline{X}$ can be lifted to an A^∞ section of $TE|_{\overline{X}}$. Pick now $n = \dim E - \dim X$ A^∞-sections, $\sigma'_1, \sigma'_2, \ldots, \sigma'_n$, of $N_E\overline{X}$ that span $N_E\overline{X}$ at every point, and denote their lifting $\sigma_1, \sigma_2, \ldots, \sigma_n \in H^0(TE|_{\overline{X}})$. One can extend σ_i in a $\mathcal{C}^\infty$-fashion to sections of $TE|_Y$ (provided that Y is small). We shall keep the same notations for these extended sections.

Embed next E (which is a Stein manifold) into a euclidean space $\mathbb{C}^k$. A neighbourhood $G \subset \mathbb{C}^k$ of E can be holomorphically retracted on E (see [4]). Denote this retraction by r.

By the embedding, tangent vectors of E are identified with tangent vectors of $\mathbb{C}^k$; hence the sections $\sigma_1, \ldots, \sigma_n$ can be regarded as sections of the bundle $T\mathbb{C}^k|_Y \simeq Y \times \mathbb{C}^k$. Projecting on the second factor we obtain mappings $\hat{\sigma}_1, \ldots, \hat{\sigma}_n : Y \longrightarrow \mathbb{C}^k$. Define now a mapping Φ from a neighbourhood of $\overline{X}$ in N_EY to E in the following way:

if $z \in Y \subset \mathbb{C}^k$ and w is in the fiber $(N_E Y)_z$, define $t_1, \ldots, t_n \in \mathbb{C}$ by:

$$w = \sum_{j=1}^{n} t_j \sigma_j(z)$$

and put:

$$\Phi(w) := r\left(z + \sum_{j=1}^{n} t_j \cdot \hat{\sigma}_j(z)\right). \tag{6.4}$$

It is then straighforward to check that Φ (restricted to a small neighbourhood of $\overline{X}$) has the required properties.

PROPOSITION 6.3. *There are a neighbourhood $\mathcal{V}$ of $\overline{X}$ in E and $\mathcal{C}^\infty$-functions $u_1, \ldots, u_n$ on $\mathcal{V}$ such that:*

$$Y \cap \mathcal{V} = \{z \in \mathcal{V} : u_1(z) = \ldots = u_n(z) = 0\}$$

$$\partial u_1 \wedge \ldots \wedge \partial u_n(z) \neq 0 \textit{ if } z \in \mathcal{V}, \textit{ and}$$

$$\overline{\partial} u_1, \ldots, \overline{\partial} u_n \textit{ vanish on } \overline{X} \textit{ up to infinite order.}$$

PROOF. The dual bundle $N_E^* \overline{X}$ of $N_E \overline{X}$ is also trivial so that it is spanned by n A^∞-sections $u_1', \ldots, u_n'$. Extend these sections to $\mathcal{C}^\infty$-sections of $N_E^* Y$. Then u_i' can also be considered as functions on the total space $N_E Y$. Composing them with the inverse of the diffeomorphism Φ of Proposition 6.2. we obtain the desired functions $u_1, \ldots, u_n$.

Construct now a small, strictly pseudo-convex and smooth domain Ω_1 in $E|_{\overline{\Omega}}$ such that:

$$\overline{X} \subset \overline{\Omega}_1, \ \partial\Omega_1 \text{ and } \mathcal{U} \text{ are transverse,}$$

and in a neighbourhood of $\overline{X} \cap \partial\Omega$, Ω and $\Omega_1 \cap \mathcal{U}$ coincide.

With this Ω_1 replacing Ω, we are in the situation of Theorem 5.1; so, after an eventual shrinking of Ω_1 we have therefore:

PROPOSITION 6.4. *There are $v_1, \ldots, v_n \in A^\infty(\Omega_1)$ such that:*

$$\overline{X} = \{z \in \overline{\Omega}_1, \ v_1(z) = \ldots = v_n(z) = 0\}$$

and

$$\partial v_1 \wedge \ldots \wedge \partial v_n(z) \neq 0$$

The functions v_j thus constructed will be used for two purposes: first, starting with them, for every k we shall construct functions $w_i \in A^k(\Omega)$ that define X; second, they can be used to modify the proof of Theorem 4.1. to yield approximation results.

PROPOSITION 6.5. *There are functions* $w_1, \ldots, w_s \in A^k(\Omega)$ *such that:*

$$\overline{X} = \{z \in \overline{\Omega} : w_1(z) = \ldots = w_s(z) = 0\}$$

and for every $z \in \overline{X}$ *there are* $j_1, \ldots, j_t$ *such that:*

$$\partial w_{j_1} \wedge \ldots \wedge \partial w_{j_t}(z) \neq 0.$$

Here k is an arbitrary integer.

PROOF. Extend the function $\sum_{j=1}^{n} |v_j|_{\overline{\Omega}_1 \cap \overline{\Omega}}|^2$ to a smooth function $\psi \in C^\infty(\overline{\Omega})$ in such a way that ψ is positive off $\overline{X}$. Extend furthermore the functions $v_j|_{\overline{\Omega}_1 \cap \Omega}$ to smooth functions $u_j \in C^\infty(\overline{\Omega})$. Define a weight function:

$$\varphi(z) := \log \psi(z) + \Lambda |z|^2$$

with Λ sufficiently large to ensure that φ is p.s.h. Then, solving the equation:

$$\overline{\partial} \eta_j = \overline{\partial} u_j$$

in suitably weighted $L^2(\Omega,\ e^{-k'\varphi})$ spaces, k' large enough, we obtain exactly as in §2, functions $w_1, \ldots, w_n \in A^k(\overline{\Omega})$, $w_j = u_j - \eta_j$, vanishing on $\overline{X}$ and such that for every $z \in \overline{X}$, there are $j_1, \ldots, j_t$ so that: $\partial w_{j_1} \wedge \ldots \wedge \partial w_{j_t}(z) \neq 0$.

These n functions may, however, have common zeroes outside $\overline{X}$ (more precisely, outside a tubular neighbourhood of $\overline{X}$ in $\overline{\Omega}$.). To get rid of these additional zeroes, pick a point $z_0 \in \overline{\Omega} \backslash \overline{X}$ and construct a smooth continuation u of, say, $v_1|_{\overline{\Omega}_1 \cap \overline{\Omega}}$ which does not vanish in z_0. Introduce the weight function:

$$\varphi' := \log \psi(z) + \Lambda |z|^2 + \log |z - z_0|.$$

Solving now:

$$\overline{\partial} \eta = \overline{\partial} u$$

in $L^2(\Omega,\ e^{-k'\varphi'})$, we obtain $w = w_{z_0} = u - \eta \in A^k(\overline{\Omega})$ such that $w|_{\overline{X}} = 0$ but $w(z_0) \neq 0$. This of course also means that w does not vanish in a neighbourhood of z_0. Using the compactness of $\overline{\Omega}\backslash$tubular neighbourhood of $\overline{X}$, we can complete $w_1, \ldots, w_n$, by adding finitely many functions $w_{n+1}, \ldots w_s$, to a family of functions that has the required properties.

PROPOSITION 6.6. *In Proposition* 6.5. $s = n + 1$ *can always be assumed.*

PROOF. Suppose $s > n + 1$, and let $L : \mathbb{C}^s \longrightarrow \mathbb{C}^{n+1}$ be a linear mapping. Put $(w'_1, \ldots, w'_{n+1}) = L(w_1, \ldots, w_s)$; it is then easy to check that, for a generic L the functions w'_j will do.

An essential ingredient in the proof of the local result was Theorem 4.1. Its proof used the fact that $\overline{X}$ was a complete intersection (in a neighbourhood of $\overline{X}$). Nevertheless, the corresponding approximation result holds in our general context, too. Namely, it is possible to divide by $v_1, \ldots, v_n$ in the following way: let $h \in A^m(\Omega')$ (Ω' as slightly larger domain than Ω, as in §4). Then by the bundle projection $\pi : E \longrightarrow \mathcal{U}$ we can pull it back to a function $h' := h o \pi$ defined in a neighbourhood of $X \subset E$. Now $\overline{X}$ is a complete intersection in E, so that if $h|_{\overline{X}} = 0$, then $h' = \sum h'_i v_i$ with h'_i of class A^{m-s} on E near $\overline{X}$.

Restricting to Ω, we get $\sum h_i v_i$ on $\overline{\Omega}$ near $\overline{X}$. From here on, the proof of Theorem 4.1. is valid without any changes.

Hence the proof of §5 can also be repeated in the global case, which then concludes the proof of Theorem 6.1.

Appendix

Our intention here is to show how one can quickly prove $H^1(F) = 0$, where F is an A^∞ vector bundle over a Stein manifold $\overline{X}$ with strictly pseudo-convex boundary.

We shall represent $H^1(F)$ by the $\overline{\partial}$-Dolbeault cohomology. (One could also use the Cech-cohomology). Thus we have to prove that if ω is a $\overline{\partial}$-closed F-valued $(0,1)$-form of class $\mathcal{C}^\infty(\overline{X})$, then there is a section $u : \overline{X} \longrightarrow F$, of class $\mathcal{C}^\infty(\overline{X})$, such that $\omega = \overline{\partial} u$.

When F is trivial, this follows from Kohn's result (see [8]). As a matter of fact, Kohn there considers only domains in a Stein manifold, rather than Stein manifold with boundary; nevertheless the same proof yields this more general result.

We shall now show to reduce the case of a non trivial bundle F to that of a trivial (line-) bundle.

Let G denote the dual bundle of F, π, resp. ρ, the bundle projection in F, resp. in G.

Given a smooth F-valued k-form ω on $\overline{X}$, we shall associate with it a smooth $\mathbb{C}$-valued k-form $\Omega = T\omega$ on the total space of G in the following way.

Let $g \in G$, $V_1, \ldots, V_k \in T_g(G)$. Put:

$$\Omega(V_1 \otimes \ldots \otimes V_k) = < \omega(\rho_* V_1 \otimes \ldots \otimes \rho_* V_k),\ g >_{\rho(g)}$$

where $<, >_x$ denotes, for $x \in \overline{X}$ the duality between the fibers F_x, G_x.

PROPOSITION. *The transformation T preserves bigrading and commutes with $\overline{\partial}$. That is, if ω is an F-valued (p,q)-form, then $\Omega = T\omega$ is a (p,q)-form and $\overline{\partial}\Omega = T(\overline{\partial}\omega)$.*

PROOF. The statement is local, so that to verify it one can assume that F, hence also G, is trivial. In that case a straightforward computation in local coordinates proves the claim.

Next we shall define a left inverse of T acting on sections of F. Let U be a smooth function on G, or even only on a neighbourhood of X in G. Let us linearize it along the fibers of G. This means a function $U_\ell : G \longrightarrow \mathbb{C}$, linear on the fiber $G_x(x \in \overline{X})$, such that :

$$U(g) = U(x) + U_\ell(g) + o(|g - x|)$$

as $g \longrightarrow x$, $g \in G_x$. Here $|\cdot|$ stands for an arbitrary norm on G_x.

Now this linearized function U_ℓ corresponds to a section $u = tU$ of the bundle dual to G, i.e. F. Clearly, if $U = Tu$, then $u = tU$.

Let now a smooth, $\overline{\partial}$-closed F-valued $(0,1)$-form ω be given. By the Proposition, $\Omega = T\omega$ is also $\overline{\partial}$-closed on G. Construct a strictly pseudo-convex smooth domain $D \subset G$ so that $D \supset X$. By Kohn's theorem there is a function $U \in \mathcal{C}^\infty(\overline{D})$ such that $\overline{\partial} U = \Omega$. Put $u = tU$: u is clearly a smooth section of F. Furthermore, a straightforward computation in local coordinates shows that $\overline{\partial} u = \omega$. This proves that $H^1(F) = 0$.

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