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Algebraic Methods in the Theory of Theta Functions

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The functions of theta type were introduced for the first time in 1968 by I. Barsotti [1] as a generalization of the classical theta functions. This generalization consists in considering formal power series over an algebraically closed field k: a non-zero element $\vartheta(t) \in k[[t]]$ is called a theta type if

$$F(t_1, t_2, t_3) = \frac{\vartheta(t_1 + t_2 + t_3)\vartheta(t_1)\vartheta(t_2)\vartheta(t_3)}{\vartheta(t_1 + t_2)\vartheta(t_1 + t_3)\vartheta(t_2 + t_3)}$$

belongs to the quotient field of the tensor product over k, $k[[t_1]] \otimes k[[t_2]] \otimes k[[t_3]]$ (for a more detailed description see Section 1).

The first construction of theta types was strongly geometric and could not be generalized to characteristic $p > 0$. Only several years later (cfr. [2] and [7]) the true cohomological nature of F was discovered, and this allowed the direct construction of ϑ from the function F. The new technique, which is called the "F method", applies in quite different situations, and in particular in the case of positive characteristic.

More recently (cfr. [3]), the introduction of another function, called g, was proposed. This is simply a specialization of the function F, by means of which a simpler and more useful definition of theta types can be given; but the proof of this fact is once more geometric.

In this paper we propose first of all to develop the "g method" and to show that it is perfectly equivalent to the previous "F method", and finally to give an algebraic proof of the following fundamental result: the so called "prosthaferesis formula"

$$\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$$

is sufficient to define theta types ([3], Theorem 3.7).

We begin, in Section 1, by recalling some basic definitions and results on the theory of theta types; then, in Section 2, we introduce the function g and

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show that there exists a functional relation which is a necessary and sufficient condition for a power series $g(t_1, t_2)$ to split as

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}.$$

When g splits, we give a completely algebraic way to construct ϑ starting from g.

Finally, in Section 3, we show that the definition of theta type can be given in terms of the function g, thus proving the complete equivalence of the two methods. The proof we give here is almost completely algebraic: more precisely, we will show in a purely algebraic way that ϑ^2 is a theta type but, to conclude that also ϑ is a theta type, we must use a geometric argument, involving the group variety and the divisor of ϑ.

The section ends with some remarks on the hyperfield C of a theta type ϑ: more precisely, we show that C is finitely generated over k by the coefficients of the Taylor expansion of g, together with their first order partial derivatives.

1. - Preliminaries

We recall some basic facts on functions of theta type, refering the reader to the fundamental works of I. Barsotti [1] and [3] for an introduction and a detailed treatment of the subject.

Let k be an algebraically closed field of characteristic zero and $k[[t]]$, $t = (t^{(1)}, \dots, t^{(n)})$, the ring of formal power series in n variables over k. If I is an integral domain, we denote by $Q(I)$ its quotient field. A non-zero element $\vartheta(t) \in Q(k[[t]])$ is called a *function of theta type*, or simply a *theta type*, if the function

$$F(t_1, t_2, t_3) = \frac{\vartheta(t_1 + t_2 + t_3)\vartheta(t_1)\vartheta(t_2)\vartheta(t_3)}{\vartheta(t_1 + t_2)\vartheta(t_1 + t_3)\vartheta(t_2 + t_3)}$$

belongs to the quotient field of the tensor product over k, $k[[t_1]]\otimes k[[t_2]]\otimes k[[t_3]]$. Two theta types are *associate* if their ratio is a quadratic exponential, i.e. a factor of the form $c \exp q(t)$, where $c \in k$ and $q(t)$ is a polynomial of degree ≤ 2 with vanishing constant term. To a theta type ϑ, one can associate a hyperfield C in the following way: C is the smallest subfield of $Q(k[[t]])$, containing k, such that $F \in Q(C\otimes C\otimes C)$; the coproduct $\mathbf{P}$ of C is induced by the coproduct of $k[[t]]$,

$$\mathbf{P} : k[[t]] \longrightarrow k[[t]]\hat{\otimes}k[[t]] \cong k[[t, t']]$$

$$t^{(i)} \longrightarrow t^{(i)}\hat{\otimes}1 + 1\hat{\otimes}t^{(i)}$$

(for the definition of hyperfield, see the brief exposition in [1] or the more detailed treatment in [8]).

We define the *transcendency* of ϑ, in symbols transc ϑ, as transc (C/k) and the *dimension* of ϑ, dim ϑ, as the least positive integer m such that there exists a theta type θ, associate to ϑ, and linear combinations $u^{(1)}, \ldots, u^{(m)}$ of $t^{(1)}, \ldots, t^{(n)}$, with coefficients in k, such that $\theta(t) \in Q(k[[u]])$. We always have $\dim \vartheta \leq n$, and ϑ is called *degenerate* if $\dim \vartheta < n$. Moreover it is $\dim \vartheta \leq$ transc ϑ, and ϑ is a *theta function* if the equality holds.

A fundamental result, on the hyperfield C of a theta type ϑ, states that it is finitely generated over k by the logarithmic derivatives of ϑ from the seconds on, hence it is the function field $C = k(A)$ of a commutative group variety A over k, called the group variety of ϑ. By definition $F \in Q(C \otimes C \otimes C) = k(A \times A \times A)$, so it defines a divisor on $A \times A \times A$. It can be shown that there exists a unique divisor X on A such that the divisor of F on $A \times A \times A$ is

$$(p_1+p_2+p_3)^*X + p_1^*X + p_2^*X + p_3^*X(p_1+p_2)^*X - (p_1+p_3)^*X - (p_2+p_3)^*X,$$

where $p_i : A \times A \times A \to A$, denotes the i-th canonical projection, $i = 1, 2, 3$. This divisor X on A, which is automatically on $A - A_d$, where A_d denotes the degeneration locus of the group variety A, is the divisor of the theta type $\vartheta : X = \operatorname{div} \vartheta$. If ϑ and θ are associated theta types, they define the same hyperfield C, the same variety A and the same divisor X. Moreover the following properties hold: if $X = \operatorname{div} \vartheta_X$ and $Y = \operatorname{div} \vartheta_Y$, then $\operatorname{div}(\vartheta_X \vartheta_Y) = X + Y; X = 0$ if and only if $\vartheta_X = 1$ and $X \sim 0$ if and only if $\vartheta_X \in k(A)$, where all equalities between theta types are considered modulo substitution of a theta type with an associate one. It can also be shown that, if ϑ is non-degenerate, its divisor X has the property that $T_P^*X = X$ if and only if $P = 0$, the identity point of A, where $T_P : A \to A$ denotes translation by P, and a necessary and sufficient condition, for X to be an effective divisor, is that ϑ satisfy the following relation, called *holomorphic prosthaferesis*:

$$\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in k[[t_1]] \otimes k[[t_2]],$$

in this case we say that ϑ is a holomorphic theta type (if $k = \mathbb{C}$, the complex field, a holomorphic theta type is precisely an entire function).

To conclude, we just mention a result which explains the relationships between theta types and theta functions; it asserts that a theta type is just a theta whose arguments are replaced by "generic" linear combinations of fewer arguments, precisely:

THEOREM 1.1. *If $\vartheta(u) \in Q(k[[u_1, \ldots, u_n]])$ is a non-degenerate theta type, then there exists a non-degenerate theta $\theta(\nu) \in Q(k[[\nu_1, \ldots, \nu_m]])$ and elements $c_{ij} \in k$ $(i = 1, \ldots, m; j = 1, \ldots, n)$ such that $m \geq n$, the matrix (c_{ij}) has rank n and $\vartheta(u) = \theta(x_1, \ldots, x_m)$, where $x_i = \sum_j c_{ij}u_j$. The homomorphism of $k[[\nu]]$ onto $k[[u]]$, which sends ν_i to x_i, induces an isomorphism σ of C_θ into C_ϑ, such that $\sigma^*(\operatorname{div} \vartheta) = \operatorname{div} \theta$.*

Conversely if $\theta(\nu) \in Q(k[[\nu_1, \ldots, \nu_m]])$ is a non-degenerate theta and if the homomorphism just described, with rank $(c_{ij}) = n$, induces an isomorphism *of C_θ into $Q(k[[u]])$, then $\theta(x)$ is a non-degenerate theta type with hyperfield C_θ.*

2. - The function g

For a given $\vartheta(t) \in Q(k[[t]]), t = (t^{(1)}, \dots, t^{(n)}), \vartheta(t) \neq 0$, let us denote by g the following function

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}. \tag{2.1}$$

In the sequel we will always assume that $\vartheta(t) \in k[[t]]$ and $\vartheta(0) = 1$ (this is not restrictive if $\vartheta(0) \neq 0$, i.e. if ϑ is a unit in $k[[t]]$) and we will call such an element *normalized.* Under these hypotheses, we have $g(t_1, t_2) \in k[[t_1, t_2]], g(t_1, 0) = g(0, t_2) = 1$ and, in particular, we note that $g(t_1, t_2) = F(t_1, t_2, -t_2)^{-1}$.

By a simple calculation, we can check that g satisfies the following functional relation:

$$\begin{aligned} &g(t_1 + t_2, t_3 + t_4)g(t_1 - t_2, t_3 - t_4)g(t_1, t_2)^2 g(t_3, t_4)g(-t_3, t_4) = \\ &= g(t_1 + t_3, t_2 + t_4)g(t_1 - t_3, t_2 - t_4)g(t_1, t_3)^2 g(t_2, t_4)g(-t_2, t_4), \end{aligned} \tag{2.2}$$

which states the invariance of the left hand side under the mutual exchange of t_2 and t_3.

There are other properties of g which can be derived from (2.2): if we let $t_1 = t_2 = 0$, we get $g(-t_3, t_4) = g(-t_3, -t_4)$, which shows that g is an even function of the second variable; if we let $t_1 = t_4 = 0$, we have

$$g(t_2, t_3)g(-t_2, t_3) = g(t_3, t_2)g(-t_3, t_2),$$

and finally, letting $t_3 = 0$ and using the two preceding relations, we find another functional relation already pointed out by I. Barsotti in the introduction of [3]:

$$g(t_1 + t_2, t_4)g(t_1 - t_2, t_4)g(t_1, t_2)^2 = g(t_1, t_2 + t_4)g(t_1, t_2 - t_4)g(t_4, t_2)g(-t_4, t_2).$$

Now we come to the most important result of this section, i.e. to the proof that the relation (2.2) is not only necessary but also sufficient in order that a power series $g(t_1, t_2)$ splits as in (2.1).

THEOREM 2.3. *Let $g(t_1, t_2) \in k[[t_1, t_2]]$ satisfy* (2.2), *and suppose also that $g(t_1, 0) = g(0, t_2) = 1$. Then there exists a power series $\vartheta(t) \in k[[t]]$, uniquely determined up to multiplication by a quadratic exponential, such that* (2.1) *holds.*

PROOF. First of all we must introduce some notations. If $\mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ are multiindices and r is a positive integer, we let $\mu+\nu = (\mu_1+\nu_1,\ldots,\mu_n+\nu_n), r\mu = (r\mu_1,\ldots,r\mu_n)$, $|\mu| = \mu_1+\ldots+\mu_n$ and $\mu! = \mu_1!\ldots\mu_n!$; $\mu \leq \nu$ means $\mu_i \leq \nu_i$ for all i, and $\mu < \nu$ means $\mu_i \leq \nu_i$ but $\mu_j < \nu_j$ for some j. In the sequel ε_i will always denote the multiindex $(\delta_{1i},\ldots,\delta_{ni})$, where δ_{ij} is Kronecker's symbol. If $t=(t^{(1)},\ldots,t^{(n)})$, t^μ means $t^{(1)\mu_1}\cdot\ldots\cdot t^{(n)\mu_n}$, $\partial t^{(1)},\ldots,\partial t^{(n)}$ are the differentials of $t^{(1)},\ldots,t^{(n)}$ and d denotes derivation with respect to the variables t. When there are more than one set of variables, we use d_i to mean derivation with respect to the variables $t_i=(t_i^{(1)},\ldots,t_i^{(n)})$; more precisely, we let

$$\mathrm{d}_i^\mu = \frac{\partial^{|\mu|}}{\partial t_i^{(1)\mu_1}\ldots\partial t_i^{(n)\mu_n}}.$$

Let us start with $g(t_1,t_2)\in k[[t_1,t_2]]$ as in the statement of the theorem. The normalization of g assures us of the existence of $\log g(t_1,t_2)$ and from (2.2) it follows that g, and also $\log g$, is an even function of the second variable; so we can expand log g in a power series as follows:

$$\log g(t_1,t_2)=\sum_\mu A_\mu(t_1)t_2^\mu, \qquad A_\mu(t_1)\in k[[t_1]],\ A_\mu(0)=0,$$

where the sum is over all $\mu\in\mathbb{N}^n-\{0\}$ such that $|\mu|\equiv 0 \bmod 2$.

Let us consider the 1-forms

$$\omega_j=\frac{1}{2}A_{\varepsilon_1+\varepsilon_j}(t)\partial t^{(1)}+\ldots+A_{\varepsilon_j+\varepsilon_j}(t)\partial t^{(j)}+\ldots+\frac{1}{2}A_{\varepsilon_n+\varepsilon_j}(t)\partial t^{(n)},$$

for $j=1,\ldots,n$: we shall prove that they are closed.

In order for ω_j to be closed, we must have

$$\begin{aligned} &\mathrm{d}^{\varepsilon_r}\left(\tfrac{1}{2}A_{\varepsilon_s+\varepsilon_j}\right)=\mathrm{d}^{\varepsilon_s}\left(\tfrac{1}{2}A_{\varepsilon_r+\varepsilon_j}\right), && \text{if } s\neq j\neq r,\\ &\mathrm{d}^{\varepsilon_r}\left(A_{\varepsilon_j+\varepsilon_j}\right)=\mathrm{d}^{\varepsilon_j}\left(\tfrac{1}{2}A_{\varepsilon_r+\varepsilon_j}\right), && \text{if } j\neq r. \end{aligned} \tag{2.4}$$

To show this, we apply log to (2.2) and use the power series expansion of log g, getting

$$\begin{aligned} &\sum_\mu A_\mu(t_1+t_2)(t_3+t_4)^\mu+\sum_\mu A_\mu(t_1-t_2)(t_3-t_4)^\mu+2\sum_\mu A_\mu(t_1)t_2^\mu\\ &+\sum_\mu A_\mu(t_3)t_4^\mu+\sum_\mu A_\mu(-t_3)t_4^\mu=\sum_\mu A_\mu(t_1+t_3)(t_2+t_4)^\mu\\ &+\sum_\mu A_\mu(t_1-t_3)(t_2-t_4)^\mu+2\sum_\mu A_\mu(t_1)t_3^\mu\\ &+\sum_\mu A_\mu(t_2)t_4^\mu+\sum_\mu A_\mu(-t_2)t_4^\mu. \end{aligned} \tag{2.5}$$

Now if we apply $d_2^{\varepsilon_r} d_3^{\varepsilon_s} d_4^{\varepsilon_j}$ to (2.5) and let $t_2 = t_3 = t_4 = 0$, we easily obtain (2.4).

This proves that ω_j is closed, hence it is exact (remember we are in a ring of formal power series over a field of characteristic zero) and we can consider its integral η_j, normalized by letting $\eta_j(0) = 0$. Let $\varsigma = \sum_j \eta_j(t)\partial t^{(j)}$, where j ranges from 1 up to n: it follows immediately from the definition of η_j that ς is closed, so we can take its integral γ, again normalized by letting $\gamma(0) = 0$. Now let $\vartheta = \exp\gamma$: we claim this is the function we are looking for. We have only to show that

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)},$$

or equivalently:

$$\log g(t_1, t_2) = \gamma(t_1 + t_2) + \gamma(t_1 - t_2) - 2\gamma(t_1) - \gamma(t_2) - \gamma(-t_2). \tag{2.6}$$

Expanding the right hand side of (2.6) in a power series in t_2, we find

$$2 \sum_{\substack{\mu \neq 0 \\ |\mu| \equiv 0 \bmod 2}} \frac{1}{\mu!}(d^\mu\gamma(t_1) - d^\mu\gamma(0))t_2^\mu,$$

while the left hand side is simply

$$\log g(t_1, t_2) = \sum_{\substack{\mu \neq 0 \\ |\mu| \equiv 0 \bmod 2}} A_\mu(t_1)t_2^\mu.$$

This shows that (2.6) is equivalent to

$$A_\mu(t_1) = 2(\mu!)^{-1}(d^\mu\gamma(t_1) - d^\mu\gamma(0)), \qquad \text{for all } \mu \text{ s.t. } |\mu| \equiv 0 \bmod 2. \tag{2.7}$$

Now let us apply $d_3^\nu d_4^\lambda$ to (2.5) and let $t_2 = t_3 = t_4 = 0$, we get:

$$(\nu + \lambda)!A_{\nu+\lambda}(t_1) + (\nu + \lambda)!(-1)^{|\lambda|}A_{\nu+\lambda}(t_1) + \lambda!\, d^\nu A_\lambda(0) + \lambda!(-1)^{|\nu|}\, d^\nu A_\lambda(0)$$
$$= \lambda!\, d^\nu A_\lambda(t_1) + \lambda!(-1)^{|\lambda+\nu|}\, d^\nu A_\lambda(t_1).$$

From this, under the hypotheses $|\lambda| \equiv 0 \bmod 2, |\nu| \equiv 0 \bmod 2$ and $t = t_1$, we find

$$A_{\mu+\nu}(t) = \lambda!(\nu + \lambda)!^{-1}(d^\nu A_\lambda(t) - d^\nu A_\lambda(0)), \tag{2.8}$$

which becomes, by taking $\lambda = \varepsilon_i + \varepsilon_j, \nu = \mu - \varepsilon_i - \varepsilon_j$ with $i \neq j$,

$$A_\mu(t) = \frac{1}{\mu!}\left[d^{\mu-\varepsilon_i-\varepsilon_j}A_{\varepsilon_i+\varepsilon_j}(t) - d^{\mu-\varepsilon_i-\varepsilon_j}A_{\varepsilon_i+\varepsilon_j}(0)\right]. \tag{2.9}$$

This holds, however, only if μ_i and μ_j are both ≥ 1, otherwise, if there is only one $\mu_i \geq 2$ (recall that $|\mu|$ must be even), we must take $i = j$ and get

$$A_\mu(t) = \frac{2}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(0)\right]. \tag{2.10}$$

These two last relations are really meaningful. They show that all A_μ's are completely determined by $A_{\varepsilon_i+\varepsilon_j}$'s and give explicit formulas by which to construct them. Now recall that

$$\sum_{i=1}^{n} (\mathrm{d}^{\varepsilon_i}\gamma)\partial t^{(i)} = \partial\gamma = \varsigma = \sum_{i=1}^{n} \eta_i \partial t^{(i)},$$

i.e. $\mathrm{d}^{\varepsilon_i}\gamma = \eta_i$, from which it follows immediately that

$$\begin{aligned} \mathrm{d}^{\varepsilon_i+\varepsilon_j}\gamma &= \mathrm{d}^{\varepsilon_i}\eta_j = \tfrac{1}{2} A_{\varepsilon_i+\varepsilon_j}, \qquad \text{if } i \neq j, \\ \mathrm{d}^{\varepsilon_i+\varepsilon_i}\gamma &= \mathrm{d}^{\varepsilon_i}\eta_i = A_{\varepsilon_i+\varepsilon_i}. \end{aligned}$$

In order to prove (2.7), just substitute $A_{\varepsilon_i+\varepsilon_j}(t) = 2\,\mathrm{d}^{\varepsilon_i+\varepsilon_j}\gamma(t)$ in (2.9) or $A_{\varepsilon_i+\varepsilon_i}(t) = \mathrm{d}^{\varepsilon_i+\varepsilon_i}\gamma(t)$ in (2.10) according to whether there exist i, j with $i \neq j, \mu_i \geq 1$ and $\mu_j \geq 1$, or there is only one $\mu_i \geq 2$.

It is now straightforward to verify that any other solution of

$$g(t_1, t_2) = \frac{\vartheta(t_1 + t_2)\vartheta(t_1 - t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}$$

is of the form $c\ \exp(q(t))\vartheta(t)$, where $q(t)$ is a polynomial of degree ≤ 2 such that $q(0) = 0$ and c is a non-zero constant; the normalization of g then implies that $c = 1$ or $c = -1$. In the sequel, we shall always choose the normalization $\vartheta(0) = 1$. Q.E.D.

By now we have shown how to construct ϑ starting from g, then, using ϑ, we can also construct F; but we can find a more direct relation between the functions F and g.

Let us consider $\log F(t_1, t_2, t_3)$ and expand in power series, we find:

$$\log F(t_1, t_2, t_3) = \sum_{\mu,\nu} B_{\mu\nu}(t_1) t_2^\mu t_3^\nu, \qquad B_{\mu\nu}(t_1) \in k[[t_1]],\ B_{\mu\nu}(0) = 0,$$

the sum being performed over all multiindices $\mu, \nu \in \mathbb{N}^n\{0\}$.

We have already observed that $g(t_1, t_2) = F(t_1, t_2, -t_2)^{-1}$; from this, substituting the power series expansions of $\log F$ and $\log g$, by some simple calculations, we conclude that

$$A_\mu(t) = -\sum_{\substack{\alpha+\beta=\mu \\ \alpha,\beta\neq 0}} (-1)^{|\beta|} B_{\alpha\beta}(t), \tag{2.11}$$

which holds for $|\mu| \equiv 0 \bmod 2$.

We can also find an expression for the $B_{\mu\nu}$'s in terms of the A_μ's: from the proof of Theorem 2.3, we have

$$\mathrm{d}^{\varepsilon_i+\varepsilon_j} \log \vartheta(t) = \tfrac{1}{2} A_{\varepsilon_i+\varepsilon_j}(t), \qquad \text{if } i \neq j,$$
$$\mathrm{d}^{\varepsilon_i+\varepsilon_i} \log \vartheta(t) = A_{\varepsilon_i+\varepsilon_i}(t),$$

and also

$$A_\mu(t) = \frac{1}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_j} A_{\varepsilon_i+\varepsilon_j}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_j} A_{\varepsilon_i+\varepsilon_j}(0)\right], \qquad \text{if } i \neq j,$$
$$A_\mu(t) = \frac{2}{\mu!}\left[\mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(t) - \mathrm{d}^{\mu-\varepsilon_i-\varepsilon_i} A_{\varepsilon_i+\varepsilon_i}(0)\right].$$

With similar considerations, made on the function F, it can be shown that (cfr. [6], Theorem A.4):

$$\mathrm{d}^{\varepsilon_i+\varepsilon_j} \log \vartheta(t) = B_{\varepsilon_i\varepsilon_j}(t),$$

and

(2.12) $$B_{\mu\nu}(t) = \frac{1}{\mu!\nu!}\left[\mathrm{d}^{\mu+\nu-\varepsilon_i-\varepsilon_j} B_{\varepsilon_i\varepsilon_j}(t) - \mathrm{d}^{\mu+\nu-\varepsilon_i-\varepsilon_j} B_{\varepsilon_i\varepsilon_j}(0)\right].$$

From these relations it follows immediately that

(2.13) $$B_{\mu\nu}(t) = \frac{(\mu+\nu)!}{2\mu!\nu!} A_{\mu+\nu}(t), \qquad \text{if } |\mu+\nu| \equiv 0 \bmod 2.$$

Note that (2.13) holds under the restrictive condition $|\mu+\nu| \equiv 0 \bmod 2$; if we want to find an expression for $B_{\mu\nu}(t)$ in case $|\mu+\nu|$ is odd, we must use (2.12) (or other equivalent relations), and the derivatives of the A_μ's are also involved in such an expression.

3. - The prosthaferesis

For the sake of simplicity in this section we shall denote $(\mu!)^{-1}\,\mathrm{d}^\mu \log \varphi(t)$ by $\varphi_\mu(t)$, for every $\varphi(t) \in Q(k[[t]])$ and every multiindex $\mu > 0$. It can be shown that (cfr. [3], Section 3):

(3.1) $$(\mu!)^{-1}\,\mathrm{d}^\mu \varphi(t) = \varphi(t) Q_\mu(\varphi),$$

where the Q_μ's are polynomial functions with positive rational coefficients in the φ_ν's, $0 < \nu \leq \mu$. More precisely, we have:

LEMMA 3.2. *If $\varphi(t) \in Q(k[[t]])$ and $\mu = (\mu_1, \ldots, \mu_n)$ is a multiindex > 0 and if $\nu_1, \ldots, \nu_h$ are all multiindices with n components, such that $0 < \nu_i \leq \mu, i = 1, \ldots, h$, then*

$$Q_\mu(\varphi) = \sum_j (j!)^{-1} \varphi_{\nu_1}^{j_1} \cdots \varphi_{\nu_h}^{j_h},$$

where the sum is over all h-tuples $j = (j_1, \ldots, j_h)$ of non-negative integers, satisfying the condition $j_1\nu_1 + \ldots + j_h\nu_h = \mu$.

For the proof of this result see [3], Section 3.

We need one more lemma, which we cite without proof (cfr. [3], Lemma 3.3):

LEMMA 3.3. *Let $\varphi(t_1, t_2) \in k[[t_1, t_2]]$. If $\varphi(t_1, t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, the field generated over k by the derivatives $\mathrm{d}_2^\mu \varphi(t_1, 0)$ for all μ, is a finitely generated subfield $C_1 \subset Q(k[[t_1]])$. Analogously the field C_2, generated over k by the derivatives $\mathrm{d}_1^\nu \varphi(0, t_2)$, is a finitely generated subfield of $Q(k[[t_2]])$. C_1 is the smallest subfield C of $Q(k[[t_1]])$, containing k, such that $\varphi(t_1, t_2) \in Q(C[[t_2]])$, or equivalently such that $\varphi(t_1, t_2) \in Q(C \otimes Q(k[[t_2]]))$. Moreover we have $\varphi(t_1, t_2) \in Q(C_1 \otimes C_2)$.*

We can now prove the following

LEMMA 3.4. *Let $\vartheta(t) \in k[[t]]$ be a formal power series such that $\vartheta(0) = 1$. The following conditions are equivalent:*

i) $\vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$;

ii) *$g(t_1, t_2) \in Q(C \otimes C)$, where C is the subfield of $Q(k[[t]])$ generated over k by the logarithmic derivatives $\mathrm{d}^\mu \log \vartheta(t)$, for all μ such that $|\mu| \geq 2$.*

Moreover, under these hypotheses, C is a finitely generated hyperfield over k.

PROOF. That ii) $\Rightarrow$ i) is obvious; the hard part is to show that i) $\Rightarrow$ ii).

Let $\varsigma_i(t) = \mathrm{d}^{\varepsilon_i} \log \vartheta(t), i = 1, \ldots, n$. By applying $\mathrm{d}_1^{\varepsilon_i}$ log to i), we obtain

$$\varsigma_i(t_1 + t_2) + \varsigma_i(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]]),$$

while, if we apply $\mathrm{d}_2^{\varepsilon_i}$ log, we get

$$\varsigma_i(t_1 + t_2) - \varsigma_i(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]]);$$

from these relations it follows that $\varsigma_i(t_1 + t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, for $i = 1, \ldots, n$.

We are now under the hypotheses of Lemma 3.3, therefore there exists a subfield C of $Q(k[[t]])$ such that $\varsigma_i(t_1 + t_2) \in Q(C \otimes C)$. C is finitely generated over k by the derivatives of $\varsigma_i(t)$, i.e. by the derivatives $\mathrm{d}^\mu \log \vartheta(t)$ with $|\mu| \geq 2$, hence $\mathbf{P}(C)$ is generated by $\mathrm{d}^\mu \log \vartheta(t_1 + t_2)$, actually by a finite number of them. This shows that $\mathbf{P}(C) \subset Q(C \otimes C)$.

Let C' be the field generated over k by $\mathrm{d}^\mu \log \vartheta(-t), |\mu| \geq 2$, considered as functions of t: the same reasoning proves that $\mathbf{P}(C') \subset Q(C' \otimes C')$. Now let L be the smallest subfield of $Q(k[[t]])$ containing both C and C': we have $\mathbf{P}(L) \subset Q(L \otimes L)$ and also $\rho(L) \subset L$, where ρ denotes the inversion of $k[[t]]$, moreover L is the quotient field of $k[[t]] \cap L$, since $\mathrm{d}^\mu \log \vartheta(t)$ and $\mathrm{d}^\mu \log \vartheta(-t)$ are in $k[[t]]$. This sufficies to conclude that L is a finitely generated hyperfield over k (cfr. [1], Section 2). Now, from [1], Lemma 2.1, it follows that C is also a finitely generated hyperfield over k. To complete the proof we need only check that $g(t_1, t_2) \in Q(C \otimes C)$.

Let $\varphi(t_1, t_2) = \vartheta(t_1 + t_2)\vartheta(t_1 - t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$: from Lemma 3.3, it follows that $\varphi(t_1, t_2) \in Q(C_1 \otimes C_2)$, where C_1 and C_2 are the subfields of $Q(k[[t_1]])$ and $Q(k[[t_2]])$ generated over k by $\mathrm{d}_2^\mu \varphi(t_1, 0)$ and $\mathrm{d}_1^\nu \varphi(0, t_2)$ respectively.

Lemma 3.2 states that

$$(\mu!)^{-1}\, \mathrm{d}_2^\mu \varphi(t_1, t_2) = \varphi(t_1, t_2) Q_\mu(\varphi),$$

where the $Q_\mu(\varphi)$'s are polynomials in $\mathrm{d}_2^\nu \log \varphi(t_1, t_2)$, with $0 < \nu \leq \mu$, and recalling the definition of $\varphi(t_1, t_2)$, we can immediately check that

$$(\mu!)^{-1}\, \mathrm{d}_2^\mu \varphi(t_1, 0) = \vartheta(t_1)^2 Q'_\mu(\varphi),$$

where the $Q'_\mu(\varphi)$'s are obtained from the $Q_\mu(\varphi)$'s by replacing $\mathrm{d}_2^\nu \log \varphi(t_1, t_2)$ with $2\,\mathrm{d}^\nu \log \vartheta(t_1)$, if $|\nu|$ is even and with 0 if $|\nu|$ is odd. This shows that all $Q'_\mu(\varphi)$'s are elements of C, hence $\mathrm{d}_2^\mu \varphi(t_1, 0)$ is written as a product of $\vartheta(t_1)^2$ by an element of C.

In a similar way we have:

$$(\mu!)^{-1}\, \mathrm{d}_1^\mu \varphi(t_1, t_2) = \varphi(t_1, t_2) Q_\mu(\varphi),$$

where now the $Q_\mu(\varphi)$'s are polynomials in $\mathrm{d}_1^\nu \log \varphi(t_1, t_2)$, with $0 < \nu \leq \mu$, and we can easily prove that

$$(\mu!)^{-1}\, \mathrm{d}_1^\mu \varphi(0, t_2) = \vartheta(t_2)\vartheta(-t_2) Q'_\mu(\varphi),$$

where the $Q'_\mu(\varphi)$'s are obtained from the $Q_\mu(\varphi)$'s by replacing $\mathrm{d}_1^\nu \log \varphi(t_1, t_2)$ with $\mathrm{d}^\nu \log \vartheta(t_2) + \mathrm{d}^\nu \log \vartheta(-t_2)$. As before, these are all elements of C, except at most those with $|\nu| = 1$, i.e. $\varsigma_i(t_2) + \varsigma_i(-t_2)$; but recall that $\varsigma_i(t_1 + t_2) \in Q(C \otimes C)$, hence $\varsigma_i(t_1 + t_2) - \varsigma_i(t_1) - \varsigma_i(t_2) \in Q(C \otimes C)$, and if we let $t_1 = -t_2$ in this last expression, we find that $\varsigma_i(t_2) + \varsigma_i(-t_2) \in C$. Thus we have shown that $\mathrm{d}_1^\mu \varphi(0, t_2)$ is the product of $\vartheta(t_2)\vartheta(-t_2)$ by an element of C, therefore we can conclude that

$$g(t_1,t_2)=\frac{\vartheta(t_1+t_2)\vartheta(t_1-t_2)}{\vartheta^2(t_1)\vartheta(t_2)\vartheta(-t_2)}\in Q(C\otimes C),$$

Q.E.D.

Now we come to the main result of this section:

THEOREM 3.5. *Let $\vartheta(t)\in k[[t]]$ be a normalized power series (i.e. $\vartheta(0)=1$). $\vartheta(t)$ is a theta type if and only if it satisfies the prosthaferesis formula*

$$\vartheta(t_1+t_2)\vartheta(t_1-t_2)\in Q(k[[t_1]]\otimes k[[t_2]]).$$

PROOF. The necessity of this condition is straightforward: just recall that $g(t_1,t_2)=F(t_1,t_2,-t_2)^{-1}$ and ϑ is a theta type if $F(t_1,t_2,t_3)\in Q(k[[t_1]]\otimes k[[t_2]]\otimes k[[t_3]])$.

In order to prove that it is also sufficient, we recall that the prosthaferesis formula is equivalent, by Lemma 3.4, to the fact that $g(t_1,t_2)\in Q(C\otimes C)$, where C is a finitely generated hyperfield over k. From this, it follows immediately that

$$g(t_1+t_2,t_3)g(t_1,t_3)^{-1}g(t_2,t_3)^{-1}\in Q(C\otimes C\otimes C).$$

Recalling the definition of F, we can easily check that

$$g(t_1+t_2,t_3)g(t_1,t_3)^{-1}g(t_2,t_3)^{-1}=F(t_1,t_2,t_3)F(t_1,t_2,-t_3),$$

hence

$$F(t_1,t_2,t_3)F(t_1,t_2,-t_3)\in Q(C\otimes C\otimes C). \tag{3.6}$$

In the same way, using

$$g(t_1+t_3,t_2)g(t_1,t_2)^{-1}g(t_3,t_2)^{-1}\in Q(C\otimes C\otimes C)$$

and

$$g(t_1,t_2+t_3)g(t_1,t_2)^{-1}g(t_1,t_3)^{-1}\in Q(C\otimes C\otimes C),$$

we get respectively

$$F(t_1,t_2,t_3)F(t_1,-t_2,t_3)\in Q(C\otimes C\otimes C) \tag{3.7}$$

and

$$F(t_1,t_2,t_3)F(t_1,-t_2,-t_3)\in Q(C\otimes C\otimes C). \tag{3.8}$$

Now, if we divide (3.6) by (3.8), we find that

$$F(t_1,t_2,-t_3)F(t_1,-t_2,-t_3)^{-1}\in Q(C\otimes C\otimes C),$$

i.e.

$$F(t_1, t_2, t_3)F(t_1, -t_2, t_3)^{-1} \in Q(C \otimes C \otimes C),$$

and multiplying this last relation by (3.7), we finally get

$$F(t_1, t_2, t_3)^2 \in Q(C \otimes C \otimes C),$$

which proves that $\vartheta^2(t)$ is a theta type.

To show that $\vartheta(t)$ is also a theta type, we recall that C is a finitely generated hyperfield over k, i.e. it is the function field of a group variety A over k, hence $\vartheta^2(t)$, being a theta type, has a divisor X on A.

But we have shown that $g(t_1, t_2) \in Q(C \otimes C)$, so it defines a divisor Y on $A \times A$, and

$$g(t_1, t_2)^2 = \frac{\vartheta^2(t_1 + t_2)\vartheta^2(t_1 - t_2)}{\vartheta^4(t_1)\vartheta^2(t_2)\vartheta^2(-t_2)},$$

hence we must have:

$$2Y = (p_1 + p_2)^*X + (p_1 - p_2)^*X - 2p_1^*X - p_2^*X - (-p_2)^*X,$$

where $p_i : A \times A \to A$, denotes the i-th canonical projection, $i = 1, 2$. This implies that $X = 2V$, for some divisor V on A.

Let $\vartheta_V(u)$ be the non-degenerate theta function of the divisor V (see [1]), $\vartheta_V(u) \in Q(k[[u]]) = Q(k[[u_1, \ldots, u_m]])$, where $k[[u_1, \ldots, u_m]]$ is the completion of the local ring of the identity point of A. We know that C is embedded in $Q(k[[u]])$, but also $C \subset Q(k[[t]])$; this gives a homomorphism

$$\sigma : k[[u]] \to k[[t]],$$

which induces an isomorphism on the hyperfields, $C \cong C$.

From $X = 2V$, it follows that $\vartheta^2(t)$ is associated to $\vartheta_V(\sigma u)^2$, hence $\vartheta(t)$ is associated to $\vartheta_V(\sigma u)$. Now use Theorem 1.1 to conclude that $\vartheta(t)$ is a theta type. Q.E.D.

We end this section with a remark on the hyperfield C. Let us recall that the hyperfield C of a theta type ϑ is the smallest subfield of $Q(k[[t]])$, containing k, such that $F \in Q(C \otimes C \otimes C)$. It can be shown that such a C exists, and is generated over k by $\mathrm{d}^{\mu} \log \vartheta(t)$, with $|\mu| \geq 2$. At this point, we may ask what are the relationships between the hyperfield C and the function g. The answer is given by the following

PROPOSITION 3.9. *Let $g(t_1, t_2) \in k[[t_1, t_2]]$ and $\vartheta(t) \in k[[t]]$ be as in the statement of* Theorem 2.3. *Consider the power series expansion of g:*

$$g(t_1, t_2) = 1 + \sum_{\mu} D_{\mu}(t_1)t_2^{\mu}, \qquad D_{\mu}(t_1) \in k[[t_1]], \ D_{\mu}(0) = 0.$$

Then the fields C, generated over k by $\mathrm{d}^{\mu}\log\vartheta(t)$, with $|\mu| \geq 2$, and C', generated over k by $D_\mu(t)$ and $\mathrm{d}^{\varepsilon_i}D_\mu(t)$, for every $\mu \neq 0$ with $|\mu| \equiv 0 \bmod 2$ and $i = 1, \dots, n$, coincide.

Moreover if ϑ is a theta type, i.e. if $g(t_1, t_2) \in Q(k[[t_1]] \otimes k[[t_2]])$, then $C = C'$ is a finitely generated hyperfield over k, with the coproduct $\mathbf{P}$ and the inversion ρ induced by those of $k[[t]]$.

PROOF. Let $\log g(t_1,t_2) = \sum_\mu A_\mu(t_1)t_2^\mu$, where the sum is over all $\mu \in \mathbb{N}^n - \{0\}$, with $|\mu| \equiv 0 \bmod 2$. From the proof of Theorem 2.3, we know that

$$A_\mu(t) = \frac{2}{\mu!}\left[\mathrm{d}^\mu \log\vartheta(t) - \mathrm{d}^\mu\log\vartheta(0)\right], \qquad |\mu| \equiv 0 \bmod 2.$$

Therefore it is clear that the fields $k(A_\mu(t), \mathrm{d}^{\varepsilon_i}A_\mu(t))$, where $\mu \in \mathbb{N}^n - \{0\}, |\mu| \equiv 0 \bmod 2$ and $i = 1, \dots, n$, and $k(\mathrm{d}^\nu \log\vartheta(t))$ where $|\nu| \geq 2$, are equal. Thus we have only to show that $k(A_\mu(t), \mathrm{d}^{\varepsilon_i}A_\mu(t)) = k(D_\mu(t), \mathrm{d}^{\varepsilon_i}D_\mu(t))$.

Let $\varphi(t_1,t_2) = \sum_\mu D_\mu(t_1)t_2^\mu$, hence $g(t_1,t_2) = 1 + \varphi(t_1,t_2)$ and

$$\log g(t_1,t_2) = \varphi(t_1,t_2) - \frac{1}{2}\varphi(t_1,t_2)^2 + \frac{1}{3}\varphi(t_1,t_2)^3 - \dots.$$

Now if we substitute the power series expansion of $\varphi(t_1,t_2)^n$ and compare with that of $\log g(t_1,t_2)$, we can easily conclude that

$$A_\mu(t) = D_\mu(t) + (\text{poly. in } D_\nu(t), \text{ with } \nu < \mu).$$

In a similar way, letting $\Psi(t_1,t_2) = \sum_\mu A_\mu(t_1)t_2^\mu = \log g(t_1,t_2)$, we have

$$g(t_1,t_2) = \exp\Psi(t_1,t_2) = 1 + \Psi(t_1,t_2) + \frac{1}{2!}\Psi(t_1,t_2)^2 + \dots$$

and finally

$$D_\mu(t) = A_\mu(t) + (\text{poly. in } A_\nu(t), \text{ with } \nu < \mu).$$

This proves what we wanted. The last statement of the proposition, being included in Lemma 3.4, is now obvious. Q.E.D.

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