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Periodic Solutions of Perturbed Superquadratic Hamiltonian Systems

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1. - Introduction and main results

We consider the existence of periodic solutions of a perturbed Hamiltonian system

$$\dot{z} = J(H'(z) + f(t)), \tag{1.1}$$

where $z,\ f : \mathbb{R} \to \mathbb{R}^{2N},\ \dot{z} = \frac{dz}{dt},\ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the identity matrix on $\mathbb{R}^N$, $H : \mathbb{R}^{2N} \to \mathbb{R}$, H' is its gradient. Let $a \cdot b$ and $|\cdot|$ denote the usual inner product and norm on $\mathbb{R}^{2N}$. H will be required to satisfy the following conditions,

(H1) $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$.

(H2) There exist $\mu > 2,\ r_0 \geq 0$ such that $0 < \mu H(z) \leq H'(z) \cdot z$, for every $|z| \geq r_0$.

(H3) There exist $0 < \frac{q_2}{2} < q_1 \leq q_2 < 2$ and $\alpha_i,\ \tau_i > 0,\ \beta_i \geq 0$, for $i = 1, 2$, such that

$$\alpha_1 e^{\tau_1 |z|^{q_1}} - \beta_1 \leq H(z) \leq \alpha_2 e^{\tau_2 |z|^{q_2}} + \beta_2, \text{ for every } z \in \mathbb{R}^{2N},$$

or

(H4) There exist $1 < p_1 \leq p_2 < 2p_1 + 1,\ \alpha_i > 0,\ \beta_i \geq 0$, for $i = 1, 2$, such that

$$\alpha_1 |z|^{p_1+1} - \beta_1 \leq H(z) \leq \alpha_2 |z|^{p_2+1} + \beta_2, \text{ for every } z \in \mathbb{R}^{2N}.$$

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Our main results are

THEOREM 1.2. *Let H satisfy conditions* (H1)-(H3). *Then, for any given* $T > 0$ *and* T*-periodic function* $f \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2N})$, (1.1) *possesses an unbounded sequence of* T*-periodic solutions.*

THEOREM 1.3. *The conclusion of Theorem* 1.2 *holds under conditions* (H1), (H2) *and* (H4).

Such global existence problems have been studied extensively in recent years. For the autonomous case of (1.1) (i.e. $f \equiv 0$), the result was proved by Rabinowitz under the conditions (H1) and (H2) only ([17], [19]). His proof is based on a group symmetry possessed by the corresponding variational formulation. When one considers the forced vibration problem (1.1) (f does depend on t), such a symmetry breaks down. Bahri and Berestycki [2] studied the perturbed problem (1.1) and proved the conclusion of Theorem 1.3 by assuming (H2), (H4) and (H1′): $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$. Our Theorem 1.3 weakens their condition on the smoothness of H and Theorem 1.2 allows H to increase faster at infinity.

Our proof extends Rabinowitz' basic ideas used in [18], [19] and ideas used in [12], [13]. In order to get the smoothness and compactness of corresponding functionals, we introduce a sequence of truncation functions $\{H_n\}$ of H in C^1 and corresponding modified functionals $\{J_n\}$ and J of I, where

$$(1.4) \qquad I(z) = \int_0^{2\pi} \left(\frac{1}{2}\, \dot{z} \cdot Jz - H(z) - f \cdot z \right) dt.$$

We make $\{H_n\}$ be monotone increasing to H and $\{J_n\}$ be monotone decreasing to J as n increases. These monotonicities also allow us to get L^∞-estimates for the critical points of J_n we found. We modify the treatment of the S^1-action on $W^{1/2,2}(S^1, \mathbb{R}^{2N})$ by introducing a simpler S^1-action on it to get upper estimates for certain minimax values. Combining with applications of Fadell-Rabinowitz cohomological index, we get the multiple existence of periodic solutions of (1.1).

In §2 we define $\{H_n\}$, $\{J_n\}$ and J. With the aid of an auxiliary space X, we define sequences of minimax values $\{a_k(n)\}$, $\{b_k(n)\}$ of J_n, and $\{a_k\}$, $\{b_k\}$ of J in §3, and discuss their properties in §4. §5 and §6 contain estimates from above and below for $\{a_k\}$. We prove the existence of critical values of J_n in §7. Then in §8, by showing that the critical points of J_n, for large n, yield solutions of (1.1), we complete the proofs of our main theorems. Finally in §9 we discuss more general forced Hamiltonian systems.

Since the proofs of Theorems 1.2 and 1.3 are similar, we shall carry out the details for the first one only and make some comments on the second in §8. For the details of the proof of Theorem 1.3, we refer to [13].

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2. - Modified functionals

By rescaling time, if necessary, we can assume $T = 2\pi$. Let $E = W^{1/2,2}(S^1, \mathbb{R}^{2N})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-1/2,2}(S^1, \mathbb{R}^{2N})$. Thus for $z \in E$, the actional integral $\frac{1}{2} A(z)$ is well defined, where

$$A(z) = \int_0^{2\pi} \dot{z} \cdot Jz \, dt. \tag{2.1}$$

For $k \in \mathbb{N}$, write $\overline{k} = \left[\frac{k}{N}\right]$, $\tilde{k} = k - \left[\frac{k}{N}\right]$, where $\left[\frac{k}{N}\right]$ is the integer part of k/N. Let $e_1, \ldots, e_{2N}$ denote the usual orthonormal basis in $\mathbb{R}^{2N}$. We write $i = \sqrt{-1}$ and denote, for $k \in \mathbb{N}$,

$$\begin{cases} \hat{\varphi}_k = (\sin \overline{k}t)e_{\tilde{k}} - (\cos \overline{k}t)e_{\tilde{k}+N}, \\ i\hat{\varphi}_k = (\cos \overline{k}t)e_{\tilde{k}} + (\sin \overline{k}t)e_{\tilde{k}+N}, \\ \hat{\psi}_k = (\sin \overline{k}t)e_{\tilde{k}} + (\cos \overline{k}t)e_{\tilde{k}+N}, \\ i\hat{\psi}_k = (\cos \overline{k}t)e_{\tilde{k}} - (\sin \overline{k}t)e_{\tilde{k}+N}. \end{cases} \tag{2.2}$$

Let

$$E^+_{m,n} = \operatorname{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid m \le k \le n\},$$

$$E^-_{m,n} = \operatorname{span}\{\hat{\psi}_k, i\hat{\psi}_k \mid m \le k \le n\},$$

for $m,\ n \in \mathbb{N},\ N \le m \le n.$

$$E^+ = E^+_{N+1,+\infty},\ E^- = E^-_{N+1,+\infty} \text{ and}$$

$$E^0 = \operatorname{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid 1 \le k \le N\}.$$

Then $E = E^+ \oplus E^- \oplus E^0$ and $A(z)$ is positive definite, negative definite and null on E^+, E^- and E^0 respectively. For

$$z = z^+ + z^- + z^0 \in E^+ \oplus E^- \oplus E^0 = E,$$

we take as a norm for E

$$\|z\|_E^2 \equiv A(z^+) - A(z^-) + |z^0|^2.$$

Under this norm, E becomes a Hilbert space and E^+, E^-, E^0 are orthogonal subspaces of E with respect to the inner product associated with this norm, as well as with the L^2 inner product. (2.2) gives a basis for E.

A result of Brézis and Wainger [6] implies that E is compactly embedded into $L^p(S^1, \mathbb{R}^{2N})$, $1 \le p < +\infty$, and the Orlicz space L_M^* with

$$M(z) = e^{\tau |z|^q} - \sum_{k=0}^{n} \frac{\tau^k}{k!} |z|^{kq},$$

for all $\tau > 0$, $0 < q < 2$, $n \in \mathbb{N}$, $nq > 1$, and $E \subset L_M$. Note that $q = 2$ is the critical embedding value (cf. [6], [10]). We shall use the following version of their embedding theorem.

LEMMA 2.3. *For $\tau > 0$, $0 < q < 2$, there exist constant $C_1, C_2 > 0$, depending only on τ and q, such that*

$$\int_0^{2\pi} \exp(\sigma\tau |z|^q) dt \le C_1 \exp\left(C_2(\sigma \|z\|_E)^{\frac{q}{2-q}}\right),$$

for every $\sigma > 0$, $z \in E$.

PROOF. We use the notations in [6] and only prove the Lemma for $E = W^{1/2,2}(S^1, \mathbb{C})$.

For $z \in E$, write

$$z(t) = C_0 + \sum_{n \neq 0} \frac{C_n}{\sqrt{|n|}} e^{\text{int}},$$

where $C_n \in \mathbb{C}$. Then $z = k * g$, where

$$k(t) = \sum_{n \neq 0} \frac{1}{\sqrt{|n|}} e^{\text{int}} \in L(2, \infty),$$

$$g(t) = \sum_{n \in \mathbf{Z}} C_n e^{\text{int}} \in L(2, 2).$$

By (8) of [6],

$$\begin{aligned} \sigma\tau |z^*|^q &\le (\varepsilon\sigma\tau)^{\frac{2}{q}} |z^*|^2 + \varepsilon^{-\frac{2}{2-q}} \\ &\le (\varepsilon\sigma\tau)^{\frac{2}{q}} C[1 + |\log t|] \|k\|_{L(2,\infty)}^2 \|g\|_{L(2,2)}^2 + \varepsilon^{-\frac{2}{2-q}}, \end{aligned}$$

for $0 < t < 1$.

Choose $\varepsilon = \frac{1}{\sigma T}\left(qC\|k\|^2_{L(2,\infty)}\ \|g\|^2_{L(2,2)}\right)^{-\frac{q}{2}}$, then from $\|g\|_{L(2,2)} \leq C\|z\|_E$ and $\int_0^{2\pi} e^{\sigma\tau|z|^q}dt = \int_0^{2\pi} e^{\sigma\tau|z^*|^q}dt$, we get the lemma. □

For $z \in E$ and $\theta \in [0, 2\pi] \simeq S^1$, we define an S^1-action on E by

$$(T_\theta z)(t) = z(t+\theta), \qquad \text{for all } t \in [0, 2\pi].$$

We say a subset B of E is $S^1(E)$-invariant if $T_\theta z \in B$, for all $z \in B$, $\theta \in [0, 2\pi]$. Note that $\mathrm{Fix}\{T_\theta\} \equiv \{z \in E \mid T_\theta z = z,\ \forall\ \theta \in [0, 2\pi]\} = E^0$.

By Lemma 2.3 and (H3), $I(z)$ defined in (1.4) is a continuous functional on E and formally the critical points of I correspond to the solutions of (1.1).

Without loss of generality, we assume $r_0 \geq 1$. Set $\alpha_0 = \min\limits_{|z|=r_0} H(z)$, $\beta_0 = \beta_1 + \max\limits_{|z|\leq r_0} |H(z)|$, where β_1 is given by (H3). Conditions (H1) and (H2) imply that, for some $\beta_3 \geq 0$,

$$(2.4)\qquad \begin{cases} \alpha_0|z|^\mu \leq H(z), & \text{for every } |z| \geq r_0, \\ \alpha_0|z|^\mu \leq H(z)+\beta_0 \leq \dfrac{1}{\mu}\ (H'(z)\cdot z+\beta_3), & \text{for every } z \in \mathbb{R}^{2N}. \end{cases}$$

Choose $\sigma \in (0,1)$ such that $\mu\sigma > 2$. We have

PROPOSITION 2.5. *Assume conditions* (H1) *and* (H2). *Then there exist a sequence* $\{K_n\} \subset \mathbb{R}$ *and a sequence of functions* $\{H_n\}$ *such that*

1°. $0 < K_0 < K_n < K_{n+1}$, $\forall\ n \in \mathbb{N}$, *and* $K_n \to +\infty$ *as* $n \to +\infty$, *where*

$$K_0 = \max\left\{1,\ r_0,\ \frac{\beta_0}{\alpha_0(1-\sigma)}\right\};$$

2°. $H_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, *for every* $n \in \mathbb{N}$;

3°. $H_n(z) = H(z)$, *for every* $n \in \mathbb{N}$ *and* $|z| \leq K_n$;

4°. $H_n(z) \leq H_{n+1}(z) \leq H(z)$, *for every* $n \in \mathbb{N}$ *and* $z \in \mathbb{R}^{2N}$;

5°. $0 < \mu\sigma H_n(z) \leq H_n'(z)\cdot z$, *for every* $n \in \mathbb{N}$ *and* $|z| \geq r_0$;

6°. *For* $n \in \mathbb{N}$, *there exist a constant* $\lambda_0 > 1$, *independent of* n, *and* $C(n) > 0$ *such that*

$$|H_n'(z)|^{\lambda_0} \leq C(n)\ (H_n'(z)\cdot z+1), \text{ for every } z \in \mathbb{R}^{2N}.$$

Since the proof of this proposition is rather technical and lengthy, we put it in the Appendix.

Similar to (2.4), there is a constant $\beta_4 \geq 0$ independent of n such that

$$\begin{cases} \alpha_0|z|^{\mu\sigma} \leq H_n(z), \text{ for all } n \in \mathbb{N} \text{ and } |z| \geq r_0, \\ \alpha_0|z|^{\mu\sigma} \leq H_n(z) + \beta_0 \\ \qquad \leq \frac{1}{\mu\sigma} (H_n'(z) \cdot z + \beta_4), \text{ for all } n \in \mathbb{N} \text{ and } z \in \mathbb{R}^{2N}. \end{cases} \tag{2.6}$$

For $n \in \mathbb{N}$, we define

$$I_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z)\mathrm{d}t - \int_0^{2\pi} f \cdot z\mathrm{d}t, \text{ for every } z \in E.$$

LEMMA 2.7.

1°. *$I_n \in C^1(E, \mathbb{R})$, for every $n \in \mathbb{N}$.*

2°. *$I(z) \leq I_{n+1}(z) \leq I_n(z)$, for every $n \in \mathbb{N}$ and $z \in E$.*

PROOF. For 1° we refer to [5], [17]. 2° follows from 4° of Proposition 2.5. □

From now on, in this section, we define $\tau = \frac{1}{4}\sqrt{3\mu\sigma + 10}$.

LEMMA 2.8. *There is $\beta_5 > 0$, independent of n, such that*

$$\frac{\tau(3\mu\sigma + 2)}{2(\tau - 1)} \|f\|_{L^2}\|z\|_{L^2} \leq \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_0)\,\mathrm{d}t + \beta_5, \tag{2.9}$$

for every $n \in \mathbb{N}$ and $z \in E$.

PROOF. By (2.6)

$$\frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_0)\,\mathrm{d}t - \frac{\tau(3\mu\sigma + 2)}{2(\tau - 1)} \|f\|_{L^2}\|z\|_{L^2}$$
$$\geq \frac{3\mu\sigma + 2}{8} \alpha_0 \|z\|_{L^{\mu\sigma}}^{\mu\sigma} - \frac{\tau(3\mu\sigma + 2)}{2(\tau - 1)} \|f\|_{L^2}(2\pi)^{\frac{\mu\sigma - 2}{2\mu\sigma}} \|z\|_{L^{\mu\sigma}}.$$

Since $\mu\sigma > 2$, (2.9) holds. □

LEMMA 2.10. *There is $\beta_6 > 0$, independent of n, such that for any $n \in \mathbb{N}$, $z \in E$, if $I_n'(z) = 0$ then*

$$\frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_0)\,\mathrm{d}t + \beta_5 \leq \left[\left(\frac{1}{2} A(z)\right)^2 + 1\right]^{1/2} + \beta_6. \tag{2.11}$$

PROOF. From $< I_n'(z), z > = 0$, we get that

$$\frac{1}{2} A(z) = \frac{1}{2} \int_0^{2\pi} \left(H_n'(z) \cdot z + \beta_4\right) dt + \frac{1}{2} \int_0^{2\pi} f \cdot z dt - \pi\beta_4$$

$$\tag{2.12} \begin{aligned} &\geq \frac{\mu\sigma}{2} \int_0^{2\pi} (H_n(z) + \beta_0)\, dt \\ &+ \frac{1}{2} \int_0^{2\pi} f \cdot z dt - \pi\beta_4, \quad \text{[by (2.6)]}, \\ &\geq \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_0)\, dt \\ &+ \frac{\mu\sigma - 2}{16} \alpha_0 \|z\|_{L^{\mu\sigma}}^{\mu\sigma} - \|f\|_{L^2} \|z\|_{L^2} - \pi\beta_4 - 1. \end{aligned}$$

Since $\mu\sigma > 2$, (2.11) holds. □

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ if $s \leq 1$, $\chi(s) = 0$ if $s \geq \tau$, and $-\frac{2}{\tau - 1} < \chi'(s) < 0$ if $1 < s < \tau$, where τ is defined before Lemma 2.8.

For $n \in \mathbb{N}$ and $z \in E$, we define

$$\varphi_0(z) = \left[\left(\frac{1}{2} A(z)\right)^2 + 1\right]^{1/2} + \beta_6,$$

$$\varphi(z) = \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H(z) + \beta_0)\, dt + \beta_5,$$

$$\varphi_n(z) = \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_0)\, dt + \beta_5,$$

$$\psi(z) = \chi\left(\frac{\varphi(z)}{\tau\varphi_0(z)}\right), \quad \psi_n(z) = \chi\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right),$$

$$J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z) dt - \psi(z) \int_0^{2\pi} f \cdot z \, dt,$$

$$J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z) dt - \psi_n(z) \int_0^{2\pi} f \cdot z \, dt.$$

For these functionals on E, we have

LEMMA 2.13. 1°. $\psi_n \in C^1(E,\mathbb{R})$, $J_n \in C^1(E,\mathbb{R})$, *for every* $n \in \mathbb{N}$.
2°. $\psi \in C(E,\mathbb{R})$, $J \in C(E,\mathbb{R})$.

PROOF. For 1° we refer to [5], [17]; 2° follows from (H3) and Lemma 2.3. □

LEMMA 2.14. *For any* m, $n \in \mathbb{N}$, $m \geq n$ *and* $z \in E$,

1°.
$$0 \leq \frac{1}{2}\int_0^{2\pi} (H_m(z) - H_n(z))\, dt \leq J_n(z) - J_m(z) \leq \frac{3}{2}\int_0^{2\pi} (H_m(z) - H_n(z))\, dt,$$

2°.
$$0 \leq \frac{1}{2}\int_0^{2\pi} (H(z) - H_n(z))\, dt \leq J_n(z) - J(z) \leq \frac{3}{2}\int_0^{2\pi} (H(z) - H_n(z))\, dt.$$

PROOF. We only prove 1°. The proof of 2° is similar. Let supp ψ_n be the closure of $\{z \in E \mid \psi_n(z) \neq 0\}$ in E.

If $z \notin \operatorname{supp} \psi_n \cup \operatorname{supp} \psi_m$, then by 4° of Proposition 2.5

$$J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z) - H_n(z))\, dt \geq 0.$$

If $z \in \operatorname{supp} \psi_n \cup \operatorname{supp} \psi_m$, then $\varphi_j(z) \leq \tau^2\varphi_0(z)$ for $j = n$ or $j = m$. By Lemma 2.8,

$$\varphi_j(z) \geq \frac{\tau(3\mu\sigma + 2)}{2(\tau - 1)} \|f\|_{L^2}\|z\|_{L^2}.$$

So

$$\frac{\frac{2}{\tau-1} \cdot \frac{3\mu\sigma+2}{8} \|f\|_{L^2}\|z\|_{L^2}}{\tau\varphi_0(z)} \leq \frac{\varphi(z)}{2\tau^2\varphi_0(z)} \leq \frac{1}{2}. \tag{2.15}$$

On the other hand

$$J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z) - H_n(z))\, dt + \chi'(\varsigma)\, \frac{\varphi_m(z) - \varphi_n(z)}{\tau\varphi_0(z)} \int_0^{2\pi} f \cdot z\, dt,$$

where we have used the mean value theorem with a number ς between $\frac{\varphi_m(z)}{\tau\varphi_0(z)}$

and $\frac{\varphi_n(z)}{\tau\varphi_0(z)}$. By the definition of φ_m and φ_n, we get

$$J_n(z) - J_m(z) = \left(1 + \frac{\chi'(\varsigma)}{\tau\varphi_0(z)} \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} f \cdot z \, dt\right) \int_0^{2\pi} (H_m(z) - H_n(z))\, dt. \tag{2.16}$$

Since $|\chi'(\xi)| \leq \frac{2}{\tau - 1}$, for every $\xi \in \mathbb{R}$, by (2.15) we get that

$$\frac{1}{2} \leq 1 + \frac{\chi'(\varsigma)}{\tau\varphi_0(z)} \frac{3\mu\sigma + 2}{8} \int_0^{2\pi} f \cdot z \, dt \leq \frac{3}{2}.$$

Combining with 4° of Proposition 2.5 and (2.16) we get the proof of 1°. □

COROLLARY 2.17. *For any $n \in \mathbb{N}$ and $z \in E$*

$$J_n(z) \geq J_{n+1}(z) \geq J(z).$$

LEMMA 2.18. *There exists $\beta_7 > 0$, independent of n, such that for any $n \in \mathbb{N}$, $z \in E$ and $M \geq \beta_8$, if $J_n(z) \geq M$ then $\varphi_0(z) \geq \frac{M}{2}$.*

PROOF. Since $\mu\sigma > 2$ and

$$J_n(z) \leq \frac{1}{2} A(z) - \alpha_0 \|z\|_{L^{\mu\sigma}}^{\mu\sigma} + \|f\|_{L^2}\|z\|_{L^2} + 2\pi\beta_0,$$

there exists $C > 0$, independent of n, such that

$$\varphi_0(z) = \left[\left(\frac{1}{2} A(z)\right)^2 + 1\right]^{1/2} + \beta_6 \geq J_n(z) + \beta_6 - C$$

and this yields the Lemma. □

LEMMA 2.19. *There exists a constant $\beta_8 > 0$, independent of n, such that for any $n \in \mathbb{N}$ and $z \in E$, if $J_n(z) \geq \beta_8$ and $< J_n'(z), z > = 0$, then $J_n(z) = I_n(z)$ and $J_n'(z) = I_n'(z)$.*

PROOF. For any z, $\varsigma \in E$, we have that

$$< J_n'(z), \varsigma > = \left(1 + T_{n,1}(z)\right) \overline{A}(z, \varsigma) - \left(1 + T_{n,2}(z)\right) \int_0^{2\pi} H_n'(z) \cdot \varsigma \, dt - \psi_n(z) \int_0^{2\pi} f \cdot \varsigma \, dt, \tag{2.20}$$

where

$$\overline{A}(z,\varsigma) = \frac{1}{2}\int_0^{2\pi} (\dot{z}\cdot J\varsigma + \dot{\varsigma}\cdot Jz)\,dt,$$

$$T_{n,1}(z) = \chi'\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right)\ \frac{A(z)\varphi_n(z)}{2\tau\varphi_0^2(z)\ (\varphi_0(z)-\beta_6)}\int_0^{2\pi} f\cdot z\ dt$$

$$T_{n,2}(z) = \chi'\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right)\ \frac{3\mu\sigma+2}{8\tau\varphi_0(z)}\int_0^{2\pi} f\cdot z\ dt.$$

If, for large enough β_8, we have

$$|T_{n,1}(z)| \le \frac{\mu\sigma-2}{16\mu\sigma},\quad |T_{n,2}(z)| \le \frac{\mu\sigma-2}{16\mu\sigma}, \tag{2.21}$$

then from (2.20), with $\varsigma = z$, we get

$$\frac{1}{2}\ A(z) \ge \frac{3\mu\sigma+2}{8}\int_0^{2\pi} (H_n(z)+\beta_0)\,dt$$
$$+\frac{\mu\sigma-2}{16}\ \alpha_0\|z\|_{L^{\mu\sigma}}^{\mu\sigma} - \|f\|_{L^2}\|z\|_{L^2}\beta_4 - 1.$$

This is (2.12). So $\varphi_0(z)\ge\varphi_n(z)$, thus $\psi_n(z)=1$ and $\psi_n'(z)=0$. This yields the lemma. Therefore we reduce to the proof of (2.21).

If $z\notin \operatorname{supp}\psi_n$, $T_{n,1}(z)=T_{n,2}(z)=0$. If $z\in\operatorname{supp}\psi_n$, then

$$\tau^2\varphi_0(z) \ge \varphi_n(z) \ge \frac{3\mu\sigma+2}{8}\ \alpha_0\|z\|_{L^{\mu\sigma}}^{\mu\sigma}.$$

Thus

$$|T_{n,1}(z)| \le \frac{\tau}{\tau-1}\ \|f\|_{L^2}\ (2\pi)^{(\mu\sigma-2)/2\mu\sigma}\ \frac{\|z\|_{L^{\mu\sigma}}}{\varphi_0(z)} \le M\,(\varphi_0(z))^{\frac{1}{\mu\sigma}-1}.$$

Similarly

$$|T_{n,2}(z)| \le M\,(\varphi_0(z))^{\frac{1}{\mu\sigma}-1},$$

for some constant $M>0$, independent of n and z; by Lemma 2.18, this implies (2.21) and completes the proof of the Lemma. □

We say J_n satisfies the Palais-Smale condition (PS) if, whenever a sequence $\{z_j\}$ in E satisfies that $\{J_n(z_j)\}$ is bounded and $J_n'(z_j)\to 0$ as $j\to+\infty$, then $\{z_j\}$ possesses a convergent subsequence.

LEMMA 2.22. *For any $n \in \mathbb{N}$, J_n satisfies (PS) on*

$$[J_n]_{\beta_8} \equiv \{z \in E \mid J_n(z) \geq \beta_8\},$$

where the constant $\beta_8 > 0$, independent of n, is defined in Lemma 2.19.

PROOF. Since $J_n'(z_j) \to 0$, we may assume $| < J_n'(z_j),\ z > | \leq \|z\|_E$ for every $z \in E$. By the assumption $\beta_8 \leq J_n(z_j) \leq M$, from (2.20) and (2.21), we get

$$M + \|z_j\|_E \geq J_n(z_j) - \frac{1}{2\left(1 + T_{n,1}(z_j)\right)} < J_n'(z_j),\ z_j >$$

$$= \frac{1 + T_{n,2}(z_j)}{2\left(1 + T_{n,1}(z_j)\right)} \int_0^{2\pi} H_n'(z_j) \cdot z_j \ dt - \int_0^{2\pi} H_n(z_j)dt$$

$$- \left(1 - \frac{1}{2\left(1 + T_{n,1}(z_j)\right)}\right) \psi_n(z_j) \int_0^{2\pi} f \cdot z_j \ dt$$

$$\geq \frac{\mu\sigma - 2}{16\mu\sigma} \int_0^{2\pi} H_n'(z_j) \cdot z_j dt + \frac{1}{4} (\mu\sigma - 2) \int_0^{2\pi} \left(H_n(z_j) + \beta_0\right) dt$$

$$- \frac{2}{3} \|f\|_{L^2}\|z_j\|_{L^2} - 2\pi(\beta_4 - \beta_0).$$

Therefore there are $M_1 > 0$, independent of n, and j such that

$$\|z_j\|_{L^{\mu\sigma}}^{\mu\sigma} + \int_0^{2\pi} \left(H_n'(z_j) \cdot z_j + \beta_4\right) dt \leq M_1(\|z_j\|_E + 1). \tag{2.23}$$

Write $z_j = z_j^+ + z_j^- + z_j^0$. Since $z_j^0 = \frac{1}{2\pi} \int_0^{2\pi} z_j \ dt$, there is $M_2 > 0$, independent of $n,\ j$, such that

$$|z_j^0| \leq \|z_j\|_{L^{\mu\sigma}} \leq M_2 \left(\|z_j\|_E^{\frac{1}{\mu\sigma}} + 1\right). \tag{2.24}$$

From (2.20) for $< J_n'(z_j),\ z_j^+ >$, we get

$$\frac{1}{2} \|z_j^+\|_E^2 \leq \left(1 + T_{n,1}(z_j)\right) \ A(z_j^+) \leq \frac{3}{2} \int_0^{2\pi} |H_n'(z_j)| \ |z_j^+|dt$$

$$+\|f\|_{L^2}\|z_j^+\|_{L^2} + \|z_j^+\|_E.$$

By 6° of Proposition 2.5 and (2.23), we get

$$\int_0^{2\pi} |H_n'(z_j)|\ |z_j^+|dt \le \left(\int_0^{2\pi} |H_n'(z_j)|^{\lambda_0}dt\right)^{1/\lambda_0} \|z_j^+\|_L \frac{\lambda_0}{\lambda_0-1}$$
$$\le M_3(n) \left(\|z_j\|_E^{1+1/\lambda_0}+1\right),$$

for some constant $M_3(n) > 0$ independent of j. Thus there is $M_4(n) > 0$, independent of j, such that

$$\|z_j^+\|_E^2 \le M_4(n) \left(\|z_j\|_E^{1+1/\lambda_0}+1\right).$$

Similarly

$$\|z_j^-\|_E^2 \le M_5(n) \left(\|z_j\|_E^{1+1/\lambda_0}+1\right),$$

for some $M_5(n) > 0$ independent of j. Combining with (2.24), we get a constant $M_6(n) > 0$, independent of j, such that

$$\|z_j\|_E \le M_6(n). \tag{2.25}$$

Let $P^\pm : E \to E^\pm$ be the orthogonal projections. From (2.20)

$$P^\pm J_n'(z_j) = \pm\left(1+T_{n,1}(z_j)\right)\ z_j^\pm \pm \mathcal{P}_n(z_j),$$

where $\mathcal{P}_n$ is a compact operator by (H1), (H2) and Proposition 2.5. Since $|T_{n,1}(z_j)| \le \frac{1}{16}$,

$$\pm\ z_j^\pm = \left(1+T_{n,1}(z_j)\right)^{-1} P^\pm J_n'(z_j) - \left(1+T_{n,1}(z_j)\right)^{-1} P^\pm \mathcal{P}_n(z_j).$$

By (2.25), this shows that $\{z_j^+\}$ and $\{z_j^-\}$ are precompact in E. By (2.25), $\{z_j^0\}$ is also precompact, therefore $\{z_j\}$ is precompact in E, and the proof is complete. □

LEMMA 2.26. *There exists a constant $\beta_9 > 0$ such that*

$$|J(z) - J(T_\theta z)| \le \beta_9 \left[\log^{1/q_1}\left(|J(z)|+1\right)+1\right], \text{ for all } z \in E, \tag{2.27}$$

where q_1 is defined in (H3).

PROOF. A direct application of Hölder inequality shows that there exists $c_1 > 0$ such that

$$\exp\left[\tau_1\left(\frac{1}{2\pi}\int_0^{2\pi} |z|dt\right)^{q_1}\right] \le \frac{1}{\pi}\int_0^{2\pi} \exp\left(\tau_1|z|^{q_1}\right)dt + c_1, \text{ for all } z \in E.$$

If $z \in \operatorname{supp} \psi$,

$$|J(z)| \geq \left(\frac{3\mu\sigma + 2}{8\tau^2} - 1\right) \int_0^{2\pi} (H(z) + \beta_0)\,\mathrm{d}t - c_2 \int_0^{2\pi} |z|\mathrm{d}t - c_3$$
$$\geq c_4 \int_0^{2\pi} \exp\left(\tau_1 |z|^{q_1}\right) \mathrm{d}t - c_5.$$

Here we used (H3), and c_j's denote positive constants. Therefore

$$|J(z) - J(T_\theta z)| \leq 2\ \psi(z) \int_0^{2\pi} |f \cdot z|\mathrm{d}t \leq c_6 \psi(z) \int_0^{2\pi} |z|\mathrm{d}t$$
$$\leq \beta_9 \left[\log^{1/q_1}\left(|J(z)| + 1\right) + 1\right],$$

for some $\beta_9 > 0$, and the proof is complete. □

3. - A minimax structure

For $k \in \mathbb{N}$, $k \geq N + 1$, we define $V_k(E) = E^+_{N+1,k} \oplus E^- \oplus E^0$. By (2.6) and Corollary 2.17, there exists R_k, for $k \geq N + 1$, such that $1 < R_k < R_{k+1}$ and $J(z) \leq J_n(z) \leq 0$ for all $n \in \mathbb{N}$, $z \in V_k(E)$ with $\|z\|_E \geq R_k$.

Let $D_k(E) = V_k(E) \cap B_k(E)$, $B_k(E) = \{z \in E \mid \|z\|_E \leq R_k\}$.

For $z = z^0 + z^+ + z^- \in E$, we write

$$\text{(3.1)} \qquad z^+ = \sum_{k \geq N+1} \rho_k e^{i\hat{\alpha}_k}\ \hat{\varphi}_k, \quad z^- = \sum_{k \geq N+1} \sigma_k e^{i\hat{\beta}_k} \hat{\psi}_k,$$

where $\rho_k, \sigma_k \geq 0$, $\hat{\alpha}_k, \hat{\beta}_k \in [0, 2\pi]$.

Then we have

$$\|z\|_E^2 = |z^0|^2 + 2\pi \sum_{k \geq N+1} \bar{k}\left(\rho_k^2 + \sigma_k^2\right), \text{ and}$$

$$T_\theta z = z^0 + \sum_{k \geq N+1} \left[\rho_k e^{i(\hat{\alpha}_k + \bar{k}\theta)}\ \hat{\varphi}_k + \sigma_k e^{i(\hat{\beta}_k + \bar{k}\theta)} \hat{\psi}_k\right], \text{ for all } \theta \in [0, 2\pi].$$

We define a new S^1-action on E by

$$\hat{T}_\theta z = z^0 + \sum_{k \geq N+1} \left[\rho_k e^{i(\hat{\alpha}_k + \theta)}\ \hat{\varphi}_k + \sigma_k e^{i(\hat{\beta}_k + \theta)} \hat{\psi}_k\right], \text{ for all } \theta \in [0, 2\pi],$$

for z with expression (3.1) and denote E with $\hat{T}_\theta$ by X. We define an S^1-action on $X \times E$ by

$$\tilde{T}_\theta(x, z) = (\hat{T}_\theta x, T_\theta z), \text{ for all } (x, z) \in X \times E \text{ and } \theta \in [0, 2\pi].$$

We also define S^1-invariant sets, equivariant maps, invariant functionals for X, E, $X \times E$ in a usual way (cf. [9]). Let $\mathcal{E}$ (or $\mathcal{X}, \mathcal{F}$) denote the family of closed, in E (or X, $X \times E$) S^1-invariant subsets in $E\backslash\{0\}$ (or $X\backslash\{0\}$, $X \times E\backslash\{0\}$). Then $\mathcal{F}$ contains $\mathcal{X} \times \{0\}$ and $\{0\} \times \mathcal{E}$. For $B \in \mathcal{X}$, we say a map $h : B \to E$ is S^1-equivariant if $h(\hat{T}_\theta x) = T_\theta h(x)$ for all $x \in B$ and $\theta \in [0, 2\pi]$. We also denote by $V_k(X)$, $D_k(X)$, $B_k(X)$ the sets in X corresponding to $V_k(E)$, $D_k(E)$, $B_k(E)$, etc. We introduce the Fadell-Rabinowitz cohomological index theory on $\mathcal{F}$ (cf. [9]).

LEMMA 3.2. *There is an index theory on $\mathcal{F}$, i.e. a mapping $\gamma : \mathcal{F} \to \{0\} \cup \mathbb{N} \cup \{+\infty\}$ such that, if $A, B \in \mathcal{F}$,*

1°. *$\gamma(A) \le \gamma(B)$ if there exists $h \in C(A, B)$ with h being S^1-equivariant.*

2°. *$\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.*

3°. *If $B \subset (X \times E)\backslash(X^0 \times E^0)$ and B is compact, then $\gamma(B) < \infty$ and there is a constant $\delta > 0$ such that $\gamma(\mathcal{N}_\delta(B, X \times E)) = \gamma(B)$, where $\mathcal{N}_\delta(B, X \times E) = \{z \in X \times E \mid \|z - B\|_{X \times E} \le \delta\}$.*

4°. *If $S \subset (X \times E)\backslash(X^0 \times E^0)$ is a $2n - 1$ dimensional invariant sphere, then $\gamma(S) = n$.*

By identifying $\mathcal{X}$ with $\mathcal{X} \times \{0\}$, and $\mathcal{E}$ with $\{0\} \times \mathcal{E}$, we may view that the index theory γ is defined on both $\mathcal{X}$ and $\mathcal{E}$.

For $x \in X$, $z \in E$, we write $z \sim x$ if $z^0 = x^0$, $\rho_k(z) = \rho_k(x)$, $\sigma_k(z) = \sigma_k(x)$, $\hat{\alpha}_k(z) = \hat{\alpha}_k(x)$ and $\hat{\beta}_k(z) = \hat{\beta}_k(x)$ for all $k \ge N + 1$.

For $x = x^0 + \sum_{k \ge N+1} \left[\rho_k e^{i\hat{\alpha}_k} \hat{\varphi}_k + \sigma_k e^{i\hat{\beta}_k} \hat{\psi}_k\right] \in X$, we define

$$h(x) = x^0 + \sum_{k \ge N+1} \left[\rho_k e^{i\bar{k}\hat{\alpha}_k} \hat{\varphi}_k + \sigma_k e^{i\bar{k}\hat{\beta}_k} \hat{\psi}_k\right].$$

About this map h, we have

LEMMA 3.3. 1°. *$h \in C(X, E)$ and is surjective.*

2°. *h is S^1-equivariant.*

3°. *$h(\partial B_\rho(X) \cap V_k(X)) = \partial B_\rho(E) \cap V_k(E)$, for all $\rho > 0$, $k \ge N + 1$.*

4°. *$h(x) = z$, if $x \in X^0$ and $x \sim z$.*

5°. *If $\{h(x_n)\}$ is convergent in E, $\{x_n\}$ is precompact in X.*

PROOF. 1°... 4° are direct consequences of the definition of h. Suppose $\{x_n\} \subset X$ and $h(x_n) \to z$ in E as $n \to \infty$. Write

$$x_n = \left(x_n^0;\ \rho_k(n),\ \alpha_k(n);\ \sigma_k(n),\ \beta_k(n)\right).$$

Since $2\pi\overline{k}\,(\rho_k(n) - \rho_k(m))^2 \leq \|h(x_n) - h(x_m)\|_E^2$, we get that $\{\rho_k(n)\}$, similarly $\{\sigma_k(n)\}$, is convergent for each $k \in \mathbb{N}$. Since $\{x_n^0\}$ is precompact, and $\alpha_k(n),\ \beta_k(n) \in [0, 2\pi]$, we can choose a sequence $\{n_j\}$ in $\mathbb{N}$ such that $\{x_{n_j}^0\}$, $\{\alpha_k(n_j)\}$ and $\{\beta_k(n_j)\}$ are convergent for every fixed $k \in \mathbb{N}$. Then $\{x_{n_j}\}$ is convergent in X. This proves 5°. □

We name the above map $h : X \to E$ by "id". With the aid of "id", we can define minimax structures now.

DEFINITION 3.4. For $j \in \mathbb{N}$, $j \geq N+1$, define Γ_j to be the family of such maps h, which satisfy the following conditions:

1°. $h \in C(D_j(X), E)$ and is S^1-equivariant.

2°. $h = id$ on $\left(\partial B_j(X) \cap V_j(X)\right) \cup \left(X^0 \cap D_j(X)\right) \equiv F_j(X)$.

3°. $P^- h(x) = \alpha(x) P^- id(x) + \beta(x)$, for all $x \in D_j(X)$, where $\alpha \in C\left(D_j(X), [1, \overline{\alpha}]\right)$, with $1 \leq \overline{\alpha} < +\infty$ depending on h, and α is S^1-invariant, $\beta \in C(D_j(X), E^-)$ is compact and S^1-equivariant, and $\beta = 0$ on $F_j(X)$.

4°. $h(D_j(X))$ is bounded in E.

DEFINITION 3.5. For $j \in \mathbb{N}$, $j \geq N+1$, define Λ_j to be the family of maps h, which satisfy

1°. $h \in C\left(D_{j+1}(X), E\right)$ and $h|_{D_j(X)} \in \Gamma_j$.

2°. $h = id$ on $\left(\partial B_{j+1}(X) \cap V_{j+1}(X)\right) \cup \left([B_{j+1}(X) \backslash B_j(X)] \cap V_j(X)\right) \equiv G_j(x)$.

3°. $P^- h(x) = \alpha(x) P^- id(x) + \beta(x)$, for any $x \in D_{j+1}(X)$, where $\alpha \in C\left(D_{j+1}(X),\ [1, \tilde{\alpha}]\right)$, with $1 \leq \tilde{\alpha} < +\infty$ depending on h, $\beta \in C\left(D_{j+1}(X),\ E^-\right)$ is compact and $\beta = 0$ on $G_j(X)$, and α, β are extensions of the corresponding maps defined in 1° via the definition of Γ_j.

4°. $h\left(D_{j+1}(X)\right)$ is bounded in E.

REMARK. $id \in \Gamma_j \cap \Lambda_j$ for any $j \geq N+1$.

LEMMA 3.6. *For $j \geq N+1$, any $h \in \Gamma_j$ can be extended to a map in Λ_j.*

PROOF. For $h \in \Gamma_j$, define $h = id$ on $G_j(X)$. This also extends α and β in 3° of Definition 3.4 of h, by $\alpha = 1$ and $\beta = 0$ on $G_j(X)$. Now we use Dugundji extension theorem [7] to extend α, β to the whole $D_{j+1}(X)$. Since by this theorem the image of the extension mapping is contained in the closed convex hull of the original image, $\alpha \in C\left(D_{j+1}(X),\ [1, \overline{\alpha}]\right)$ and $\beta \in C\left(D_{j+1}(X),\ E^-\right)$ is compact and 4° of Definition 3.5 holds. We use this theorem again to extend

P^+h, P^0h to the whole $D_{j+1}(X)$. ($P^0 : E \to E^0$ is the orthogonal projection). Then define $h = P^+h + P^-h + P^0h$. It is easy to check that $h \in \Lambda_j$. □

Now for $k \in \mathbb{N}$, $k \geq N+1$, we define

$$\mathcal{A}_k = \left\{\overline{h(D_j(X)\backslash Y)} \mid j \geq k,\ h \in \Gamma_j,\ Y \in \mathcal{X} \text{ with } \gamma(Y) \leq j-k\right\},$$

$$\mathcal{B}_k = \left\{\overline{h(D_{j+1}(X)\backslash Y)} \mid j \geq k,\ h \in \Lambda_j,\ Y \in \mathcal{X} \text{ with } \gamma(Y) \leq j-k\right\},$$

$$a_k = \inf_{A \in \mathcal{A}_k} \sup_{z \in A} J(z), \quad b_k = \inf_{B \in \mathcal{B}_k} \sup_{z \in B} J(z)$$

$$a_k(n) = \inf_{A \in \mathcal{A}_k} \sup_{z \in A} J_n(z), \quad b_k(n) = \inf_{B \in \mathcal{B}_k} \sup_{z \in B} J_n(z).$$

Since $id \in \Gamma_j \cap \Lambda_j$ and $0 \in A \cap B$ for all $A \in \mathcal{A}_j$, $B \in \mathcal{B}_\ell$,

$$-\infty < a_k,\ b_k,\ a_k(n),\ b_k(n) < +\infty.$$

4. - Properties of sequences of minimax values

We have

LEMMA 4.1. 1°. *$\{a_k\}$ and $\{b_k\}$ are increasing sequences.*

2°. *$\{a_k(n)\}$ and $\{b_k(n)\}$ are increasing sequences for fixed $n \in \mathbb{N}$.*

3°. *$a_k \leq b_k$, $a_k(n) \leq b_k(n)$, for all n, $k \in \mathbb{N}$.*

4°. *$a_k \leq a_k(n+1) \leq a_k(n)$, $b_k \leq b_k(n+1) \leq b_k(n)$, for all n, $k \in \mathbb{N}$.*

PROOF. 1° and 2° are due to the fact that $\mathcal{A}_{k+1} \subseteq \mathcal{A}_k$ and $\mathcal{B}_{k+1} \subseteq \mathcal{B}_k$. For any $B = \overline{h(D_{j+1}(X)\backslash Y)} \in \mathcal{B}_k$, let $A = \overline{h(D_j(X)\backslash Y)}$, then $A \subset B$ and $A \in \mathcal{A}_k$, so 3° holds. 4° follows from Corollary 2.17. □

LEMMA 4.2. *For any fixed $k \in \mathbb{N}$, $k \geq N+1$, we have*

1°. $\lim_{n\to\infty} a_k(n) = a_k$

2°. $\lim_{n\to\infty} b_k(n) = b_k$.

PROOF. We only prove 1°. 2° can be done similarly.

Given any $\varepsilon > 0$, by the definition of a_k, there is $A_0 \in \mathcal{A}_k$ such that $\sup_{z \in A_0} J(z) \leq a_k + \frac{\varepsilon}{3}$. So

$$a_k(n) \leq \sup_{z \in A_0} J_n(z) \leq a_k + \frac{\varepsilon}{2} + \sup_{z \in A_0} D_n(z), \text{ for all } n \in \mathbb{N}, \tag{4.3}$$

where $D_n(z) = J_n(z) - J(z)$. Since A_0 is bounded in E, by Lemma 2.3, $\sup_{z\in A_0} D_n(z) < +\infty$. Thus, for every $n \in \mathbb{N}$, there exists $z_n \in A_0$ such that

$$\sup_{z\in A_0} D_n(z) \le D_n(z_n) + \frac{\varepsilon}{3}. \tag{4.4}$$

Since E is compactly embedded into $L^1(S^1, \mathbb{R}^{2N})$, $\{z_n\}$ possesses a subsequence $\{z_{n_j}\}$ which converges to some z_0 in $L^1(S^1, \mathbb{R}^{2N})$. From real analysis (cf. [14]), $\{z_{n_j}\}$ has a subsequence which converges to z_0 almost everywhere. We still denote it by $\{z_{n_j}\}$.

Let

$$Q = \{t \in [0, 2\pi] \ |z_0(t)| < \infty\} \cap \left\{t \in [0, 2\pi] \ z_{n_j}(t) \to z_0(t) \text{ as } j \to \infty\right\},$$

then Q has Lebesgue measure 2π. For any $t \in Q$, there is $N_1(t) > 0$ such that $|z_{n_j}(t)| \le |z_0(t)| + 1$, for every $j \ge N_1(t)$. Choose $N_2(t) \ge N_1(t)$ such that $K_{n_j} \ge |z_0(t)| + 1$, for every $j \ge N_2(t)$, where $\{K_n\}$ is defined in Proposition 2.5. Then $H_{n_j}\left[z_{n_j}(t)\right] = H\left[z_{n_j}(t)\right]$, for every $j \ge N_2(t)$. This shows that $H(z_{n_j}) - H_{n_j}(z_{n_j}) \to 0$ almost everywhere as $j \to \infty$. Thus $H(z_{n_j}) - H_{n_j}(z_{n_j}) \to 0$ in measure as $j \to \infty$.

Since $\{z_{n_j}\}$ are bounded in E, by (H3) and Lemma 2.3,

$$\{H(z_{n_j}) - H_{n_j}(z_{n_j}) \mid j \in \mathbb{N}\}$$

are bounded in $L^2(S^1, \mathbb{R}^{2N})$. So by a theorem of De La Vallée-Poussin (Theorem VI.3.7 [14]) with $\Phi(u) = u$, $\{H(z_{n_j}) - H_{n_j}(z_{n_j}) \mid j \in \mathbb{N}\}$ have equi-absolute continuous integrals.

Now we can apply D. Vitali's theorem (Theorem VI.3.2 [14]) and get a constant $N_3 > 0$ such that

$$\left| \int_0^{2\pi} \left[H(z_{n_j}) - H_{n_j}(z_{n_j})\right] \mathrm{d}t \right| < \frac{2\varepsilon}{9}, \qquad \text{for every } j \ge N_3.$$

Combining with (4.4) and Lemma 2.14, we get

$$0 \le D_{n_j}(z_{n_j}) < \frac{\varepsilon}{3}, \qquad \text{for every } j \ge N_3.$$

Let $N_0 = n_{N_3}$, then combining with (4.3) yields $a_k(N_0) \le a_k + \varepsilon$. By Lemma 4.1 we get

$$a_k \le a_k(n) \le a_k(N_0) \le a_k + \varepsilon, \qquad \text{for every } n \ge N_0.$$

This completes the proof. □

5. - An upper estimate for the growth rate of $\{a_k\}$

By (2.27), we get

$$J(T_\theta z) \leq J(z) + \beta_9 \left[\log^{1/q_1}\left(|J(z)|+1\right)+1\right], \tag{5.1}$$

for all $z \in E$ and $\theta \in [0, 2\pi]$.

So there is $M_1 > 0$, depending only on β_9 and q_1, such that

$$J(z) + \beta_9 \left[\log^{1/q_1}\left(|J(z)|+1\right)+1\right] \leq 0, \text{ if } J(z) \leq -M_1. \tag{5.2}$$

We shall prove the following claim in §6: $a_k \to +\infty$ as $k \to \infty$. So there is a $k_0 \in \mathbb{N}$, $k_0 \geq N+1$, such that

$$a_k \geq M_1 + 1, \quad \text{for every } k \geq k_0. \tag{5.3}$$

PROPOSITION 5.4. *Assume that there is $k_1 \geq k_0$ such that $b_k = a_k$, for every $k \geq k_1$. Then there is $M = M(k_1) > 0$ such that*

$$a_k \leq Mk\,(\log\ k)^{1/q_1}, \quad \text{for every } k \geq k_1. \tag{5.5}$$

PROOF. Assuming the following inequality for a moment

$$\inf_{A\in\mathcal{A}_{k+1}} \sup_{z\in A} \left(\max_{\theta\in[0,2\pi]} J(T_\theta z)\right) \leq \inf_{B\in\mathcal{B}_k} \sup_{z\in B} \left(\max_{\theta\in[0,2\pi]} J(T_\theta z)\right), \tag{5.6}$$

for $k \geq k_1$, we get

$$a_{k+1} \leq \inf_{B\in\mathcal{B}_k} \sup_{z\in B} \left(\max_{\theta\in[0,2\pi]} J(T_\theta z)\right).$$

For any $\varepsilon > 0$, by the definition of b_k, there is a $B \in \mathcal{B}_k$ such that $\sup_{z\in B} J(z) \leq b_k + \varepsilon = a_k + \varepsilon$. For this B, using (5.1), (5.2), (5.3), we get that

$$J(T_\theta z) \leq a_k + \varepsilon + \beta_9 \left[\log^{1/q_1}(a_k+\varepsilon+1)+1\right],$$

for all $z \in B$ and $\theta \in [0, 2\pi]$.

Therefore there is $M_2 > 0$, depending only on β_9 and q_1, such that

$$a_{k+1} \leq a_k + \varepsilon + M_2 \left[\log^{1/q_1}(a_k+\varepsilon)+1\right].$$

Letting $\varepsilon \to 0$ yields

$$a_{k+1} \le a_k + M_2 \left(\log^{1/q_1} a_k + 1\right). \tag{5.7}$$

Write $\delta_k = a_k(k\ \log^p k)^{-1}$, $p = \frac{1}{q_1}$. We need to prove $\{\delta_k\}$ is bounded. Now (5.7) becomes

$$\begin{aligned}\delta_{k+1}(k+1)\log^p(k+1) \le \delta_k\ k\ \log^p k \\ +M_2\left\{\left[\log\ \delta_k + \log(k\ \log^p k)\right]^p + 1\right\}.\end{aligned} \tag{5.8}$$

If $\delta_{k+1} > \delta_k$, we get

$$\delta_k\ \log^{-p}\delta_k \le M_2 \cdot \frac{\left[1 + \frac{\log(k\ \log^p k)}{\log\ \delta_k}\right]^p + \frac{1}{\log\ \delta_k}}{k[\log^p(k+1) - \log^p k] + \log^p(k+1)}.$$

If $\delta_k > e$, then

$$\delta_k\ \log^{-p}\delta_k \le M_2\ \frac{(1 + \log\ k + p\ \log\ \log\ k)^p + 1}{\log^p(k+1)}.$$

Thus there is $M_3 > 0$, depending only on p and M_2, such that $\delta_k \le M_3$. Then, from (5.8), it is easy to see that there is a constant $M_4 > 0$, depending only on M_3 and p, such that $\delta_{k+1} \le M_4$.

Therefore $\delta_{k+1} \le \max\{\delta_k, M_4\}$, for every $k \ge k_1$. So $\delta_k \le \max\{\delta_{k_1}, M_4\}$, for every $k \ge k_1$. Let $M = \max\{\delta_{k_1}, M_4\}$. This yields (5.5). Therefore we reduce to the proof of (5.6), i.e.

LEMMA 5.9. *If L is a continuous S^1-invariant functional on E, then*

$$\inf_{A \in \mathcal{A}_{k+1}} \sup_{z \in A} L(z) \le \inf_{B \in \mathcal{B}_k} \sup_{z \in B} L(z), \qquad \textit{for every } k \ge N+1. \tag{5.10}$$

PROOF. Given any $B \in \mathcal{B}_k$, by the definition, there are $j \ge k$, $h_1 \in \Lambda_j$, $Y \in \mathcal{X}$, with $\gamma(Y) \le j - k$, such that $B = \overline{h_1(D_{j+1}(X)\backslash Y)}$. Let

$$U_j(X) = \left\{x \in D_{j+1}(X) \mid x = x' + \rho_{j+1}\hat{\varphi}_{j+1},\ x' \in V_j(X),\ \rho_{j+1} \ge 0 \right. \\ \left. \text{and } \|x\|_E \le R_{j+1}\right\}.$$

By the definition of h_1, for any $x \in U_j(X)$, $P^- h_1(x) = \alpha_1(x)P^- id(x) + \beta_1(x)$, where α_1 and β_1 are given by 3° of Definition 3.4. We define

$$\begin{aligned} \alpha(x) &= \alpha_1(x), & \beta(x) &= \beta_1(x), && \text{for all } x \in U_j(x) \\ \alpha(\hat{T}_\theta x) &= \alpha_1(x), & \beta(\hat{T}_\theta x) &= T_\theta\beta(x), && \text{for all } x \in U_j(X) \text{ and } \theta \in [0, 2\pi). \end{aligned}$$

Note that $D_{j+1}(X) = \bigcup_{\theta\in[0,2\pi]} \hat{T}_\theta U_j(X)$ and, for any given $y \in D_{j+1}(X)$, $\hat{T}_\theta x = y$ possesses a unique solution (x,θ) in $U_j(X) \times [0,2\pi)$. So α and β are well defined, $\alpha \in C\left(D_{j+1}(X),\ [1,\overline{\alpha}_1]\right)$ is S^1-invariant, $\beta \in C(D_{j+1}(X), E^-)$ is compact and S^1-equivariant.

Define

$$\begin{aligned} h^-(x) &= \alpha(x)P^-id(x) + \beta(x), && \text{for all } x \in D_{j+1}(X),\\ h^+(x) &= P^+h_1(x),\ h^0(x) = P^0h_1(x), && \text{for all } x \in U_j(x),\\ h^+(\hat{T}_\theta x) &= T_\theta h^+(x),\ h^0(\hat{T}_\theta x) = h^0(x), && \text{for all } x \in U_j(x) \\ &&& \text{and } \theta \in [0,2\pi), \end{aligned}$$

and

$$h(x) = h^+(x) + h^-(x) + h^0(x), \qquad \text{for all } x \in D_{j+1}(X).$$

Then $h \in \Gamma_{j+1}$, $h = h_1$ on $U_j(X)$, and $h(\hat{T}_\theta x) = T_\theta h(x)$ for all $x \in U_j(X)$ and $\theta \in [0,2\pi]$. Let $A = \overline{h(D_{j+1}(X)\backslash Y)}$ then $A \in \mathcal{A}_{k+1}$ and we have that

$$\sup_{z\in A} L(z) = \sup_{z\in\overline{h_1(U_j(X)\backslash Y)}} \left(\max_{\theta\in[0,2\pi]} L(T_\theta x)\right) \leq \sup_{z\in B} L(z).$$

This completes the proof of Lemma 5.9 and then Proposition 5.4. □

REMARK. Proposition 5.4 is a variant of Lemma 1.64 [18] in S^1-setting. Lemma 5.9 is new. The space X is introduced to get the unique expression $y = \hat{T}_\theta x$, for given $y \in D_{j+1}(X)$, in terms of (x,θ) in $U_j(X) \times [0,2\pi)$, which is crucial in the proof of Lemma 5.9.

6. - A lower estimate for the growth rate of $\{a_k\}$

In this section, we shall prove the following estimate on $\{a_k\}$.

PROPOSITION 6.1. *There are constants* $\lambda > 0$, $k_0 \geq N+1$ *such that*

$$a_k \geq \lambda\ k\ (\log\ k)^{2/q_2}, \qquad \textit{for every } k \geq k_0. \tag{6.2}$$

PROOF. We shall carry out the proof in several steps.

STEP 1. We consider a Hamiltonian system

$$\dot{z} = J\nabla F(|z|) \equiv \frac{F'(|z|)}{|z|}\ Jz \tag{6.3}$$

and its corresponding Lagrangian functional

$$\Phi_F(z) = \frac{1}{2}\ A(z) - \int_0^{2\pi} F(|z|)dt, \qquad \text{for all } z \in C^1(S^1, \mathbb{R}^{2N}).$$

By direct computation we have

LEMMA 6.4. *If the function F satisfies*

(F1) $F \in C^1([0,+\infty),\mathbb{R})$.

(F2) *Let* $g_F(t) = \frac{F'(t)}{t}$, *then* $g_F(0) = 0$, $\lim_{t\to+\infty} g_F(t) = +\infty$, *and* $g_F(t)$ *is strictly increasing.*

(F3) *Let* $h_F(t) = F'(t)t - 2F(t)$, *then* $h_F\left(g_F^{-1}(1)\right) > 0$ *and* $h_F(t)$ *is strictly increasing for* $t \geq g_F^{-1}(1)$.

Then

1°. *The solutions of* (6.3) *are all in* E^+ *and of the following form*

$$z_k(t) = \begin{pmatrix} \cos(kt)I & \sin(kt)I \\ \sin(kt)I & -\cos(kt)I \end{pmatrix} v_k,$$

for any $v_k \in \mathbb{R}^{2N}$, *with* $|v_k| = \gamma_k(F)$, $k \in \mathbb{N}$ *and* $v_0 = 0$, *where* $\gamma_k(F) = g_F^{-1}(k)$ *satisfies* $\gamma_k(F) \to +\infty$ *as* $k \to +\infty$ *and* $0 = \gamma_0(F) < \gamma_k(F) < \gamma_{k+1}(F)$ *for* $k \in \mathbb{N}$, I *is the identity matrix on* $\mathbb{R}^N$.

2°. *Let* $d_0(F) = 0$, $d_k(F) = \Phi_F(z_k)$, *for all* $k \in \mathbb{N}$, *then* $d_k(F) = \pi h_F(\gamma_k) > 0$, *is strictly increasing in* k.

STEP 2. We define a function $G : [0,+\infty) \to \mathbb{R}$ by

$$G(t) = \alpha \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!}\, t^{kq} = \alpha\, \exp(\tau t^q) - \alpha \sum_{k=0}^{n} \frac{\tau^k}{k!}\, t^{kq},$$

where $\alpha = 2\alpha_2$, $\tau = \tau_2$, $q = q_2$, α_2, τ_2, q_2 are given by (H3), and n is the smallest positive integer such that $n \geq \left[\frac{5}{q}\right] + 1$ and $\tau\alpha q \left(\frac{\tau q}{4}\right)^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!}\left(\frac{4}{q}\right)^k < 1$.

Then it is easy to see G satisfies (F1) and (F2). Since

$$\text{(6.5)} \qquad h_G(t) \equiv G'(t)t - 2G(t) = \alpha q\, \frac{\tau^{n+1}}{n!}\, t^{(n+1)q} + \alpha(\tau q t^q - 2) \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!}\, t^{kq},$$

when $t \geq \left(\frac{2}{\tau q}\right)^{1/q}$, $h_G(t) > 0$ and is strictly increasing.

Write $\gamma_k = \gamma_k(G)$ for $k \in \mathbb{N}$, we claim that $\gamma_1 > \left(\frac{4}{\tau q}\right)^{1/q}$. For otherwise, we have the following contradiction

$$1 = g_G(\gamma_1) = \alpha q \sum_{k=n+1}^{\infty} \frac{\tau^k}{(k-1)!}\, \gamma_1^{kq-2} \leq \tau\alpha q \left(\frac{\tau q}{4}\right)^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{4}{q}\right)^k < 1.$$

Therefore G satisfies (F3), and we also have that

$$G'(t)t - 4G(t) = \alpha q \frac{\tau^{n+1}}{n!} t^{(n+1)q} + \alpha(\tau q t^q - 4) \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!} t^{kq} > 0 \tag{6.6}$$

and is strictly increasing for $t \geq \left(\frac{4}{\tau q}\right)^{1/q}$. It is also easy to see that there is $t_1 > 0$ such that

$$0 \leq G(t) \leq t^4, \qquad \text{for any } t \in [0, t_1]. \tag{6.7}$$

We consider the Hamiltonian system

$$\dot{z} = J\nabla G(|z|) \equiv \frac{G'(|z|)}{|z|} Jz \tag{6.8}$$

and write $\Phi = \Phi_G$, $d_k = d_k(G)$, for all $k \in \{0\} \cup \mathbf{N}$. Besides properties described in Lemma 6.4, we also have

LEMMA 6.9. *There are* $\lambda_1 > 0$, $k_1 \in \mathbf{N}$ *such that*

$$d_k \geq \lambda_1 \; k \; (\log \; k)^{2/q}, \qquad \textit{for any } k \geq k_1. \tag{6.10}$$

PROOF. From $g_G(\gamma_k) = k$, we have

$$\tau\alpha q \; \gamma_k^{q-2} \; \exp(\tau\gamma_k^q) - \alpha q \sum_{j=1}^{n} \frac{\tau^j}{(j-1)!} \gamma_k^{jq-2} = k,$$

so

$$\tau\gamma_k^q \geq \log \; k + (2-q)\log \; \gamma_k - \log(\alpha\tau q).$$

Thus there is $k_2 \in \mathbf{N}$ such that

$$\gamma_k \geq \left(\frac{1}{2\tau} \log \; k\right)^{1/q}, \qquad \text{for any } k \geq k_2.$$

Since $G(t) = \left[\frac{G'(t)}{t} - \tau\alpha q t^{(n+1)q-2} \frac{\tau^n}{n!}\right] \frac{1}{\tau q} t^{2-q}$, we get

$$d_k = \pi\left(G'(\gamma_k)\gamma_k - 2G(\gamma_k)\right) \geq \pi\left(k\gamma_k^2 - \frac{2k}{\tau q}\gamma_k^{2-q}\right) = \pi k\gamma_k^2\left(1 - \frac{2}{\tau q}\gamma_k^{-q}\right).$$

So there is $k_1 \geq k_2$ such that, for any $k \geq k_1$,

$$d_k \geq \frac{\pi}{2} k\gamma_k^2 \geq \lambda_1 \; k \; (\log \; k)^{2/q},$$

where $\lambda_1 = \frac{\pi}{2}\left(\frac{1}{2\tau}\right)^{2/q}$. □

For $k \in \mathbb{N}$, $k \geq N+1$, we define

$$c_k = \inf_{A \in \mathcal{A}_k} \sup_{z \in A} \Phi(z).$$

Since $\Phi \in C(E, \mathbb{R})$, (6.6) and $0 \in \bigcap_{A \in \mathcal{A}_k} A$, we get $-\infty < c_k < +\infty$. From $\mathcal{A}_{k+1} \subset \mathcal{A}_k$, we get

$$c_k \leq c_{k+1}, \qquad \text{for any } k \geq N+1. \tag{6.11}$$

By (H3) and the definition of G, there is a constant $\varsigma_1 > 0$ such that

$$H(z) + \frac{1}{2}\,|z|^2 \leq G(|z|) + \varsigma_1, \qquad \text{for all } z \in \mathbb{R}^{2N}.$$

For $z \in E$, by Hölder's inequality, we get, for $\varsigma_2 = \varsigma_1 + \frac{1}{2}\,\|f\|_{L^2}^2$,

$$J(z) \geq \frac{1}{2}\,A(z) - \int_0^{2\pi} \left(H(z) + \frac{1}{2}\,|z|^2\right) dt - \frac{1}{2}\,\|f\|_{L^2}^2 \geq \Phi(z) - \varsigma_2.$$

So we get

$$a_k \geq c_k - \varsigma_2, \qquad \text{for any } k \geq N+1. \tag{6.12}$$

STEP 3. Let $m_0 = [\gamma_1] + 1$ and define, for $m \geq m_0$,

$$G_m(t) = \begin{cases} G(t), & \text{if } 0 \leq t \leq m \\ \dfrac{G'(m)}{4m^3}\, t^4 + G(m) - \frac{m}{4}\, G'(m), & \text{if } m < t. \end{cases}$$

Then G_m satisfies (F1) and (F2). For $n < t$,

$$G_m'(t)t - 2G_m(t) = \frac{G'(m)}{m^3}\, t^4 + \frac{1}{2}\,[G'(m)m - 4G(m)]. \tag{6.13}$$

By (6.6) and the properties of G, G_m satisfies (F3). So Lemma 6.4 holds for G_m. Since

$$G_m'(t)t - 4G_m(t) = \frac{G'(m)}{m^3}\, t^4 + \left(G'(m)m4G(m)\right), \qquad \text{for any } t > m,$$

we get that

$$0 < G_m(t) \leq G_{m+1}(t) \leq G(t), \qquad \text{for any } t \geq 0$$

and

$$0 < 4G_m(t) \leq G_m'(t)t, \qquad \text{for any } t \geq r_1 \equiv \left(\frac{4}{\tau q}\right)^{1/q}.$$

From the definition of G_m and (6.7), there is a constant $\sigma_m > 0$, depending on m, such that

$$G_m(t) \leq \sigma_m t^4, \qquad \text{for every } t \geq 0. \tag{6.14}$$

We consider

$$\dot{z} = J\nabla G_m(|z|) \equiv \frac{G'_m(|z|)}{|z|}\, Jz \tag{6.15}$$

and write $\Phi_m = \Phi_{G_m}$, $d_k(m) = d_k(G_m)$. Then $\Phi_m \in C^1(E, \mathbb{R})$ and satisfies (P.S) condition. Define

$$c_k(m) = \inf_{A \in \mathcal{A}_k} \sup_{z \in A} \Phi_m(z), \qquad \text{for all } k \geq N+1,\ m \geq m_0.$$

Then we have $-\infty < c_k(m) < +\infty$. To get more accurate estimates on $c_k(m)$, we need

LEMMA 6.16. *Let* $N+1 \leq j \leq m$, $0 < \rho < R_m$, $h \in \Gamma_m$ *and*

$$Q \equiv \{x \in D_m(X) \mid h(x) \in \partial B_\rho(E) \cap V_{j-1}^{\perp}(E)\}.$$

Then Q *is compact and* $\gamma(Q) \geq m - j + 1$.

PROOF. This Lemma is a variant of Proposition 1.19 [19]. Note that firstly $id(x_i^-) = P^- id(x_i)$ by 5° of Lemma 3.3, $\{P^- id(x_i)\}$ being convergent implies that $\{x_i^-\}$ has a convergent subsequence. This yields that Q is compact. Secondly, from the definition of $h \in \Gamma_m$, if we let $E_k = E^+_{N+1,k} \oplus E^-_{N+1,k} \oplus E^0$, $X_k = X^+_{N+1,k} \oplus X^-_{N+1,k} \oplus X^0$ and $P_k : E \to E_k$ be the orthogonal projection, then $P_k h(x) = z$ for $z \sim x \in X^0 \cap D_m(X)$ and

$$\begin{aligned} P_k h[\partial B_m(X) \cap V_m(X) \cap X_k] &= P_k id[\partial B_m(X) \cap V_m(X) \cap X_k] \\ &= \partial B_m(E) \cap V_m(E) \cap E_k. \end{aligned}$$

This allows us to apply Borsuk-Ulam theorem [8]. Therefore we can go through the proof of Proposition 1.19 [9]. We omit the details here. □

COROLLARY 6.17. *Let* $N+1 \leq j \leq m$, $0 < \rho < R_m$, $h \in \Gamma_m$. *For any* $Y \in \mathcal{X}$, *with* $\gamma(Y) \leq m - j$,

$$\overline{h(D_m(X)\backslash Y)} \cap \partial B_\rho(E) \cap V_{j-1}^{\perp}(E) \neq \emptyset.$$

LEMMA 6.18. $c_k(m) > 0$, *for any* $k \geq N+1$, $m \geq m_0$.

PROOF. Fix $m \geq m_0$, $k \geq N+1$, by Corollary 6.17, for any $A \in \mathcal{A}_k$ and $0 < \rho < R_k$, there is a $z \in A \cap \partial B_\rho \cap E^+$. Let C denote the embedding constant from E into L^4, then by (6.14)

$$\Phi_m(z) \geq \frac{1}{2} A(z) - \sigma_m \int_0^{2\pi} |z|^4 dt \geq \frac{1}{2} \rho^2 - \sigma_m C\rho^4 = \frac{1}{2} \rho^2(1 - \sigma_m C\rho^2).$$

Choose $\rho_m = \min\{1, (2\sigma_m C)^{-1/2}\}$, we get $c_k(m) \geq \frac{1}{2} \rho_m^2 > 0$. □

LEMMA 6.19. *For any* $k \geq N+1$, $m \geq m_0$,

1°. $c_k(m)$ *is a critical value of* Φ_m.

2°. *Any critical point of* Φ_m, *corresponding to* $c_k(m)$, *lies in* $E \backslash E^0$.

3°. *If* $c_{k+1}(m) = \ldots = c_{k+j}(m) \equiv c$ *and* $\mathcal{K} \equiv (\Phi_m')^{-1}(0) \cap \Phi_m^{-1}(c)$, *then* $\gamma(\mathcal{K}) \geq j$.

PROOF. 1° and 3° follows from the standard argument, we refer to [19]. 2° follows from 1°, Lemma 6.4 and Lemma 6.18. We omit details here. □

LEMMA 6.20. *For* $k \in \mathbb{N}$, $k \geq N+1$, $m \geq m_0$, *we have* $c_k \leq c_k(m+1) \leq c_k(m)$ *and* $\lim_{m\to\infty} c_k(m) = c_k$.

PROOF. The Lemma follows from the proofs of Lemma 4.1 and 4.2. □

STEP 4. PROOF OF PROPOSITION 6.1.

Fix $k \geq N+1$, for any $m \geq m_0$, by 1° of Lemma 6.19 and Lemma 6.18, $c_k(m) = d_j(m)$ for some $j > 0$. So

$$\begin{aligned} c_k(m) &= \Phi_m(z_j) = \pi[G_m'(\gamma_j)\gamma_j - 2G_m(\gamma_j)] \\ &\geq \min\left\{\pi \frac{G'(m)}{m^3} \gamma_j^4,\ \pi\alpha q \frac{\tau^{n+1}}{n!} \gamma_j^{(n+1)q}\right\}; \end{aligned}$$

here we used (6.5) and (6.13). Since by definition of G, $\frac{G'(t)}{t^3}$ is strictly increasing for $t > 0$, by Lemma 6.20 we get

$$c_k(m_0) \geq c_k(m) \geq \min\left\{\pi \frac{G'(m_0)}{m_0^3} \gamma_j^4,\ \pi\alpha q \frac{\tau^{n+1}}{n!} \gamma_j^{(n+1)q}\right\}.$$

So there exists $M_1 > 0$, independent of m, such that if z is a critical point of Φ_m corresponding to $c_k(m)$ with $m \geq m_0$, then $\|z\|_C \leq M_1$. Thus $G_m(|z|) = G(|z|)$ for $m \geq m_1(k) \equiv \max\{m_0, [M_1]+1\}$, and then there exists $j(m) \in \mathbb{N}$, depending on m, such that

$$c_k(m) = d_{j(m)}, \qquad \text{for any } m \geq m_1(k). \tag{6.21}$$

By Lemma 6.20, $0 < d_k \leq d_{k+1}$, and (6.10), we get $c_k = d_j$ for some $j \in \mathbb{N}$. Therefore $\{c_k\}$ is a subset of $\{d_k\}$.

We claim that $c_{k+N} > c_k$ for all $k \geq N+1$. If not, by (6.11), we get $c \equiv c_k = \ldots = c_{k+N}$. By (6.21) there exists $m \geq m_0$, depending on k, such that $c = c_k(m) = \ldots = c_{k+N}(m)$. Let $\mathcal{K} = (\Phi'_m)^{-1}(0) \cap \Phi_m^{-1}(c)$. By 3° of Lemma 6.19, $\gamma(\mathcal{K}) \geq N+1$. But 1° of Lemma 6.4 and 4° of Lemma 3.2 show that $\gamma(\mathcal{K}) = N$. This contradiction proves the claim.

Assume $c_{N+1} = d_\ell$ for some $\ell \in \mathbb{N}$, then by the above discussion and (6.10), for $k \geq \max\{k_1, 6N\}$,

$$\begin{aligned} c_k &\geq c_{N+1+[\frac{k-N-2}{N}]N} \geq d_{\ell+[\frac{k-N-2}{N}]} \\ &\geq \lambda_1 \left(\ell + \left[\frac{k-N-2}{N}\right]\right) \log^{2/q}\left(\ell + \left[\frac{k-N-2}{N}\right]\right) \\ &\geq \lambda_1 \left(\ell + \frac{k}{N} - 3\right) \log^{2/q} \left(\ell + \frac{k}{N} - 3\right) \geq \lambda_1 \, \frac{k}{2N} \, \log^{2/q} \left(\frac{k}{2N}\right) \\ &\geq \lambda \; k \; (\log \; k)^{2/q} \end{aligned}$$

for some $\lambda > 0$. Combining with (6.12), we get (6.2).

The proof of Proposition 6.1 is complete. □

7. - The existence of critical values of J_n

Fix $n,\ k \in \mathbb{N},\ k \geq N+1$, we have

PROPOSITION 7.1. *Suppose* $b_k(n) > a_k(n) \geq \beta_8$. *Let* $\delta_k(n) \in (0, b_k(n) - a_k(n))$ *and*

$$\mathcal{B}_k[n, \delta_k(n)] = \left\{ \overline{h[D_{j+1}(X)\backslash Y]} \in \mathcal{B}_k \mid J_n[h(x)] \leq a_k(n) + \delta_k(n), \right. \\ \left. \text{for } x \in D_j(X)\backslash Y \right\}.$$

Let

$$b_k[n, \delta_k(n)] = \inf_{B \in \mathcal{B}_k[n,\delta_k(n)]} \sup_{z \in B} J_n(z).$$

Then $b_k[n, \delta_k(n)]$ *is a critical value of* J_n.

REMARK. $b_k[n, \delta_k(n)] \geq b_k(n)$. By Lemma 3.6, $\mathcal{B}_k[n, \delta_k(n)] \neq \emptyset$, and $b_k[n, \delta_k(n)] < +\infty$.

For the proof of Proposition 7.1, we need the following "Deformation Theorem", which was proved in [19].

LEMMA 7.2. *Let* J_n *be as above, then if* $b > \beta_8$, $\bar{\varepsilon} > 0$ *and* b *is not a critical value of* J_n, *there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta \in C([0,1] \times E,\ E)$ *such that*

1°. $\eta(t,z) = z$, *if* $z \notin J_n^{-1}(b - \bar{\varepsilon}, b + \bar{\varepsilon})$.

2°. $\eta(0,z) = z$, *for any* $z \in E$.

3°. $\eta(1, [J_n]^{b+\varepsilon}) \subset [J_n]^{b-\varepsilon}$, *where* $[J_n]^a = \{z \in E \mid J_n(z) \leq a\}$.

4°. $P^-\eta(1,z) = \alpha_1(z)z^- + \beta_1(z)$, *for any* $z \in E$, *where* $\alpha_1 \in C(E,[1,e^2])$, $\beta_1 \in C(E,E^-)$ *and* β *is compact.*

5°. $\eta(t,\cdot)$ *is a bounded map from* E *to* E *for* $t \in [0,1]$.

PROOF OF PROPOSITION 7.1. Let $\bar{\varepsilon} = \frac{1}{2}\,[b_k(n) - a_k(n)] > 0$. If $b_k[n, \delta_k(n)]$ is not a critical value of J_n, then there exist ε and η as in Lemma 7.2. Choose $B \in \mathcal{B}_k[n, \delta_k(n)]$ such that

$$\sup_{z\in B} J(z) \leq b_k[n, \delta_k(n)] + \varepsilon.$$

Then there exist $j \geq k$, $h_0 \in \Lambda_j$, $Y \in \mathcal{X}$, with $\gamma(Y) \leq j - k$, such that $B = h_0[\overline{D_{j+1}(X)\backslash Y}]$. Define

$$\begin{aligned} h(x) &= \eta[1, h_0(x)], && \text{for all } x \in \overline{D_{j+1}(X)\backslash Y} \equiv Q_1\\ h(x) &= h_0(x), && \text{for all } x \in B_{j+1}(X) \cap V_j(X) \cap Y \equiv Q_2\\ h(x) &= id(x), && \text{for all } x \in \partial B_{j+1}(X) \cap V_{j+1}(X) \equiv Q_3. \end{aligned}$$

Denote $Q = Q_1 \cup Q_2 \cup Q_3$.

For $x \in \overline{D_j(X)\backslash Y}$,

$$J_n[h_0(x)] \leq a_k(n) + \delta_k(n) \leq b_k(n) - 2\bar{\varepsilon} < b_k[n, \delta_k(n)] - \bar{\varepsilon}.$$

Thus

$$h(x) \equiv \eta[1, h_0(x)] = h_0(x), \qquad \text{for any } x \in \overline{D_j(X)\backslash Y}. \tag{7.3}$$

For $x \in Q_3 \cup \{[B_{j+1}(X)\backslash B_j(X)] \cap V_j(X)\}$,

$$J_n[h_0(x)] \leq 0 < b_k[n, \delta_k(n)] - \bar{\varepsilon},$$

thus $\eta[1, h_0(x)] = h_0(x) = id(x)$. So $h \in C(Q, E)$.

For $x \in Q_4 \equiv [D_{j+1}(X) \cap V_j(X)] \cup Q_3$,

$$P^-[h(x)] = P^-h_0(x) = \alpha_0(x)P^-id(x) + \beta_0(x),$$

where α_0, β_0 are defined for h_0 in 3° of Definition 3.5. For $x \in Q\backslash Q_4$,

$$P^-h(x) = \alpha_1[h_0(x)]\alpha_0(x)P^-id(x) + \alpha_1[h_0(x)]\beta_0(x) + \beta_1[h_0(x)].$$

Define

$$\alpha(x) = \begin{cases} \alpha_0(x), & \text{if } x \in Q_4, \\ \alpha_1[h_0(x)]\alpha_0(x), & \text{if } x \in Q\backslash Q_4, \end{cases}$$

$$\beta(x) = \begin{cases} \beta_0(x), & \text{if } x \in Q_4, \\ \alpha_1[h_0(x)]\beta_0(x) + \beta_1[h_0(x)], & \text{if } x \in Q\backslash Q_4. \end{cases}$$

Then $\alpha \in C(Q, [1, e^2\tilde{\alpha}_0])$, $\beta \in C(Q, E^-)$ is compact and

$$P^-h(x) = \alpha(x)P^-id(x) + \beta(x), \qquad \text{for any } x \in Q.$$

Let $W = [D_{j+1}(X) \cap Y]\backslash V_j(X)$, then $\partial W \subset Q$, where "∂" is taken within $V_{j+1}(X)$. Since α, β, P^+h, and P^0h are continuously defined on ∂W, we may use the Dugundji extension theorem [7] to extend them to $\overline{W}$, then define $P^-h(x) = \alpha(x)P^-id(x) + \beta(x)$, and $h(x) = P^+h(x) + P^-h(x) + P^0h(x)$. We have $h \in \Lambda_j$. Thus $D \equiv \overline{h[D_{j+1}(X)\backslash Y]} \in \mathcal{B}_k$. By (7.3)

$$J_n[h(x)] = J_n[h_0(x)] \le a_k(n) + \delta_k(n), \qquad \text{for all } x \in \overline{D_j(X)\backslash Y}.$$

Thus $D \in \mathcal{B}_k[n, \delta_k(n)]$. Now 3° of Lemma 7.2 yields

$$\sup_{z \in D} J_n(z) \le b_k[n, \delta_k(n)] - \varepsilon.$$

This contradicts to the definition of $b_k[n, \delta_k(n)]$. Therefore the proof is complete. □

8. - The proofs of the main theorems

PROOF OF THEOREM 1.2. We prove Theorem 1.2 by contradiction. Assume that the functional I is bounded from above by $M_1 > 0$ on S, the solution set of (1.1).

Since $q_2 < 2q_1$, Propositions 5.4 and 6.1 show that there exists $k \in \mathbb{N}$ such that

$$b_k > a_k \ge \max\{\beta_8, M_1\}.$$

Let $\varepsilon = \frac{1}{5}(b_k - a_k)$. By Lemmas 4.1 and 4.2, there exists $n_1 > 0$ such that

$$b_k(n) - a_k(n) \ge 4\varepsilon \text{ and } a_k(n) \le a_k + \varepsilon, \qquad \text{for any } n \ge n_1.$$

Let $\delta_k(n_1) = \varepsilon$, and $\delta_k(n) = 2\varepsilon$ for $n > n_1$. Then by Proposition 7.1, $b_k[n, \delta_k(n)]$ is a critical value of J_n for $n \ge n_1$.

If $\overline{h[D_{j+1}(X)\backslash Y]} \in \mathcal{B}_k(n_1, \varepsilon)$ then for any $x \in D_j(X)\backslash Y$

$$J_n[h(x)] \le J_{n_1}[h(x)] \le a_k(n_1) + \varepsilon \le a_k + 2\varepsilon \le a_k(n) + 2\varepsilon, \text{ for any } n > n_1.$$

So $\mathcal{B}_k(n_1,\varepsilon) \subset \mathcal{B}_k(n,2\varepsilon)$, for all $n > n_1$. Therefore for $n > n_1$,

$$b_k(n,2\varepsilon) \le \inf_{B\in\mathcal{B}_k(n_1,\varepsilon)} \sup_{z\in B} \left[\frac{1}{2}\, A(z) - \int_0^{2\pi} P_0(z)dt + \int_0^{2\pi} |f\cdot z|dt\right] \equiv b < +\infty,$$

where $P_0(z) = \alpha_0|z|^{\mu\sigma} - \beta_0$ and we used (2.6).

Let z_n be a critical point of J_n corresponding to $b_k(n,2\varepsilon)$ for $n > n_1$. Using (2.6), $f \in W^{1,2}(S^1,\mathbb{R}^{2N})$ and the proof of Lemma 5.3 [2], we get

$$\|z_n\|_{L^\infty} \le M_2, \qquad \text{for every } n > n_1, \tag{8.1}$$

where the constant $M_2 > 0$ depending on b, but independent of n. Now we choose $n_2 > n_1$ such that $K_{n_2} > M_2$, where $\{K_n\}$ is defined in Proposition 2.5. From (8.1) and Lemma 2.19, we get that $H'_{n_2}(z_{n_2}) = H'(z_{n_2})$ on $[0,2\pi]$ and z_{n_2} is a solution of (1.1), i.e. $z_{n_2} \in S$. But

$$I(z_{n_2}) = I_{n_2}(z_{n_2}) = J_{n_2}(z_{n_2}) = b_k(n_2,2\varepsilon) \ge b_k(n_2) > a_k(n_2) \ge a_k > M_1.$$

This contradicts to the definition of M_1, and completes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. The proof of Theorem 1.3 is similar. Instead of (5.5) and (6.2), we shall have "$a_k \le Mk^{(p_1+1)/p_1}$, for all $k \ge k_1$" and "$a_k \ge \alpha k^{(p_2+1)/(p_2-1)}$", by (H4) they yield "$b_k > a_k$ for infinitely many k". The proof is rather simpler than that of Theorem 1.2. For example, the corresponding

$$\Phi(z) = \frac{1}{2}\, A(z) - \left(\alpha_2 + \frac{1}{2}\right) \int_0^{2\pi} |z|^{p_2+1}dt$$

is C^2 and satisfies (P.S.) condition. So the lower estimate for a_k is quite straightforward. For the details we refer to [13]. □

In [15], Pisani and Tucci gave a result for (1.1):

THEOREM 8.2 (Theorem 1.1 [15]). *Let H satisfy* (H1) *and the following conditions*:

(H5)
$$\lim_{|z|\to+\infty} \frac{H'(z)\cdot z}{|z|^2} = +\infty;$$

(H6) *There are constants* $p \ge 1$, α_1, $\beta_1 > 0$ *such that*

$$\frac{1}{2}\, H'(z)\cdot z - H(z) \ge \alpha_1|z|^{p+1} - \beta_1, \qquad \text{for any } z \in \mathbb{R}^{2N};$$

(H7) *There are constants* $q \in [p, p+1)$ *and* α_2, $\beta_2 > 0$ *such that*

$$|H'(z)| \leq \alpha_2 |z|^q + \beta_2, \qquad \text{for any } z \in \mathbb{R}^{2N}.$$

Then the conclusion of Theorem 1.3 *holds for given* $T > 0$ *and* T*-periodic function* $f \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2N})$.

One difficulty in the proof of this theorem is caused by the S^1-action on $W^{1/2,2}(S^1, \mathbb{R}^{2N})$. Using the minimax idea introduced in §3, this difficulty can be overcome as in the proof of our Lemma 5.9. Condition (H7) allows us to carry out the proof without doing any truncation on H, so the function f can be allowed only in L^2 and the proof becomes rather simpler. We omit the details here.

We also refer readers to a related density result proved earlier.

THEOREM 8.3 (Theorem 1.5 [11]). *Let* H *satisfy* (H1) *and* (H5), *then for any* $T > 0$, *there exists a dense set* D *in the space of* T*-periodic functions in* $L^2([0,T], \mathbb{R}^{2N})$ *such that, for every* $f \in D$, (1.1) *is solvable.*

This theorem poses a natural question whether the condition (H3) or (H4) is necessary in Theorems 1.2 or 1.3.

9. - Results for general forced systems

In this section we consider the general Hamiltonian system

$$\dot{z} = J\hat{H}_z(t, z). \tag{9.1}$$

Firstly we consider (9.1) with bounded perturbations. That is

THEOREM 9.2. *Let* $\hat{H}$ *satisfy the following conditions*:

(G1) $\hat{H} \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ *and* $\hat{H}(t, z)$ *is* T*-periodic in* t;

(G2) *There exist* $H : \mathbb{R}^{2N} \to \mathbb{R}$, *satisfying* (H1), (H2), *and constants* $0 < q < 2$, α, $\tau > 0$, $\beta \geq 0$ *such that*

1°. $H(z) \leq \alpha e^{\tau |z|^q} + \beta$, *for every* $z \in \mathbb{R}^{2N}$;

2°. $|\hat{H}(t, z) - H(z)| \leq \alpha$, *for every* $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$;

3°. $|\hat{H}_z(t, z) - H_z(z)| \leq \alpha(|z|^{p-1} + 1)$, *for every* $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,
$|\hat{H}_t(t, z)| \leq \alpha(|z|^p + 1)$, *for every* $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,
where $1 \leq p < \mu$ *and* $\mu > 2$ *is defined in* (H2).

Then the system (9.1) *possesses infinitely many distinct* T*-periodic solutions.*

REMARK. Theorem 9.2 weakened conditions of Bahri and Berestycki's corresponding result, Theorem 10.1 [2], which required $\hat{H}$ satisfying the following conditions:

1°. $\hat{H} \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and is T-periodic in t;

2°. There exist $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ satisfying (H2), and constants $q > 1$, $\alpha > 0$ such that

$$H(z) \le \alpha(|z|^{q+1} + 1), \qquad \text{for every } z \in \mathbb{R}^{2N}$$

and

$$\|\hat{H} - H\|_{C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})} < \infty.$$

In order to prove Theorem 9.2, we let $G(t, z) = \hat{H}(t, z) - H(z)$, and consider functionals

$$J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z)\mathrm{d}t - \psi(z) \int_0^{2\pi} G(t, z)\mathrm{d}t$$

and

$$J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z)\mathrm{d}t - \psi_n(z) \int_0^{2\pi} G(t, z)\mathrm{d}t,$$

where H_n, ψ_n are defined in Section 2, and we can go through the proofs in Sections 2-7 with the following estimates for $\{a_k\}$ from above.

LEMMA 9.3. *If $b_k = a_k$, for every $k \ge k_1$, then there is $M = M(k_1) > 0$ such that*

$$a_k \le Mk, \qquad \text{for every } k \ge k_1. \tag{9.4}$$

PROOF. Using 2° of (G2), instead of (2.27), we get that

$$|J(z) - J(T_\theta z)| \le 4\pi\alpha, \qquad \text{for every } z \in E.$$

So, as in the proof of Proposition 5.4, we get that

$$\begin{aligned} a_{k+1} &\le \inf_{A \in \mathcal{A}_{k+1}} \sup_{z \in A} \left(\max_{\theta \in [0, 2\pi]} J(T_\theta z) \right) \\ &\le \inf_{B \in \mathcal{B}_k} \sup_{z \in B} \left(\max_{\theta \in [0, 2\pi]} J(T_\theta z) \right) \\ &\qquad\qquad\qquad \text{(by Lemma 5.9)} \\ &\le \inf_{B \in \mathcal{B}_k} \sup_{z \in B} J(z) + 4\pi\alpha = b_k + 4\pi\alpha = a_k + 4\pi\alpha, \text{ for every } k \ge k_1. \end{aligned}$$

Let $\delta_k = \frac{a_k}{k}$. If $\delta_{k+1} > \delta_k$, then from the above inequality

$$(k+1)\delta_{k+1} \leq k\delta_k + 4\pi\alpha \leq k\delta_{k+1} + 4\pi\alpha,$$

so

$$\delta_{k+1} \leq 4\pi\alpha.$$

This shows that

$$\delta_{k+1} \leq \max\{\delta_k,\ 4\pi\alpha\}, \qquad \text{for every } k \geq k_1.$$

Thus

$$\delta_k \leq \max\{\delta_{k_1},\ 4\pi\alpha\}, \qquad \text{for every } k \geq k_1.$$

Let $M = \max\left\{\frac{a_{k_1}}{k_1},\ 4\pi\alpha\right\}$, we get (9.4), and this completes the proof of Lemma 9.3. □

Now the arguments in §8 yield Theorem 9.2.

Secondly, it is not difficult to get direct generalizations of Theorems 1.2 and 1.3 for (9.1).

THEOREM 9.5. *Let* $\hat{H}$ *satisfy conditions* (G1) *and*

(G3) *There exists* $H : \mathbb{R}^{2N} \to \mathbb{R}$ *satisfying* (H1), (H2) *and* $\alpha, p, q > 0$ *such that*

$$\begin{aligned}
|\hat{H}(t,z) - H(z)| &\leq \alpha(|z|^p + 1), && \textit{for every } (t,z) \in \mathbb{R} \times \mathbb{R}^{2N},\\
|\hat{H}_z(t,z) - H_z(z)| &\leq \alpha(|z|^{q-1} + 1), && \textit{for every } (t,z) \in \mathbb{R} \times \mathbb{R}^{2N},\\
|\hat{H}_t(t,z)| &\leq \alpha(|z|^q + 1), && \textit{for every } (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}.
\end{aligned}$$

where $0 < q < \mu$, *and either*

1°. H *satisfying* (H3) *and* $1 \leq p < \min\ \{2q_1/q_2, \mu\}$,

or

2°. H *satisfying* (H4) *and* $1 \leq p < \min\ \{2(p_1+1)/(p_2+1), \mu\}$.

Then the system (9.1) *possesses infinitely many distinct* T*-periodic solutions.*

We omit the details here.

Appendix - Monotone truncations of H in $C^1(\mathbb{R}^{2N}, \mathbb{R})$

In this appendix, we give a proof of Proposition 2.5.

Recall that $\sigma \in (0,1)$, $\mu\sigma > 2$, and $r_0 \geq 1$ (see §2). Choose $\lambda \in (\sigma, 1)$ such that $\mu(\lambda - \sigma) < 1$. Define $K_1 = K_0 + 2$, $\tau_0 = 1$. For $n \in \mathbb{N}$, define inductively

$$\tau_n = \max\left\{\tau_{n-1} + 2,\ \alpha_0 + \frac{1}{K_n^{\mu\sigma}} \max_{K_n \leq |z| \leq K_n+1} H(z)\right\}, \tag{A.1}$$

(A.2) $$K_{n+1} = \max\left\{ K_n + 2,\ \left(\frac{\tau_n}{\alpha_0}\right)^{\frac{1}{\mu(1-\lambda)}} \right\}.$$

Here $\alpha_0 = \min_{|z|=r_0} H(z) > 0$. For $K \in \mathbb{R}$, take $\chi(\cdot, K) \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\chi(s,K) = 1$ for $s \leq K$, $\chi(s,K) = 0$ for $s \geq K+1$ and $\chi'(s,K) < 0$ for $s \in (K, K+1)$. Then for $n \in \mathbb{N}$ set

$$M_n(z) = \chi(|z|, K_n)H(z) + [1 - \chi(|z|, K_n)]\tau_n|z|^{\mu\lambda}, \text{ for all } z \in \mathbb{R}^{2N}.$$

This kind of truncation functions was used by Rabinowitz in [17]. Since the M_n's do not satisfy 4° of Proposition 2.5, we need to modify them. Direct computations (cf. [17]) show that

LEMMA A.3. *For $n \in \mathbb{N}$, $M_n \in C^1(\mathbb{R}^{2N},\mathbb{R})$ has the following properties,*

(A.4) $$M_n(z) = H(z), \text{ if } |z| \leq K_n;$$

(A.5) $$M_n(z) = \tau_n|z|^{\mu\lambda},\ M_n'(z)\cdot z = \tau_n\mu\lambda|z|^{\mu\lambda}, \text{ if } |z| \geq K_n + 1;$$

(A.6) $$0 < \mu\lambda M_n(z) \leq M_n'(z)\cdot z, \text{ for } |z| \geq r_0.$$

Integrating (A.6) we get that

(A.7) $$\alpha_0|z|^{\mu\lambda} \leq M_n(z), \text{ for } |z| \geq r_0.$$

LEMMA A.8. *For $\rho \geq K_{n+1}$, we have that*

(A.9) $$\max_{z\in S^{2N-1}} M_n'(\rho z)\cdot z \leq \min_{z\in S^{2N-1}} H'(\rho z)\cdot z$$

and

(A.10) $$\max_{z\in S^{2N-1}} M_n'(\rho z)\cdot z \leq \min_{z\in S^{2N-1}} M_{n+1}'(\rho z)\cdot z.$$

PROOF. For any ς, $z \in S^{2N-1}$, by (H2) and (2.4)

$$\begin{aligned} H'(\rho\varsigma)\cdot\varsigma &\geq \frac{\mu}{\rho} H(\rho\varsigma) \geq \mu\alpha_0\rho^{\mu-1} \\ &\geq \mu\lambda\rho^{\mu-1}\alpha_0\rho^{\mu(1-\lambda)} \\ &\geq \mu\lambda\rho^{\mu\lambda-1}\ \tau_n \qquad \text{[by (A.2)]} \\ &= M_n'(\rho z)\cdot z \qquad \text{[by (A.5)]}. \end{aligned}$$

This proves (A.9).

For any $\varsigma,\ z \in S^{2N-1}$,

$$\begin{aligned} M'_{n+1}(\rho\varsigma)\cdot\varsigma &= \chi(\rho, K_{n+1})H'(\rho\varsigma)\cdot\varsigma \\ &+ [1-\chi(\rho,K_{n+1})]\tau_{n+1}\mu\lambda\rho^{\mu\lambda-1} + \chi'(\rho,K_{n+1})\ [H(\rho\varsigma)-\tau_{n+1}\rho^{\mu\lambda}] \\ &\geq \min\left\{H'(\rho\varsigma)\cdot\varsigma,\ \tau_{n+1}\mu\lambda\rho^{\mu\lambda-1}\right\} \\ &\quad \text{[by (A.1) and the definition of } \chi] \\ &\geq M'_n(\rho z)\cdot z \quad \text{[by (A.9) and (A.1)].} \end{aligned}$$

This proves (A.10). □

We now introduce the spherical coordinates (r,θ) on $\mathbb{R}^{2N}$. For

$$z = (z_1,\ldots,z_{2N}) \in \mathbb{R}^{2N},$$

write $z = r\overline{z}(\theta),\ r = |z|,\ \overline{z}(\theta) = \frac{z}{|z|},\ \theta = (\theta_1,\ldots,\theta_{2N-1})$,

$$\text{(A.11)}\qquad \begin{cases} z_1 = r\ \cos\ \theta_1 \\ z_2 = r\ \sin\ \theta_1\ \cos\ \theta_2 \\ \qquad\cdots\cdots \\ z_{2N-1} = r\ \sin\ \theta_1\ldots\sin\ \theta_{2N-2}\ \cos\ \theta_{2N-1} \\ z_{2N} = r\ \sin\ \theta_1\ldots\sin\ \theta_{2N-2}\ \sin\ \theta_{2N-1}, \end{cases}$$

where $r\geq 0,\ \theta_1 \in [0,\pi],\ \theta_i\in\mathbb{R}$ for $i = 2,\ldots,2N-1$. We also write

$$\begin{aligned} &d\theta = d\theta_1\ldots d\theta_{2N-1}, \text{ and} \\ &d\hat{\theta}_i = d\theta_1\ldots d\theta_{i-1}d\theta_{i+1}\ldots d\theta_{2N-1}, \text{ for } i = 1,\ldots,2N-1. \end{aligned}$$

Let $\Omega = [0,\pi]\times\mathbb{R}^{2N-2}$. For $\theta\in\Omega,\ \rho > 0$, define

$$U(\theta,\rho) = ([\theta_1 - \pi\sqrt{\rho}, \theta_1 + \pi\ \sqrt{\rho}] \cap [0,\pi]) \times \prod_{i=2}^{2N-1}\ [\theta_i - \sqrt{\rho}, \theta_i + \sqrt{\rho}].$$

Since H and M_n are uniformly continuous on $\{K_n \leq |z| \leq K_{n+1}\}$, there is a constant $\delta_n \in (0,1]$ such that

$$\text{(A.12)}\qquad |H(z) - H(\tilde{z})| + |M_n(z) - M_n(\tilde{z})| \leq (\lambda-\sigma)\alpha_0 K_n^{\mu\lambda},$$

for any $z,\ \tilde{z}\in\{K_n \leq |z| \leq K_{n+1}\}$ and $|z-\tilde{z}|\leq\delta_n$. There is a constant $\varepsilon_n \in (0,1]$ such that

$$\text{(A.13)}\qquad |r\overline{z}(\theta) - r\overline{z}(\xi)| \leq \delta_n,$$

for any $r \in [K_n, K_{n+1}]$, $\theta \in \Omega$ and $\xi \in U(\theta, \varepsilon_n)$. Define

$$\nu_n(t) = \min\{\sqrt{t},\ \sqrt{\varepsilon_n}\}, \text{ for } t \geq 0. \tag{A.14}$$

For $n \in \mathbb{N}$; $i = 2, \ldots, 2N-1$; $k = 1, 2$; $\theta \in \Omega$; $\rho \geq K_n$ define

$$\omega_{n,1}(\theta_1, \rho) = \left[\theta_1 - \frac{1}{2}\ \theta_1 \nu_n(\rho - K_n),\ \theta_1 + \frac{1}{2}\ (\pi - \theta_1)\nu_n(\rho - K_n)\right],$$

$$\omega_{n,i}(\theta_i, \rho) = \left[\theta_i - \frac{1}{2}\ \nu_n(\rho - K_n),\ \theta_i + \frac{1}{2}\ \nu_n(\rho - K_n)\right],$$

$$\omega_{n,1}^{(k)}(\theta_1, \rho) = \left[\theta_1 - \frac{k}{2}\ \pi\nu_n(\rho - K_n),\ \theta_1 + \frac{k}{2}\ \pi\nu_n(\rho - K_n)\right] \cap [0, \pi],$$

$$\omega_{n,i}^{(k)}(\theta_i, \rho) = \left[\theta_i - \frac{k}{2}\ \nu_n(\rho - K_n),\ \theta_i + \frac{k}{2}\ \nu_n(\rho - K_n)\right],$$

and for $j = 1, \ldots, 2N-1$, define

$$\Omega_n(\theta, \rho) = \prod_{j=1}^{2N-1} \omega_{n,j}(\theta_j, \rho),$$

$$\Omega_{n,j}(\theta, \rho) = \prod_{\substack{1 \leq \ell \leq 2N-1 \\ \ell \neq j}} \omega_{n,\ell}(\theta_\ell, \rho),$$

$$\Omega_n^{(k)}(\theta, \rho) = \prod_{j=1}^{2N-1} \omega_{n,j}^{(k)}(\theta_j, \rho),$$

$$\Omega_{n,j}^{(k)}(\theta, \rho) = \prod_{\substack{1 \leq \ell \leq 2N-1 \\ \ell \neq j}} \omega_{n,\ell}(\theta_\ell, \rho).$$

Then direct computations show that

$$\begin{cases} \omega_{n,1}(\theta_1, \rho) \subseteq [0, \pi], \\ \Omega_n(\theta, \rho) \subseteq \Omega_n^{(1)}(\theta, \rho) \subseteq \Omega_n^{(2)}(\theta, \rho) \subseteq U(\theta, \rho - K_n), \\ |\omega_{n,1}(\theta_1, \rho)| = \frac{\pi}{2}\ \nu_n(\rho - K_n), \\ |\omega_{n,i}(\theta, \rho)| = \nu_n(\rho - K_n),\ i = 2, \ldots, 2N-1, \\ |\Omega_n(\theta, \rho)| = \frac{\pi}{2}\ [\nu_n(\rho - K_n)]^{2N-1}. \end{cases} \tag{A.15}$$

To simplify the notations we write $V_n(\rho) = |\Omega_n(\theta, \rho)|$. These sets satisfy:

LEMMA A.16. *For* $n \in \mathbb{N}$, $\theta \in \Omega$, $\rho \geq K_n$,

1°. $\beta \in \Omega_n(\theta, \rho)$ *implies* $\theta \in \Omega_n^{(1)}(\beta, \rho)$.

2°. *$\beta \in \Omega_n(\theta,\rho)$ and $\gamma \in \Omega_n^{(1)}(\beta,\theta)$ imply $\gamma \in \Omega_n^{(2)}(\theta,\rho)$.*

PROOF.

1°. If $\beta_1 \in \omega_{n,1}(\theta_1,\rho)$, we have

$$\theta_1 - \frac{1}{2}\,\theta_1\nu_n(\rho - K_n) \le \beta_1 \le \theta_1 + \frac{1}{2}\,(\pi - \theta_1)\nu_n(\rho - K_n),$$

then

$$\beta_1 - \frac{1}{2}\,\nu_n(\rho - K_n) \le \theta_1\left[1 - \frac{1}{2}\,\nu_n(\rho - K_n)\right] \le \theta_1$$
$$\le \theta_1 + \frac{1}{2}\,(\pi - \theta_1)\nu_n(\rho - K_n) \le \beta_1 + \frac{\pi}{2}\,\nu_n(\rho - K_n).$$

Since $\theta_1 \in [0,\pi]$, $\theta_1 \in \omega_{n,1}^{(1)}(\beta_1,\rho)$. Similarly $\beta_i \in \omega_{n,i}(\theta_i,\rho)$ implies $\theta_i \in \omega_{n,i}^{(1)}(\beta_i,\rho)$ for $i = 2,\ldots,2N-1$. Therefore 1° holds.

2°. If $\beta \in \Omega_n(\theta,\rho)$, $\gamma \in \Omega_n^{(1)}(\beta,\rho)$, then

$$\theta_1 - \frac{1}{2}\,\theta_1\nu_n(\rho - K_n) \le \beta_1 \le \theta_1 + \frac{1}{2}\,(\pi - \theta_1)\nu_n(\rho - K_n)$$

and

$$\beta_1 - \frac{\pi}{2}\,\nu_n(\rho - K_n) \le \gamma_1 \le \beta_1 + \frac{\pi}{2}\,\nu_n(\rho - K_n).$$

So

$$\theta_1 - \pi\nu_n(\rho - K_n) \le \theta_1 - \frac{1}{2}\,(\pi + \theta_1)\nu_n(\rho - K_n)$$
$$\le \beta_1 - \frac{\pi}{2}\,\nu_n(\rho - K_n) \le \gamma_1,$$

and

$$\gamma_1 \le \beta_1 + \frac{\pi}{2}\,\nu_n(\rho - K_n) \le \theta_1 + \pi\nu_n(\rho - K_n) - \frac{1}{2}\,\theta_1\nu_n(\rho - K_n)$$
$$\le \theta_1 + \pi\nu_n(\rho - K_n).$$

Since $\gamma_1 \in [0,\pi]$, we get that $\gamma_1 \in \omega_{n,1}^{(2)}(\theta_1,\rho)$. Similarly $\gamma_i \in \omega_{n,i}^{(2)}(\theta_i,\rho)$ for $i = 2,\ldots,2N-1$. Thus 2° holds and the proof is complete. □

For $n \in \mathbb{N}$, $\rho \ge K_n$, $\bar{z} \in S^{2N-1}$, define

$$F_n(\rho,\bar{z}) = \min\left\{M_n'(\rho\bar{z})\cdot\bar{z},\ H'(\rho\bar{z})\cdot\bar{z}\right\}.$$

Note that F_n is continuous in its arguments. Define for $z = r\bar{z}(\theta) \in \mathbb{R}^{2N}$, $r = |z|$, $\bar{z}(\theta) = \frac{z}{|z|}$,

$$G_n(z) = \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n[\rho,\bar{z}(\gamma)]\,\mathrm{d}\beta\,\mathrm{d}\rho$$

and

$$\hat{H}_n(z) = \begin{cases} H(z), & \text{if } |z| \le K_n, \\ G_n(z) + H[K_n \bar{z}(\theta)], & \text{if } K_n < |z|. \end{cases}$$

Note that when $r = |z| > K_{n+1}$, by (A.5) and (A.9),

(A.17)
$$G_n(z) = \int_{K_n}^{K_{n+1}} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma \in \Omega_n^{(1)}(\beta,\rho)} F_n[\rho, \bar{z}(\gamma)] \mathrm{d}\beta \mathrm{d}\rho + \tau_n \left(r^{\mu\lambda} - K_{n+1}^{\mu\lambda} \right).$$

LEMMA A.18. *For $n \in \mathbb{N}$, we have $\hat{H}_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$.*

PROOF. Since H, $M_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ and, in the formula of G_n, all the variables r, θ only appear linearly in the integration limits, $\hat{H}_n$ is C^1-continuous on $\{|z| \le K_n\}$ and $\{|z| > K_n\}$. We only need to verify the C^1-continuity of $\hat{H}_n$ at every $\varsigma \in \mathbb{R}^{2N}$, with $|\varsigma| = K_n$, $\bar{\varsigma} = \frac{\varsigma}{|\varsigma|}$.

For $z = r\bar{z}(\theta)$, with $r = |z| > K_n$, $\bar{z}(\theta) = \frac{z}{|z|}$,

(A.19)
$$\frac{\partial \hat{H}_n(z)}{\partial r} = \frac{\partial G_n(z)}{\partial r} = \frac{1}{V_n(r)} \int_{\Omega_n(\theta,r)} \min_{\gamma \in \Omega_n^{(1)}(\beta,r)} F_n[r, \bar{z}(\gamma)] \mathrm{d}\beta = \min_{\gamma \in \Omega_n^{(1)}(\xi,r)} F_n[r, \bar{z}(\gamma)],$$

for some $\xi \in \Omega_n(\theta, r)$, by the mean value theorem of integration. Thus

$$\lim_{\substack{z \to \varsigma \\ |z| > |\varsigma| = K_n}} \frac{\partial \hat{H}_n(z)}{\partial r} = \lim_{\substack{z \to \varsigma \\ |z| > |\varsigma| = K_n}} \min_{\gamma \in \Omega_n^{(1)}(\xi,r)} F_n[r, \bar{z}(\gamma)] = F_n(K_n, \bar{\varsigma}) = H'(K_n \bar{\varsigma}) \cdot \bar{\varsigma} = \frac{\partial H(\varsigma)}{\partial r}.$$

From the definition of G_n, we get that

(A.20)
$$\frac{\partial G_n(z)}{\partial \theta_1} = \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_{n,1}(\theta,\rho)} \left(1 - \frac{1}{2} \nu_n(\rho - K_n)\right) \left[\min_{\gamma \in \Delta_1} F_n[\rho, \bar{z}(\gamma)] - \min_{\gamma \in \Delta_2} F_n[\rho, \bar{z}(\gamma)] \right] \mathrm{d}\hat{\beta}_1 \mathrm{d}\rho,$$

where

$$\Delta_1 = \omega_{n,1}^{(1)} \left(\theta_1 + \frac{1}{2}\,(\pi\theta_1)\nu_n(\rho - K_n), \rho\right) \times \Omega_{n,1}^{(1)}(\beta, \rho),$$

$$\Delta_2 = \omega_{n,1}^{(1)} \left(\theta_1 - \frac{1}{2}\,\theta_1\nu_n(\rho - K_n), \rho\right) \times \Omega_{n,1}^{(1)}(\beta, \rho).$$

So, for $K_n < r \leq K_n + \varepsilon_n$, we get that

$$\left|\frac{\partial G_n(z)}{\partial \theta_1}\right| \leq \frac{2}{\pi}\, 2\overline{M}_n \int_{K_n}^{r} \frac{1}{\sqrt{\rho - K_n}}\, \mathrm{d}\rho \quad \text{[by (A.15)]}$$

$$= \frac{8}{\pi}\, \overline{M}_n\, \sqrt{r - K_n} \to 0, \ \text{as } r \to K_n,$$

where $\overline{M}_n = \max\limits_{(\theta,\rho)\in\Omega\times[K_n,K_n+1]} F_n[\rho, \overline{z}(\theta)]$. Thus

$$\lim_{\substack{z\to\varsigma \\ |z|>|\varsigma|=K_n}} \frac{\partial \hat{H}_n(z)}{\partial \theta_1} = \lim_{\substack{z\to\varsigma \\ |z|>|\varsigma|=K_n}} \left(\frac{\partial G_n(z)}{\partial \theta_1} + \frac{\partial H[K_n\overline{z}(\theta)]}{\partial \theta_1}\right) = \frac{\partial H(\varsigma)}{\partial \theta_1}.$$

Similarly, we have that

$$\lim_{\substack{z\to\varsigma \\ |z|>|\varsigma|=K_n}} \frac{\partial \hat{H}_n(z)}{\partial \theta_i} = \frac{\partial H(\varsigma)}{\partial \theta_i}, \ \text{for } i = 2, \ldots, 2N-1.$$

This completes the proof. □

LEMMA A.21. *For $n \in \mathbb{N}$ and $z \in \mathbb{R}^{2N}$, we have*

(A.22) $$\hat{H}_n(z) \leq \hat{H}_{n+1}(z) \leq H(z).$$

PROOF.

1°. We prove that $\hat{H}_n(z) \leq H(z)$.
If $|z| \leq K_n$, this is true, since $\hat{H}_n(z) = H(z)$.
If $K_n < |z|$, write $z = r\overline{z}(\theta)$, then

$$\hat{H}_n(z) \leq \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} H'[\rho\overline{z}(\gamma)]\cdot\overline{z}(\gamma)\mathrm{d}\beta\mathrm{d}\rho$$
$$+ H(K_n\overline{z}(\theta)).$$

By 1° of Lemma A.16, $\beta \in \Omega_n(\theta, \rho)$ implies that $\theta \in \Omega_n^{(1)}(\beta, \rho)$, so

$$\hat{H}_n(z) \le \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} H'[\rho\bar{z}(\theta)] \cdot \bar{z}(\theta) \mathrm{d}\beta \mathrm{d}\rho + H[K_n \bar{z}(\theta)]$$

$$= \int_{K_n}^{r} H'[\rho\bar{z}(\theta)] \cdot \bar{z}(\theta) \mathrm{d}\rho + H[K_n \bar{z}(\theta)] = H(z).$$

2°. We prove that $\hat{H}_n(z) \le \hat{H}_{n+1}(z)$.
If $|z| \le K_{n+1}$, this is a consequence of 1°, since $\hat{H}_{n+1}(z) = H(z)$.
If $K_{n+1} \le |z|$, write $z = r\bar{z}(\theta)$, then by the definition of $\hat{H}_n$,

$$\hat{H}_n(z) = \int_{K_{n+1}}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma \in \Omega_n^{(1)}(\beta,\rho)} F_n[\rho, \bar{z}(\gamma)] \mathrm{d}\beta \mathrm{d}\rho + \hat{H}_n[K_{n+1}\bar{z}(\theta)]$$

$$\le \int_{K_{n+1}}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \max_{\gamma \in \Omega} M_n'[\rho\bar{z}(\gamma)] \cdot \bar{z}(\gamma) \mathrm{d}\beta \mathrm{d}\rho + \hat{H}_n[K_{n+1}\bar{z}(\theta)]$$

$$= \int_{K_{n+1}}^{r} \frac{1}{V_{n+1}(\rho)} \int_{\Omega_{n+1}(\theta,\rho)} \max_{\gamma \in \Omega} M_n'[\rho\bar{z}(\gamma)] \cdot \bar{z}(\gamma) \mathrm{d}\beta \mathrm{d}\rho + \hat{H}_n[K_{n+1}\bar{z}(\theta)]$$

$$\le \int_{K_{n+1}}^{r} \frac{1}{V_{n+1}(\rho)} \int_{\Omega_{n+1}(\theta,\rho)} \min_{\gamma \in \Omega} F_{n+1}[\rho, \bar{z}(\gamma)] \mathrm{d}\beta \mathrm{d}\rho + H[K_{n+1}\bar{z}(\theta)],$$

here we used (A.9), (A.10), and 1° of this lemma. Thus

$$\hat{H}_n(z) \le \hat{H}_{n+1}(z), \text{ if } K_{n+1} \le |z|,$$

and this completes the proof. □

LEMMA A.23. *For every $n \in \mathbb{N}$, we have*

$$0 < \mu\sigma \hat{H}_n(z) \le \hat{H}_n'(z) \cdot z, \quad \textit{for every } |z| \ge r_0. \tag{A.24}$$

PROOF. Write $z = r\bar{z}(\theta)$, $r = |z|$, $\bar{z}(\theta) = \frac{z}{|z|}$.
If $r_0 \le |z| \le K_n$, (A.24) holds by (H2) and $\hat{H}_n(z) = H(z)$.
If $K_{n+1} \le |z|$, then by the definition of $\hat{H}_n$ and (A.17), $\hat{H}_n'(z) \cdot z = \tau_n \mu\lambda r^{\mu\lambda}$.

So

$$\hat{H}_n'(z)\cdot z = \mu\sigma\hat{H}_n(z) - \mu\sigma\int_{K_n}^{r}\frac{1}{V_n(\rho)}\int_{\Omega_n(\theta,\rho)}\min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n[\rho,\bar{z}(\gamma)]\mathrm{d}\beta\mathrm{d}\rho$$

$$- \mu\sigma H[K_n\bar{z}(\theta)] + \tau_n\mu\lambda r^{\mu\lambda}$$

$$\geq \mu\sigma\hat{H}_n(z) - \mu\sigma\int_{K_n}^{r} M_n'[\rho\bar{z}(\theta)]\cdot\bar{z}(\theta)\mathrm{d}\rho$$

$$- \mu\sigma H[K_n\bar{z}(\theta)] + \tau_n\mu\lambda r^{\mu\lambda} \qquad \text{(by 1° of Lemma A.16)}$$

$$= \mu\sigma\hat{H}_n(z) + \tau_n\mu\lambda r^{\mu\lambda} - \mu\sigma\tau_n r^{\mu\lambda}$$

$$+ \mu\sigma M_n[K_n\bar{z}(\theta)]\mu\sigma H[K_n\bar{z}(\theta)]$$

$$\geq \mu\sigma\hat{H}_n(z),$$

since $\lambda > \sigma$ and $M_n[K_n\bar{z}(\theta)] = H[K_n\bar{z}(\theta)]$.

If $K_n < |z| < K_{n+1}$, by 2° of Lemma A.16,

$$\hat{H}_n'(z)\cdot z = \frac{r}{V_n(r)}\int_{\Omega_n(\theta,r)}\min_{\gamma\in\Omega_n^{(1)}(\beta,r)} F_n[r,\bar{z}(\gamma)]\mathrm{d}\beta$$

$$\geq \min_{\gamma\in\Omega_n^{(2)}(\theta,r)} rF_n[r,\bar{z}(\gamma)]$$

$$= rF_n[r,\bar{z}(\xi)],$$

for some $\xi\in\Omega_n^{(2)}(\theta,r)$, by the compactness of $\Omega_n^{(2)}(\theta,r)$. So we get that, by 1° of Lemma A.16,

$$\hat{H}_n'(z)\cdot z \geq \mu\sigma\hat{H}_n(z) + rF_n[r,\bar{z}(\xi)]$$

$$\text{(A.25)} \qquad - \mu\sigma\int_{K_n}^{r} F_n[\rho,\bar{z}(\theta)]\mathrm{d}\rho - \mu\sigma H[K_n\bar{z}(\theta)].$$

We consider two cases:

CASE 1. $F_n[r,\bar{z}(\xi)] = M_n'[r\bar{z}(\xi)]\cdot\bar{z}(\xi)$.
Then, from (A.25) and 1° of Lemma A.16,

$$\hat{H}_n'(z)\cdot z \geq \mu\sigma\hat{H}_n(z) + M_n'[r\bar{z}(\xi)]\cdot r\bar{z}(\xi)$$

$$- \mu\sigma\int_{K_n}^{r} M_n'[\rho\bar{z}(\theta)]\cdot\bar{z}(\theta)\mathrm{d}\rho - \mu\sigma H[K_n\bar{z}(\theta)]$$

$$\geq \mu\sigma \hat{H}_n(z) + \mu\lambda M_n[r\bar{z}(\xi)] - \mu\sigma M_n[r\bar{z}(\theta)]$$

[by (A.4) and (A.6)]

$$\geq \mu\sigma \hat{H}_n(z) + \mu(\lambda - \sigma)M_n[r, \bar{z}(\xi)]\mu\sigma \left|M_n[r\bar{z}(\xi)] - M_n[r\bar{z}(\theta)]\right|$$

$$\geq \mu\sigma \hat{H}_n(z) + \mu(\lambda - \sigma)\alpha_0 K_n^{\mu\lambda} - \mu\sigma \left|M_n[r\bar{z}(\xi)] - M_n[r\bar{z}(\theta)]\right|$$

[by (A.7)]

$$\geq \mu\sigma \hat{H}_n(z).$$

In the last inequality, we used (A.12), (A.13), (A.15), and that $\xi \in \Omega_n^{(2)}(\theta, r)$.

CASE 2. $F_n[r, \bar{z}(\xi)] = H'[r\bar{z}(\xi)] \cdot \bar{z}(\xi)$.

Then, from (A.25) and 1° of Lemma A.16,

$$\hat{H}_n'(z) \cdot z \geq \mu\sigma \hat{H}_n(z) + H'[r\bar{z}(\xi)] \cdot r\bar{z}(\xi)$$

$$- \mu\sigma \int_{K_n}^{r} H'[\rho\bar{z}(\theta)] \cdot \bar{z}(\theta) d\rho - \mu\sigma H[K_n\bar{z}(\theta)]$$

$$\geq \mu\sigma \hat{H}_n(z) + \mu H[r\bar{z}(\xi)] - \mu\sigma H[r\bar{z}(\theta)]$$

[by (H2)]

$$\geq \mu\sigma \hat{H}_n(z) + \mu(1 - \sigma)\alpha_0 K_n^{\mu} - \mu\sigma \left|H[r\bar{z}(\xi)] - H[r\bar{z}(\theta)]\right|$$

[by (2.4)]

$$\geq \mu\sigma \hat{H}_n(z),$$

here we used (A.12), (A.13), (A.15), and that $\xi \in \Omega_n^{(2)}(\theta, r)$.

Thus $\hat{H}_n'(z) \cdot z \geq \mu\sigma \hat{H}_n(z)$, if $K_n < |z| < K_{n+1}$.

Finally from (A.6) and (H2), $\hat{H}_n(z) > 0$ for $|z| \geq r_0$, and this completes the proof of (A.24). □

To get 6° of Proposition 2.5, we modify $\hat{H}_n$'s again.

For $n \in \mathbb{N}$, $z = r\bar{z}(\theta)$, from (A.17), if $r > K_{n+1}$, we get that

$$\hat{H}_n(z) = G_n[K_{n+1}\bar{z}(\theta)] + \tau_n(r^{\mu\lambda} - K_{n+1}^{\mu\lambda}) + H[K_n\bar{z}(\theta)].$$

Set

$$C_n = \max_{z \in S^{2N-1}} \left|G_n(K_{n+1}z) + H(K_n z) - \tau_n K_{n+1}^{\mu\lambda}\right| + 1.$$

Then we have that

$$\begin{aligned} \hat{H}_n(z) &\leq \tau_n |z|^{\mu\lambda} + C_n \\ &\leq (\tau_n + 1)|z|^{\mu\lambda}, \quad \text{for every } |z| \geq \max\{K_{n+1}, C_n\}, \end{aligned} \tag{A.26}$$

and

$$\text{(A.27)} \qquad \begin{aligned} \hat{H}_{n+1}(z) &\geq \tau_{n+1}|z|^{\mu\lambda} - C_{n+1} \\ &\geq (\tau_n + 1)|z|^{\mu\lambda} + |z|^{\mu\lambda} - C_{n+1} \\ &\qquad\qquad [\text{by (A.1)}] \\ &\geq (\tau_n + 1)|z|^{\mu\lambda}, \qquad \text{for every } |z| \geq \max\{K_{n+2}, C_{n+1}\}. \end{aligned}$$

Let $\hat{K}_n = \max\{K_{n+2}, C_n, C_{n+1}\}$ and define

$$H_n(z) = \chi(|z|, \hat{K}_n)\hat{H}_n(z) + [1 - \chi(|z|, \hat{K}_n)](\tau_n + 1)|z|^{\mu\lambda}, \text{ for all } z \in \mathbb{R}^{2N}.$$

Then we have that $H_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ possesses the following properties,

$$\text{(A.28)} \qquad H_n(z) = \hat{H}_n(z), \text{ for } |z| \leq \hat{K}_n,$$

$$\text{(A.29)} \qquad H_n(z) = (\tau_n + 1)|z|^{\mu\lambda}, \text{ for } |z| \geq \hat{K}_n + 1,$$

$$\text{(A.30)} \qquad 0 < \mu\sigma H_n(z) \leq H_n'(z) \cdot z, \text{ for } |z| \geq r_0.$$

From (A.26)-(A.29) and the definition of H_n, we also have that

$$\text{(A.31)} \qquad \hat{H}_n(z) \leq H_n(z) \leq \hat{H}_{n+1}(z), \qquad \text{for every } z \in \mathbb{R}^{2N}.$$

Now we can give the

PROOF OF PROPOSITION 2.5. 1°-3° are true from the definitions of K_n, $\hat{H}_n$, H_n and Lemma A.18. Lemma A.21 and (A.31) yield 4°. (A.30) gives 5°. 6° is a consequence of (A.29), by letting $\lambda_0 = \frac{\mu\lambda}{\mu\lambda - 1}$.

The proof of Proposition 2.5 is complete. □

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