ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

BERNHARD RUF

Singularity theory and the geometry of a nonlinear elliptic equation

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Singularity Theory and the Geometry of a Nonlinear Elliptic Equation

BERNHARD RUF

1. - Introduction

In this paper we apply singularity theory (in the sense of H. Whitney [11] and R. Thom [10]) to obtain a complete description of the solution structure of the equation

$$(1.1)\qquad \begin{cases} \Delta u - \lambda u + u^3 = h, & \text{in } \Omega \subset \mathbb{R}^n \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

where λ is a positive constant, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and $h(x)$ is a given data function.

Let $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ denote the eigenvalues of the equation

$$(1.2)\qquad \begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that (1.1) has a unique solution in

$$E = \left\{ u \in C^{2,\alpha}(\Omega),\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \right\}, \alpha \in (0,1) \text{ fixed},$$

for every given $h \in C^{0,\alpha}(\Omega)$ if $\lambda < 0$. In fact, in this case the linearization of the mapping $\Phi = -\Delta - \lambda + (\cdot)^3$, namely

$$(1.3)\qquad \Phi'(u)[v] = -\Delta v - \lambda v + 3u^2 v,$$

is regular in every point $u \in E$, that is Φ is locally invertible in any given point u. From this the above statement follows easily, since Φ is proper.

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On the other hand, if $\lambda > \lambda_1 = 0$, then Φ has *singular points*, that is there exist functions u such that the equation

$$(1.4)\qquad \begin{cases} -\Delta v - \lambda v + 3u^2 v = 0, & \text{in } \Omega \subset \mathbb{R}^n \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution v.

The aim is to obtain a complete description of the solution behaviour of equation (1.1) through the study of the *singular set* of Φ. Roughly speaking we will show that, for $0 < \lambda < \frac{\lambda_2}{12}$, the mapping Φ is a *global cusp*. More precisely, we will prove the following theorem.

THEOREM 1.1. *For $0 < \lambda < \frac{\lambda_2}{12}$ there exists in $F = C^{0,\alpha}(\Omega)$ a C^0-manifold of codimension 1 such that $F \backslash G = F_1 \cup F_3$ consists of two open components F_1, F_3, with $0 \in F_3$, and such that*

if $h \in F_3$, then equation (1.1) has exactly 3 solutions,

if $h \in F_1$, then equation (1.1) has exactly 1 solution.

We note that, for $h(x) \equiv 0$, equation (1.1) can be viewed as a bifurcation problem in $(\lambda, u) \in \mathbb{R} \times E$; in fact, it follows from the classical results in bifurcation theory [7,9] that $(\lambda_1, 0)$ is a bifurcation point. Therefore, for small $\lambda > \lambda_1 = 0$, equation (1.1) has, for $h \equiv 0$, at least 3 solutions, namely the trivial solution $(\lambda, 0)$ and a positive and a negative solution. In fact, for Neumann boundary conditions, these solutions can be calculated explicitely: let 1 denote the constant function (equal 1) on Ω. Then we have for $s \cdot 1$, $s \in \mathbb{R}$, the equation

$$(-\Delta - \lambda)\ s \cdot 1 + (s \cdot 1)^3 = -\lambda s \cdot 1 + s^3 1 = 0,$$

which has the solutions $s_0 = 0$, $s_{1,2} = \pm\sqrt{\lambda}$.

The theorem stated above describes the *global structure* of this bifurcation. If the parameter λ crosses the first eigenvalue λ_1, then a set $F_3(\lambda)$ appears in the space F which is covered three times by the mapping $-\Delta - \lambda + (\cdot)^3$. We note that the proof given below gives a precise description of the geometric structure of the set $F_3(\lambda)$, which can be visualized as follows.

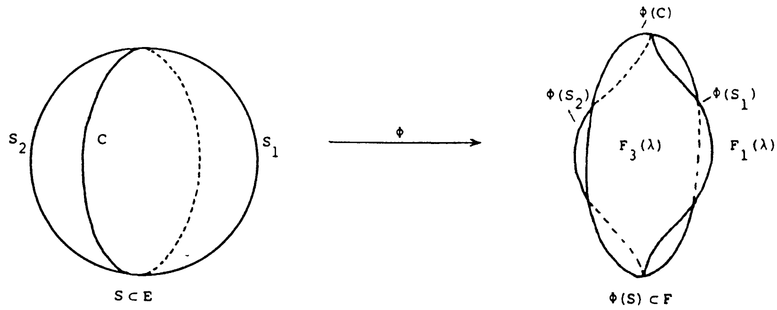

Figure 1

As already mentioned, the proof of this theorem relies on singularity theory in Banach space.

More precisely, one studies the nonlinear mapping

$$-\Delta - \lambda + (\cdot)^3 : E \to F, \tag{1.5}$$

where E and F are as before. For $\lambda > 0$ this mapping has singular points u, that is the Fréchet derivative in u of (1.3) vanishes in some direction v:

$$-\Delta v - \lambda v + 3u^2 v = 0. \tag{1.6}$$

The aim is to characterize completely the singular set S (that is the set of singular points) and the image of the singular set $\Phi(S)$. Then one can use the following well-known proposition to obtain the desired information about the solution structure of equation (1.1).

PROPOSITION 1.2. *Let X, Y be Banach spaces, and let $\Phi : X \to Y$ be a Fréchet-differentiable and proper mapping (i.e. for every $K \subset Y$ compact, the set $\Phi^{-1}(K) \subset X$ is compact). Furthermore, let*

$$N(y) = \# \{u \in X | \Phi(u) = y\}.$$

Then $N(y)$ is constant on the components of $Y \backslash \Phi(S)$ (for a proof, see e.g. Ambrosetti-Prodi [1]).

The characterization of the singular set involves two steps. First, the characterization of the form of the singular set, that is the description of the singular set S as a subset in the given space. Second, the classification of the singular points on S (according to the classification of singularities of R. Thom).

We remark that the singularities considered here lie in infinite dimensional Banach spaces. However, under appropriate assumptions these singularities can be handled like their finite dimensional analogues (see e.g. Berger-Church [2,3], Berger-Church-Timourian [4]).

In general, the task of classifying the singular points is difficult. However, in the case of the mapping (1.3) this can be done, and we will prove in this paper the following classification result.

THEOREM 1.3. *Let* $0 < \lambda < \frac{\lambda_2}{7}$. *Then the singular set* S *is a smooth manifold of codimension* 1 *such that* $E \backslash S$ *has two components* E_1 *and* E_3. *Furthermore,* S *contains a smooth submanifold* C *of codimension* 1 (*with respect to* S) *such that*

$S \backslash C = S_1 \cup S_2$ *has two components consisting of fold points*

C *consists of cusp points* (*see figure* 1).

For the notions of fold points and cusp points we refer to section 2. We just note that folds and cusps are the two most simple singularities in the classification of R. Thom.

The result in this paper should be compared with a result of A. Ambrosetti-G. Prodi. They showed in [1] that the equation

$$\begin{cases} -\Delta u - f(u) = h, & \text{in } \Omega \\ \qquad\quad u = h, & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

with $f \in C^2(\mathbb{R})$, $f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2$ and $f''(t) > 0$, for all $t \in \mathbb{R}$, has the form of a *global fold*, i.e. the singular set of $\Phi = -\Delta - f : E_0 \to F$, where $E_0 = \{u \in C^{2,\alpha}(\Omega),\ u = 0 \text{ on } \partial\Omega\}$, is a smooth manifold S of codimension 1 consisting of fold points. Furthermore, $M = \Phi(S)$ is a smooth manifold of codimension 1 in F, and $F \backslash M$ has two components M_0 and M_2 such that

if $h \in M_0$, then (1.5) has no solutions,

if $h \in M_2$, then (1.5) has exactly two solutions.

In the result of Ambrosetti-Prodi, as well as in our result, it is crucial to know exactly what kind of singularities occur. In fact, the limitation $\lambda < \frac{\lambda_2}{7}$ is used to exclude the existence of higher singularities than cusp singularities. The existence of such higher singularities would correspond to the occurrence of secondary (and higher) bifurcation, which would complicate the structure of the equation.

We mention that V. Cafagna-F. Donati [6] exhibited a similar structure as the one described in theorem 1.3 for the equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\, u(x) + au(x) + bu^2(x) + cu^{2k+1}(x) = h(x) \\[2ex] u(0) = u(1) \end{cases}$$

where $k \in \mathbb{N}\backslash\{0\}$, $a \geq 0$, $a^2 + b^2 > 0$ and $c < 0$.

Finally, we remark that by topological methods one can easily establish the existence of a region in F such that, for h in this region, equation (1.1) has at least three solutions. Thus, the importance of theorem 1.3 lies in the assertion of the exact number of solutions and the precise description of the region where the equation has three solutions.

We remark that our results could also be useful for numerical applications. In fact, the singular set S is found constructively in our proof and is therefore accessible to numerical methods. Furthermore, the operator Φ maps S one to one onto $\Phi(S)$, and hence also $\Phi(S)$ can be represented numerically. This then allows to localize the solutions to a given data function h.

2. - Singularity theory in Banach space

In this section we give a short account of singularity theory in Banach space. For more details and the proofs we refer to [2,3,4,5,8].

In this section we assume that X and Y are Banach subspaces of a Hilbert space H, and that

$$F : X \to Y$$

is a smooth Fredholm mapping of index 0 (i.e. the Fréchet derivative $F'(u)$ of F in $u \in X$ is a Fredholm operator of index 0, that is, Ker $F'(u)$ and Im $F'(u)$ are closed and have closed complements, and the index of $F'(u)$ is zero, i.e. $\mathrm{i}(F'(u)) :=$ dim Ker $F'(u)-$ codim Im $F'(u) = 0$. It is easy to see that $\mathrm{i}(F'(u))$ is independent of $u \in X$).

DEFINITION 2.1. A point $u \in X$ is a *singular point* of F, if the equation

$$F'(u)v = 0$$

has a nontrivial solution v. Otherwise, u is a *regular point* of F. We denote by S the set of singular points of F.

We now give a condition such that the set of singular points of F is locally nice.

PROPOSITION 2.2. *Let $u \in X$ be a singular point of F, and assume that*

$$\text{dim Ker } F'(u) = 1. \tag{2.1}$$

$$(F'(u)v, w)_H = (v, F'(u)w)_H, \qquad \textit{for every } v, w \in X. \tag{2.2}$$

$$\textit{there exists } w \in X \textit{ such that } (F''(u)(w,e),e)_H \neq 0, \tag{2.3}$$
$$\textit{where } 0 \neq e \in \operatorname{Ker} F'(u).$$

Then there exists a neighbourhood U of u in X such that $S \cap U$ is a smooth hypersurface.

In order to describe the behaviour of F in a neighbourhood of a singular point, we need a classification of the singular points.

DEFINITION 2.3. Assume that $u \in X$ is a singular point of F satisfying (2.1-3). We call u a *fold point*, if the following condition holds

$$(F''(u)(e,e),e)_H \neq 0, \text{ where } 0 \neq e \in \operatorname{Ker} F'(u). \tag{2.4}$$

The term fold point is justified by the following *Normal Form theorem.*

THEOREM 2.4. *Let $u \in X$ be a fold point of F. Then there exists a Banach space Z such that F is locally equivalent at u to the map $G : \mathbb{R} \times Z \to \mathbb{R} \times Z$ given by*

$$G(t,z) = (t^2, z), \ t \in \mathbb{R}, \ z \in Z.$$

More precisely, there exist neighbourhoods U of u in X and V of $F(u)$ in Y and diffeomorphisms $\alpha : U \to \mathbb{R} \times Z$, $\beta : V \to \mathbb{R} \times Z$ such that

$$G = \beta \circ F \circ \alpha^{-1}.$$

In case that u satisfies (2.1-3), but not (2.4), we say that u is a *higher singularity*. We give again first a condition such that the set C of higher singularities is locally a nice subset of S.

PROPOSITION 2.5. *Let $u \in X$ be a higher singularity, and assume that there exists a $w \in X$ with*

$$(F''(u)(w,e),e)_H = 0$$

(2.5) *and such that*

$$(F'''(u)(w,e,e),e)_H + 3(F''(u)(e,z(w)),e)_H \neq 0,$$

where $0 \neq e \in \operatorname{Ker} F'(u)$ and $z = z(w)$ is a solution of the equation

$$F'(u)z = -F''(u)(w,e). \tag{2.6}$$

Then there exists a neighbourhood U of u in X such that $C \cap U$ is a smooth hypersurface in S, and hence a smooth submanifold of X of codimension 2.

A sharpening of condition (2.5) leads to the following definition.

DEFINITION 2.6. We call $u \in X$ a *cusp point* of F if u is a higher singular point of F and if the following condition holds

$$(F'''(u)(e,e,e),e)_H + 3(F''(u)(e,z(e)),e)_H \neq 0, \tag{2.7}$$

where $0 \neq e \in \operatorname{Ker} F'(u)$, and $z(e)$ is given by (2.6).

The term *cusp point* is again justified by a Normal Form theorem.

THEOREM 2.7. *Let $u \in X$ be a cusp point of F. Then there exists a Banach space Z such that F is locally equivalent at u to the map $G : \mathbb{R} \times \mathbb{R} \times Z \to \mathbb{R} \times \mathbb{R} \times Z$ given by*

$$G(t,s,z) = (t^3 + ts, s, z), \qquad t,s \in \mathbb{R}, \ \ z \in Z.$$

For later purposes, we give conditions (2.4) and (2.7) for the mapping $F(u) = -\Delta u - \lambda u + u^3$ with $0 < \lambda < \lambda_2$.

PROPOSITION 2.8. *Let $F(u) = -\Delta u - \lambda u + u^3$.*

a) *The fold condition* (2.4) *then is given by*

$$\int_\Omega u \cdot v_1^3(u) \neq 0 \tag{2.8}$$

where $v_1(u)$ is the solution of

$$\begin{cases} -\Delta v - \lambda v + 3u^2 v = 0, & \text{in } \Omega \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

b) *The cusp condition* (2.7) *is given by*

$$\int_\Omega v_1^4(u) - 18 \int uv_1^2(u) \ \left[-\Delta - \lambda + 3u^2\right]^{-1} \ \left[uv_1^2(u)\right] \neq 0. \tag{2.10}$$

PROOF. a) Since

$$F'(u)[v_1(u)] = -\Delta v_1(u) - \lambda v_1(u) + 3u^2 v_1(u) = 0$$

we have $e = v_1(u)$, and hence

$$F''(u)[v_1(u), v_1(u)] = 6uv_1^2(u), \tag{2.11}$$

from which (2.8) follows.

b) From (2.11) we get

$$F'''(u)[v_1(u), v_1(u), v_1(u)] = 6v_1^3(u).$$

Furthermore, from (2.6) we get

$$(-\Delta - \lambda + 3u^2)z(v_1(u)) = -6uv_1^2(u).$$

Since $(uv_1^2(u), v_1(u)) = 0$ by assumption, we can invert the operator $-\Delta - \lambda + 3u^2$ and get

$$z(v_1(u)) = -[-\Delta - \lambda + 3u^2]^{-1}(6uv_1^2(u)).$$

From (2.7) we now obtain the condition

$$\left(v_1^3(u), v_1(u)\right) - 18\left(uv_1(u)[-\Delta - \lambda + 3u^2]^{-1}[uv_1^2(u)], v_1(u)\right) \neq 0,$$

hence (2.10); here $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-inner product. ■

3. - The form of the singular set

In this section we describe the form of the singular set of the mapping

$$\Phi = -\Delta - \lambda + (\cdot)^3 : E \to F,$$

with

$$E = \left\{ u \in C^{2,\alpha}(\Omega), \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}, \quad F = C^{0,\alpha}(\Omega), \text{ for some } \alpha \in (0,1).$$

The Fréchet-derivative of Φ is given by

$$\Phi'(u) = -\Delta - \lambda + 3u^2 : E \to F.$$

The *singular set* S of Φ is defined as

$$S = \{u \in E \mid \exists v \in E, \ v \neq 0 : -\Delta v - \lambda v + 3u^2 v = 0\}.$$

We denote by $\mu_i(u)$, $i \in \mathbb{N}$, the spectrum of $\Phi'(u)$ with corresponding L^2-normalized eigenfunctions $v_i(u)$:

$$\Phi'(u)v_i(u) \equiv -\Delta v_i(u) - \lambda v_i(u) + 3u^2 v_i(u) = \mu_i(u)v_i(u), \ i \in \mathbb{N}. \tag{3.1}$$

LEMMA 3.1. *Let $0 < \lambda < \lambda_2$. Then the singular set S of Φ is given by*

$$S = \{u \in E \mid \mu_1(u) = 0\}. \tag{3.2}$$

Furthermore, we have

$$\mu_2(u) \geq \lambda_2 - \lambda, \qquad \textit{for every } u \in E. \tag{3.3}$$

PROOF. The statement (3.2) follows from (3.3), since for $\lambda < \lambda_2$ we have $\mu_2(u) > 0$, for all $u \in E$. The inequality (3.3) follows from the variational characterization of the eigenvalues. Let $\tilde{\mu}_2(u) = \mu_2(u) + \lambda$. We denote by

$$K(x, y) = \{\alpha x + \beta y;\ \alpha^2 + \beta^2 = 1\},$$

where $(x, y) \in V_2$ with

$$V_2 = \left\{(x, y) \in H^1(\Omega) \times H^1(\Omega); \|x\| = \|y\| = 1,\ (x, y) = 0\right\}.$$

Here and below $(\cdot, \cdot)$ denotes the L^2-inner product and $\|\cdot\|$ the L^2-norm. Then we have

$$\begin{aligned}
\tilde{\mu}_2(u) &= \min_{V_2} \max_{K(x,y)} \left\{\|\nabla v\|^2 + 3 \int u^2 v^2\right\} \\
&= \max_{K(\bar{x},\bar{y})} \left\{\|\nabla v\|^2 + 3 \int u^2 v^2\right\} \\
&\geq \max_{K(\bar{x},\bar{y})} \|\nabla v\|^2 \\
&\geq \min_{V_2} \max_{K(x,y)} \|\nabla v\|^2 = \tilde{\mu}_2(0) = \lambda_2.
\end{aligned}$$

Hence $\mu_2(u) = \tilde{\mu}_2(u) - \lambda \geq \tilde{\mu}_2(0) - \lambda = \lambda_2 - \lambda$. ∎

For the next considerations we distinguish between the Sturm-Liouville problem and the equation in higher dimensions.

PROPOSITION 3.2. *Let* $\Omega = (0, 1)$, *and let* D_1 *denote the unit* L^2*-sphere in* E. *Then, if* $0 < \lambda < \frac{\lambda_2}{4}$, *there exists a radial diffeomorphism*

$$\rho : D_1 \to S,\ u \mapsto \rho(u) \cdot u \in S.$$

PROOF. Let $u \in D_1$ be given. We consider the function

$$\mu_1(\rho \cdot u), \quad \rho \in \mathbb{R}^+.$$

This function is strictly monotone in ρ, since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\ \mu_1(\rho) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\ \left[\left\|\frac{\mathrm{d}}{\mathrm{d}x}\ v_1(\rho u)\right\|^2 - \lambda\|v_1(\rho u)\|^2 + 3\int \rho^2 u^2 v_1^2(\rho u)\right] \\
&= 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\ v_1(\rho u),\ \frac{\mathrm{d}}{\mathrm{d}x}\ \frac{\mathrm{d}}{\mathrm{d}\rho}\ v_1(\rho u)\right) - 2\lambda\left(v_1(\rho u),\ \frac{\mathrm{d}}{\mathrm{d}\rho}\ v_1(\rho u)\right) \\
&\quad + 6\int \rho^2 u^2 v_1(\rho u)\ \frac{\mathrm{d}}{\mathrm{d}\rho}\ v_1(\rho u) + 6\int \rho u^2 v_1^2(\rho u) \\
&= 2\mu_1(\rho u)\ \left(v_1(\rho u),\ \frac{\mathrm{d}}{\mathrm{d}\rho}\ v_1(\rho u)\right) + 6\int \rho u^2 v_1^2(\rho u) \\
&= 6\int \rho u^2 v_1^2(\rho u) > 0, \ \text{ for } \rho > 0,
\end{aligned}$$

since $v_1(\rho u)$ does not change sign. Furthermore, we have $\mu_1(0) = -\lambda < 0$, since $v_1(0)$ is the constant function equal to $+1$.

From the following lemma 3.3 we have that

$$\lim_{\rho\to+\infty}\ \mu_1(\rho u) \geq \frac{\lambda_2}{4} - \lambda, \qquad \text{for every } u \in D_1.$$

Hence, if $\lambda < \frac{\lambda_2}{4}$, there exists on each ray $\{\rho u;\ \rho \in \mathbb{R}^+\}$ a unique $\overline{\rho} = \overline{\rho}(u)$ such that $\mu_1(\overline{\rho}(u)u) = 0$.

The proposition now follows easily, either by showing directly that $\rho(u)$ is depending smoothly on u, or by using the implicit function theorem (we have shown that u is a transversal vector to S in the point $\overline{\rho}(u)\cdot u$). ■

Before stating lemma 3.3 we introduce some notation. Let I denote a union of open and disjoint subintervals of $(0,1)$, and denote by $\lambda_1(I)$ the first eigenvalue of the equation

$$(3.4)\qquad \begin{cases} -\quad v'' = \lambda v, & \text{in } (0,1)\backslash\overline{I} \\ \quad v'(0) = v'(1) = 0 & \\ \quad v = 0, & \text{on } \overline{I}; \end{cases}$$

we set $\lambda_1(I) = +\infty$, if $I = (0,1)$. Note that

$$(3.5)\qquad \inf_{I\neq\emptyset}\ \lambda_1(I) = \frac{\lambda_2}{4}.$$

For this, one observes that

$$\inf_{I\neq\emptyset}\ \lambda_1(I) = \lim_{\varepsilon\to 0}\ \lambda_1((0,\varepsilon)) = \frac{\left\|\frac{\mathrm{d}}{\mathrm{d}x}\ \sin\ \frac{\pi}{2}\ x\right\|^2}{\left\|\sin\ \frac{\pi}{2}\ x\right\|^2} = \frac{\lambda_2}{4}.$$

LEMMA. 3.3. *Let* $\Omega = (0,1)$. *Then*

$$\lim_{\rho\to+\infty} \mu_1(\rho u) \geq \frac{\lambda_2}{4} - \lambda, \quad \textit{for every } u \in S_1.$$

PROOF. For $u \in D_1$, let $I(u) = \{x \in (0,\pi) | u(x) \neq 0\}$. We claim that

$$\lim_{\rho\to+\infty} \mu_1(\rho u) = \lambda_1(I(u)) - \lambda, \quad \text{for all } u \in D_1. \tag{3.6}$$

To prove (3.6) we note first that we have $\mu_1(\rho u) \leq \lambda_1(I(u)) - \lambda$ for all $\rho \in \mathbb{R}^+$. This follows from the variational characterization (we set again $\tilde{\mu}_1(u) = \mu_1(u) + \lambda$):

$$\begin{aligned}\tilde{\mu}_1(\rho u) &= \min_{v\in H^1(\Omega), \|v\|=1} \left[\left\| \frac{\mathrm{d}}{\mathrm{d}x} v \right\|^2 + 3\int \rho^2 u^2 v^2 \right] \\ &\leq \min_{v \in H^1, \|v\|=1, v|_{I(u)}=0} \left\| \frac{\mathrm{d}}{\mathrm{d}x} v \right\|^2 = \lambda_1 \ (I(u)).\end{aligned} \tag{3.7}$$

We now show that $\mu_1(\rho u) \underset{\rho\to+\infty}{\longrightarrow} \lambda_1(I(u)) - \lambda$. In fact, (3.7) implies that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} v_1(\rho) \right\|^2 \leq \lambda_1(I(u)) < \infty, \qquad \text{for every } \rho \in \mathbb{R}^+,$$

and hence $(v_1(\rho))_{\rho\in\mathbb{R}^+}$ is relatively compact in C^α, $0 < \alpha < \frac{1}{2}$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence with $\rho_n \to +\infty$, and such that $v_1(\rho_n) \xrightarrow{C^\alpha} \overline{v}_1$. We claim that $\overline{v}_1|_{I(u)} = 0$ in $L^2(I(u))$. If not, we would have

$$\tilde{\mu}_1(\rho_n u) = \left\| \frac{\mathrm{d}}{\mathrm{d}x} v_1(\rho_n) \right\|^2 + 3\int \rho_n^2 u^2 v_1^2(\rho_n) \to +\infty, \tag{3.8}$$

contradicting (3.7).

This implies that $\overline{v}_1$ solves the equation (3.4) with $I = I(u)$, and hence $\lim_{n\to\infty} \mu_1(\rho_n u) = \lambda_1(I(u)) - \lambda$.

By (3.5) we have $\lambda_1(I(u)) - \lambda \geq \frac{\lambda_2}{4} - \lambda$, and hence the lemma is proved. ■

For $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, the situation is more complicated, since in this case there exist rays which do not meet S.

LEMMA 3.4. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *and assume that* $\lambda > 0$. *Then there exist functions* $u \in D_1$ *such that*

$$\{\rho u | \rho \in \mathbb{R}^+\} \cap S = \emptyset. \tag{3.9}$$

PROOF. Let $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset \Omega$, where

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n;\ |y - x| < \varepsilon\}.$$

Let $u_\varepsilon \in E$ with supp $u_\varepsilon \subset B_\varepsilon$ and $\|u_\varepsilon\| = 1$. We claim that, for $\varepsilon > 0$ sufficiently small, the function u has the property (3.9).

In fact, we note first that

$$\mu_1(\rho u_\varepsilon) \leq \mu_1(B_\varepsilon), \qquad \text{for every } \rho \in \mathbb{R}^+$$

where $\mu_1(B_\varepsilon)$ is the first eigenfunction of the equation

$$\begin{cases} -\Delta v - \lambda v = \mu v, & \text{in } \Omega\backslash\overline{B}_\varepsilon(x_0) \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \\ v = 0, & \text{on } \overline{B}_\varepsilon(x_0). \end{cases} \tag{3.10}$$

This follows by the analogue of (3.7) in higher dimensions (replacing $I(u)$ by B_ε). We now estimate $\tilde{\mu}_1(B_\varepsilon) = \mu_1(B_\varepsilon) + \lambda$. Let

$$H_\varepsilon = \{u \in H^1(\Omega),\ u \equiv 0 \text{ on } \overline{B}_\varepsilon\}.$$

Then

$$\tilde{\mu}_1(B_\varepsilon) = \inf_{0 \neq u \in H_\varepsilon} \frac{\|\nabla u\|^2}{\|u\|^2} \leq \frac{\|\nabla \phi_\varepsilon\|^2}{\|\phi_\varepsilon\|^2}$$

where

$$\phi_\varepsilon(x) = \begin{cases} 0, & x \in B_\varepsilon(0) \\ \dfrac{(|x| - \varepsilon)^2}{(\sqrt{\varepsilon} - \varepsilon)^2}, & x \in B_{\sqrt{\varepsilon}}(0)\backslash B_\varepsilon(0) \\ 1, & x \in \Omega\backslash B_{\sqrt{\varepsilon}}(0) \end{cases}$$

(here we assume, without loss of generality, that $x_0 = 0 \in \Omega$ and $\varepsilon > 0$ such that $B_{\sqrt{\varepsilon}}(0) \subset \Omega$). Then $\phi_\varepsilon \in H_\varepsilon$. We now estimate $\|\nabla\phi_\varepsilon\|^2/\|\phi_\varepsilon\|^2$, setting

$$r^2 = |x|^2 = \sum_{i=1}^n |x_i|^2 \qquad \text{and} \qquad \omega_n = \int_{B_1(0)} 1 dx :$$

$$\begin{aligned} \|\nabla\phi_\varepsilon\|^2 &= n \cdot \omega_n \int_\varepsilon^{\sqrt{\varepsilon}} \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} \frac{(r - \varepsilon)^2}{(\sqrt{\varepsilon} - \varepsilon)^2} \right|^2 r^{n-1} dr \\ &= n \cdot \omega_n \int_\varepsilon^{\sqrt{\varepsilon}} 4\, \frac{(r - \varepsilon)^2 r \cdot r^{n-1}}{(\sqrt{\varepsilon} - \varepsilon)^4}\, dr \\ &\leq c \cdot \sqrt{\varepsilon}, \qquad \text{for } 0 < \varepsilon \leq \varepsilon_0,\ \varepsilon_0 \text{ sufficiently small.} \end{aligned}$$

Finally, since $\|\phi_\varepsilon\| \geq d > 0$, for $0 < \varepsilon \leq \varepsilon_0$, we get the estimate $\tilde{\mu}_1(B_\varepsilon) \leq c \cdot \sqrt{\varepsilon}$, and hence

$$\mu_1(B_\varepsilon) \leq -\lambda + c \cdot \sqrt{\varepsilon}. \qquad \blacksquare$$

We now set $\Sigma := \{u \in S_1;\ \lim\limits_{\rho + \infty} \mu_1(\rho u) > 0\}$. Then we have

PROPOSITION 3.5. *There exists a radial diffeomorphism*

$$\rho : \Sigma \to S, \quad \mu \to \rho(u)u.$$

Furthermore, if $(u_n) \subset \Sigma$ *is a sequence converging to a boundary point of* Σ, *then* $\rho(u_n) \to +\infty$ *for* $n \to \infty$.

PROOF. As in proposition 3.2, we get that $\mu_1(\rho u)$ is strictly monotone increasing in ρ. Hence, if $u \in \Sigma$ there exists a unique $\rho(u)$ such that $\mu_1(\rho(u)u) = 0$. Clearly, $\rho(u)$ is smooth in u.

Let now $(u_n) \subset \Sigma$ be a sequence converging to the boundary of Σ and assume that, contrary to our claim, we have $\rho_n(u_n)u_n \in S$ with $\rho_n(u_n) \leq c$, for all $n \in \mathbb{N}$. Then we find subsequences $\rho_n u_n \to \overline{\rho} u$, $\mu_1(\rho_n u_n) = 0$ and $v_1(\rho_n) \to v_1$ and hence we obtain the limiting equation

$$-\Delta v_1 - \lambda v_1 + 3\ \overline{\rho}^2 u^2 v_1 = \mu_1(\overline{\rho} u) v_1 = 0.$$

But since $\mu_1(\rho u)$ is strictly increasing in ρ, we find

$$\lim\ \mu_1(\rho u) > \mu_1(\overline{\rho} u) = 0,$$

thus contradicting our assumption. $\blacksquare$

We conclude this section with the following remark.

LEMMA 3.6. *The normal vector* $n(u)$ *to* S *in the point* $u \in S$ *is given by* $n(u) = 6uv_1^2(u)$.

PROOF. By Lemma 3.1 we have $S = \{u \in E;\ \mu_1(u) = 0\}$. Hence $D\mu_1(u)[z] = 6(uv_1^2(u), z)$, i.e. $n(u) = 6uv_1^2(u)$. $\blacksquare$

4. - The structure of the singular set

In this section we classify the singular points of the mapping Φ. More precisely, we will show that the singular set S contains a submanifold C of codimension 1 (relative to S) of cusp points, and that $S \backslash C$ consists of fold points.

First we show that, for $0 < \lambda < \dfrac{\lambda_2}{7}$, the singular set contains only fold points and cusp points.

PROPOSITION 4.1. *Assume that* $0 < \lambda < \frac{\lambda_2}{7}$. *Then* S *contains only fold points and cusp points.*

PROOF. Let $u \in S$ be a higher singularity. We claim that then

$$\int v_1^4(u) - 18 \int u v_1^2(u)[-\Delta - \lambda + 3u^2]^{-1}(u v_1^2(u)) > 0. \tag{4.1}$$

In fact, assuming to the contrary that (4.1) does not hold, we obtain the estimate

$$\int v_1^4(u) \leq 18 \int u v_1^2(u) \; [-\Delta - \lambda + 3u^2]^{-1}(u v_1^2(u))$$
$$= 18 \sum_{i \geq 2} \frac{\left(u v_1^2, v_i(u)\right)^2}{\mu_i(u)}.$$

Since, by (3.3), $\mu_i(u) \geq \mu_2(u) \geq \lambda_2 - \lambda$, for every $u \in S$, $i \geq 2$, we obtain

$$\int v_1^4(u) \leq 18 \, \frac{\int u^2 v_1^4(u)}{\lambda_2 - \lambda}. \tag{4.2}$$

On the other hand, we obtain from

$$-\Delta v_1(u) - \lambda v_1(u) + 3u^2 v_1(u) = 0$$

by scalar multiplication with $v_1^3(u)$

$$3\left(\nabla v_1(u), v_1^2(u) \nabla v_1(u)\right) - \lambda \int v_1^4(u) + 3 \int u^2 v_1^4(u) = 0$$

and hence

$$3 \int u^2 v_1^4(u) \leq \lambda \int v_1^4(u). \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\int v_1^4(u) \leq \frac{6\lambda}{\lambda_2 - \lambda} \int v_1^4(u),$$

and hence

$$\lambda_2 - \lambda \leq 6\lambda \qquad \text{that is} \qquad \frac{\lambda_2}{7} \leq \lambda.$$

Therefore, if $\lambda < \frac{\lambda_2}{7}$, we have a contradiction and hence (4.1) holds. By proposition (2.10), it follows that u is a cusp point. ∎

Next we show that there exist higher singularities on S.

PROPOSITION 4.2. *Let* $0 < \lambda < \lambda_2$. *Then* S *contains higher singularities.*

PROOF. Let $e_1 =$ const. and e_2 denote the first and second eigenfunction of $-\Delta$, let

$$\Gamma = \left\{ u(\alpha) \in S_1;\ u(\alpha) = \alpha e_1 + (1-\alpha^2)^{1/2} e_2,\ |\alpha| \le 1 \right\}$$

and let $\rho(\alpha) \in \mathbb{R}^+$ such that $\rho(\alpha)u(\alpha) \in S$. Since $u(\alpha)$ vanishes only on sets of measure zero, it follows that $\rho(\alpha) < +\infty$, for all $|\alpha| \le 1$. Hence, $\{\rho(\alpha)u(\alpha);\ -1 \le \alpha \le 1\}$ is a continuous path with $\rho(\pm 1)u(\pm 1) = \pm\rho(e_1)e_1$. Noting that $v_1(\pm\rho(e_1)e_1) = e_1$, we deduce from $\pm \int \rho(\pm e_1)e_1^4 \gtrless 0$ that there exists a $\alpha \in (-1,1)$ such that $\int \rho(\alpha)u(\alpha)v_1^3(u(\alpha)) = 0$. ■

COROLLARY 4.3. *Assume that* $0 < \lambda < \dfrac{\lambda_2}{7}$. *Then the singular set* S *contains a codimension* 1 *(relative to* S*) manifold* C *consisting of cusp points and* $S \backslash C$ *consists of fold points.*

PROOF. By proposition 4.2, $C \neq \emptyset$ and, by propositions 2.7 and 2.4, C is a codimension 1 submanifold of S. By proposition 4.1, $S \backslash C$ consists of fold points. ■

5. - The image

We now come to the characterization of the image of $\Phi = -\Delta - \lambda + (\cdot)^3$. One way to achieve this would be to prove that $\Phi|_S$ is one to one. This would imply that $F \backslash \Phi(S)$ has exactly two components. By the application of the topological proposition 1.2, one could then deduce that the number of solutions is constant in each of these components: in fact, 3 in one component and 1 in the other.

We choose here a more direct and constructive method to prove this result, namely the Lyapunov-Schmidt reduction. That is, we decompose equation (1.1) into a pair of equations, namely

$$\Delta y - \lambda y + P(s1 + y)^3 = Ph \equiv h_1 \tag{5.1}$$

$$\lambda s1 + Q(s1 + y)^3 = Qh, \tag{5.2}$$

where

$$P : F \to F_1 = \left\{ u \in F;\ \int u \,\mathrm{d}x = 0 \right\}$$

and

$$Q = \mathrm{Id} - P : F \to \{ s \cdot 1; s \in \mathbb{R},\ 1 = \text{ constant function equal 1 on } \Omega \}$$

denote orthogonal projections (with respect to the L^2-scalar product), and $u = s \cdot 1 + y = Qu + Pu$ denotes the decomposition of u according to these projections (restricted to E). One now shows that (5.1) has a unique solution $y(s, h_1)$ for each fixed $s \in \mathbb{R}$ and $h_1 \in F_1$, which depends smoothly on h_1 and s. The problem is then reduced to study the functions $\Gamma(\cdot, h_1) : \mathbb{R} \to \mathbb{R}$ given by

$$\Gamma(s, h_1)1 = -\lambda s \cdot 1 + Q(s \cdot 1 + y(s, h_1))^3. \tag{5.3}$$

PROPOSITION 5.1. *Assume that $\lambda < \lambda_2$. Then equation* (5.1) (*with Neumann boundary conditions*) *has, for each fixed s and given data function h_1, exactly one solution which is smoothly depending on the arguments s and h_1.*

PROOF. The existence of a solution follows from the Leray-Schauder principle (see e.g. [12, p. 65]) for the equivalent equation

$$y - (-\Delta)^{-1}\left[\lambda y - P(s \cdot 1 + y)^3 + h_1\right] = 0.$$

We have to show that there exists a $R > 0$ such that, if $(\tau, y) \in [0, 1] \times E_1$ solves

$$y - \tau(-\Delta)^{-1} \left(\lambda y - P(s \cdot 1 + y)^3 + h_1\right) = 0,$$

then $\|y\|_E \leq R$. In fact, multiplying this equation by y, we get

$$\|\nabla y\|^2 - \tau\lambda\|y\|^2 + \tau\int(s1 + y)^4 - \tau\int(s1 + y)^3 s1 = \mathcal{T}(h, y).$$

Since

$$\int(s1 + y)^4 - \int(s1 + y)^3 s1 \geq \int(s1 + y)^4 - c\left(\int(s1 + y)^4\right)^{3/4} \geq -c,$$

we conclude that

$$(\lambda_2 - \lambda)\|y\|^2 \leq c\|y\| + d.$$

From this we infer that $\|y\|_{H^1} \leq c$, and then, for $h_1 \in C^{0,\alpha}(\Omega)$, it follows by standard regularity results that $\|y\|_E \leq R$.

To prove uniqueness, assume that there exist solutions $y_1, y_2 \in E_1 := PE$ of (5.1), i.e.

$$(-\Delta - \lambda)\ (y_1 - y_2) + P\ (s \cdot 1 + y_1)^3 - P(s \cdot 1 + y_2)^3 = 0.$$

Multiplying this equation by $y_1 - y_2$, we get the estimate:

$$\begin{aligned} 0 &= ((-\Delta - \lambda)\ (y_1 - y_2), y_1 - y_2) + \left((s \cdot 1 + y_1)^3 - (s \cdot 1 + y_2)^3, y_1 - y_2\right) \\ &\geq (\lambda_2 - \lambda)\ \|y_1 - y_2\|^2, \end{aligned}$$

since we have, setting $u_i = s \cdot 1 + y_i,\ i = 1,2,$

$$(u_1^3 - u_2^3, y_1 - y_2) = (u_1^3 - u_2^3,\ u_1 - u_2) = \left(\frac{u_1^3 - u_2^3}{u_1 - u_2},\ (u_1 - u_2)^2\right)$$

$$= \left(\frac{1}{2}\ (u_1^2 + u_2^2) + \frac{1}{2}\ (u_1 + u_2)^2,\ (u_1 - u_2)^2\right) \geq 0.$$

By assumption we have $\lambda_2 > \lambda$ and hence $y_1 = y_2$.

The smooth dependence of y on s and h_1 is easily verified. ∎

We note that for given $h_1 \in F_1$ the mappings $s \mapsto u(s, h_1) = s \cdot 1 + y(s, h_1)$ are curves in E which are disjoint for $h_1 \neq g_1$. We use these curves to give an interpretation of fold points and cusp points.

PROPOSITION 5.2.

a) *A point $u(s, h_1) = s \cdot 1 + y(s, h_1) \in E$ is a fold point if $u(s, h_1)$ is a nondegenerate zero of the functon $s \to \eta(s, h_1) = \frac{d}{ds}\ \Gamma(s, h_1)$, i.e.*

$$\eta(s, h_1) = 0 \qquad \textit{and} \qquad \frac{d}{dt}\ \eta(t, h_1)\Big|_{t=s} \neq 0.$$

b) *A point $u(s, h_1) \in E$ is a cusp point if $u(s, h_1)$ is a nondegenerate extremum at level zero of the function $\eta(s, h_1)$, i.e.*

$$\eta(s, h_1) = 0, \qquad \frac{d}{dt}\ \eta(t, h_1)\Big|_{t=s} = 0$$

and

$$\frac{d^2}{dt^2}\ \eta(t, h_1)\Big|_{t=s} \neq 0.$$

PROOF. a) Taking the derivative of equations (5.1) and (5.3) with respect to s and writing $y_s = \frac{d}{ds}\ y(s, h_1)$, we obtain

$$- \Delta y_s - \lambda y_s + 3P(s \cdot 1 + y)^2\ (1 + y_s) = 0, \tag{5.4}$$

$$- \lambda 1 + 3Q(s \cdot 1 + y)^2\ (1 + y_s) = \eta(s, h_1)1. \tag{5.5}$$

Hence, $1 + y_s =: v_1(s \cdot 1 + y)$ is the first eigenfunction to the eigenvalue $\mu_1(s \cdot 1 + y) = 0$ if and only if $\eta(s, h_1) = 0$, since then

$$(-\Delta - \lambda)(1 + y_s) + 3(s \cdot 1 + y)^2 (1 + y_s) = 0. \tag{5.6}$$

Taking the derivative $\frac{d}{ds}\ \eta(s, h_1)1$, we get

$$\frac{d}{ds}\ \eta(s, h_1)1 = 3Q(s \cdot 1 + y)^2 y_{ss} + 6Q(s \cdot 1 + y)(1 + y_s)^2. \tag{5.7}$$

From (5.4), we obtain

$$-\Delta y_{ss} - \lambda y_{ss} + 3P(s \cdot 1 + y)^2 y_{ss} + 6P(s \cdot 1 + y)(1 + y_s)^2 = 0. \tag{5.8}$$

Adding (5.7) and (5.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \eta(s, h_1)1 = \left(-\Delta - \lambda + 3(s \cdot 1 + y)^2\right) y_{ss} + 6(s \cdot 1 + y)(1 + y_s)^2. \tag{5.9}$$

Multiplying this by $v_1 = 1 + y_s$, we obtain by (5.6)

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \eta(s, h_1) = 6\left(s \cdot 1 + y, (1 + y_s)^3\right) = 6\left(s \cdot 1 + y, v_1^3(s \cdot 1 + y)\right). \tag{5.10}$$

Hence, if $\frac{\mathrm{d}}{\mathrm{d}s}\, \eta(s, h_1) \neq 0$, the condition (2.8) is satisfied, that is

$$u(s, h_1) = s \cdot 1 + y(s, h_1)$$

is a fold point.

b) Taking the derivative with respect to s of (5.10), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\, \eta(s, h_1) = 6 \int v_1^4(s \cdot 1 + y) + 18\left((s \cdot 1 + y) v_1^2(s \cdot 1 + y), y_{ss}\right). \tag{5.11}$$

Since $\frac{\mathrm{d}}{\mathrm{d}s}\, \eta(s, h_1) = 0$ by assumption, we get from (5.9)

$$y_{ss} = -\left(-\Delta - \lambda + 3(s \cdot 1 + y)^2\right)^{-1} \left(6(s \cdot 1 + y) v_1^2(s \cdot 1 + y)\right).$$

Substituting this into (5.11), we see that $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\, \eta(s, h_1) \neq 0$ iff (2.10) holds. ∎

REMARK 5.3. The reason that the singular points can be characterized by the solution curves $s \cdot 1 + y(s, h_1)$ is that, in a singular point

$$u(s_0, h_1) = s_0 1 + y(s_0, h_1) \in S,$$

the tangent to the curve $u(s, h_1)$ has the direction $v_1(u(s_0, h_1))$, i.e.

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\, u(s, h_1)\right|_{s = s_0} = 1 + y_s(s_0, h_1) = v_1(u(s_0, h_1)).$$

We now derive a differential inequality for the function η.

LEMMA 5.4. *For every $h_1 \in F_1$, the function $\eta(s, h_1)$ satisfies the inequality*

$$\eta_{ss}(s) \geq 6 \int v_1(s)^4 \left(1 - \frac{6\lambda}{\lambda_2 - \lambda}\right) - \frac{36}{\lambda_2 - \lambda}\, \eta(s) \int v_1(s)^3, \tag{5.12}$$

for all $s \in \mathbb{R}$.

PROOF. The proof follows the idea of proposition 4.1. By equation (5.11) we have, setting $u = s1 + y(s, h_1)$ and $v_1 = v_1(s) = 1 + y_s(s, h_1)$,

$$\eta_{ss}(s) = 6\int v_1^4 + 18\int uv_1^2 y_{ss}. \tag{5.13}$$

Let e_i, with $\left.\frac{\partial e_i}{\partial n}\right|_{\partial\Omega} = 0$, $i \in \mathbb{N}$, denote the L^2-normalized eigenfunctions of $-\Delta$. Since $(y_{ss}, e_1) = 0$, we have

$$(uv_1^2, y_{ss}) = \sum_{i\geq 2} (uv_1^2, e_i)(e_i, y_{ss}). \tag{5.14}$$

By (5.9), we have

$$\big((-\Delta - \lambda + 3u^2)y_{ss}, e_i\big) = -6(uv_1^2, e_i), \quad i = 2, 3, \dots$$

and hence

$$(\lambda_i - \lambda)(y_{ss}, e_i) + 3(u^2 y_{ss}, e_i) = -6(uv_1^2, e_i)$$

$$(y_{ss}, e_i) = \frac{-3(u^2 y_{ss}, e_i) - 6(uv_1^2, e_i)}{\lambda_i - \lambda}, \quad i = 2, 3, \dots \quad .$$

Inserting this in (5.14), we get

$$\begin{aligned}
(uv_1^2, y_{ss}) &= -\sum_{i\geq 2} (uv_1^2, e_i)\ \frac{3(e_i, u^2 y_{ss})}{\lambda_i - \lambda} + \frac{6(e_i, uv_1^2)}{\lambda_i - \lambda} \\
&= \frac{1}{6}\sum_{i\geq 2} \big((-\Delta - \lambda + 3u^2)y_{ss}, e_i\big)\ \frac{3(e_i, u^2 y_{ss})}{\lambda_i - \lambda} - 6\sum_{i\geq 2} \frac{(uv_1^2, e_i)^2}{\lambda_i - \lambda} \\
&= \frac{1}{2}\sum_{i\geq 2} (y_{ss}, e_i)\ (e_i, u^2 y_{ss}) \\
&\quad + \frac{3}{2}\sum_{i\geq 2} \frac{(u^2 y_{ss}, e_i)^2}{\lambda_i - \lambda} - \frac{6}{\lambda_i - \lambda}\sum_{i\geq 2} (uv_1^2, e_i)^2 \\
&\geq -\frac{6}{\lambda_i - \lambda}\int u^2 v_1^4.
\end{aligned}$$

Hence we can estimate in (5.13)

$$\eta_{ss}(s, h_1) \geq 6\int v_1^4 - 6 \cdot 18\int u^2 v_1^4. \tag{5.15}$$

We now use that by (5.4) and (5.5)

$$(-\Delta - \lambda)v_1 + 3u^2 v_1 = \eta(s, h_1)1. \tag{5.16}$$

Multiplying this equation by v_1^3, we get

$$3\int v_1^2|\nabla v_1|^2 - \lambda\int v_1^4 + 3\int u^2 v_1^4 = \eta(s,h_1)\int v_1^3$$

and hence

$$3\int u^2 v_1^4 \le \lambda\int v_1^4 + \eta(s,h_1)\int v_1^3. \tag{5.17}$$

Using this in (5.15), we get (5.12). ∎

PROPOSITION 5.5. *Let* $0 < \lambda < \frac{\lambda_2}{7}$. *Then the set* $M \cap S$ *consists of strict minimum points for* $\eta(s,h_1)$.

PROOF. For $u = s + y \in M \cap S$, we have $\eta(s) = 0 = \eta_s(s)$, and then by (5.12) $\eta_{ss}(s) > 0$. ∎

To characterize the image of Φ, it is crucial to know that $\Phi|_S$ is injective. For this it is clearly sufficient to prove that Φ is injective on

$$S_{h_1} := \{s + y(s,h_1);\ s \in \mathbb{R}\} \cap S,$$

for every $h_1 \in F_1$, since

$$\Phi(S_{h_1}) \subset \{s + h_1;\ s \in \mathbb{R}\}.$$

Note that the set

$$S_{h_1} = \{s + y(s,h_1);\ \eta(s,h_1) = 0\}$$

consists of discrete points, since by proposition 5.5 for a point $\bar{s} \in S_{h_1}$, with $\eta_s(\bar{s}) = 0$, one has $\eta_{ss}(\bar{s}) > 0$. Note furthermore that in intervals where $\eta(s,h_1) = \frac{d}{ds}\,\Gamma(s,h_1) \ge 0$ the function $\Gamma(s,h_1)$ is strictly increasing, while in intervals where $\eta(s,h_1) < 0$ the function Γ is strictly decreasing.

To prove injectivity, it suffices to consider four subsequent points $s_1 < t_1 < s_2 < t_2$ in S_{h_1} such that

$$\eta_s(s_i,h_1) < 0, \quad \eta_s(t_i,h_1) > 0, \quad i = 1,2$$

and

$$\eta(s,h_1) < 0, \qquad s \in (s_1,t_1) \cup (s_2,t_2) \tag{5.18}$$

$$\eta(s,h_1) \ge 0, \qquad s \in (t_1,s_2) \tag{5.19}$$

(the cases that S_{h_1} contains one or two points are trivial, while the case that S_{h_1} contains three points follows easily from the above case).

Since h_1 remains fixed in the argument, we suppress it in the sequel. From (5.18) it follows that

$$\Gamma(t_1) < \Gamma(s_1), \qquad \Gamma(t_2) < \Gamma(s_2).$$

To obtain injectivity of $\Phi|_{S_{h_1}}$ it is therefore sufficient to prove that

$$\Gamma(s_1) < \Gamma(t_2). \tag{5.20}$$

Qualitatively it is easy to see that this follows from inequality (5.12) for $\lambda < 0$ with $|\lambda|$ sufficiently small. In fact, we have $\eta(s) \geq -\lambda$ for all $s \in \mathbb{R}$, since, setting $\omega = \int_\Omega 1 \, dx$,

$$\eta\omega = \big((-\Delta - \lambda + 3u^2)(1 + y_s), 1 + y_s\big) \geq ((-\Delta - \lambda)y_s, y_s) - \lambda\omega \geq -\lambda\omega.$$

Hence we see that

$$\Gamma(s_i) - \Gamma(t_i) = \int_{t_i}^{s_i} \eta(s) ds, \qquad i = 1, 2,$$

is $O(\lambda)$ for λ small. On the other hand, by (5.12), $\eta_{ss} \geq d - c\eta(s)$ (see below), and hence it follows that $\Gamma(s_2) - \Gamma(t_1) = \int_{t_1}^{s_2} \eta(s)\, ds$ is bounded away from zero for λ tending to zero, say $\Gamma(s_2) - \Gamma(t_1) \geq c > 0$. Hence, for λ small, we have

$$\begin{aligned}\Gamma(t_2) - \Gamma(s_1) &= \Gamma(s_2) - \Gamma(t_1) + \Gamma(t_2) - \Gamma(s_2) + \Gamma(t_1) - \Gamma(s_1) \\ &\geq c - 2 \cdot O(\lambda) > O.\end{aligned}$$

In the sequel we estimate the size of λ.

PROPOSITION 5.6. *Let* $0 < \lambda \leq \frac{\lambda_2}{12}$. *Then* $\Phi|_{S_{h_1}}$ *is injective.*

PROOF. The proof proceeds in several steps.

1. Denoting again $\omega = \int_\Omega 1 \, dx$, we have

$$\left|\int (1+y_s)^3\right| \leq \left(\int 1^4\right)^{1/4} \left(\int (1+y_s)^4\right)^{3/4} = \omega^{1/4} \left(\int (1+y_s)^4\right)^{3/4},$$

$$\begin{aligned}\int v_1^4 &= \int 1 + 6\int y_s^2 + 4\int y_s^3 + \int y_s^4 \\ &\geq \omega + 6\int y_s^2 + \int y_s^4 - 4\left(3\int y_s^2\right)^{1/2} \left(\frac{1}{3}\int y_s^4\right)^{1/2} \\ &\geq \omega + 6\int y_s^2 + \int y_s^4 - 6\int y_s^2 - \frac{2}{3}\int y_s^4 \geq \omega + \frac{1}{3}\int y_s^4.\end{aligned} \tag{5.21}$$

2. If $\eta \geq 0$, then $v_1 = 1 + u_s > 0$ in Ω. In fact, note that $\eta \geq 0$ if and only if the first eigenvalue μ_1 of

$$(5.22) \qquad \begin{aligned} (-\Delta - \lambda + 3u^2)v &= \mu\, v \qquad \text{in } \Omega, \\ \frac{\partial v}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

is non-negative (and $\eta > 0$ if and only if $\mu_1 > 0$). Hence, if $\eta = 0$, then $1 + u_s$ is the first eigenfunction of (5.22), and therefore it is positive. If $\eta > 0$, then $\mu_1 > 0$, and therefore the equation

$$(-\Delta - \lambda + 3u^2)(1 + y_s) = \eta > 0$$

implies by the maximum-principle that $1 + y_s > 0$ in Ω.

3. We now consider (5.12) separately, where η is smaller than $\frac{\lambda_2 - 7\lambda}{6}$ and where it is larger.

a)
$$-\lambda \leq \eta(s) \leq \frac{\lambda_2 - 7\lambda}{6}.$$

Here we can estimate (5.12) further as follows. In lemma 5.7 below, we show that, for $\eta < 0$, $\int v_1^3 \geq \sigma(\lambda)\ \omega$, with $\sigma(\lambda) \geq 0.81$ for $0 < \lambda \leq \frac{\lambda_2}{12}$. Hence we get, for $\eta(s) < 0$,

$$\eta_{ss} \geq \sigma\omega\ \frac{6}{\lambda_2 - \lambda}\ (\lambda_2 - 7\lambda - 6\eta).$$

If $0 < \eta(s)$ then $\int v_1^3 \geq \omega$, since then $v_1 > 0$ (see above) and hence

$$\begin{aligned} \int v_1^3 = \int (1 + y_s)^2 &= \omega + \int 3y_s^2 + \int y_s^3 \\ &= \omega + 2\int y_s^2 + \int y_s^2(1 + y_s) \geq \omega. \end{aligned}$$

Hence we get, by (5.12) and step 1 above,

$$\eta_{ss} \geq \frac{6\ \omega^{1/4}}{\lambda_2 - \lambda} \left(\int v^4\right)^{3/4} (\lambda_2 - 7\lambda - 6\eta) \geq \frac{6\sigma\omega}{\lambda_2 - \lambda}\ (\lambda_2 - 7\lambda - 6\eta),$$

where the last estimate is done for simplification. Hence we have on the whole interval $\eta_{ss} \geq \sigma\ \frac{6\omega}{\lambda_2 - \lambda}\ (\lambda_2 - 7\lambda - 6\eta)$. We solve the equation

$$(5.23) \qquad \begin{aligned} &\varsigma_{ss} = \sigma\ \frac{6\omega}{\lambda_2 - \lambda}\ (\lambda_2 7\lambda - 6\varsigma) \\ &\varsigma(0) = -\lambda, \quad \varsigma'(0) = 0. \end{aligned}$$

The solution is given by

$$\varsigma(s) = -\frac{\lambda_2 - \lambda}{6} \cos \left[6 \left(\frac{\sigma\omega}{\lambda_2 - \lambda} \right)^{1/2} s \right] + \frac{\lambda_2 - 7\lambda}{6},$$

with $-\lambda \leq \varsigma(s) \leq \dfrac{\lambda_2 - 7\lambda}{6}$ for $s \in \left[0, \frac{\pi}{12} \left(\frac{\lambda_2 - \lambda}{\sigma\omega} \right)^{1/2} \right] =: [0, a]$.

Note that $\varsigma^{-1}(0) = \dfrac{1}{6} \left(\dfrac{\lambda_2 - \lambda}{\sigma\omega} \right)^{1/2} \arccos \dfrac{\lambda_2 - 7\lambda}{\lambda_2 - \lambda}$.

b)
$$\eta(s) \geq \frac{\lambda_2 - 7\lambda}{6}.$$

Let $M \geq 1$ be such that $\int (1 + u_s)^3 \leq M\omega$, for $\eta(s) \leq A$ (see lemma 5.7 below). Then we can estimate (5.12): $\eta_{tt} \geq \dfrac{6\omega}{\lambda_2 - \lambda} (\lambda_2 - 7\lambda - 6M\eta(t))$.

We solve the equation

$$\xi_{tt} = \frac{6\omega}{\lambda_2 - \lambda} [\lambda_2 - 7\lambda - 6M\xi(t)], \qquad t \leq \frac{\pi}{12} \left(\frac{\lambda_2 - \lambda}{M\omega} \right)^{1/2} =: b \tag{5.24}$$

$$\xi(b) = A, \quad \xi'(b) = 0.$$

The solution is given by

$$\xi_A(t) = \left(A - \frac{\lambda_2 - 7\lambda}{6M} \right) \sin \left[6 \left(\frac{M\omega}{\lambda_2 - \lambda} \right)^{1/2} t \right] + \frac{\lambda_2 - 7\lambda}{6M}.$$

c) We now join the solutions ς and ξ as follows:

$$z(t) = \begin{cases} \varsigma(t), & t \in [0, a] \\ \xi_A(t - a + \alpha), & t \in (a, a + b - \alpha] =: (a, c], \end{cases} \tag{5.25}$$

where $\alpha > 0$ is such that $\xi_A(\alpha) = \varsigma(a) = \dfrac{\lambda_2 - 7\lambda}{6}$. This yields

$$\left(A - \frac{\lambda_2 - 7\lambda}{6M} \right) \sin \left[6 \left(\frac{M\omega}{\lambda_2 - \lambda} \right)^{1/2} \alpha \right] = \left(1 - \frac{1}{M} \right) \frac{\lambda_2 - 7\lambda}{6}. \tag{5.26}$$

Moreover, we choose $A > 0$ such that $z'(t)$ is continuous in $t = a$: $\varsigma'(a) = \xi_A'(\alpha)$. This yields

$$\left(A - \frac{\lambda_2 - 7\lambda}{6M} \right) \cos \left[6 \left(\frac{\omega M}{\lambda_2 - \lambda} \right)^{1/2} \alpha \right] = \frac{\lambda_2 - \lambda}{6} \left(\frac{\sigma}{M} \right)^{1/2}. \tag{5.27}$$

Adding the squares of (5.26) and (5.27), and setting $B := \frac{A}{\lambda_2}$ and $\rho := \frac{\lambda}{\lambda_2}$ yields

$$B^2 = \frac{(1-7\rho)B}{3M} + (M^2 - 2M)\left(\frac{1-7\rho}{6M}\right)^2 + \left(\frac{1-\rho}{6}\right)^2 \frac{\sigma}{M}. \tag{5.28}$$

We will use later this relation to estimate B.

4. Next we show that the solution $\eta(t)$ of (5.12) lies above the function $z(t)$ defined in (5.25), more precisely, if

$$z(0) = \eta(0), \qquad \text{then } \eta(t) > z(t), \qquad \text{for } t \in (0, c],$$

respectively, if

$$z(c) = \eta(c), \qquad \text{then } \eta(t) > z(t), \qquad \text{for } t \in [0, c).$$

To see this, note that

$$w(s) = \begin{cases} \sin\left[6\left(\frac{\omega\sigma}{\lambda_2 - \lambda}\right)^{1/2} s\right], & s \in [0, a] \\ \cos\left[6\left(\frac{M\omega}{\lambda_2 - \lambda}\right)^{1/2} (s - a)\right], & s \in (a, a + b] \end{cases}$$

is the first eigenfunction (with eigenvalue $\theta_1 = 1$) of the equation

$$-v'' = \theta\psi(x)v, \qquad \text{in } (0, a+b), \quad v(0) = v(a+b) = 0,$$

with

$$\psi(x) = \begin{cases} \frac{36\omega\sigma}{\lambda_2 - \lambda}, & x \in (0, a] \\ \frac{36M\omega}{\lambda_2 - \lambda}, & x \in (a, +b]. \end{cases}$$

Note that, if $\eta(0) = z(0)$, then $\eta(t) > z(t)$ for t near 0, since $\eta'(0) = z'(0) = 0$ and $\eta''(0) > z''(0)$, and similarly, if $\eta(c) = z(c)$, then $\eta(t) > z(t)$ near c. Assume now that $\eta(t_0) = z(t_0)$ for some $t_0 \in (0, c)$. Then $\eta - z$ solves

$$-(\eta - z)'' = \psi(t)(\eta - z) - g(t), \qquad \text{for some function } g(t) \geq 0,$$

$$(\eta - z)(0) = (\eta - z)(t_0) = 0, \quad [\text{resp. } (\eta - z)(t_0) = (\eta - z)(b) = 0].$$

Let w_1 denote the positive first eigenfunction with corresponding eigenvalue $\theta_1(t_0)$ of $-v'' = \theta\psi(t)v$, $v(0) = v(t_0) = 0$ [resp. $v(t_0) = v(b) = 0$]. Since $t_0 \in (0, a + b - \alpha)$, with $\alpha > 0$, we have $\theta_1(t_0) > 1$. Multiplying the above equation with w_1 yields

$$\theta_1(t_0)\int \psi(\eta - z)w_1 = \int \psi(\eta - z)w_1 - \int g(t)w_1,$$

contradicting

$$\theta_1(t_0) > 1, \ \psi > 0, \ (\eta - z) > 0, \ w_1 > 0, \ \text{and } g \geq 0.$$

5. We have seen in (5.20) that for proving injectivity it suffices to show that $\Gamma(t_2) - \Gamma(s_1) = \int_{s_1}^{t_2} \eta(t)dt > 0$. This holds, if

$$\int_{s_1}^{t_1} |\eta(t)| + \int_{s_2}^{t_2} |\eta(t)| < \int_{t_1}^{s_2} \eta(t). \tag{5.29}$$

Since $\eta(t)$ lies above $\varsigma(t)$, we can estimate the left hand side from above by $4 \int_0^{\varsigma^{-1}(0)} |\varsigma(t)|dt$. Furthermore, since $\eta(s) \geq z(s)$, we know that

$$\overline{A} := \sup_{[t_1,s_2]} \eta(s) \geq \max \ z(s) = A.$$

The integral on the right can therefore be estimated from below by

$$\int_{t_1}^{s_2} \eta(t) > 2 \int_{\varsigma^{-1}(0)}^{a} \varsigma(t) + 2 \int_{\alpha}^{b} \xi_A(t)$$

(it is seen from the calculations below that $\int \xi_A$ increases for A increasing). Hence (5.29) holds, if

$$2 \int_0^{\varsigma^{-1}(0)} |\varsigma(t)|dt < \int_{\alpha}^{b} \xi_A(t)dt + \int_{\varsigma^{-1}(0)}^{a} \varsigma(t)dt.$$

We have

$$I_1 := 2 \int_0^{\varsigma^{-1}(0)} |\varsigma(t)| = 2 \int_0^{\varsigma^{-1}(0)} \left[\frac{\lambda_2 - \lambda}{6} \cos \left[6 \left(\frac{\omega\sigma}{\lambda_2 - \lambda}\right)^{1/2} s\right] - \frac{\lambda_2 - 7\lambda}{6}\right]$$

$$= \frac{(\lambda_2)^{3/2}}{(\omega\sigma)^{1/2}} \frac{(1-\rho)^{1/2}}{18} \left[(1-\rho) \sin \operatorname{arc} \cos \frac{1-7\rho}{1-\rho} - (1-7\rho) \operatorname{arc} \cos \frac{1-7\rho}{1-\rho}\right].$$

It is shown in lemma 5.7 below that, for $0 < \rho \leq \frac{1}{12}$, one has $\sigma \geq 0.81$. With these values the number I_1 can be estimated: $I_1 < \frac{(\lambda_2)^{3/2}}{\omega^{1/2}} \, 0.022$.

Similarly we have

$$I_2 := \int_{\varsigma^{-1}(0)}^{a} \varsigma(t) = \frac{\lambda_2^{3/2}(1-\rho)^{1/2}}{(\omega\sigma)^{1/2}\ 36}$$

$$\left[(1-7\rho)\left(\frac{\pi}{2} - \text{arc cos } \frac{1-7\rho}{1-\rho}\right) - (1-\rho)\left(1 - \text{sin arc cos } \frac{1-7\rho}{1-\rho}\right)\right]$$

$$> \frac{\lambda_2^{3/2}}{\omega^{1/2}}\ 0.002, \qquad \text{for } 0 < \rho \leq \frac{1}{12}.$$

In lemma 5.7 it will also be shown that, for $0 < \rho \leq \frac{1}{12}$, one has $M \leq 1.54$ and $B = \frac{A}{\lambda_2} \geq 0.15$. With this we estimate

$$I_3 := \int_{\alpha}^{b} \xi_A(t)dt = \frac{\lambda_2 - 7\lambda}{6M}\ (b - \alpha)$$

$$+ \left(A - \frac{\lambda_2 - 7\lambda}{6M}\right) \frac{1}{6} \left(\frac{\lambda_2 - \lambda}{M\omega}\right)^{1/2} \cos 6\left(\frac{M\omega}{\lambda_2 - \lambda}\right)^{1/2} \alpha$$

$$> \frac{\lambda_2^{3/2}}{\omega^{1/2}}\ 0.020.$$

Hence we find that $I_1 < I_2 + I_3$. ■

To complete the proof of the proposition, it remains to prove the following lemma.

LEMMA 5.7.

1.a) *For all s such that* $\overline{\eta}(s) := \frac{\eta(s)}{\lambda_2} \geq 0$ *holds*

$$\int v_1^3(s) \leq \omega \left[\left(1 + \frac{\rho}{3.5-\rho}\right)\left(\frac{1+\overline{\eta}}{1-\rho}\right)^{3/2} + \frac{\overline{\eta}}{3.5-\rho}\ \frac{1+\overline{\eta}}{1-\rho}\right]. \tag{5.30}$$

b) *For* $0 < \lambda \leq \frac{\lambda_2}{12}$, *one has* $M \leq 1.54$ *and* $B \geq 0.15$.

2.a) *For* $-\frac{1}{12} < -\rho \leq \overline{\eta}(s) < 0$ *holds*

$$\int v_1^3(s) \geq \omega \left[1 - 2\rho\left(\frac{3}{(1-\rho)(3-4\rho)}\right)^{1/2}\right].$$

b) *For* $0 < \rho \leq \frac{1}{12}$, *one has* $\frac{1}{\omega} \int v_1^3(s) \geq 0.81 =: \sigma$.

PROOF. 1.a) Multiplying (5.16) by v_1^2, we get

$$\int \nabla v_1 \nabla v_1^2 - \lambda \int v_1^3 + 3 \int u^2 v_1^2 = \eta \int v_1^2. \tag{5.31}$$

Using that $v_1(s) > 0$ for $\overline{\eta}(s) \geq 0$, we have

$$\int \nabla v_1 \nabla v_1^2 = 2 \int v_1 |\nabla v_1|^2 = 2 \int v_1 (2 v_1^{1/2}\ \nabla v_1^{1/2})^2$$

$$= 8\ \frac{4}{9} \int \left|\frac{3}{2}\ v_1\right|^2 \left|\nabla v_1^{1/2}\right|^2 = \frac{32}{9} \int \left|\nabla v_1^{3/2}\right|^2.$$

Hence we derive from (5.31)

$$\frac{32}{9}\ \lambda_2 \left[\int v_1^3 - \frac{1}{\omega} \left(\int v_1^{3/2}\right)^2\right]$$

$$\leq \frac{32}{9} \int \left|\nabla v_1^{3/2}\right|^2 \leq \lambda \int v_1^3 + \eta \int v_1^2,$$

$$\left(\frac{32}{9} \lambda_2 - \lambda\right) \int v_1^3 \leq \frac{32}{9}\ \frac{\lambda_2}{\omega} \left(\int v_1^{3/2}\right)^2 + \eta \int v_1^2$$

$$\leq \frac{32}{9}\ \lambda_2\ \frac{1}{\omega^{1/2}} \left(\int v_1^2\right)^{3/2} + \eta \int v_1^2.$$

Since $\int |\nabla u_s|^2 - \lambda \int u_s^2 \leq \lambda\omega + \eta\omega$, we have furthermore

$$\int v_1^2 = \int (1+u_s)^2 \leq \omega \left(1 + \frac{\lambda+\eta}{\lambda_2 - \lambda}\right) = \omega\ \frac{\lambda_2 + \eta}{\lambda_2 - \lambda}. \tag{5.32}$$

Using this in the above inequality, we find, setting again $\overline{\eta} = \dfrac{\eta}{\lambda_2}$, $\rho = \dfrac{\lambda}{\lambda_2}$,

$$\left(\frac{32}{9} - \rho\right) \int v_1^3 \leq \omega \left[\frac{32}{9} \left(\frac{1+\overline{\eta}}{1-\rho}\right)^{3/2} + \overline{\eta}\ \frac{1+\overline{\eta}}{1-\rho}\right].$$

This yields (5.30).

1.b) Setting $B = \max\ \overline{\eta}(s)$, we obtain (see proof of prop. 5.6, section 3b)

$$M = \left(1 + \frac{\rho}{3.5-\rho}\right) \left(\frac{1+B}{1-\rho}\right)^{3/2} + \frac{B}{3.5-\rho}\ \frac{1+B}{1-\rho}. \tag{5.33}$$

The equations (5.28) and (5.33), considered as a system, have unique solutions $M > 1$ and $B > 0$. A numerical calculation yields for $\rho = \frac{1}{12}$ the values in

1.b). Finally, we see that in (5.33), for B fixed and ρ decreasing, we have M decreasing, while in (5.28), for M fixed and ρ decreasing, we have B increasing. This implies that the estimates in 1.b) hold for all $\rho \in \left(0, \frac{1}{12}\right]$.

2.a) Multiplying (5.16) by v_1^3 gives

$$\begin{aligned} \frac{3}{4}\lambda_2\left[\int v_1^4 - \frac{1}{\omega}\left(\int v_1^2\right)^2\right] &\le \frac{3}{4}\int |\nabla v_1^2|^2 \\ &\le \lambda\int v_1^4 + |\eta|\left|\int v_1^3\right| \le (\lambda + |\eta|)\int v_1^4 \end{aligned} \tag{5.34}$$

and hence, since $\mu(s) \ge -\lambda$, for all $s \in \mathbb{R}$,

$$\int v_1^4 \le \left(1 + \frac{8\lambda}{3\lambda_2 - 8\lambda}\right)\frac{1}{\omega}\left(\int v_1^2\right)^2 = \left(1 + \frac{8\rho}{3 - 8\rho}\right)\frac{1}{\omega}\left(\int v_1^2\right)^2. \tag{5.35}$$

Together with (5.21), we obtain

$$\begin{aligned} \omega + \frac{1}{3}\int y_s^4 &\le \left(1 + \frac{8\rho}{3 - 8\rho}\right)\frac{1}{\omega}\left(\int v_1^2\right)^2 \\ &= \left(1 + \frac{8\rho}{3 - 8\rho}\right)\left[\omega + 2\int y_s^2 + \frac{1}{\omega}\left(\int y_s^2\right)^2\right] \\ &\le \omega + \frac{8\rho}{3 - 8\rho}\,\omega \\ &\quad + \left(1 + \frac{8\rho}{3 - 8\rho}\right)\left(2 + \frac{\rho + \overline{\eta}}{1 - \rho}\right)\int y_s^2, \end{aligned}$$

where we have used (5.32). Setting

$$C(\rho) = \frac{8\rho}{3 - 8\rho}, \quad D(\rho) = \left(1 + \frac{8\rho}{3 - 8\rho}\right)\left(2 + \frac{\rho + \overline{\eta}}{1 - \rho}\right),$$

we conclude

$$\begin{aligned}\left|\int y_s^3\right| &\leq \left(\int y_s^2\right)^{1/2} \left(\int y_s^4\right)^{1/2} \\ &\leq \left(\int y_s^2\right)^{1/2} \left[3\left(\omega C(\rho) + D(\rho)\int y_s^2\right)\right]^{1/2} \\ &\leq 3^{1/2}\left(\int y_s^2\right)^{1/2} \left[\omega^{1/2} C(\rho)^{1/2} + D(\rho)^{1/2}\left(\int y_s^2\right)^{1/2}\right] \\ &\leq \omega 3^{1/2}\left(\frac{\rho+\overline{\eta}}{1-\rho}\right)^{1/2} C(\rho)^{1/2} + 3^{1/2} D(\rho)^{1/2} \int y_s^2.\end{aligned}$$

We use this in

$$\begin{aligned}\int (1+y_s)^3 &= \omega + 3\int y_s^2 + \int y_s^3 \\ &\geq \omega \left[1 - \left(3\,\frac{\rho+\overline{\eta}}{1-\rho}\,C(\rho)\right)^{1/2}\right] + \left[3 - (3D(\rho))^{1/2}\right]\int y_s^2.\end{aligned}$$

One calculates that, for $-\frac{1}{12} \leq -\rho \leq \overline{\eta} \leq 0$, the terms on the right are positive and hence $\int (1+y_s)^3 > 0$. With this information we can improve the estimate by omitting in (5.34), on the right hand side, the term $\eta \int v_1^3$. This then yields, proceeding as before, instead of (5.35):

$$\int v_1^4 \leq \left(1 + \frac{4\rho}{3-4\rho}\right) \frac{1}{\omega} \left(\int v_1^2\right)^2$$

and then

$$\left|\int y_s^3\right| \leq \left(\frac{\rho}{1-\rho}\,\frac{12\rho}{3-4\rho}\right)^{1/2} \omega + \left[3\left(1+\frac{4\rho}{3-4\rho}\right)\left(2+\frac{\rho}{1-\rho}\right)\right]^{1/2} \int y_s^2.$$

This proves 2.a). 2.b) is a simple calculation. ∎

It is now easy to give a complete characterization of the image of Φ. Let $S = S_1 \cup C \cup S_2$, where $S_{1/2} = \left\{ u(s,h_1) \in S \,\middle|\, \frac{\mathrm{d}}{\mathrm{d}s}\,\eta(s,h_1) \gtrless 0 \right\}$.

THEOREM 5.8. *Let $0 < \lambda \leq \frac{\lambda_2}{12}$. Then the restriction of $\Phi = -\Delta - \lambda + (\cdot)^3$ to $S \subset E$ is one to one, and the restrictions of Φ to C, S_1 and S_2 are diffeomorphisms.*

Furthermore, $F \backslash \Phi(S)$ has exactly two components, say F_1 and F_3, with $0 \in F_3$.

The equation (1.1) *has for* $h \in F_3$ *exactly* 3 *solutions, and for* $h \in F_1 \cup \Phi(C)$ *exactly one solution. Finally, for* $h \in \Phi(S) \backslash \Phi(C)$ *equation* (1.1) *has exactly two solutions.*

PROOF. By proposition 5.6 the mapping $\Phi|_S$ is one to one. It then follows by theorems 2.4 and 2.6 that $\Phi|_{S_1 \cup S_2}$, resp. $\Phi|_C$, are diffeomorphisms. Note that this can also be obtained directly from the proof of proposition 5.6.

Let now $F_3 = \Phi\{u(s, h_1) \in E;\ \eta(s, h_1) < 0\}$. Then F_3 is connected and $\partial F_3 = \Phi(S)$. Since $\Phi(S)$ is a codimension 1 manifold which separates F locally into two components, it follows that $F \backslash \Phi(S)$ has exactly two components, see [1, prop. 2.7]. Let $F_1 = F \backslash \overline{F}_3$. We could now apply proposition 1.2 to complete the proof of the theorem. However, due to the Lyapunov-Schmidt reduction, we can with little effort complete the proof directly.

For this, note that the curves $u(s, h_1)$, $s \in \mathbb{R}$, get mapped by Φ onto the lines $\{t1 + h_1;\ t \in \mathbb{R}\}$; in fact, we have $\Gamma(s, h_1) \underset{s \to \pm\infty}{\longrightarrow} \pm\infty$, since $-c \leq \Gamma(s, h_1) \leq c$, for all $s \in \mathbb{R}$, would imply

$$\int |\nabla(s+y)|^2 + \int (s+y)^4 - \lambda \int (s+y)^2 = \int [\Gamma(s, h_1) + h_1]\ (s+y) \leq c_1 \|s+y\|,$$

and hence $s^2 + \int y^2$ bounded.

If $\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s, h_1) = \eta(s, h_1) \geq 0$, then there exists clearly exactly one solution $u(s, h_1)$ for every given $h = t1 + h_1$ (note that, in points with $\eta(s, h_1) = 0 = \eta_s(s, h_1)$, we have $\eta_{ss}(s, h_1) > 0$ by proposition 5.5).

Suppose now that for a given $h_1 \in F_1$ there exist points $s_1(h_1) < t_1(h_1)$ such that $\eta(s, h_1) < 0$, for $s_1 < s < t_1$, and $\eta(s, h_1) > 0$, for $s < s_1$, $s > t_1$. Then the function $\Gamma(s, h_1)$ is monotone increasing for $s < s_1$, monotone decreasing for $s_1 < s < t_1$, and then again monotone increasing for $s > t_1$. Hence, if $t \in (\Gamma(t_1, h_1), \Gamma(s_1, h_1))$, then there exist exactly 3 solutions for $t1 + h_1$, while if $t \in \mathbb{R} \backslash [\Gamma(t_1, h_1), \Gamma(s_1, h_1)]$ there exists exactly 1 solution, and if $t = \Gamma(t_1, h_1)$ or $t = \Gamma(s_1, h_1)$ there exist exactly 2 solutions.

Finally, if there exist two (or more) pairs $s_1 < t_1 < s_2 < t_2$, with $\eta(s, h_1) < 0$ for $s_i < s < t_i$, $i = 1, 2$, and $\eta(s, h_1) > 0$ for $t_1 < s < s_2$, then we see by the relation (5.20) that $\Gamma(s_1, h_1) < \Gamma(t_2, h_1)$. Hence we find by the above argument exactly 3 solutions for $t \in (\Gamma(t_1, h_1), \Gamma(s_1, h_1))$, $(\Gamma(t_2, h_1), \Gamma(s_2, h_1))$ and exactly 1 solution for $t < \Gamma(t_1, h_1)$, $\Gamma(s_1, h_1) < t < \Gamma(t_2, h_1)$, $t > \Gamma(s_2, h_1)$, and exactly 2 solutions for $t = \Gamma(s_i, h_1)$, $i = 1, 2$, $t = \Gamma(t_i, h_1)$, $i = 1, 2$.

This completes the proof. ∎

6. - Remarks and generalizations

1. *Dirichlet boundary conditions*

All the arguments work with some alterations also for Dirichlet boundary conditions. But there are some changes in the results.

First, we remark that for the Sturm-Liouville problem we now have (as in the higher dimensional case for the Neumann-problem) rays $\{\alpha u | \alpha \in \mathbb{R}^+\}$ which do not meet the singular set S.

PROPOSITION 6.1. *Let* $\Omega = (0,1)$ *and consider equation* (1.1) *with Dirichlet boundary conditions. Let* $\lambda_1 < \lambda < \lambda_2$, *where* λ_1, λ_2 *denote the first and the second eigenvalue of* $-v'' = \lambda v$, $v(0) = v(1) = 0$. *Then there exist*

$$u \in E_0 = \{u \in C^{2,\alpha}(0,1),\ u(0) = u(1) = 0\} \text{ such that } \{\rho u; \rho \in \mathbb{R}^+\} \cap S = \emptyset.$$

PROOF. Let u such that supp $u \subset (0, \varepsilon)$. Then

$$\mu_1(\rho u) = \inf_{v \in H^1, \|v\|_{L^2}=1} \|\nabla v\|^2 - \lambda \|v\|^2 + 3 \int (\rho u)^2 v^2$$

$$\leq \frac{\left\| \frac{d}{dx} \sin \frac{\pi(x-\varepsilon)^+}{1-\varepsilon} \right\|^2}{\left\| \sin \frac{\pi(x-\varepsilon)^+}{1-\varepsilon} \right\|^2} - \lambda = \left(\frac{\pi}{1-\varepsilon}\right)^2 - \lambda, \qquad \text{for every } \rho \in \mathbb{R}^+,$$

where $y^+ = \max\{y, 0\}$. Since $\lambda_1 = \pi^2$, we have $\left(\frac{\pi}{1-\varepsilon}\right)^2 - \lambda < 0$ for $\varepsilon > 0$ small. ■

One has an analogue of proposition 3.5 for this situation.

Second, we note that the condition $0 < \lambda < \frac{\lambda_2}{7}$ becomes more complicated for Dirichlet boundary conditions. The analogue of the crucial proposition 4.1 is

PROPOSITION 6.2. *Consider equation* (1.1) *with Dirichlet boundary conditions. Let* λ_1, λ_2 *denote the first and the second eigenvalue of*

$$-\Delta v = \lambda v, \text{ in } \Omega, \qquad v|_{\partial\Omega} = 0,$$

and assume that

$$\lambda_2 > 2.5\, \lambda_1 \tag{6.1}$$

$$\lambda_1 < \lambda < \frac{\lambda_2 + 4.5\lambda_1}{7}. \tag{6.2}$$

Then the singular set S *contains only fold points and cusp points.*

PROOF. One proceeds as in proposition 4.1 and derives as there that, if (4.1) does not hold, then

$$\int v_1^4(u) \leq 18\, \frac{\int u^2 v_1^4(u)}{\lambda_2 - \lambda}. \tag{6.3}$$

Then, from

$$-\Delta v_1(u) - \lambda v_1(u) + 3u^2 v_1(u) = 0$$

one gets by multiplication with $v_1^3(u)$

$$\frac{3}{4}\ \left\|\nabla(v_1^2(u))\right\|^2 - \lambda \int v_1^4(u) + 3 \int u^2 v_1^4(u) = 0$$

and, since $\|\nabla(v_1^2)\|^2 \geq \lambda_1 \|v_1^2\|^2$,

$$\left(\frac{3}{4}\ \lambda_1 - \lambda\right) \int v_1^4(u) + 3 \int u^2 v_1^4(u) \leq 0. \tag{6.4}$$

Combining (6.3) and (6.4) yields

$$\int v_1^4(u) \leq \frac{18}{\lambda_2 - \lambda} \int u^2 v_1^4(u) \leq \frac{6}{\lambda_2 - \lambda} \left(\lambda - \frac{3}{4}\ \lambda_1\right) \int v_1^4(u),$$

$$\lambda_2 - \lambda \leq 6\lambda - 4.5\lambda_1 \qquad \text{that is} \qquad \frac{\lambda_2 + 4.5\lambda_1}{7} \leq \lambda.$$

Hence, if $\lambda < \dfrac{\lambda_2 + 4.5\lambda_1}{7}$, then (4.1) holds. ∎

We remark that for $\Omega = (0,1)$ we have $\lambda_1 = \pi^2$ and $\lambda_2 = 4\pi^2$, and hence $\lambda_2 > 2.5\ \lambda_1$. However, for $\Omega \subset \mathbb{R}^n$, $n \geq 2$, it is possible that (6.1) is not satisfied. In fact, take for example $\Omega = (0,1) \times (0,1)$. Then $\lambda_1 = 2\pi^2$ and $\lambda_2 = 5\pi^2$, i.e. $\lambda_2 = 2.5\ \lambda_1$.

Based on proposition 6.2, similar estimates as in proposition 5.6 can be done to obtain a statement as in theorem 1.1 (for a different range of λ).

The estimates for the parameter λ given in this paper are certainly not optimal. However, numerical investigations seem to indicate that higher singularities than cusps appear for values λ smaller than λ_2.

2. *The form of the nonlinearity*

It has been used at several places that the nonlinearity has the form $u \mapsto u^3$, most crucially in the propositions 4.1 and 6.1. It is easy to see that we can allow an arbitrary positive constant times $u^3 : u \mapsto \alpha u^3$, $\alpha > 0$. Also, it is readily verified that we can perturb u^3 by a nonlinearity which is small in $C^3(\mathbb{R})$ and still obtain the same result (for a different range of λ). Whether a similar result holds for a wider class of nonlinearities is an open question.

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Mathematisches Institut
der Universität zu Köln,
Germany

Current address:
Dipartimento di Matematica
Via C. Saldini, 50
20133 MILANO