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Integral formulas for the $\bar{\partial}$-equation on complex projective algebraic manifolds

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Introduction

This paper deals primarily with integral formulas for the $\bar{\partial}$-equation on domains in algebraic submanifolds of the complex projective space $\mathbb{P}^n$ which are complete intersections. We consider p homogeneous polynomials $h_i = h_i(z_0, \ldots, z_n)$, $i = 1, \ldots, p$ ($(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ are homogeneous coordinates on $\mathbb{P}^n$) and we let

$$M = \{[z] \in \mathbb{P}^n : h_i(z) = 0, i = 1, \ldots, p\}.$$

We assume that $dh_1 \wedge \ldots \wedge dh_p \neq 0$ on M so that M is a smooth manifold. Suppose $D \subset M$ is a domain on M with smooth boundary ∂D. If $U_j = \{[z] : z_j \neq 0\}$, $0 \leq j \leq n$, is the standard cover of $\mathbb{P}^n$, let $O(\ell)$ be the line bundle with transition functions

$$g_{jk} : U_j \cap U_k \to \mathbb{C},$$
$$g_{jk}([z]) = \left(\frac{z_k}{z_j}\right)^\ell,$$

ℓ being an integer. Let $C_{(0,q)}(\overline{D}, O(\ell))$ denote the set of $(0,q)$-forms whose coefficients are continuous sections of $O(\ell)$ over $\overline{D}$. The version of the $\bar{\partial}$-equation, we are concerned with, is the following: given an $f \in C_{(0,q)}(\overline{D}, O(\ell))$, with $\bar{\partial} f = 0$ (in the sense of distributions), find $g \in C_{(0,q-1)}$ $(\overline{D}, O(\ell))$ so that $\bar{\partial} g = f$. If the domain D is s-pseudoconcave (see §II.3 for the definition), then we show, in a constructive manner, that the above $\bar{\partial}$-equation is solvable if $q \leq s-1$ and $\ell < 0$. In fact an explicit solution is obtained. If $q = s$ and $\ell < 0$, then we obtain a necessary and sufficient condition for the solvability of that $\bar{\partial}$-equation. This follows from the integral formula of Theorem 3 which is one of the main results of the paper. Theorem 2 is, in a sense, a more general

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integral representation formula for differential forms from $C_{(0,q)}(\overline{D}, O(\ell))$; on it (in fact on its proof) the proof of Theorem 3 is based. Theorems 2 and 3 are our main results. The method we use, in order to prove Theorem 2, is via the $\overline{\partial}_\tau$-equation (the tangential Cauchy-Riemann equations) on $\pi^{-1}(D) \cap S^{2n+1}$, where $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ is the natural projection and S^{2n+1} is the unit sphere of $\mathbb{C}^{n+1}$, that is $S^{2n+1} = \{\varsigma \in \mathbb{C}^{n+1} : |\varsigma| = 1\}$. The map π is used to bring results back and forth between S^{2n+1} and $\mathbb{P}^n$ and between appropriate submanifolds of domains on them. The study of the $\overline{\partial}_\tau$-equation is carried out in part I; in fact we work in a slightly more general setting in which the sphere S^{2n+1} is replaced by a strictly convex hypersurface $\{\varsigma \in \mathbb{C}^{n+1} : \rho(\varsigma) = 0\}$ and the homogeneous polynomials $h_1, \dots, h_p$ are replaced by holomorphic functions is some neighbourhood of the hypersurface (there is no complication, at this point, to work in this setting; in fact we could have worked with a strictly pseudoconvex hypersurface in place of S^{2n+1}). The main result of part I is theorem 1 which is an integral formula for the $\overline{\partial}_\tau$-equation; this formula is, of course, of independent interest. Let us point out that some other versions of the above results are possible to obtain by replacing the space of $C_{(0,q)}$-forms by other spaces of forms, for example $L^1_{(0,q)}$-forms or spaces of forms with measure coefficients; in fact one can even prove estimates for the operators involved in various norms; we found, however, the space of continuous forms a reasonable space to carry out the constructions and to explain the main ideas. Finally we point out that the presentation in this paper has been influenced by the papers of Henkin [5] and Henkin-Polyakov [6] (where we also refer for background material as well as for references on related topics); we think, however, that this paper is justified by its main results, theorems 1 and 2 and especially theorem 3 which generalizes [6, theorem 2.2, page 562] from concave domains in $\mathbb{P}^n$ to pseudoconcave domains on complete intersection algebraic submanifolds of $\mathbb{P}^n$.

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Part I: The $\overline{\partial}_\tau$-equation

I.1. - Notation

This notation is to be used throughout part I. Let ρ be a strictly convex function and let $\Omega \subset \mathbb{C}^n$ be a neighbourhood of the strictly convex hypersurface $\{\rho = 0\}$. Let $h = (h_1, \dots, h_p) : \Omega \to \mathbb{C}^p$ be a holomorphic map, $p < n$, and let

$h_{ij}(\varsigma, z)$ be holomorphic functions in $(\varsigma, z) \in \Omega \times \Omega$, so that

$$\sum_{j=1}^{n} h_{ij}(\varsigma, z)(\varsigma_j - z_j) = h_i(\varsigma) - h_i(z), \quad i = 1, \dots, p,$$

and $h_{ij}(\varsigma, z) = h_{ij}(z, \varsigma)$ for $(\varsigma, z) \in \Omega \times \Omega$.

Let $V = \{z \in \Omega : h(z) = 0\}$ and $M = \{\varsigma \in V : \rho(\varsigma) = 0\}$. Assume that $\partial\rho \wedge dh_1 \wedge \dots \wedge dh_p \not= 0$ on M; thus V is a smooth complex manifold of (complex) dimension $m := n - p$ (we may have to shrink Ω for that); also M is a smooth manifold of (real) dimension $2m - 1$. Let us also fix a domain $D \subset M$ with smooth boundary ∂D.

Let $C_{(0,q)}(\overline{D})$ denote the set of $(0,q)$-forms whose coefficients are continuous functions on $\overline{D}$; two forms $f, g \in C_{(0,q)}(\overline{D})$ are considered equal if

$$\int_D f \wedge \varphi = \int_D g \wedge \varphi$$

for every $(m, m-q-1)$-form φ with C^∞ coefficients in a neighbourhood (in V) of M with $\varphi = 0$ on a neighbourhood of $M - D$ (the role of such φ's will be that of test forms; call them $(m, m-q-1)$-test forms). If $f \in C_{(0,q)}(\overline{D})$ we say that $\overline{\partial}_\tau f \in C_{(0,q+1)}(\overline{D})$ if there exists $g \in C_{(0,q+1)}(\overline{D})$ with

$$\int_D g \wedge \varphi = (-1)^{q+1} \int_D f \wedge \overline{\partial}\varphi$$

for every $(m, m-q-2)$-test form φ; in that case we write $\overline{\partial}_\tau f = g$.

I.2. - A calculus of Cauchy-Fantappiè forms on V

In this paragraph we collect some background material from [2] and [3] and obtain some consequences of them which we will be using. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a smooth map defined for (ς, z) in a subset of $\Omega \times \Omega$ so that

$$(\gamma, \varsigma - z) := \sum_{j=1}^{n} (\varsigma_j - z_j)\gamma_j = 1.$$

Consider the differential form

$$(1) \qquad c \det[H, \gamma, \overbrace{(\overline{\partial}_\varsigma + \overline{\partial}_z)\gamma}^{m-1}]$$

where $H = H(\varsigma, z)$ stands for the matrix

$$H = \begin{bmatrix} h_{11} & \dots & h_{p1} \\ \cdot & \dots & \cdot \\ \cdot & & \cdot \\ \cdot & \dots & \cdot \\ h_{1n} & \dots & h_{pn} \end{bmatrix}$$

and γ (inside the determinant) is written in a column form $\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$; also the column

$$(\overline{\partial}_\varsigma + \overline{\partial}_z)\gamma = \begin{pmatrix} (\overline{\partial}_\varsigma + \overline{\partial}_z)\gamma_1 \\ \vdots \\ (\overline{\partial}_\varsigma + \overline{\partial}_z)\gamma_n \end{pmatrix}$$

in the determinant (1) is repeated $(m-1)$-times so that (1) is an $n \times n$ determinant; (for properties of such determinants see [1, p. 8]); also c is a constant

$$c = \frac{(-1)^{\frac{m(m-1)}{2}}}{m!(2\pi i)^m};$$

this is a normalizing constant for proposition 5 below.

Let $\alpha_q(\gamma) = \alpha_q(\gamma)(\varsigma, z)$ be the part of (1) which is $(0, q)$-form in z and $(0, m-q-1)$-form in ς, i.e.

$$\alpha_q(\gamma) = c \binom{m-1}{q} \det[H, \gamma, \overbrace{\overline{\partial}_z\gamma,}^{q} \overbrace{\overline{\partial}_\varsigma\gamma}^{m-q-1}], \quad 0 \le q \le m-1;$$

also define $\alpha_{-1}(\gamma) = \alpha_m(\gamma) = 0$.

Now if $\gamma^1 = (\gamma_1^1, \dots, \gamma_n^1)$ and $\gamma^2 = (\gamma_1^2, \dots, \gamma_n^2)$ are two maps so that $(\gamma^1, \varsigma - z) = (\gamma^2, \varsigma - z) = 1$, consider the differential form

$$\eta = c \det[H, (1-\lambda)\gamma^1 + \lambda\gamma^2, \overbrace{(\overline{\partial}_\varsigma + d_\lambda + \overline{\partial}_z)((1-\lambda)\gamma^1 + \lambda\gamma^2)}^{m-1}]$$

and let $\eta_q = \eta_q(\varsigma, z, \lambda)$ be the part of η which is $(0, q)$-form in z and $(m-q-1)$-form in $d\overline{\varsigma}_1, \dots, d\overline{\varsigma}_n$, $d\lambda$ $(0 \le \lambda \le 1)$, i.e.

$$\eta_q = c \binom{m-1}{q} \det[H, (1-\lambda)\gamma^1 + \lambda\gamma^2, \overbrace{(1-\lambda)\overline{\partial}_z\gamma^1 + \lambda\overline{\partial}_z\gamma^2}^{q},$$

$$\overbrace{(1-\lambda)\overline{\partial}_\varsigma\gamma^1 + \lambda\overline{\partial}_\varsigma\gamma^2 + (\gamma^2 - \gamma^1)d\lambda}^{m-q-1}].$$

Then $\eta = \sum_{q=0}^{m-1} \eta_q$ and $(\overline{\partial}_\varsigma + \mathrm{d}_\lambda + \overline{\partial}_z)\eta = 0$ (for the proof of this see [2, p. 88]; from now on, differential forms are restricted in $(\varsigma, z) \in V \times V$ wherever they are defined); it follows that

$$\overline{\partial}_\varsigma \eta_q + \overline{\partial}_z \eta_{q-1} + d_\lambda \eta_q = 0. \tag{2}$$

If $\tilde{\eta}_q$ is the part of η_q which contains $\mathrm{d}\lambda$, then (2) gives

$$\overline{\partial}_\varsigma \tilde{\eta}_q + \overline{\partial}_z \tilde{\eta}_{q-1} + \mathrm{d}_\lambda \eta_q = 0$$

which implies

$$\int_0^1 \mathrm{d}_\lambda \eta_q = \overline{\partial}_\varsigma \left(\int_0^1 \tilde{\eta}_q \right) + \overline{\partial}_z \left(\int_0^1 \tilde{\eta}_{q-1} \right);$$

but

$$\int_0^1 \mathrm{d}_\lambda \eta_q = \eta_q|_{\lambda=1} - \eta_q|_{\lambda=0} = \alpha_q(\gamma^2) - \alpha_q(\gamma^1).$$

Thus setting

$$\alpha_q(\gamma^1, \gamma^2) = \alpha_q(\gamma^1, \gamma^2)(\varsigma, z) := \int_0^1 \tilde{\eta}_q, \quad 0 \le q \le m-1,$$

and

$$\alpha_{-1}(\gamma^1, \gamma^2) = \alpha_m(\gamma^1, \gamma^2) = 0,$$

we have proved

PROPOSITION 1. *With differential forms restricted in ς and z to the manifold V, we have*

$$\alpha_q(\gamma^2) - \alpha_q(\gamma^1) = \overline{\partial}_\varsigma \alpha_q(\gamma^1, \gamma^2) + \overline{\partial}_z \alpha_{q-1}(\gamma^1, \gamma^2)$$

for $0 \le q \le m$.

Similarly let

$$\theta = c \det[H, (1 - \lambda - \mu)\gamma^1 + \lambda\gamma^2 + \mu\gamma^3, \overbrace{(\overline{\partial}_\varsigma + \mathrm{d}_{\lambda,\mu} + \overline{\partial}_z)((1 - \lambda - \mu)\gamma^1 + \lambda\gamma^2 + \mu\gamma^3)}^{m-1}]$$

where $\gamma^3 = (\gamma_1^3, \dots, \gamma_n^3)$ is a third map with $(\gamma^3, \varsigma - z) = 1$.

Let $\theta_q = \theta_q(\varsigma, z, \lambda, \mu)$ be the part of θ which is $(0,q)$-form in z and $(m-q-1)$-form in $d\overline{\varsigma}_1, \dots, d\overline{\varsigma}_n, d\lambda, d\mu$, i.e.

$$\theta_q = c\binom{m-1}{q}\det[H, (1-\lambda-\mu)\gamma^1 + \lambda\gamma^2 + \mu\gamma^3,$$

$$\overbrace{(1-\lambda-\mu)\overline{\partial}_z\gamma^1 + \lambda\overline{\partial}_z\gamma^2 + \mu\overline{\partial}_z\gamma^3,}^{q}$$

$$\overbrace{(1-\lambda-\mu)\overline{\partial}_\varsigma\gamma^1 + \lambda\overline{\partial}_\varsigma\gamma^2 + \mu\overline{\partial}_\varsigma\gamma^3 + (\gamma^2-\gamma^1)d\lambda + (\gamma^3-\gamma^1)d\mu}^{m-q-1}].$$

Since $\theta = \sum_{q=0}^{m-1}\theta_q$ and $(\overline{\partial}_\varsigma + d_{\lambda,\mu} + \overline{\partial}_z)\theta = 0$, it follows that

$$\overline{\partial}_\varsigma\theta_q + \overline{\partial}_z\theta_{q-1} + d_{\lambda,\mu}\theta_q = 0. \tag{3}$$

Let $\widetilde{\theta}_q$ be the part of θ_q which contains the term $d\lambda \wedge d\mu$ and $\widetilde{\widetilde{\theta}}_q$ be the part of θ_q which contains $d\lambda$ or $d\mu$. Then (3) gives

$$\overline{\partial}_\varsigma\widetilde{\theta}_q + \overline{\partial}_z\widetilde{\theta}_{q-1} + d_{\lambda,\mu}\widetilde{\widetilde{\theta}}_q = 0. \tag{4}$$

Integrating (4) over $\Delta = \{(\lambda,\mu) \in \mathbb{R}^2 : \lambda \geq 0, \mu \geq 0, \lambda + \mu \leq 1\}$, we obtain

$$\overline{\partial}_\varsigma\left(\int_{(\lambda,\mu)\in\Delta}\widetilde{\theta}_q\right) + \overline{\partial}_z\left(\int_{(\lambda,\mu)\in\Delta}\widetilde{\theta}_{q-1}\right) = -\int_{(\lambda,\mu)\in\Delta} d_{\lambda,\mu}\widetilde{\widetilde{\theta}}_q = -\int_{(\lambda,\mu)\in\partial\Delta}\widetilde{\widetilde{\theta}}_q.$$

Since

$$\int_{(\lambda,\mu)\in\partial\Delta}\widetilde{\widetilde{\theta}}_q = \alpha_q(\gamma^1,\gamma^2) + \alpha_q(\gamma^2,\gamma^3) + \alpha_q(\gamma^3,\gamma^1)$$

setting

$$\alpha_q(\gamma^1,\gamma^2,\gamma^3) = \alpha_q(\gamma^1,\gamma^2,\gamma^3)(\varsigma,z) := \int_{(\lambda,\mu)\in\Delta}\widetilde{\theta}_q, \quad 0 \leq q \leq m-1,$$

and

$$\alpha_{-1}(\gamma^1,\gamma^2,\gamma^3) = \alpha_m(\gamma^1,\gamma^2,\gamma^3) = 0,$$

we have proved

PROPOSITION 2. *With differential forms restricted in ς and z to the manifold V, we have*

$$\alpha_q(\gamma^1,\gamma^2) + \alpha_q(\gamma^2,\gamma^3) + \alpha_q(\gamma^3,\gamma^1) = -\overline{\partial}_\varsigma\alpha_q(\gamma^1,\gamma^2,\gamma^3) - \overline{\partial}_z\alpha_{q-1}(\gamma^1,\gamma^2,\gamma^3)$$

for $0 \leq q \leq m$.

Next, computing the integral which defines $\alpha_q(\gamma^1, \gamma^2)$, we obtain

$$(5) \quad \alpha_q(\gamma^1,\gamma^2) = \sum_{\substack{0\le k_1\le q\\ 0\le k_2\le m-q-2}} c(k_1,k_2)\det[H,\gamma^1,\gamma^2,\overbrace{\overline{\partial}_z\gamma^1}^{k_1},\overbrace{\overline{\partial}_z\gamma^2}^{q-k_1},\overbrace{\overline{\partial}_\varsigma\gamma^1}^{k_2},\overbrace{\overline{\partial}_\varsigma\gamma^2}^{m-q-k_2-2}]$$

where

$$c(k_1,k_2) = c\binom{m-1}{q}\binom{q}{k_1}\binom{m-q-1}{k_2}\int_0^1(1-\lambda)^{k_1+k_2}\cdot\lambda^{m-k_1-k_2-2}d\lambda.$$

Similarily, computing the integral which defines $\alpha_q(\gamma^1, \gamma^2, \gamma^3)$, we obtain

$$(6) \quad \alpha_q(\gamma^1,\gamma^2,\gamma^3) = \sum_{\substack{0\le k_1+k_2\le q\\ 0\le k_3+k_4\le m-q-3}} c(k_1,k_2,k_3,k_4)\det[H,\gamma^1,\gamma^2,\gamma^3,\overbrace{\overline{\partial}_z\gamma^1}^{k_1},\overbrace{\overline{\partial}_z\gamma^2}^{k_2},$$

$$\overbrace{\overline{\partial}_z\gamma^3}^{q-k_1-k_2},\overbrace{\overline{\partial}_\varsigma\gamma^1}^{k_3},\overbrace{\overline{\partial}_\varsigma\gamma^2}^{k_4},\overbrace{\overline{\partial}_\varsigma\gamma^3}^{m-q-k_3-k_4-3}]$$

where

$$c(k_1,k_2,k_3,k_4) = c\frac{(m-1)!}{k_1!k_2!k_3!k_4!(q-k_1-k_2)!(m-q-k_3-k_4-3)!}$$
$$\times\int_{(\lambda,\mu)\in\Delta}(1-\lambda-\mu)^{k_1+k_3}\lambda^{k_2+k_4}\mu^{m-k_1-k_2-k_3-k_4 3}d\lambda\wedge d\mu.$$

Now we consider the more general case in which the maps γ^1 and γ^2 satisfy simply $(\gamma^1, \varsigma - z) \neq 0$ and $(\gamma^2, \varsigma - z) \neq 0$; then define

$$\alpha_q(\gamma^1,\gamma^2) := \alpha_q(\tilde{\gamma}^1,\tilde{\gamma}^2)$$

where $\tilde{\gamma}^1 = \dfrac{1}{(\gamma^1,\varsigma - z)}\gamma^1$ and $\tilde{\gamma}^2 = \dfrac{1}{(\gamma^2,\varsigma - z)}\gamma^2$.

Similarily define

$$\alpha_q(\gamma^1,\gamma^2,\gamma^3) := \alpha_q(\tilde{\gamma}^1,\tilde{\gamma}^2,\tilde{\gamma}^3) \quad \text{and} \quad \alpha_q(\gamma) := \alpha_q(\tilde{\gamma}).$$

Now parts (i), (ii) and (iii) of the following proposition follow from (5), (6) and properties of determinants; part (iv) follows from propositions 1 and 2 and the relevant definitions.

PROPOSITION 3. *If* $(\gamma, \varsigma - z) \neq 0$ *and* $(\gamma^i, \varsigma - z) \neq 0$, $i = 1,2,3$, *then*

(i) $$\alpha_q(\gamma) = c\binom{m-1}{q}\frac{1}{(\gamma,\varsigma - z)^m}\det[H, \gamma, \overbrace{\overline{\partial}_z\gamma}^{q}, \overbrace{\overline{\partial}_\varsigma\gamma}^{m-q-1}]$$

(ii) $$\alpha_q(\gamma^1,\gamma^2) = \sum_{k_1,k_2} c(k_1,k_2)\frac{\det[H,\gamma^1,\gamma^2,\overbrace{\overline{\partial}_z\gamma^1}^{k_1},\overbrace{\overline{\partial}_z\gamma^2}^{q-k_1},\overbrace{\overline{\partial}_\varsigma\gamma^1}^{k_2},\overbrace{\overline{\partial}_\varsigma\gamma^2}^{m-q-k_2-2}]}{(\gamma^1,\varsigma - z)^{k_1+k_2+1}\cdot(\gamma^2,\varsigma - z)^{m-k_1-k_2-1}}$$

($\sum$ *is as in* (5))

(iii) $$\alpha_q(\gamma^1,\gamma^2,\gamma^3) = \sum_{k_1,k_2,k_3,k_4} c(k_1,k_2,k_3,k_4)$$

$$\times\frac{\det[H,\gamma^1,\gamma^2,\gamma^3,\overbrace{\overline{\partial}_z\gamma^1}^{k_1},\overbrace{\overline{\partial}_z\gamma^2}^{k_2},\overbrace{\overline{\partial}_z\gamma^3}^{q-k_1-k_2},\overbrace{\overline{\partial}_\varsigma\gamma^1}^{k_3},\overbrace{\overline{\partial}_\varsigma\gamma^2}^{k_4},\overbrace{\overline{\partial}_\varsigma\gamma^3}^{m-q-k_3-k_4-3}]}{(\gamma^1,\varsigma - z)^{k_1+k_3+1}\cdot(\gamma^2,\varsigma - z)^{k_2+k_4+1}\cdot(\gamma^3,\varsigma - z)^{m-k_1-k_2-k_3-k_4-2}}$$

($\sum$ *as in* (6)).

(iv) *The identities of propositions* 1 *and* 2 *are valid for the above forms.*

Next define

$$\beta(\varsigma) = \|dh_1 \wedge \ldots \wedge dh_p\|^{-2}\det\left[\overline{\frac{\partial h_1}{\partial\varsigma_j}},\ldots,\overline{\frac{\partial h_p}{\partial\varsigma_j}},\overbrace{d\varsigma_j}^{m}\right]$$

where

$$\|dh_1 \wedge \ldots \wedge dh_p\|^2 = \sum_{1\le j_1<\ldots<j_p\le n}\left|\frac{\partial(h_1,\ldots,h_p)}{\partial(\varsigma_{j_1},\ldots,\varsigma_{j_p})}(\varsigma)\right|^2.$$

The following proposition follows from a result in [2, p. 76].

PROPOSITION 4. (i) *If* $\dfrac{\partial(h_1,\ldots,h_p)}{\partial(\varsigma_{m+1},\ldots,\varsigma_n)}(\varsigma^0) \neq 0$ *for some point* $\varsigma^0 \in V$, *then* $\beta(\varsigma)$ *restricted to* V *locally at* ς^0 *can be written as*

$$\beta(\varsigma) = (\text{constant})\left(\frac{\partial(h_1,\ldots,h_p)}{\partial(\varsigma_{m+1},\ldots,\varsigma_n)}(\varsigma)\right)^{-1} d\varsigma_1 \wedge \ldots \wedge d\varsigma_m.$$

In particular $d\beta = 0$.

(ii) *If* $\varphi \in C^\infty_{(m,q)}(V)$ *then there exists* $\tilde{\varphi} \in C^\infty_{(0,q)}(V)$ *so that* $\varphi = \tilde{\varphi} \wedge \beta$.

The following proposition is a special case of [3, Theorem 1, p. 336]; it is a version of the Bochner-Martinelli-Koppelman formula on V.

PROPOSITION 5. *Let U be an open subset of V with smooth boundary. If $g \in C^1_{(0,q)}(\overline{U})$ and $z \in U$ then*

$$(-1)^q g(z) = \int_{\varsigma\in\partial U} g(\varsigma) \wedge \alpha_q(b) \wedge \beta(\varsigma)$$

$$- \int_{\varsigma\in U} \overline{\partial} g(\varsigma) \wedge \alpha_q(b) \wedge \beta(\varsigma) + \overline{\partial}\left[\int_{\varsigma\in U} g(\varsigma) \wedge \alpha_{q-1}(b) \wedge \beta(\varsigma)\right]$$

where $b = \overline{\varsigma} - \overline{z}$ and $\alpha_q(b) = \alpha_q(b)(\varsigma, z)$.

I.3. - A Bochner-Martinelli type transform and its jump behaviour

Let $V^+ = \{\varsigma \in V : \rho(\varsigma) < 0\}$ and $V^- = \{\varsigma \in V : \rho(\varsigma) > 0\}$. If $f \in C_{(0,q)}(\overline{D})$ define $f^+(z)$, $z \in V^+$, and $f^-(z)$, $z \in V^-$, as follows:

$$\int_{\varsigma\in D} f(\varsigma) \wedge \alpha_q(b)(\varsigma, z) \wedge \beta(\varsigma) = \begin{cases} f^+(z) & \text{for } z \in V^+ \\ f^-(z) & \text{for } z \in V^-. \end{cases}$$

For $\varepsilon > 0$ small, let $M_\varepsilon = \{\varsigma \in V : \rho(\varsigma) = -\varepsilon\}$ and $M^\varepsilon = \{\varsigma \in V : \rho(\varsigma) = \varepsilon\}$. With this notation we will prove the following Plemelj type formula.

PROPOSITION 6. *If $f \in C_{(0,q)}(\overline{D})$, with $\overline{\partial}_\tau f \in C_{(0,q+1)}(\overline{D})$, and φ is an $(m, m-q-1)$-test form, then*

$$\lim_{\varepsilon\to 0}\left(\int_{M_\varepsilon} f^+ \wedge \varphi - \int_{M^\varepsilon} f^- \wedge \varphi\right) = (-1)^q \int_D f \wedge \varphi.$$

For the proof we will need the following

LEMMA 1. *The integral*

$$\int_{(\varsigma,z)\in M\times U_\varepsilon} \frac{\mathrm{d}\sigma(\varsigma)\mathrm{d}v(z)}{|\varsigma - z|^{2m-1}} < \infty$$

where $\mathrm{d}\sigma$ and $\mathrm{d}v$ are the volume elements of M and V and $U_\varepsilon = \{z \in V : -\varepsilon < \rho(z) < \varepsilon\}$.

PROOF OF PROPOSITION 6. By definition of f^+ and f^- and Fubini's theorem, we have

$$(1) \qquad \int_{M_\varepsilon} f^+ \wedge \varphi - \int_{M^\varepsilon} f^- \wedge \varphi = \int_{\varsigma\in D} \int_{z\in\partial U_\varepsilon} f(\varsigma) \wedge \alpha_q(b) \wedge \beta(\varsigma) \wedge \varphi(z)$$

(since $\partial U_\varepsilon = M^\varepsilon - M_\varepsilon$).

Now recall that

$$(2)\qquad \alpha_q(b)(\varsigma, z) = (-1)^q c \binom{m-1}{q} \frac{1}{|\varsigma - z|^{2m}} \det[H, \bar\varsigma - \bar z, \overbrace{d\bar z}^{q}, \overbrace{d\bar\varsigma}^{m-q-1}].$$

Since $H(\varsigma, z) = H(z, \varsigma)$, we obtain from (2) that

$$(3)\qquad \alpha_q(b)(\varsigma, z) = (-1)^m \alpha_{m-q-1}(b)(z, \varsigma).$$

On the other hand, by Proposition 4(ii), we may write

$$(4)\qquad \varphi(z) = \tilde\varphi(z) \wedge \beta(z)$$

for some smooth $(0, m-q-1)$-form $\tilde\varphi$. It follows from (3) and (4) that

$$(5)\qquad \int_{z\in\partial U_\varepsilon} f(\varsigma) \wedge \alpha_q(b)(\varsigma, z) \wedge \beta(\varsigma) \wedge \varphi(z) = (-1)^{m(m-q-1)}(-1)^{m-1} f(\varsigma) \wedge \beta(\varsigma)$$
$$\wedge \int_{z\in\partial U_\varepsilon} \tilde\varphi(z) \wedge \alpha_{m-q-1}(b)(z, \varsigma) \wedge \beta(z).$$

But, by Proposition 5 applied to the domain U_ε and the $(0, m-q-1)$-form $\tilde\varphi(z)$, we have (for a fixed ς)

$$(6)\qquad \int_{z\in\partial U_\varepsilon} \tilde\varphi(z) \wedge \alpha_{m-q-1}(b)(z, \varsigma) \wedge \beta(z) = (-1)^{m-q-1}\tilde\varphi(\varsigma)$$
$$+ \int_{z\in U_\varepsilon} \bar\partial\tilde\varphi(z) \wedge \alpha_{m-q-1}(b)(z, \varsigma) \wedge \beta(z) - \bar\partial_\varsigma \left[\int_{z\in U_\varepsilon} \tilde\varphi(z) \wedge \alpha_{m-q-2}(b)(z, \varsigma) \wedge \beta(z) \right].$$

Substituting (6) into (5) we obtain, in view of (1) and (4),

$$(7)\qquad \int_{M_\varepsilon} f^+ \wedge \varphi - \int_{M^\varepsilon} f^- \wedge \varphi = (-1)^q \int_{\varsigma\in D} f(\varsigma) \wedge \varphi(\varsigma) + r_1(\varepsilon) + r_2(\varepsilon)$$

where

$$r_1(\varepsilon) = \pm \int_{\varsigma\in D} \int_{z\in U_\varepsilon} f(\varsigma) \wedge \beta(\varsigma) \wedge \bar\partial\tilde\varphi(z) \wedge \alpha_{m-q-1}(b)(z, \varsigma) \wedge \beta(z)$$

and

$$r_2(\varepsilon) = \pm \int_{\varsigma\in D} f(\varsigma) \wedge \beta(\varsigma) \wedge \bar\partial_\varsigma \left[\int_{z\in U_\varepsilon} \tilde\varphi(z) \wedge \alpha_{m-q-2}(b)(z, \varsigma) \wedge \beta(z) \right].$$

Since $(\mathrm{d}\sigma \times \mathrm{d}v)(D \times U_\varepsilon) \to 0$, as $\varepsilon \to 0$, it follows from lemma 1 that

$$\lim_{\varepsilon \to 0} r_1(\varepsilon) = 0. \tag{8}$$

Similarly, integrating by parts, we see that

$$r_2(\varepsilon) = \pm \int\limits_{\varsigma \in D} \overline{\partial}_\tau f \wedge \beta \wedge \int\limits_{z \in U_\varepsilon} \tilde{\varphi} \wedge \alpha_{m-q-2}(b) \wedge \beta$$

$$\pm \int\limits_{\varsigma \in \partial D} f \wedge \beta \wedge \int\limits_{z \in U_\varepsilon} \tilde{\varphi} \wedge \alpha_{m-q-2}(b) \wedge \beta.$$

Therefore, by lemma 1 again,

$$\lim_{\varepsilon \to 0} r_2(\varepsilon) = 0. \tag{9}$$

Now the proposition follows from (7), (8) and (9).

PROOF OF LEMMA 1. We have to show that

$$\lim_{\delta \to 0} \int\limits_{\{(\varsigma, z) \in M \times U_\varepsilon : |\varsigma - z| > \delta\}} \frac{\mathrm{d}\sigma(\varsigma)\mathrm{d}v(z)}{|\varsigma - z|^{2m-1}} < \infty$$

(since $(\mathrm{d}\sigma \times \mathrm{d}v)(M \times M) = 0$).

But the above limit is equal to

$$\lim_{\delta \to 0} \int\limits_{\varsigma \in M} g_\delta(\varsigma)\mathrm{d}\sigma(\varsigma) \tag{10}$$

where

$$g_\delta(\varsigma) = \int\limits_{z \in U_\varepsilon - B(\varsigma, \delta)} \frac{\mathrm{d}v(z)}{|\varsigma - z|^{2m-1}}$$

and $B(\varsigma, \delta) = \{z \in \mathbb{C}^n : |z - \varsigma| < \delta\}$.

Furthermore

$$0 \le g_\delta(\varsigma) \le \int\limits_{z \in U_\varepsilon} \frac{\mathrm{d}v(z)}{|\varsigma - z|^{2m-1}} = O\left(\int\limits_{u \in \mathbb{C}^m \cap (\text{compact})} \frac{\mathrm{d}u}{|u|^{2m-1}} \right) < \infty$$

uniformly in δ and ς.

Hence the limit (10) is finite; this completes the proof of the lemma.

I.4. - An auxiliary integral formula for the $\overline{\partial}_\tau$-equation

Now we will use the strict convexity of ρ; set $P_j = P_j(\varsigma, z) := \frac{\partial \rho}{\partial \varsigma_j}(\varsigma)$, $j = 1, \ldots, n$; by the strict convexity of ρ it follows that

$$P = P(\varsigma, z) := \sum_{j=1}^{n} P_j \cdot (\varsigma_j - z_j) \neq 0$$

provided that ς and z satisfy $\rho(\varsigma) = 0$, $\rho(z) \leq 0$ and $\varsigma \neq z$. Also denote by $P = (P_1, \ldots, P_n)$ (P will play the role of a γ). Similarly, setting $Q_j = Q_j(\varsigma, z) := \frac{\partial \rho}{\partial \varsigma_j}(z)$, $j = 1, \ldots, n$, we have that $Q = Q(\varsigma, z) := \sum_{j=1}^{n} Q_j(\varsigma_j - z_j) \neq 0$ for ς and z with $\rho(\varsigma) = 0$, $\rho(z) \geq 0$ and $\varsigma \neq z$; also set $Q := (Q_1, \ldots, Q_n)$. (We use here P to denote two different objects but it will be clear from the context which one we mean in each case; similarly for Q).

Now for $f \in C_{(0,q)}(\overline{D})$ define

$$(T_q f)(z) = \int_{\varsigma \in D} f(\varsigma) \wedge \alpha_{q-1}(b, P) \wedge \beta(\varsigma), \qquad \text{if } z \in V^+.$$

(here $\alpha_{q-1}(b, P) = \alpha_{q-1}(\gamma^1, \gamma^2)(\varsigma, z)$ with $\gamma^1 = b = \overline{\varsigma} - \overline{z}$ and $\gamma^2 = P = (P_1, \ldots, P_n)$).

Notice that $(T_q f)(z)$ is well-defined and $T_q f \in C^{\infty}_{(0,q-1)}(V^+)$. Similarly we define

$$(S_q f)(z) = \int_{\varsigma \in D} f(\varsigma) \wedge \alpha_{q-1}(b, Q) \wedge \beta(\varsigma), \qquad \text{if } z \in V^-;$$

then $S_q f \in C^{\infty}_{(0,q-1)}(V^-)$.

PROPOSITION 7. *For $f \in C_{(0,q)}(\overline{D})$, $1 \leq q \leq m-2$, with $\overline{\partial}_\tau f \in C_{(0,q+1)}(\overline{D})$, we have*

(i) $$(-1)^q f^+ = \overline{\partial}(T_q f) + T_{q+1}(\overline{\partial}_\tau f) - \int_{\varsigma \in \partial D} f(\varsigma) \wedge \alpha_q(b, P) \wedge \beta(\varsigma), \qquad \textit{on } V^+;$$

(ii) $$(-1)^q f^- = \overline{\partial}(S_q f) + S_{q+1}(\overline{\partial}_\tau f) - \int_{\varsigma \in \partial D} f(\varsigma) \wedge \alpha_q(b, Q) \wedge \beta(\varsigma), \qquad \textit{on } V^-.$$

PROOF. (i) Fix a $z \in V^+$. By Stokes' theorem

(1) $$\int_{\varsigma \in \partial D} f(\varsigma) \wedge \alpha_q(b, P) \wedge \beta(\varsigma) = \int_{\varsigma \in D} (\overline{\partial}_\tau f)(\varsigma) \wedge \alpha_q(b, P) \wedge \beta(\varsigma)$$
$$+(-1)^q \int_D f(\varsigma) \wedge \overline{\partial}_\varsigma \alpha_q(b, P) \wedge \beta(\varsigma).$$

But by Proposition 1

(2) $$\overline{\partial}_\varsigma \alpha_q(b, P) = -\overline{\partial}_z \alpha_{q-1}(b, P) + \alpha_q(P) - \alpha_q(b);$$

furthermore

(3) $$\alpha_q(P) = 0$$

since $q \geq 1$ and $\overline{\partial}_z P_j = 0$.

Now (1), in view of (2) and (3), becomes

$$\int_{\varsigma \in \partial D} f(\varsigma) \wedge \alpha_q(b, P) \wedge \beta(\varsigma) = \int_{\varsigma \in D} (\overline{\partial}_\tau f)(\varsigma) \wedge \alpha_q(b, P) \wedge \beta(\varsigma)$$

$$+(-1)^{q+1} \int_{\varsigma \in D} f(\varsigma) \wedge \overline{\partial}_z \alpha_{q-1}(b, P) \wedge \beta(\varsigma) + (-1)^{q+1} f^+(z)$$

which is equivalent to (i).

The proof of (ii) is similar; here we use the fact that $\alpha_q(Q) = 0$ since $q \leq m - 2$ and $\overline{\partial}_\varsigma Q_j = 0$. Next we extend the definition of $T_q f(z)$ and $S_q f(z)$ for $z \in D$; since these integrals become singular at $\varsigma = z$, we have to examine their singularity; and this is done by using a version of Levi coordinates which we describe next.

First it follows from the strict convexity of ρ and Taylor's theorem that

$$2 \text{ Re } P(\varsigma, z) \geq \rho(\varsigma) - \rho(z) + c_o |\varsigma - z|^2$$

for some constant $c_o > 0$ as long as ς and z are sufficiently close to each other; this, together with the assumption $\partial\rho \wedge \partial h_1 \wedge \ldots \wedge \partial h_p \neq 0$ on M, gives the following lemma (for its proof see [4, p. 299])

LEMMA 2. *Fix a point $z \in M$. Then*

(i) *There are (real) coordinates $t_1, \ldots, t_{2n}$ for points $\varsigma \in \mathbb{C}^n$ close to z so that*

$$t_1 + it_2 = (\rho(\varsigma) - \rho(z)) + i \text{ Im } P(\varsigma, z)$$

$$t_{2(m+k)-1} + it_{2(m+k)} = h_k(\varsigma) - h_k(z), \quad 1 \leq k \leq p;$$

moreover

$$|P(\varsigma, z)| \approx |\text{ Re } P(\varsigma, z)| + |\text{ Im } P(\varsigma, z)| \gtrsim |t_2| + t_2^2 + \ldots + t_{2m}^2,$$

$$|\varsigma - z|^2 \approx t_2^2 + \ldots + t_{2m}^2, \quad d\sigma(\varsigma) \approx dt_2 \ldots dt_{2m}$$

uniformly for $\varsigma \in M$, $z \in M$ and ς close to z.

(ii) *There are (real) coordinates* $s_1, \ldots, s_{2n}$ *for points* $\varsigma \in \mathbb{C}^n$ *close to* z *so that*

$$s_1 + is_2 = (\rho(\varsigma) - \rho(z)) + i \ \text{Im} \ Q(\varsigma, z)$$

$$s_{2(m+k)-1} + is_{2(m+k)} = h_k(\varsigma) - h_k(z), \quad 1 \leq k \leq p;$$

moreover

$$|Q(\varsigma, z)| \gtrsim |s_2| + s_2^2 + \ldots + s_{2m}^2, \ |\varsigma - z|^2 \approx s_2^2 + \ldots + s_{2m}^2, \ \mathrm{d}\sigma(\varsigma) \approx \mathrm{d}s_2 \ldots \mathrm{d}s_{2m}$$

for $\varsigma, z \in M$, ς *close to* z *(uniformly in* ς *and* z*).*

PROPOSITION 8. *Let* $f \in C_{(0,q)}(\overline{D})$. *Then the integrals which define* $T_q f(z)$ *and* $S_q f(z)$ *converge absolutely for* $z \in D$ *defining forms (denoted by* $T_q f$ *and* $S_q f$ *again) which belong to* $C_{(0,q-1)}(\overline{D})$. *Moreover*

$$\int\limits_{M_\varepsilon} T_q f \wedge \varphi \to \int\limits_D T_q f \wedge \varphi$$

and

$$\int\limits_{M^\varepsilon} S_q f \wedge \varphi \to \int\limits_D S_q f \wedge \varphi$$

as $\varepsilon \to 0^+$ *for each* $(m, m-q)$*-test-form* φ.

For the proof we need the following lemmas.

LEMMA 3. (i) *For a continuous function* g *on* $\mathbb{R}^2$, *we have*

$$\int\limits_{\{x \in \mathbb{R}^N : x_1 > 0, |x| < \delta\}} g(x_1, |x|^2) \mathrm{d}x_1 \ldots \mathrm{d}x_N = (\text{constant})$$

$$\int\limits_0^\delta \int\limits_0^{\pi/2} g(r \ \cos \ \theta, r^2) r^{N-1} (\sin \ \theta)^{N-2} \mathrm{d}\theta \mathrm{d}r.$$

(ii) *For a continuous function* g *on* $\mathbb{R}$, *we have*

$$\int\limits_{\{x \in \mathbb{R}^N : x_1 > 0, |x| = \varepsilon\}} g(x_1) \mathrm{d}\sigma_\varepsilon(x) = (\text{constant}) \int\limits_0^{\pi/2} g(\varepsilon \ \cos \ \theta) \varepsilon^{N-1} (\sin \ \theta)^{N-2} \mathrm{d}\theta,$$

where $\mathrm{d}\sigma_\varepsilon$ *denotes surface area element on* $\{|x| = \varepsilon\}$.

The elementary proof of it is omitted.

LEMMA 4.

(i) $$\int\limits_{\varsigma\in M\cap B(z,\delta)} |\alpha_{q-1}(b,P)|\mathrm{d}\sigma(\varsigma) = O\left(\delta\log\frac{1}{\delta}\right)$$

as $\delta \to 0^+$ uniformly in $z \in M$.

(ii) $$\int\limits_{\varsigma\in M\cap B(\varsigma,\delta)} |\alpha_{q-1}(b,Q)|\mathrm{d}\sigma(\varsigma) = O\left(\delta\log\frac{1}{\delta}\right)$$

as $\delta \to 0^+$ uniformly in $z \in M$.

PROOF. (i) Taking into account the fact that $|P| \gtrsim |\varsigma - z|^2$ (for $\varsigma, z \in M$), we obtain from proposition 3(ii) that

$$|\alpha_{q-1}(b,P)| \lesssim \frac{|\varsigma - z|}{|P(\varsigma,z)|\cdot|\varsigma - z|^{2m-2}}$$

uniformly in $\varsigma, z \in M$.

Hence, using the t-coordinates of lemma 2(i), we obtain

$$\int\limits_{\varsigma\in M\cap B(z,\delta)} |\alpha_{q-1}(b,P)|\mathrm{d}\sigma(\varsigma) \lesssim \int\limits_{|t'|<\delta} \frac{\mathrm{d}t_2\ldots\mathrm{d}t_{2m}}{|t'|^{2m-3}\cdot[|t_2|+|t'|^2]}$$

where $t' = (t_2,\ldots,t_{2m})$ and $|t'| = (t_2^2+\ldots+t_{2m}^2)^{1/2}$; therefore, by lemma 3(i),

$$\int\limits_{\varsigma\in M\cap B(z,\delta)} |\alpha_{q-1}(b,P)|\mathrm{d}\sigma(\varsigma) \lesssim \int_0^{\delta}\int_0^{\pi/2} \frac{r^{2m-2}\sin\,\theta\mathrm{d}\theta\mathrm{d}r}{r^{2m-3}(r\cos\,\theta + r^2)} = O\left(\delta\log\frac{1}{\delta}\right).$$

This proves (i); the proof of (ii) is similar, using the s-coordinates of lemma 2(ii). This completes the proof.

LEMMA 5. *We have*

$$\lim_{\varepsilon\to 0}\left[\varepsilon\int\limits_{z\in M}\frac{\mathrm{d}\sigma(z)}{(\varepsilon+|\varsigma - z|)^{2m-1}}\right] = 0$$

uniformly in $\varsigma \in M$.

PROOF. We have

$$\int_{z\in M} \frac{d\sigma(z)}{(\varepsilon+|\varsigma-z|)^{2m-1}} \approx \int_{|t'|<1} \frac{dt_2\dots dt_{2m}}{(\varepsilon^2+|t'|^2)^{m-\frac{1}{2}}}$$

$$\approx \int_0^1 \frac{r^{2m-2}dr}{(\varepsilon^2+r^2)^{m-\frac{1}{2}}} \approx \int_0^1 \frac{r^{2m-2}}{(\varepsilon+r)^{2m-1}}dr$$

$$\lesssim \int_0^1 \frac{dr}{r+\varepsilon} = O\left(\log\frac{1}{\varepsilon}\right);$$

this implies the lemma.

PROOF OF PROPOSITION 8. For $g \in L^\infty_{(0,q)}(M)$, let

$$(T_q g)(z) = \int_{\varsigma\in M} g(\varsigma)\wedge\alpha_{q-1}(b,P)\wedge\beta(\varsigma), \quad z\in\overline{V^+}.$$

It is clear that $T_q g \in C^\infty_{(0,q-1)}(V^+)$ and that the integral converges absolutely for $z \in M$, by lemma 4(i). Also for $\varepsilon > 0$, let

$$T_q^\varepsilon g(z) := (T_q g)(w(z,-\varepsilon)), \quad z\in M,$$

where $w(z,-\varepsilon)$ is the point on M_ε "closest" to z (closest on V). We claim that

$$\lim_{\varepsilon\to 0} \|T_q^\varepsilon g - T_q g\|_\infty = 0. \tag{4}$$

Indeed, setting $k(\varsigma,z) := \alpha_{q-1}(b,P)(\varsigma,z)\wedge\beta(\varsigma)$, we obtain

$$(T_q^\varepsilon g - T_q g)(z) = \int_{\varsigma\in M} g(\varsigma)\wedge[k(\varsigma,z)-k(\varsigma,w(z,-\varepsilon))]$$

and

$$\|T_q^\varepsilon g - T_q g\|_\infty \leq \|g\|_\infty \sup_{z\in M}\int_{\varsigma\in M} |k(\varsigma,z)k(\varsigma,w(z,-\varepsilon))|d\sigma(\varsigma).$$

But it follows from lemma 4 (i) that

$$\lim_{\varepsilon\to 0}\left[\sup_{z\in M}\int_{\varsigma\in M} |k(\varsigma,z)-k(\varsigma,w(z,-\varepsilon))|d\sigma(\varsigma)\right] = 0.$$

This implies (4).

Now, since $T_q^\varepsilon g \in C_{q-1}(M)$ for $\varepsilon > 0$, (4) implies that $T_q g \in C_{(0,q-1)}(M)$. It also follows from (4) that

$$\int_M T_q^\varepsilon g \wedge \varphi \to \int_M T_q g \wedge \varphi, \quad (\varepsilon \to 0^+),$$

for $\varphi \in C^\infty_{(m,m-q)}(M)$.

Now observe that

$$(5) \qquad \int_{M_\varepsilon} (T_q g) \wedge \varphi = \int_{z\in M} (T_q g)\,[w(z,-\varepsilon) \wedge \varphi(w(z,-\varepsilon))]$$

$$= \int_{z\in M} (T_q^\varepsilon g)(z) \wedge \varphi(z) + \int_{z\in M} (T_q g)(w(z,-\varepsilon)) \wedge [\varphi(w(z,-\varepsilon)) - \varphi(z)].$$

But

$$\varphi(w(z,-\varepsilon)) - \varphi(z) = O(\varepsilon).$$

Moreover

$$(T_q g)(w(z,-\varepsilon)) = \int_{\varsigma\in M} g(\varsigma) \wedge k(\varsigma, w(z,-\varepsilon)).$$

Since $2 \operatorname{Re} P(\varsigma,z) \underset{\sim}{>} \rho(\varsigma) - \rho(z) + c_o|\varsigma - z|^2$, it follows that

$$|P(\varsigma, w(z,-\varepsilon))| \underset{\sim}{>} -\rho(w(z,-\varepsilon)) + c_o|\varsigma - w(z,-\varepsilon)|^2 \qquad (\text{for } \varsigma \in M)$$

$$\underset{\sim}{>} |\varsigma - w(z,-\varepsilon)|^2 \approx (\varepsilon + |\varsigma - z|)^2.$$

Hence, by the definition of k and proposition 3(ii),

$$|k(\varsigma, w(z,-\varepsilon))| \leq \frac{1}{(\varepsilon + |\varsigma - z|)^{2m-1}}$$

uniformly in $\varsigma, z \in M$.

Therefore, in view of lemma 5,

$$(6) \qquad \lim_{\varepsilon\to 0} \int_{z\in M} (T_q g)(w(z,-\varepsilon)) \wedge [\varphi(w(z,-\varepsilon)) - \varphi(z)] = 0.$$

Now (5) and (6) imply that

$$\lim_{\varepsilon\to 0} \int_{M_\varepsilon} T_q g \wedge \varphi = \int_M T_q g \wedge \varphi.$$

Applying the above results to the form

$$g = \begin{cases} f & \text{on } D \\ 0 & \text{on } M - D \end{cases} \in L^\infty,$$

we obtain the part of proposition 8 about the operator T_q; the part about the operator S_q is proved similarly (using lemma 4(ii)). This completes the proof.

PROPOSITION 9. *Let* $\delta_q = \delta_q(\varsigma, z) = \alpha_q(b,P) - \alpha_q(b,Q)$. *Then for* $f \in C_{(0,q)}(\overline{D})$, $1 \leq q \leq m-2$, *with* $\overline{\partial}_\tau f \in C_{(0,q+1)}(\overline{D})$, *we have the following decomposition of* f, *in* $C_{(0,q)}(D)$:

$$f = \overline{\partial}_\tau (J_q f) + J_{q+1}(\overline{\partial}_\tau f) - \int\limits_{\varsigma \in \partial D} f(\varsigma) \wedge \delta_q(\varsigma, \cdot) \wedge \beta(\varsigma),$$

where $(J_q f)(z) = \int\limits_{\varsigma \in D} f(\varsigma) \wedge \delta_{q-1}(\varsigma, z) \wedge \beta(\varsigma)$.

PROOF. By proposition 6

$$(7) \qquad (-1)^q \int\limits_D f \wedge \varphi = \lim_{\varepsilon \to 0} \left(\int\limits_{M_\varepsilon} f^+ \wedge \varphi - \int\limits_{M^\varepsilon} f^- \wedge \varphi \right)$$

for an $(m, m-q-1)$-test-form φ. Substituting the expressions of f^+ and f^- given by proposition 7 (i) and (ii), into (7) we obtain

$$\begin{aligned}
\int\limits_D f \wedge \varphi = \lim_{\varepsilon \to 0} \Bigg[& (-1)^q \int\limits_{M_\varepsilon} T_q f \wedge \overline{\partial} \varphi - (-1)^q \int\limits_{M^\varepsilon} S_q f \wedge \overline{\partial} \varphi \\
& + \int\limits_{M_\varepsilon} T_{q+1}(\overline{\partial}_\tau f) \wedge \varphi - \int\limits_{M^\varepsilon} S_{q+1}(\overline{\partial}_\tau f) \wedge \varphi \\
& - \int\limits_{z \in D} \left(\int\limits_{\varsigma \in \partial D} f \wedge \delta_q \wedge \beta \right) \wedge \varphi \Bigg] \\
\text{(by prop. 8)} \qquad & = (-1)^q \int\limits_D (J_q f) \wedge \overline{\partial} \varphi + \int\limits_D J_{q+1}(\overline{\partial}_\tau f) \wedge \varphi \\
& - \int\limits_{z \in D} \left(\int\limits_{\varsigma \in \partial D} f \wedge \delta_q \wedge \beta \right) \wedge \varphi.
\end{aligned}$$

This proves the formula of proposition 9 in the sense of distributions. It follows now from proposition 8 that all the terms in this formula are (at least) continuous forms on D; hence the proof is complete.

I.5. - The integral formula for the $\overline{\partial}_\tau$-equation

For $f \in C_{(0,q)}(\overline{D})$ and $z \in \overline{D}$, let

$$(R_q f)(z) := \int_{\varsigma \in D} f(\varsigma) \wedge \alpha_{q-1}(P, Q) \wedge \beta(\varsigma)$$

(we do not know yet that this integral converges); here $\alpha_{q-1}(P, Q) = \alpha_{q-1}(P, Q)(\varsigma, z)$, i.e. $\alpha_{q-1}(\gamma^1, \gamma^2)$ with $\gamma^1 = P$ and $\gamma^2 = Q$. Since $\overline{\partial}_z P_j = 0$ and $\overline{\partial}_\varsigma Q_j = 0$, it follows from proposition 3 (ii) that

$$\alpha_{q-1}(P, Q) = (\text{constant}) \, \frac{\det[H, P, Q, \overbrace{\overline{\partial}_\varsigma P}^{m-q-1}, \overbrace{\overline{\partial}_z Q}^{q-1}]}{P^{m-q} \cdot Q^q}.$$

Since $P_j - Q_j = O(|\varsigma - z|)$, it follows that

$$|\alpha_{q-1}(P, Q)| \lesssim \frac{|\varsigma - z|}{|P|^{m-q}|Q|^q} \leq \max\left(\frac{|\varsigma - z|}{|P|^m}, \frac{|\varsigma - z|}{|Q|^m}\right)$$

for $\varsigma, z \in M$, $\varsigma \neq z$. Therefore, using the t- and s-coordinates of lemma 2, we see that

$$\int_{\varsigma \in M \cap B(z,\delta)} |\alpha_{q-1}(P, Q)| d\sigma(\varsigma) \lesssim \int_{|t'| < \delta} \frac{|t'| dt_2 \ldots dt_{2m}}{[|t_2| + |t'|^2]^m}$$

(recall $t' = (t_2, \ldots, t_{2m})$) uniformly in $\delta > 0$ and $z \in M$. Now estimating the above integral, using lemma 3(i), we obtain

$$(1) \qquad \int_{\varsigma \in M \cap B(z,\delta)} |\alpha_{q-1}(P, Q)| d\sigma(\varsigma) = O(\delta)$$

as $\delta \to 0$, uniformly in $z \in M$.

Also if $\rho(z) > 0$, small (i.e. $z \in V^-$ is sufficiently close to M), then

$$P^*(\varsigma, z) := P(\varsigma, z) + \frac{1}{2}\rho(z) \neq 0 \quad \text{and } \varsigma \in M.$$

Hence we may define

$$R_q f(z) = \int_{\varsigma \in D} f(\varsigma) \wedge \alpha^*_{q-1}(P, Q) \wedge \beta(\varsigma), \qquad \text{for } z \text{ with } \rho(z) > 0, \text{ small},$$

where $\alpha^*_{q-1}(P, Q)$ is obtained from $\alpha_{q-1}(P, Q)$ by replacing P (in the denominator only) by P^*. Then $R_q f$ is a $C^\infty_{(0,q-1)}$-form in $\{z : \rho(z) > 0\}$.

Now if $\varepsilon > 0$ and $z \in M$, let $w(z,\varepsilon)$ be the point on M^ε "closest" to z. Then $R_q^\varepsilon f(z) =: R_q f(w(z,\varepsilon))$ is a well defined C^∞-form in M (for sufficiently small $\varepsilon > 0$). Now using (1) and the operators R_q^ε, we prove (in the same way we proved proposition 8):

PROPOSITION 10. *Let $f \in C_{(0,q)}(\overline{D})$. Then, for $z \in D$, the integral $(R_q f)(z)$ converges absolutely defining a continuous $(0, q-1)$-form in $\overline{D}$.*

Now we can state the main result of part I.

THEOREM 1. *Let $f \in C_{(0,q)}(\overline{D})$, $1 \le q \le m-2$, with $\overline{\partial}_\tau f \in C_{(0,q+1)}(\overline{D})$. Then f can be decomposed, in $C_{(0,q)}(D)$, as follows:*

$$f = \overline{\partial}_\tau(R_q f) + R_{q+1}(\overline{\partial}_\tau f) - \int\limits_{\varsigma \in \partial D} f(\varsigma) \wedge \alpha_q(P,Q)(\varsigma,\cdot) \wedge \beta(\varsigma).$$

For its proof we need the following lemma.

LEMMA 5. *Let $1 \le q \le m-1$. Then*

(i)
$$\int\limits_{\varsigma \in M \cap \partial B(z,\varepsilon)} |\alpha_q(b,P,Q)(\varsigma,z)| \mathrm{d}\sigma_\varepsilon(\varsigma) = O(\varepsilon)$$

as $\varepsilon \to 0$, uniformly in $z \in M$, and

(ii)
$$\int\limits_{z \in M \cap \partial B(\varsigma,\varepsilon)} |\alpha_q(b,P,Q)(\varsigma,z)| \mathrm{d}\sigma_\varepsilon(z) = O(\varepsilon)$$

as $\varepsilon \to 0$, uniformly in $\varsigma \in M$. (Here $\mathrm{d}\sigma_\varepsilon$ denotes the appropriate surface area element; also $\alpha_q(b,P,Q)$ is $\alpha_q(\gamma^1,\gamma^2,\gamma^3)$ with $\gamma^1 = b, \gamma^2 = P$ and $\gamma^3 = Q$).

PROOF. (i) It follows, from proposition 3 (iii) and the fact that $P_j - Q_j = O(|\varsigma - z|)$, $|P| \gtrsim |\varsigma - z|^2, |Q| \gtrsim |\varsigma - z|^2$ for $\varsigma, z \in M$, that

$$|\alpha_q(b,P,Q)| \lesssim \frac{1}{|P||Q||\varsigma - z|^{2m-6}} \le \max\left\{\frac{1}{|P|^2|\varsigma - z|^{2m-6}}, \frac{1}{|Q|^2|\varsigma - z|^{2m-6}}\right\}.$$

Now using lemma 2 we see that

$$|\alpha_q(b,P,Q)| \lesssim \frac{1}{(\varepsilon^2 + |t_2|)^2 \cdot \varepsilon^{2m-6}}, \qquad \text{for } |\varsigma - z| = \varepsilon.$$

Hence by lemma 3(ii) we obtain

$$\int\limits_{\varsigma \in M \cap \partial B(z,\varepsilon)} |\alpha_q(b,P,Q)| \mathrm{d}\sigma_\varepsilon(\varsigma) \lesssim \int\limits_0^{\pi/2} \frac{\varepsilon^{2m-2} \sin\,\theta \mathrm{d}\theta}{(\varepsilon^2 + \varepsilon \cos\,\theta)^2 \cdot \varepsilon^{2m-6}} = O(\varepsilon);$$

this proves (i).

(ii) This follows from (i), since by interchanging ς and z in $\alpha_q(b,P,Q)(\varsigma,z)$, we do not change essentially its singularity.

PROOF OF THEOREM 1. We have to show that, for an $(m, m-q-1)$-test-form φ,

$$(2)\qquad \int_D f\wedge\varphi = (-1)^q \int_{z\in D}\int_{\varsigma\in D} f\wedge\alpha_{q-1}\wedge\beta\wedge\overline{\partial}\varphi + \int_{z\in D}\int_{\varsigma\in D}\overline{\partial}f\wedge\alpha_q\wedge\beta\wedge\varphi - \int_{z\in D}\int_{\varsigma\in\partial D} f\wedge\alpha_q\wedge\beta\wedge\varphi,$$

where we have set $\alpha_q = \alpha_q(P,Q)$; all the integrals in (2) are absolutely convergent by the remarks preceding proposition 10. Let us also point out that, in the right side of (2), $f = f(\varsigma)$, $\alpha_q = \alpha_q(\varsigma,z), \varphi = \varphi(z)$, $\overline{\partial}f = \overline{\partial}f(\varsigma)$, $\overline{\partial}\varphi = \overline{\partial}\varphi(z)$, $\alpha_{q-1} = \alpha_{q-1}(\varsigma,z)$ and $\beta = \beta(\varsigma)$ (i.e. this is how the various forms depend on ς and z).

But, by proposition 9,

$$(3)\qquad \int_D f\wedge\varphi = (-1)^q \int_{z\in D}\int_{\varsigma\in D} f\wedge\delta_{q-1}\wedge\beta\wedge\overline{\partial}\varphi + \int_{z\in D}\int_{\varsigma\in D}\overline{\partial}f\wedge\delta_q\wedge\beta\wedge\varphi - \int_{z\in D}\int_{\varsigma\in\partial D} f\wedge\delta_q\wedge\beta\wedge\varphi.$$

On the other hand, by proposition 2,

$$(4)\qquad \delta_q = \alpha_q - [\overline{\partial}_\varsigma\mu_q + \overline{\partial}_z\mu_{q-1}] \quad\text{and}\quad \delta_{q-1} = \alpha_{q-1} - [\overline{\partial}_\varsigma\mu_{q-1} + \overline{\partial}_z\mu_{q-2}],$$

where we have set

$$\mu_q = \alpha_q(b,P,Q)(\varsigma,z).$$

Substituting (4) into (3), we see that in order to prove (2) it suffices to show that

$$(5)\qquad \lim_{\varepsilon\to 0^+}[I_1^\varepsilon + I_2^\varepsilon] + I_3 = 0,$$

where

$$I_1^\varepsilon = (-1)^q \int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} f\wedge[\overline{\partial}_\varsigma\mu_{q-1} + \overline{\partial}_z\mu_{q-2}]\wedge\beta\wedge\overline{\partial}\varphi$$

$$I_2^\varepsilon = \int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} \overline{\partial}f\wedge[\overline{\partial}_\varsigma\mu_q + \overline{\partial}_z\mu_{q-1}]\wedge\beta\wedge\varphi$$

and

$$I_3 = -\int_{z\in D}\int_{\varsigma\in\partial D} f\wedge[\overline{\partial}_\varsigma\mu_q+\overline{\partial}_z\mu_{q-1}]\wedge\beta\wedge\varphi.$$

Let us consider first the part of I_1^ε which involves the term $\overline{\partial}_z\mu_{q-2}$; this term is equal (up to a sign) to

$$\int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} f\wedge\overline{\partial}_z\mu_{q-2}\wedge\beta\wedge\overline{\partial}\varphi = \pm\int_{\varsigma\in D}\int_{z\in D-B(\varsigma,\varepsilon)} f\wedge\overline{\partial}_z\mu_{q-2}\wedge\beta\wedge\overline{\partial}\varphi$$
$$= \pm\int_{\varsigma\in D}\int_{z\in D\cap\partial B(\varsigma,\varepsilon)} f\wedge\mu_{q-2}\wedge\beta\wedge\overline{\partial}\varphi$$

which tends to zero by lemma 5; (the first of the above equations is by Fubini's theorem and the second one is by Stokes' theorem since φ is zero on ∂D). Thus

$$I_1^\varepsilon = \tilde{I}_1^\varepsilon + r_1(\varepsilon)$$

where

$$\tilde{I}_1^\varepsilon = (-1)^q\int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} f\wedge\overline{\partial}_\varsigma\mu_{q-1}\wedge\beta\wedge\overline{\partial}\varphi$$

and $r_1(\varepsilon)\to 0$ as $\varepsilon\to 0^+$. But by Stokes' theorem again (in the variable $\varsigma\in D-B(z,\varepsilon)$) we have

$$\tilde{I}_1^\varepsilon = \int_{z\in D}\int_{\varsigma\in\partial D} f\wedge\mu_{q-1}\wedge\beta\wedge\overline{\partial}\varphi - \int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} \overline{\partial}f\wedge\mu_{q-1}\wedge\beta\wedge\overline{\partial}\varphi + r_2(\varepsilon)$$

where

$$r_2(\varepsilon) = -\int_{z\in D}\int_{\varsigma\in D\cap\partial B(z,\varepsilon)} f\wedge\mu_{q-1}\wedge\beta\wedge\overline{\partial}\varphi \to 0 \quad \text{(by lemma 5).}$$

Similar computations, based on Stokes' theorem (integration by parts), Fubini's theorem and the fact that $\varphi=0$ on ∂D, give

$$I_2^\varepsilon = (-1)^{q+1}\int_{z\in D}\int_{\varsigma\in\partial D} \overline{\partial}f\wedge\mu_q\wedge\beta\wedge\varphi + \int_{z\in D}\int_{\varsigma\in D-B(z,\varepsilon)} \overline{\partial}f\wedge\mu_{q-1}\wedge\beta\wedge\overline{\partial}\varphi + r_3(\varepsilon)$$

where

$$r_3(\varepsilon) = \pm\int_{z\in D}\int_{\varsigma\in D\cap\partial B(z,\varepsilon)} \overline{\partial}f\wedge\mu_q\wedge\beta\wedge\varphi \pm\int_{\varsigma\in D}\int_{z\in D\cap\partial B(\varsigma,\varepsilon)} \overline{\partial}f\wedge\mu_{q-1}\wedge\beta\wedge\varphi \to 0$$

and

$$I_3 = (-1)^q \int\limits_{z\in D}\int\limits_{\varsigma\in\partial D} \overline{\partial} f \wedge \mu_q \wedge \beta \wedge \varphi - \int\limits_{z\in D}\int\limits_{\varsigma\in\partial D} f \wedge \mu_{q-1} \wedge \beta \wedge \overline{\partial}\varphi.$$

Hence

$$I_1^\varepsilon + I_2^\varepsilon + I_3 = r_1(\varepsilon) + r_2(\varepsilon) + r_3(\varepsilon).$$

This proves (5) and completes the proof of the theorem.

Part II: The $\overline{\partial}$-equation

II.1. - Notation and preliminaries

First we establish notation for Part II which is slightly different from the one established in §I.1 and used in Part I. Let $\mathbb{P}^n = \mathbb{CP}^n$ be the n-dimensional complex projective space, its complex structure introduced as usual by the open cover:

$$\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n \qquad \text{where } U_j = \{[z] \in \mathbb{P}^n : z_j \neq 0\}$$

and the maps:

$$U_j \to \mathbb{C}^n, [z] \mapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

($(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} - \{0\}$ denoting homogeneous coordinates on $\mathbb{P}^n$). Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the natural projection $z \mapsto [z]$ and let $O(\ell)$, ℓ being an integer, denote the line bundle over $\mathbb{P}^n$ with transition functions:

$$g_{jk} : U_j \cap U_k \to \mathbb{C}, \quad U_j \cap U_k \ni [z] \mapsto g_{jk}([z]) = \left(\frac{z_k}{z_j}\right)^\ell \in \mathbb{C}.$$

Let $h_1, \ldots, h_p$ be homogeneous polynomials in $z_0, \ldots, z_n$ of degree $r_1, \ldots, r_p$. Setting

$$h_{ij}(\varsigma, z) := \frac{1}{2}\left[\int_0^1 \frac{\partial h_i}{\partial z_j}((1-t)z + t\varsigma)\mathrm{d}t + \int_0^1 \frac{\partial h_i}{\partial z_j}((1-t)\varsigma + tz)\mathrm{d}t\right]$$

we have that $h_{ij}(\varsigma, z)$ are homogeneous polynomials of degree $r_i - 1$ in $\varsigma_0, \ldots, \varsigma_n, z_0, \ldots, z_n$; moreover

$$\sum_{j=0}^n h_{ij}(\varsigma, z)(\varsigma_j - z_j) = h_i(\varsigma) - h_i(z), \quad i = 1, \ldots, p$$

(this follows from the fundamental theorem of calculus) and

$$h_{ij}(\varsigma, z) = h_{ij}(z, \varsigma).$$

Let

$$M = \{[\varsigma] \in \mathbb{P}^n : h_1(\varsigma) = \ldots = h_p(\varsigma) = 0\}$$

and assume that $dh_1 \wedge \ldots \wedge dh_p \neq 0$ on M so that M is a smooth projective algebraic variety which is a complete intersection.

Let $D \subset M$ be a domain on M with smooth boundary ∂D and set

$$\widetilde{D} = \{\varsigma \in \widetilde{M} : [\varsigma] \in D\}$$

where

$$\widetilde{M} = \{\varsigma \in S^{2n+1} : h_1(\varsigma) = \ldots = h_p(\varsigma) = 0\}$$

(S^{2n+1} is the unit sphere $\{\varsigma \in \mathbb{C}^{n+1} : |\varsigma| = 1\}$). Also let

$$V = \{z \in \mathbb{C}^{n+1} - \{0\} : h_1(z) = \ldots = h_p(z) = 0\}.$$

Thus V is a complex manifold of (complex) dimension $(n+1) - p := m+1$ (i.e. $m = n - p$) and $\widetilde{M}$ is a smooth manifold of (real) dimension $2m+1$ (notice that, by homogeneity, V meets S^{2n+1} transversally, i.e. $\partial\rho \wedge \partial h_1 \wedge \ldots \wedge \partial h_p \neq 0$ on $\widetilde{M}$, where $\rho(\varsigma) = |\varsigma|^2 - 1$ is the usual defining function of S^{2n+1}).

Now we discuss differential forms in $\mathbb{P}^n$ with coefficients in $O(\ell)$ and their pull-backs in $\mathbb{C}^{n+1}$ and S^{2n+1}. Let f be a $(0,q)$-form in an open set $U \subset \mathbb{P}^n$ whose coefficients are continuous sections of $O(\ell)$. This means that in each $U \cap U_j$, $0 \leq j \leq n$, we are given a $(0,q)$-form f_j so that

$$z_j^\ell f_j = z_k^\ell f_k \qquad \text{on } U \cap U_j \cap U_k.$$

Thus if $\pi^*(f_j)$ is the pull-back of f_j via $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$, then

$$z_j^\ell \pi^*(f_j) = z_k^\ell \pi^*(f_k) \qquad \text{on } \pi^{-1}(U \cap U_j \cap U_k);$$

hence, setting $\widetilde{f} =: z_j^\ell \pi^*(f_j)$ in $\pi^{-1}(U \cap U_j)$, we define a $(0,q)$-form $\widetilde{f}$ with continuous coefficients on $\pi^{-1}(U)$. This defines a map $\sim$: $C(0,q)(U, 0(\ell)) \to C_{(0,q)}(\pi^{-1}(U))$, $f \mapsto \widetilde{f}$; the image of this map can be described as follows:

$$(*) \qquad \widetilde{f}(tz) = t^\ell \widetilde{f}(z), \quad t \in \mathbb{C} - \{0\}$$

$$(**) \qquad L\widetilde{f} = 0$$

where

$$L\widetilde{f} = \left(\sum_{j=0}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\right) \lrcorner \widetilde{f} \qquad \text{(see [6, p. 556])}.$$

In other words a $(0,q)$-form $g(z)$ in $\pi^{-1}(U)$ admits descent to a $(0,q)$-form in U with coefficients in $O(\ell)$, i.e. $g = \tilde{f}$ for some f (as above), if and only if $g(tz) = t^\ell g(z)$ and $Lg = 0$. Analogously, if $U \subset M$ is an open subset of M and f is a $(0,q)$-form with coefficients continuous sections of $O(\ell)$ over U, then $\tilde{f}$ is a $(0,q)$-form on $\pi^{-1}(U)$ (which is an open subset of V) which satisfies $\tilde{f}(tz) = t^\ell \tilde{f}(z)$. Finally we may view $\tilde{f}$ as a differential form on $\pi^{-1}(U) \cap S^{2n+1}$ and then we have the equation $(\overline{\partial} f)^{\sim} = \overline{\partial}_\tau \tilde{f}$.

II.2. - The integral formula for the $\overline{\partial}$-equation

Using the defining function

$$\rho(\varsigma) = |\varsigma|^2 - 1 = \sum_{j=0}^{n} \varsigma_j \overline{\varsigma}_j - 1$$

for S^{2n+1}, let

$$P_j = \frac{\partial \rho}{\partial \varsigma_j}(\varsigma) = \overline{\varsigma}_j \text{ and } Q_j = \frac{\partial \rho}{\partial \varsigma_j}(z) = \overline{z}_j, \; j = 0, 1, \ldots, n$$

(notice the slight change in notation from part I since we use $\mathbb{C}^{n+1}$ instead of $\mathbb{C}^n$). Recall from §I.2 the differential forms (notice the change from n to $n+1$)

$$\alpha_q(P,Q)(\varsigma, z) = c_1 \frac{\det[h_{1j}(\varsigma, z), \ldots, h_{pj}(\varsigma, z), \overline{\varsigma}_j, \overline{z}_j, \overbrace{\mathrm{d}\overline{\varsigma}_j}^{m-q-1}, \overbrace{\mathrm{d}\overline{z}_j}^{q}]_{0 \le j \le n}}{\left(\sum_{j=0}^{n} \overline{\varsigma}_j(\varsigma_j - z_j)\right)^{m-q} \cdot \left(\sum_{j=0}^{n} \overline{z}_j(\varsigma_j - z_j)\right)^{q+1}}$$

(c_1 a constant) and

$$\beta(\varsigma) = \left[\sum_{0 \le j_1 < \ldots < j_p \le n} \left| \frac{\partial(h_1, \ldots, h_p)}{\partial(\varsigma_{j_1}, \ldots, \varsigma_{j_p})}(\varsigma) \right|^2 \right]^{-1} \det \left[\frac{\overline{\partial h_1}}{\partial \varsigma_j}, \ldots, \frac{\overline{\partial h_p}}{\partial \varsigma_j}, \overbrace{\mathrm{d}\varsigma_j}^{m+1} \right]_{0 \le j \le n}.$$

Notice that, by the homogeneity properties of the functions involved, we have for $t \in \mathbb{C} - \{0\}$:

$$\alpha_q(P,Q)(t\varsigma, tz) = t^{r_1 - 1} \ldots t^{r_p - 1} \cdot t^{-(m+1)} \cdot \alpha_q(P,Q)(\varsigma, z)$$

and

$$\beta(t\varsigma) = t^{-(r_1 - 1)} \ldots t^{-(r_p - 1)} \cdot t^{m+1} \cdot \beta(\varsigma).$$

Hence

$$\alpha_q(P,Q)(t\varsigma, tz) \wedge \beta(t\varsigma) = \alpha_q(P,Q)(\varsigma, z) \wedge \beta(\varsigma). \tag{1}$$

Now let $f \in C_{(0,q)}(\overline{D}, O(\ell))$, i.e. a $(0,q)$-form whose coefficients are continuous sections of $O(\ell)$ over $\overline{D}$, $1 \leq q \leq m-1$, and let $\tilde{f}$ be the corresponding $(0,q)$-form in $\tilde{D}$. We define

$$\tilde{\mathcal{R}}_q \tilde{f}(z) = \int_{\varsigma \in \tilde{D}} \tilde{f}(\varsigma) \wedge \alpha_{q-1}(P,Q)(\varsigma, z) \wedge \beta(\varsigma)$$

and

$$\tilde{B}_q \tilde{f}(z) = -\int_{\varsigma \in \partial \tilde{D}} \tilde{f}(\varsigma) \wedge \alpha_q(P,Q)(\varsigma, z) \wedge \beta(\varsigma).$$

Notice that the above integrals converge absolutely for z in a neighbourhood (in $\mathbb{C}^{n+1}$) of $\tilde{D}$ (this follows in part by proposition 10). Assuming furthermore that $\overline{\partial} f \in C_{(0,q+1)}(\overline{D}, O(\ell))$, we obtain from theorem 1:

$$\tilde{f} = \overline{\partial}_\tau(\tilde{\mathcal{R}}_q \tilde{f}) + \tilde{\mathcal{R}}_{q+1}(\overline{\partial}_\tau \tilde{f}) + \tilde{B}_q \tilde{f}, \quad \text{in } C_{(0,q)}(\tilde{D}). \tag{2}$$

On the other hand, we claim that $\tilde{\mathcal{R}}_q \tilde{f}(z)$ admits descent to D. Indeed if $t \in \mathbb{C}$, with $|t| = 1$, then

$$\begin{aligned}
\tilde{\mathcal{R}}_q \tilde{f}(tz) &= \int_{\varsigma \in \tilde{D}} \tilde{f}(\varsigma) \wedge \alpha_{q-1}(P,Q)(\varsigma, tz) \wedge \beta(\varsigma) \\
&= \int_{\varsigma \in \tilde{D}} \tilde{f}(t\varsigma) \wedge \alpha_{q-1}(P,Q)(t\varsigma, tz) \wedge \beta(t\varsigma) \\
&= t^\ell \int_{\varsigma \in \tilde{D}} \tilde{f}(\varsigma) \wedge \alpha_{q-1}(P,Q)(\varsigma, z) \wedge \beta(\varsigma) \quad \text{(by (1) and } (*)) \\
&= t^\ell \tilde{\mathcal{R}}_q \tilde{f}(z).
\end{aligned}$$

Hence $\tilde{\mathcal{R}}_q \tilde{f}$ satisfies $(*)$.

Moreover it follows from the definitions of $\alpha_{q-1}(P,Q)(\varsigma, z)$ and of the operator L that, for a fixed ς,

$$L(\alpha_{q-1}(P,Q)(\varsigma, \cdot)) = 0 \quad (L \text{ acts in } z);$$

therefore

$$L(\tilde{\mathcal{R}}_q \tilde{f}) = 0;$$

thus $\tilde{\mathcal{R}}_q \tilde{f}$ satisfies condition $(**)$ too, which proves the claim.

Therefore there exists $\mathcal{R}_q f \in C_{(0,q-1)}(\overline{D}, O(\ell))$ so that

$$(\mathcal{R}_q f)^{\sim} = \tilde{\mathcal{R}}_q \tilde{f}.$$

Similarly $\widetilde{B}_q\widetilde{f}$ admits descent to D, i.e. there exists $B_q f \in C_{(0,q)}(D, O(\ell))$ so that $(B_q f)^{\sim} = \widetilde{B}_q\widetilde{f}$. Hence, in view of (2), we have proved the following

THEOREM 2. *For* $f \in C_{(0,q)}(\overline{D}, O(\ell))$, $1 \leq q \leq m-1$, *with* $\overline{\partial} f \in C_{(0,q+1)}(\overline{D}, O(\ell))$, *we have the following decomposition of* f *in* $C_{(0,q)}(D, O(\ell))$:

$$f = \overline{\partial}(\mathcal{R}_q f) + \mathcal{R}_{q+1}(\overline{\partial} f) + B_q f.$$

II.3. - The $\overline{\partial}$-equation in s-pseudoconcave domains

Following Henkin and Polyakov [6, p. 560], we call a domain $\Omega \subset \mathbb{P}^n$ s-concave if for every point $z \in \Omega$ there exists an s-dimensional complex projective subspace $A(z) \subset \Omega$ which passes from z and which depends smoothly on z.

With notation as in §II.1, D is called s-pseudoconcave if there exists an $(s+p)$-concave domain $\Omega \subset \mathbb{P}^n$ so that $D = M \cap \Omega$. From this point on, we assume that D and Ω are such domains. For $z^0 \in \Omega$ there exists a neighbourhood $U(z^0) \subset \mathbb{P}^n$ of z^0 and smooth functions $a_{ij}(z)$ defined for $z \in U(z^0)$ so that if

$$A(z) = \left\{ [w] \in \mathbb{P}^n : \sum_{j=0}^{n} a_{ij}(z) w_j = 0, i = 1, \dots, m-s \right\}$$

then $A(z)$ is a $(p+s)$-dimensional complex projective subspace with $A(z) \subset \Omega$; moreover we choose $a_{ij}(z)$ so that

$$\sum_{j=0}^{n} a_{ij}(z)\overline{a}_{kj}(z) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Let $A_i(w,z) := \sum_{j=0}^{n} a_{ij}(z) w_j$, $i = 1, \dots, m-s$, and

$$\gamma_j(w,z) = \sum_{i=1}^{m-s} a_{ij}(z) A_i(w,z), \quad j = 0, q, \dots, n.$$

Also set $\gamma = (\gamma_0, \dots, \gamma_n)$ and $\Gamma(w,z) = \sum_{j=0}^{n} (w_j - z_j)\gamma_j(w,z)$. Then γ is independent of the choice of $U(z^0)$ and $a_{ij}(z)$ and thus γ is well-defined for $z \in \Omega$ and $w \in \mathbb{C}^{n+1}$; moreover $\Gamma(w,z) \neq 0$ for $z \in \Omega$ and $w \in \pi^{-1}(\partial\Omega)$ (see [6, p. 561]).

Now we will use γ to obtain an integral formula for the $\overline{\partial}$-equation in D. With $\gamma = \gamma(\varsigma, z)$ as above and P and Q as in §II.2, we have the differential forms $\alpha_q(\gamma, P, Q)(\varsigma, z)$ (introduced in §I.2). For $f \in C_{(0,q)}(\overline{D}, O(\ell))$, let $\widetilde{f}$ be the corresponding form in $C_{(0,q)}(\overline{\tilde{D}})$. Define

$$\tilde{G}_q \tilde{f}(z) = \tilde{\mathcal{R}}_q f(z) + (-1)^q \int\limits_{\varsigma \in \partial \tilde{D}} \tilde{f}(\varsigma) \wedge \alpha_{q-1}(\gamma, P, Q)(\varsigma, z) \wedge \beta(\varsigma).$$

If $\overline{\partial} f \in C_{(0,q+1)}(\overline{D}, O(\ell))$, then

$$\begin{aligned} (1) \qquad \tilde{G}_{q+1}(\overline{\partial}_T \tilde{f})(z) &= \tilde{\mathcal{R}}_{q+1}(\overline{\partial}_T \tilde{f})(z) + (-1)^{q+1} \int\limits_{\varsigma \in \partial \tilde{D}} \overline{\partial} \tilde{f}(\varsigma) \wedge \alpha_q(\gamma, P, Q) \wedge \beta(\varsigma) \\ &= \tilde{\mathcal{R}}_{q+1}(\overline{\partial}_T \tilde{f})(z) + \int\limits_{\varsigma \in \partial \tilde{D}} \tilde{f}(\varsigma) \wedge \overline{\partial}_\varsigma \alpha_q(\gamma, P, Q) \wedge \beta(\varsigma), \end{aligned}$$

and, by proposition 2,

$$(2) \qquad \overline{\partial}_\varsigma \alpha_q(\gamma, P, Q) = -\overline{\partial}_z \alpha_{q-1}(\gamma, P, Q) - \alpha_q(\gamma, P) - \alpha_q(P, Q) - \alpha_q(Q, \gamma).$$

But

$$(3) \qquad \tilde{f} = \overline{\partial}_T(\tilde{\mathcal{R}}_q \tilde{f}) + \tilde{\mathcal{R}}_{q+1}(\overline{\partial} \tilde{f}) + \tilde{B}_q \tilde{f}$$

(see the discussion before theorem 2). Substituting (2) into (1) and then taking into account (3), we obtain

$$(4) \qquad \tilde{f} = \tilde{G}_{q+1}(\overline{\partial}_T \tilde{f}) + \overline{\partial}_T(\tilde{G}_q \tilde{f}) + \tilde{\mathcal{E}}_q \tilde{f}$$

where

$$\tilde{\mathcal{E}}_q \tilde{f}(z) = -\int\limits_{\varsigma \in \partial \tilde{D}} \tilde{f}(\varsigma) \wedge [\alpha_q(\gamma, P) + \alpha_q(Q, \gamma)] \wedge \beta(\varsigma).$$

Moreover $\tilde{G}_q \tilde{f} = (G_q f)\tilde{}$ for some $G_q f \in C_{(0,q-1)}(\overline{D}, O(\ell))$ and $\tilde{G}_{q+1}(\overline{\partial}_T \tilde{f}) = (G_{q+1}(\overline{\partial} f))\tilde{}$; also $\tilde{\mathcal{E}}_q \tilde{f} = (\mathcal{E}_q f)\tilde{}$ for some $\mathcal{E}_q f \in C_{(0,q)}(D, O(\ell))$. Thus (4) proves part (i) of the following

THEOREM 3. *Let D be an s-pseudoconcave domain on M.*

(i) *For $f \in C_{(0,q)}(\overline{D}, O(\ell))$, $1 \le q \le m-1$, with $\overline{\partial} f \in C_{(0,q+1)}(\overline{D}, O(\ell))$, we have the following decomposition of f in $C_{(0,q)}(D, O(\ell))$:*

$$f = G_{q+1}(\overline{\partial} f) + \overline{\partial}(G_q f) + \mathcal{E}_q f.$$

(ii) *If $\ell < 0$, then $\mathcal{E}_q f = 0$ for $q \le s-1$ $\quad (q \le m-1)$.*

(iii) *If $\ell < 0$ and $q = s \leq m-1$, then a $\overline{\partial}$-closed form f from $C_{(0,s)}(\overline{D}, O(\ell))$ is $\overline{\partial}$-exact if and only if*

$$(*) \qquad \int_{\varsigma\in\partial\widetilde{D}} \widetilde{f}(\varsigma) \wedge \alpha_s(Q,\gamma)(\varsigma, z) \wedge \beta(\varsigma) = 0 \quad \textit{for } z \in \widetilde{D}.$$

PROOF. It remains to prove (ii) and (iii); here $\ell < 0$. First we claim that for $q \leq s-1$ we have

$$(5) \qquad \int_{\varsigma\in\partial\widetilde{D}} \widetilde{f}(\varsigma)\alpha_q(\gamma, P)(\varsigma, z) \wedge \beta(\varsigma) = 0$$

and

$$(6) \qquad \int_{\varsigma\in\partial\widetilde{D}} \widetilde{f}(\varsigma) \wedge \alpha_q(\gamma, Q)(\varsigma, z) \wedge \beta(\varsigma) = 0$$

for $z \in \widetilde{D}$.

To prove (5), apply Stokes' theorem to obtain

$$(7) \qquad \int_{\pi^{-1}(\partial D)\cap\{1<|\varsigma|<r\}} \mathrm{d}_\varsigma[\widetilde{f} \wedge \alpha_q(\gamma, P) \wedge \beta] = \int_{\partial\widetilde{D}} \widetilde{f} \wedge \alpha_q(\gamma, P) \wedge \beta - \int_{\pi^{-1}(\partial D)\cap\{|\varsigma|=r\}} \widetilde{f} \wedge \alpha_q(\gamma, P) \wedge \beta.$$

But by degree reasons the left side of (7) is zero (see [6, p. 563]). Furthermore for ς with $|\varsigma| = r$, r large positive number,

$$\left|\frac{\gamma_j}{\Gamma}\right| = O\left(\frac{1}{r}\right), \quad \left|\frac{P_j}{P}\right| = O\left(\frac{1}{r}\right), \quad \left|\frac{\overline{\partial}_z\gamma_j}{\Gamma}\right| = O\left(\frac{1}{r}\right)$$

$$\left|\frac{\overline{\partial}_\varsigma\gamma_j}{\Gamma}\right| = O\left(\frac{1}{r^2}\right), \quad \left|\frac{\overline{\partial}_\varsigma P_j}{P}\right| = O\left(\frac{1}{r^2}\right) \quad \text{(see [6, p. 564]).}$$

Moreover $\mathrm{Vol}(\pi^{-1}(\partial D) \cap \{|\varsigma| = r\}) = O(r^{2m})$ (notice that $\pi^{-1}(\partial D) \cap \{|\varsigma| = r\}$ is a smooth manifold of real dimension $2m$ for each r).

Also by homogeneity

$$|h_{ij}(\varsigma, z)\beta(\varsigma)| = O(1)$$

and

$$|\widetilde{f}(\varsigma)| = O(r^{\ell-q})$$

for $|\varsigma| = r$.

Therefore, using the expression for $\alpha_q(\gamma, P)$ as given by proposition 3(ii), we obtain

$$\int_{\pi^{-1}(\partial D)\cap\{|\varsigma|=r\}} \tilde{f} \wedge \alpha_q(\gamma, P) \wedge \beta = O(r^{\ell}) \to 0, \text{ as } r \to \infty, \text{ since } \ell < 0.$$

Thus, letting $r \to \infty$ in (7), we obtain (5).

To prove (6) recall that, by proposition 3(ii) again,

$$\alpha_q(\gamma, Q) = \sum_{k=0}^{q} c(k) \frac{\det[H, Q_j, \gamma_j, \overbrace{\overline{\partial}_\varsigma \gamma_j}^{m-q-1}, \overbrace{\overline{\partial}_z Q_j}^{k}, \overbrace{\overline{\partial}_z \gamma_j}^{q-k}]}{\Gamma^{m-k} \cdot Q^{k+1}} \tag{8}$$

for some constants $c(k)$. Recall also that

$$\gamma_j = \sum_{i=1}^{m-s} a_{ij}(z)\overline{A}_i;$$

hence

$$\overline{\partial}_\varsigma \gamma_j = \sum_{i=1}^{m-s} a_{ij}(z)\overline{\partial_\varsigma A_i} \tag{9}$$

and consequently

$$\overline{\partial}_\varsigma \gamma_{j_1} \wedge \ldots \wedge \overline{\partial}_\varsigma \gamma_{j_{m-s+1}} = 0$$

for $0 \leq j_1 < \ldots < j_{m-s+1} \leq n$. Thus for $q \leq s-2$, we have $m - q - 1 \geq m - s + 2 - 1 = m - s + 1$ and therefore

$$\det[H, Q_j, \gamma_j, \overbrace{\overline{\partial}_\varsigma \gamma_j}^{m-q-1}, \ldots] = 0.$$

If $q = s - 1$, then $m - q - 1 = m - s$ and

$$\det[H, Q_j, \gamma_j, \overbrace{\overline{\partial}_\varsigma \gamma_j}^{m-s}, \ldots] = \det\left[H, Q_j, \sum_{i=1}^{m-s} a_{ij}\overline{A}_i, a_{1j}, \ldots, a_{m-s,j}, \ldots\right]$$
$$\wedge \overline{\partial_\varsigma A_1} \wedge \ldots \wedge \overline{\partial_\varsigma A_{m-s}} = 0 \text{ (by (9))}.$$

This proves, in view of (8), that

$$\alpha_q(\gamma, Q) = 0 \quad \text{for } q \leq s-1, \tag{10}$$

and (6) follows.

To prove (iii) notice that

$$\int_{\partial\widetilde{D}} \widetilde{f} \wedge \alpha_s(\gamma, P) \wedge \beta = 0$$

(this is (5) with $q = s$ which is also valid; the same proof works). Thus condition $(*)$ is sufficient (by part (i)).

To prove that $(*)$ is necessary suppose that $f = \overline{\partial} g$ for some $g \in C_{(0,q-1)}(\overline{D}, O(\ell))$. Then integrating by parts we obtain:

$$\int_{\partial\widetilde{D}} \overline{\partial}\widetilde{g} \wedge \alpha_s(\gamma, Q) \wedge \beta = \int_{\partial\widetilde{D}} \widetilde{g} \wedge \overline{\partial}_\varsigma \alpha_s(\gamma, Q) \wedge \beta \tag{11}$$

$$= -\int_{\partial\widetilde{D}} \widetilde{g} \wedge \overline{\partial}_z \alpha_{s-1}(\gamma, Q) \wedge \beta + \int_{\partial\widetilde{D}} \widetilde{g} \wedge \alpha_s(Q) \wedge \beta - \int_{\partial\widetilde{D}} \widetilde{g} \wedge \alpha_s(\gamma) \wedge \beta$$

(the last equation in (11) follows from proposition 1).

But $\alpha_s(Q) = 0$ (since $\overline{\partial}_\varsigma Q_j = 0$ and $s \leq m - 1$), $\alpha_s(\gamma) = 0$ (this follows from proposition 3(i) and (9)), and $\alpha_{s-1}(\gamma, Q) = 0$ (by 10); thus all the integrals in (11) are zero; this proves that $(*)$ is necessary and completes the proof of the theorem.

REFERENCES

[1] I. AIZENBERG - A. YUZHAKOV, *Integral representations and residues in multidimensional complex analysis*, Translation from Russian by Amer. Math. Soc., **58**, 1983.

[2] T. HATZIAFRATIS, *Integral representation formulas on analytic varieties*, Pacific J. Math. **123** (1986), 71-91.

[3] T. HATZIAFRATIS, *An explicit Koppelman type integral formula on analytic varieties*, Michigan Math. J. **33** (1986) 335-341.

[4] T. HATZIAFRATIS, *An explicit extension formula of bounded holomorphic functions from analytic varieties to strictly convex domains*, J. Functional Anal., **70** (2) (1987), 289-303.

[5] G.M. HENKIN, *H. Lewy's equation and analysis on a pseudoconvex manifold I*, Uspehi Mat. Nauk., **32** (3) (1977), 57-118, Russian Math. Surveys, **32** (1977), 59-130.

[6] G. HENKIN - P. POLYAKOV, *Homotopy formulas for the* $\overline{\partial}$*-operator on* $\mathbb{CP}^n$ *and the Radon Penrose transform*, Math. USSR-Izv., **28** (1987), No. 3, 555-587.

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