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Induced Representations of Completely Solvable Lie Groups

RONALD L. LIPSMAN*

1. - Introduction

This paper is concerned with the continuing effort to describe the direct integral decomposition of a unitary representation of a connected Lie group G, which is induced from a connected subgroup H. Complete solutions to this problem are known when G is nilpotent (see [4], [9]). The solution is given explicitly in terms of orbital parameters. That is, the spectrum, multiplicity and spectral measure that constitute the decomposition are described completely in terms of natural objects associated to the co-adjoint orbits of G. Moreover, strong evidence is presented in [9] to indicate that the orbital solution found in the nilpotent situation has much wider applicability. In [10], the author has shown that the exact same formula (which describes the decomposition when G is nilpotent) remains valid in several important cases of exponential solvable groups. These include: many $H \subset G$ in which both are algebraic, and all $H \subset G$ in which H is normal or G/H is symmetric. In this paper, we prove that these orbital formulae are valid for *any* pair $H \subset G$, when G is completely solvable.

The basic technique of the proof is modelled after that for nilpotent groups. We insert between H and G a subgroup G_1 of co-dimension one in G, and then employ mathematical induction on $\dim G/H$. In particular, this necessitates a separate treatment of the case $H = G_1$, that is $\dim G/H = 1$. Herein arises the main difference from the nilpotent case: co-dimension one subgroups need not be normal. The analysis of co-dimension one induced representations of completely solvable groups turns out to be considerably more complicated than in the nilpotent situation. This is carried out in section three of the paper (see Theorems 3.1 and 3.3). The generalization to arbitrary co-dimension is then done in section four in the monomial case – that is, for representations induced from characters (see Theorem 4.1). This part is also much more difficult than in the nilpotent situation – in particular we must match multiplicities in two

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direct integrals, only one of which is concentrated in a G-invariant set (in the parameter space) (see Lemma 4.3 (iv) and the paragraph preceding it). Arbitrary induced representations are treated in section five (Theorem 5.1). Section six contains several examples to illustrate the main features of these theorems. The precise formulation of the orbital decomposition of an induced representation is presented in section two. Therein we set up the basic terminology (see Definition 2.1 and formulas (2.1) and (2.2)), recall a fundamental lemma in the subject (Lemma 2.4), and establish the notation of the paper.

A very primitive version of our main theorem (identifying spectra, but *not* multiplicity) is announced in [3]. As far as we know, no proof ever appeared.

2. - Statement of the main result

We recall the Kirillov-Bernat orbital parametrization (see [1], [6]). Suppose G is an *exponential solvable* group. That means G is simply connected solvable and its Lie algebra $\mathcal{G}$ has no purely imaginary eigenvalues. G is called *completely solvable* if it is exponential solvable and every eigenvalue of $\mathcal{G}$ is real. The symbol $\mathcal{G}^*$ denotes the real linear dual of $\mathcal{G}$. G acts on $\mathcal{G}$ (resp. $\mathcal{G}^*$) by the adjoint (resp. co-adjoint) action. Then the dual space $\hat{G}$, of equivalence classes of irreducible unitary representations of G, is parametrized canonically by the orbit space $\mathcal{G}^*/G$. More precisely, for $\varphi \in \mathcal{G}^*$, we may find a real polarization $\mathcal{B}$ for φ – that is a subalgebra, $\mathcal{G}_\varphi \subset \mathcal{B} \subset \mathcal{G}$, that is maximal totally isotropic for $B_\varphi(X,Y) = \varphi[X,Y]$ – which satisfies the Pukanszky condition ($B \cdot \varphi = \varphi + \mathcal{B}^\perp$, $B = \exp \mathcal{B}$). Then the representation $\pi_\varphi = \mathrm{Ind}_B^G \chi_\varphi$, $\chi_\varphi(\exp X) = e^{i\varphi(X)}$, $X \in \mathcal{B}$, is irreducible; its class is independent of the choice of $\mathcal{B}$; the Kirillov map $\varphi \to \pi_\varphi$, $\mathcal{G}^* \to \hat{G}$, is surjective and factors to a bijection $\mathcal{G}^*/G \to \hat{G}$. Given $\pi \in \hat{G}$, we write $\mathcal{O}_\pi \in \mathcal{G}^*/G$ to denote the inverse image of π under the Kirillov map.

All of the preceding is valid for any exponential solvable group, but we shall only deal with completely solvable groups in this paper. Now suppose G is completely solvable, $H \subset G$ is a (closed) connected subgroup. We recall [10, Def. 2.1] (see also [9, Thms. 3.1 & 3.5]).

DEFINITION 2.1 For $\nu \in \hat{H}$, we say that the representation $\mathrm{Ind}_H^G \nu$ obeys the *orbital spectrum formula* if

$$\mathrm{Ind}_H^G \nu = \int_{p^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \pi_\varphi d\mu_{G,H}^\nu(\varphi), \tag{2.1}$$

where $p : \mathcal{G}^* \to \mathcal{H}^*$ is the canonical projection and $\mu_{G,H}^\nu$ is the push-forward of the natural measure on $p^{-1}(\mathcal{O}_\nu)$. (The natural measure is the fiber measure with H-invariant measure on the base $\mathcal{O}_\nu = H/H_\psi$, $\psi \in \mathcal{O}_\nu$, and Lebesgue measure on the affine fiber $\mathcal{H}^\perp = \{\varphi \in \mathcal{G}^* : \varphi(\mathcal{H}) = 0\}$.)

It follows from the work of [9, §3] that, when $\mathrm{Ind}_H^G \nu$ obeys the orbital spectrum formula, we also have the multiplicity formula

$$\mathrm{Ind}_H^G \nu = \int_{G \cdot p^{-1}(\mathcal{O}_\nu)/G}^{\oplus} n_\varphi^\nu \pi_\varphi \mathrm{d}\tilde{\mu}_{G,H}^\nu(\varphi), \tag{2.2}$$

where $\tilde{\mu}_{G,H}^\nu$ is again the push-forward of the natural measure under

$$p^{-1}(\mathcal{O}_\nu) \to G \cdot p^{-1}(\mathcal{O}_\nu)/G,$$

and $n_\varphi^\nu = \#H$-orbits on $G \cdot \varphi \cap p^{-1}(\mathcal{O}_\nu)$.

The main result of this paper is the following.

THEOREM 2.2. *Let G be completely solvable, $H \subset G$ closed and connected, $\nu \in \hat{H}$. Then the induced representation $\mathrm{Ind}_H^G \nu$ satisfies the orbital spectrum formula.*

It is well known [13] that any exponential solvable group – in particular any completely solvable group – is type I. Therefore the unitary representation $\mathrm{Ind}_H^G \nu$ has a direct integral decomposition

$$\mathrm{Ind}_H^G \nu = \int_{\mathcal{Q}_\nu}^{\oplus} n_\pi^\nu \pi \mathrm{d}\mu_\nu(\pi).$$

The measure class $[\mu_\nu]$ is uniquely determined; the multiplicity function n_π^ν is uniquely determined ($[\mu_\nu]$-a.e.); and the spectrum $\mathcal{Q}_\nu$ – meaning any subset of $\hat{G}$ in which μ_ν is concentrated – is also determined ($[\mu_\nu]$-a.e.). To prove Theorem 2.2 we must verify that the triple $(\tilde{\mu}_{G,H}^\nu,\ n_\varphi^\nu,\ G \cdot p^{-1}(\mathcal{O}_\nu)/G)$ constitute these three ingredients for the induced representation $\mathrm{Ind}_H^G \nu$. The scheme of the proof will be the same as in [9]. Namely, we first handle the case that ν is a character by employing induction on $\dim G/H$. After that, we pass to an arbitrary irreducible $\nu \in \hat{H}$. The latter step is virtually identical to the corresponding part of [9, §3]. But there is a key difference in the first step. To explain, we remark that since G is completely solvable, we can always place a co-dimension one closed connected subgroup G_1 between H and G. We prove the orbital spectrum formula for representations induced from G_1 to G. Then the induction hypothesis gives us the orbital spectrum formula for representations induced from H to G_1. The heart of the argument is to combine them to obtain the orbital spectrum formula for representations induced from H to G. Now in the *nilpotent* case, G_1 will always be *normal* in G. In completely solvable groups, such normality can no longer be assured. This causes a greater number of possibilities for the structure of the induced representation $\mathrm{Ind}_{G_1}^G \nu$, $\nu \in \hat{G}_1$, than exist in the nilpotent case. (In fact, when G_1 is not normal, five – much more complicated – possibilities occur instead of two.) Our work in the next section is devoted to sorting out these possibilities.

Before beginning that effort, we cite two known results, and we establish some notation.

THEOREM 2.3. *Theorem 2.2 is true if H is normal.*

This is proven in [9, Theorem 6.1].

The second fact is the following.

LEMMA 2.4. *Let $N \subset G$ be normal and connected, $\varphi \in \mathcal{G}^*$,*

$$\theta = \varphi|_{\mathcal{N}} \in \mathcal{N}^*, \quad \gamma = \gamma_\theta \in \hat{N}.$$

The Lie algebra of the stability group G_γ is $\mathcal{G}_\gamma = \mathcal{G}_\theta + \mathcal{N}$. Then

$$N_\theta \cdot \varphi = \varphi + \mathcal{G}_\gamma^\perp.$$

See [12, Lemma 2] or [8, p. 271].

NOTATION. Whenever $\mathcal{H}$ is a subalgebra of $\mathcal{G}$, we write $p_{\mathcal{G},\mathcal{H}} : \mathcal{G}^* \to \mathcal{H}^*$ for the canonical projection, $p_{\mathcal{G},\mathcal{H}}(\varphi) = \varphi|_{\mathcal{H}}$, $\varphi \in \mathcal{G}^*$. If the algebras are clear from the context, we set $p = p_{\mathcal{G},\mathcal{H}}$. We denote

$$\mathcal{H}^\perp = p^{-1}(0) \subset \mathcal{G}^*;$$

if it is necessary to specify the superalgebra, we write

$$\mathcal{H}^\perp(\mathcal{G}) = \{\varphi \in \mathcal{G}^* : \varphi|_{\mathcal{H}} = 0\}.$$

By a generic subset of $\mathcal{G}^*$, we mean a subset, the complement of whose interior is Lebesgue null. More generally, for any manifold W, we say a statement P_w, $w \in W$, is true generically if it holds for all points of W except for a set whose interior is co-null with respect to the canonical measure class.

3. - Co-dimension one

In this section, we carry out a detailed and complete description of the decomposition of a representation of a completely solvable group G induced from a co-dimension one connected subgroup G_1. We shall give the orbital parameters as well as related information on various stabilizers and orbit correspondences. When G_1 is normal, it has Mackey parameters and those are specified by [7]. We shall see that even when G_1 is not normal – the typical situation – there is associated a canonical normal subgroup of co-dimension two. We shall present its Mackey parameters and relate them to the orbital parameters.

We start with G completely solvable, $N \triangleleft G$ a co-dimension one closed connected normal subgroup. Let $\theta \in \mathcal{N}^*$, $\gamma = \gamma_\theta \in \hat{N}$ the corresponding Kirillov-Bernat irreducible representation. The analysis of $\mathrm{Ind}_N^G \gamma_\theta$ is known in great detail (see [6] or [9]). We summarize in

THEOREM 3.1. *Let $\theta \in \mathcal{N}^*$, $\varphi \in p^{-1}(\theta) \subset \mathcal{G}^*$, $\pi_\varphi \in \hat{G}$ the corresponding Kirillov-Bernat irreducible representation. Select $\alpha \in \mathcal{N}^\perp = p^{-1}(0)$, $\alpha \neq 0$. Then $p^{-1}(\theta) = \{\varphi + t\alpha : t \in \mathbb{R}\}$ and there are two mutually exclusive possibilities:*

(i) *$G \cdot \varphi \supset p^{-1}(\theta)$. Then $G_\theta = N_\theta$, $G_\gamma = N$, $N_\theta \cdot \varphi = \varphi + \mathcal{N}^\perp$ and $G_\varphi = N_\varphi$. Moreover, $\mathrm{Ind}_N^G \gamma_\theta = \pi_\varphi$ is irreducible.*

(ii) *The orbits $\{G \cdot (\varphi + t\alpha) : t \in \mathbb{R}\}$ are all distinct. Then $N_\theta = N_\varphi$, $G_\theta = G_\varphi$ and $G_\gamma = G$. Moreover,*

$$\mathrm{Ind}_N^G \gamma_\theta = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi + t\alpha} dt.$$

Theorem 3.1 gives both the Mackey and orbital parameters for $\mathrm{Ind}_N^G \gamma_\theta$. We know by Theorem 2.3 that the orbital spectrum formula is valid, that is,

$$\mathrm{Ind}_N^G \gamma_\theta = \int_{p^{-1}(N\cdot\theta)/N}^{\oplus} \pi_{\varphi'}\ d\mu_{G,N}^{\gamma_\theta}(\varphi').$$

This says in particular that: in case (i), $G \cdot \varphi \cap p^{-1}(N \cdot \theta) = N \cdot \varphi$; and, in case (ii), $G \cdot (\varphi + t\alpha) \cap p^{-1}(N \cdot \theta) = N \cdot (\varphi + t\alpha)$, $t \in \mathbb{R}$.

Now we pass to the non-normal co-dimension one situation. We assume G is completely solvable with $G_1 \subset G$ a closed connected co-dimension one subgroup. We assume G_1 is *not* normal in G. Let $\psi \in \mathcal{G}_1^*$, $\nu_\psi \in \hat{G}_1$ the corresponding Kirillov-Bernat irreducible representation, and set $\pi = \mathrm{Ind}_{G_1}^G \nu_\psi$. We shall see that there are *five* different possibilities for the structure of π. In order to enumerate them, we need the following result.

PROPOSITION 3.2. *Let $\mathcal{G}$ be completely solvable, $\mathcal{G}_1 \subset \mathcal{G}$ a subalgebra of codimension one. Suppose $\mathcal{G}_1$ is not an ideal in $\mathcal{G}$. (In particular, $\mathcal{G}$ cannot be nilpotent). Then there exist a co-dimension one subalgebra $\mathcal{G}_0$ of $\mathcal{G}_1$ which is a co-dimension two ideal in $\mathcal{G}$, and two elements $X \in \mathcal{G}_1 \backslash \mathcal{G}_0$, $Y \in \mathcal{G} \backslash \mathcal{G}_1$ such that*

$$[X, Y] \equiv Y \bmod \mathcal{G}_0.$$

PROOF. Let $\mathcal{N}$ be the nilradical of $\mathcal{G}$. We must have $\mathcal{G} = \mathcal{G}_1 + \mathcal{N}$, for otherwise $\mathcal{G}_1 \supset \mathcal{N}$ which forces $\mathcal{G}_1$ to be an ideal (since $\mathcal{G}/\mathcal{N}$ is abelian). As vector spaces

$$\frac{\mathcal{G}}{\mathcal{G}_1} = \frac{\mathcal{G}_1 + \mathcal{N}}{\mathcal{G}_1} \cong \frac{\mathcal{N}}{\mathcal{N} \cap \mathcal{G}_1}.$$

Hence $\dim \mathcal{N}/(\mathcal{N} \cap \mathcal{G}_1) = 1$. In particular $\mathcal{N} \cap \mathcal{G}_1$ is an ideal in $\mathcal{N}$ (since $\mathcal{N}$ is nilpotent).

Consider $\mathrm{ad}_{\mathcal{G}/\mathcal{G}_1}\mathcal{G}_1$. Since $\dim \mathcal{G}/\mathcal{G}_1 = 1$, a unique Lie homomorphism $\rho : \mathcal{G}_1 \to \mathbb{R}$ is determined by selecting any $Y \notin \mathcal{G}_1$, so that

$$[W, Y] \equiv \rho(W)Y \mod \mathcal{G}_1, \qquad W \in \mathcal{G}_1.$$

Note $\rho \not\equiv 0$, since $\mathcal{G}_1$ is not an ideal. Set $\mathcal{G}_0 = \ker \rho$, an ideal in $\mathcal{G}_1$. Clearly $\dim \mathcal{G}/\mathcal{G}_0 = 2$, and $\mathcal{G}_0 = \{W \in \mathcal{G}_1 : [W, \mathcal{G}] \subset \mathcal{G}_1\}$. In fact $\mathcal{G}_0$ is actually an ideal in $\mathcal{G}$. To see that, since $\mathcal{G} = \mathcal{G}_1 + \mathcal{N}$, it is enough to show $[\mathcal{N}, \mathcal{G}_0] \subset \mathcal{G}_0$. But $[\mathcal{G}_0, \mathcal{G}] \subset \mathcal{G}_1$. Hence $[\mathcal{G}_0, \mathcal{N}] \subset \mathcal{G}_1 \cap \mathcal{N}$. We show that $\mathcal{G}_1 \cap \mathcal{N} \subset \mathcal{G}_0$. To do that, we must show $[\mathcal{G}_1 \cap \mathcal{N}, \mathcal{G}] \subset \mathcal{G}_1$. But once again $\mathcal{G} = \mathcal{G}_1 + \mathcal{N}$ and clearly $[\mathcal{G}_1 \cap \mathcal{N}, \mathcal{G}_1] \subset \mathcal{G}_1$. So it remains to prove $[\mathcal{G}_1 \cap \mathcal{N}, \mathcal{N}] \subset \mathcal{G}_1$. That is true because, as we have observed above, $\mathcal{G}_1 \cap \mathcal{N}$ is an ideal in $\mathcal{N}$.

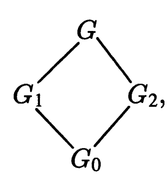
Now choose $Y \in \mathcal{N}$, $Y \notin \mathcal{G}_1$, so that $\mathcal{G} = \mathbb{R}Y + \mathcal{G}_1$. Then

$$[W, Y] = \rho(W)Y + \tau(W), \qquad W \in \mathcal{G}_1,$$

where ρ is as above and $\tau : \mathcal{G}_1 \to \mathcal{G}_1$ is linear. Actually, $[\mathcal{G}_1, Y] \subset \mathcal{N}$ implies $\tau(W) \in \mathcal{N}$. So $\tau(W) \in \mathcal{G}_1 \cap \mathcal{N} \subset \mathcal{G}_0$. Choose $X \in \mathcal{G}_1$ such that $\rho(X) = 1$. Then

$$[X, Y] = Y + \tau(X), \text{ i.e.}$$
$$[X, Y] \equiv Y \mod \mathcal{G}_0.$$

Note that $\mathcal{G}_0$ is uniquely determined by $\mathcal{G}_1$, but the elements X, Y are not. We have $\mathcal{G}_1 = \mathbb{R}X + \mathcal{G}_0$, and we set $\mathcal{G}_2 = \mathbb{R}Y + \mathcal{G}_0$ (also uniquely determined by $\mathcal{G}_1$). We write G, G_1, G_2, G_0 for the corresponding simply connected groups. Then we have the diagram



where each group is co-dimension one in any group lying on a line directly above it. Moreover, both G_0 and G_2 are normal in G. We denote by α, β the linear functionals determined by $\beta \in \mathcal{G}_2^{\perp}(\mathcal{G})$, $\beta(X) = 1$ and $\alpha \in \mathcal{G}_0^{\perp}(\mathcal{G}_2)$, $\alpha(Y) = 1$. We extend α to an element (also denoted α) of $\mathcal{G}_1^{\perp}(\mathcal{G})$ by setting $\alpha(X) = 0$.

Now we are ready to enumerate the possibilities for

$$\pi = \mathrm{Ind}_{G_1}^{G} \nu_{\psi}, \ \psi \in \mathcal{G}_1^*, \ \nu = \nu_{\psi} \in \hat{G}_1.$$

We have G_0 and G_2 according to Proposition 3.2. We set $\theta = \psi|_{\mathcal{G}_0}$ and specify $\varphi \in p_{\mathcal{G},\mathcal{G}_1}^{-1}(\psi)$ by requiring $\varphi(Y) = 0$. We set $\omega = \varphi|_{\mathcal{G}_2}$ so that $\omega(Y) = 0$, $\omega|_{\mathcal{G}_0} = \theta$.

Finally we write $\gamma = \gamma_\theta \in \hat{G}_0$, $\sigma = \sigma_\omega \in \hat{G}_2$ for the corresponding Kirillov-Bernat irreducible representations.

THEOREM 3.3. *One of the following five mutually exclusive possibilities obtains*:

(i) $G_\gamma = G_0$. *Then* $G \cdot \varphi \supset p^{-1}(G_1 \cdot \psi)$ *and* $\pi = \pi_\varphi$ *is irreducible.*

(ii) G_γ *is a non-normal co-dimension one subgroup of* G, $\neq G_1$. *Then the functionals* $\varphi + t\beta$, $t \in \mathbb{R}$, *lie in distinct G-orbits. These constitute precisely the G-orbits that meet* $p^{-1}(G_1 \cdot \psi)$ *and*

$$\pi = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+t\beta} \mathrm{d}t.$$

(iii) $G_\gamma = G_2$. *Then the functionals* $\varphi + s\alpha$, $s \in \mathbb{R}$, *lie in distinct G-orbits. These constitute the G-orbits that meet* $p^{-1}(G_1 \cdot \psi)$ *and*

$$\pi = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+s\alpha} \mathrm{d}s.$$

(iv) $G_\gamma = G_1$. *Then, as in* (i), $G \cdot \varphi \supset p^{-1}(G_1 \cdot \psi)$ *and* $\pi = \pi_\varphi$ *is irreducible.*

(v) $G_\gamma = G$. *Then there is* $s_0 \in \mathbb{R}$ *such that* $\varphi + s_1\alpha$, $\varphi + s_2\alpha$ *are in the same G-orbit* $\Longleftrightarrow$ s_1 *and* s_2 *lie on the same side of* s_0. *Fix a pair* (s_1, s_2), $s_1 < s_0 < s_2$. *Then* $G \cdot (\varphi + s_0\alpha)$, $G \cdot (\varphi + s_1\alpha)$, $G \cdot (\varphi + s_2\alpha)$ *are the only G-orbits that meet* $p^{-1}(G_1 \cdot \psi)$, *and*

$$\pi = \pi_{\varphi+s_1\alpha} \oplus \pi_{\varphi+s_2\alpha}.$$

Moreover, in every one of the five cases, we have the orbital spectrum formula

$$\pi = \mathrm{Ind}_{G_1}^{G} \nu_\psi = \int_{p^{-1}(G_1\cdot\psi)/G_1}^{\oplus} \pi_{\varphi'} \ \mathrm{d}\mu_{G,G_1}^{\psi}(\varphi').$$

NOTES. (1) The five distinct cases of Theorem 3.3 are determined by the stabilizer G_γ. But the induced representation π appears to be of only three different types: irreducible, a sum of two irreducibles, or a direct integral over a 1-parameter family of irreducibles. But cases (i) and (iv) are really distinct; so are (ii) and (iii). For the first pair, π is induced from G_0 in (i), not in (iv). For the second pair, π is induced by an irreducible of G_2 in case (ii), and that is not so in (iii). More differences will be evident from the proof and in section 4.

(2) In every case, the induced representation $\mathrm{Ind}_{G_1}^{G} \nu_\psi$ is multiplicity-free.

PROOF. The five possibilities for the stabilizer G_γ enumerated in the statement of the theorem are manifestly mutually distinct. We handle each case separately. In each we verify the orbital facts asserted and derive the direct integral decomposition of π. To substantiate the orbital spectrum formula in each case, we must identify the spectrum, multiplicity and spectral measure. We treat the first two separately in each of the five cases. We consider the measures together at the end of the proof.

In all cases we have $G_\gamma = G_\theta G_0$ and $\mathcal{G}_\gamma = \mathcal{G}_\theta + \mathcal{G}_0$.

(i) $G_\gamma = G_0$. By the Mackey Machine [11], $\mathrm{Ind}_{G_0}^G \gamma_\theta$ is irreducible. But $\mathcal{G}_0 = \mathcal{G}_\theta + \mathcal{G}_0 \Rightarrow \mathcal{G}_\theta \subset \mathcal{G}_0 \Rightarrow \mathcal{G}_\theta = (\mathcal{G}_0)_\theta$. In particular, we also have $(\mathcal{G}_1)_\theta = (\mathcal{G}_2)_\theta = (\mathcal{G}_0)_\theta$. Now apply Theorem 3.1 to the pair (ψ, θ). It says that $\nu_\psi = \mathrm{Ind}_{G_0}^{G_1} \gamma_\theta$. Therefore

$$\pi = \mathrm{Ind}_{G_1}^G \nu_\psi = \mathrm{Ind}_{G_1}^G \ \mathrm{Ind}_{G_0}^{G_1} \gamma_\theta = \mathrm{Ind}_{G_0}^G \gamma_\theta$$

is irreducible. But $(\mathcal{G}_0)_\theta = (\mathcal{G}_2)_\theta$ also says that $\sigma_\omega = \mathrm{Ind}_{G_0}^{G_2} \gamma_\theta$ is irreducible. Then

$$\pi = \mathrm{Ind}_{G_0}^G \gamma_\theta = \mathrm{Ind}_{G_2}^G \sigma_\omega.$$

Since $\varphi|_{\mathcal{G}_2} = \omega$, it must be (again by Theorem 3.1) that $\pi = \pi_\varphi$.

Now if $\varphi' \in p^{-1}(G_1 \cdot \psi)$, then for some $g_1 \in G_1$, we have $g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi$. That is, $G \cdot \varphi'$ lies over ψ, and so $G \cdot \varphi'$ lies over θ as well. But the irreducibility of both $\mathrm{Ind}_{G_0}^{G_2} \gamma_\theta$ and $\mathrm{Ind}_{G_2}^G \sigma_\omega$ means that there exists a unique G-orbit lying over θ. Hence $G \cdot \varphi$ is the only orbit meeting $p^{-1}(G_1 \cdot \psi)$. Finally $G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi$. This is because if $\varphi' = g \cdot \varphi \in p^{-1}(G_1 \cdot \psi)$, then $g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi$ for some $g_1 \in G_1$, so $g_1 g \cdot \varphi = \varphi + s\alpha$ for some $s \in \mathbb{R}$. But $(G_0)_\theta \cdot \varphi = \varphi + \mathcal{G}_0^\perp$ (Lemma 2.4). Hence there is $g_0 \in (G_0)_\theta \subset G_1$ such that $g_1 g \cdot \varphi = g_0 \cdot \varphi$. That is, $\varphi' \in G_1 \cdot \varphi$. Since the spectrum is a point, there is nothing to do to identify the spectral measure with μ_{G,G_1}^ψ, so the orbital spectrum formula is valid

$$\mathrm{Ind}_{G_1}^G \nu_\psi = \pi_\varphi = \int_{p^{-1}(G_1 \cdot \psi)/G_1}^{\oplus} \pi_{\varphi'} \ \mathrm{d}\mu_{G,G_1}^\psi(\varphi').$$

(ii) G_γ is a non-normal co-dimension one subgroup of G other than G_1. This time $(\mathcal{G}_1)_\theta = (\mathcal{G}_2)_\theta = (\mathcal{G}_0)_\theta$, but $\mathcal{G}_\theta \not\subset (\mathcal{G}_0)_\theta$. There exist real non-zero scalars x_0, y_0 so that

$$\mathcal{G}_\gamma = \mathcal{G}_0 + \mathbb{R}(x_0 X + y_0 Y).$$

Moreover,

$$\pi = \mathrm{Ind}_{G_1}^G \nu_\psi = \mathrm{Ind}_{G_0}^G \gamma_\theta = \mathrm{Ind}_{G_2}^G \sigma_\omega$$

as in (i), but this time the latter is not irreducible. Therefore,

$$\pi = \mathrm{Ind}_{G_2}^G \sigma_\omega = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi + t\beta} \mathrm{d}t$$

and the functionals $\varphi^t = \varphi + t\beta$ lie in distinct G-orbits which meet $p^{-1}_{\mathcal{G},\mathcal{G}_2}(G_2 \cdot \omega)$. But if we write $\omega_s = \omega + s\alpha$, it is also true that $\sigma_\omega \cong \sigma_{\omega_s}$ and

$$\pi = \mathrm{Ind}_{G_2}^G \sigma_{\omega_s} = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+t\beta+s\alpha} dt, \qquad \text{for any } s \in \mathbb{R}.$$

Clearly the functionals $\{\varphi + t\beta + s\alpha : s, t \in \mathbb{R}\}$ account for all G-orbits that lie over θ. But $\varphi + t_1\beta + s_1\alpha$ and $\varphi + t_2\beta + s_2\alpha$ lie in the same G-orbit iff

$$(s_1 - s_2)y_0 + (t_1 - t_2)x_0 = 0. \tag{3.1}$$

This can be seen as follows. Write $\varphi_s^t = \varphi + t\beta + s\alpha$. By Lemma 2.4 we have

$$(G_0)_\theta \cdot \varphi_s^t = \varphi_s^t + \mathcal{G}_\gamma^\perp.$$

We also know that

$$\dim \frac{\mathcal{G}}{\mathcal{G}_{\varphi_s^t}} = \dim \frac{\mathcal{G}_0}{(\mathcal{G}_0)_\theta} + 2, \qquad \dim \frac{\mathcal{G}_\theta}{(\mathcal{G}_0)_\theta} = 1$$

from which it follows that dim, $\mathcal{G}_\theta / \mathcal{G}_{\varphi_s^t} = 1$. Therefore

$$G_\theta \cdot \varphi_s^t = (G_0)_\theta \cdot \varphi_s^t = \varphi_s^t + \mathcal{G}_\gamma^\perp.$$

Hence the equation

$$g \cdot \varphi_{s_1}^{t_1} = \varphi_{s_2}^{t_2}$$

is equivalent to (3.1).

Henceforth we let $s = 0$ and pay attention only to $\varphi^t = \varphi_0^t = \varphi + t\beta$. By what we have done, these account for all the G-orbits lying over ψ; since $G \cdot \varphi^t = G \cdot \varphi_0^t = G \cdot \left(\varphi + t\frac{x_0}{y_0}\alpha\right)$. Moreover

$$G \cdot \varphi^t \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi^t.$$

Indeed $p^{-1}(G_1 \cdot \psi) \supset G_1 \cdot \varphi^t$ since there is $g_0 \in (G_0)_\theta$ satisfying

$$g_0 \cdot \varphi^t = \varphi^{\psi(X)} + s\alpha$$

where $s = (t - \psi(X))\frac{x_0}{y_0}$. Conversely if for $g \in G$ we have

$$\varphi' = g \cdot \varphi^t \in p^{-1}(G_1 \cdot \psi),$$

then $g_1 g \cdot \varphi^t|_{\mathcal{G}_1} = \psi$ for some $g_1 \in G_1$. That is, $g_1 g \cdot \varphi^t = \varphi_s^0$ where s must satisfy $s = tx_0/y_0$. Then there exists $g_0 \in (G_0)_\theta \subset G_1$ such that

$$g_1 g \cdot \varphi^t = \varphi_s^0 = g_0 \cdot \varphi^t.$$

That is $\varphi' \in G_1 \cdot \varphi^t$.

Modulo our discussion of the measure (at the end of the proof), we have derived the direct integral decomposition and the orbital spectrum formula

$$\operatorname{Ind}_{G_1}^G \nu_\psi = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+t\beta} dt = \int_{p^{-1}(G_1\cdot\psi)/G_1}^{\oplus} \pi_{\varphi'} \, d\mu^{\psi}_{G,G_1}(\varphi').$$

(iii) $G_\gamma = G_2$. Now we have $(\mathcal{G}_1)_\theta = (\mathcal{G}_0)_\theta$, but $(\mathcal{G}_2)_\theta \supsetneq (\mathcal{G}_0)_\theta$. By dimension we obtain $\mathcal{G}_\theta = (\mathcal{G}_2)_\theta$. Therefore

$$\pi = \operatorname{Ind}_{G_1}^G \nu_\psi = \operatorname{Ind}_{G_0}^G \gamma_\theta,$$

but this time

$$\operatorname{Ind}_{G_0}^{G_2} \gamma_\theta = \int_{\mathbb{R}}^{\oplus} \sigma_{\omega+s\alpha} ds.$$

Write $\omega_s = \omega + s\alpha$. I assert that $\mathcal{G}_{\omega_s} = (\mathcal{G}_2)_{\omega_s}$ for any $s \in \mathbb{R}$. To see this we use:

$$\begin{aligned}
&(\mathcal{G}_1)_\theta = (\mathcal{G}_0)_\theta, &&\text{so } \theta[aX + bW, \mathcal{G}_0] = 0 \;\Rightarrow\; a = 0,\\
& && a \in \mathbb{R},\ W \in \mathcal{G}_0;\\
&(\mathcal{G}_2)_\theta \supsetneq (\mathcal{G}_0)_\theta, &&\text{so } \theta[b_2 Y + W_2, \mathcal{G}_0] = 0\\
& && \text{for some } b_2 \neq 0,\ W_2 \in \mathcal{G}_0.
\end{aligned}$$

Then

$$\begin{aligned}
&\omega_s[aX + bY + W, \mathcal{G}_2] = 0\\
&\Rightarrow \theta[aX + bY + W, \mathcal{G}_0] = 0\\
&\Rightarrow \theta\left[aX + W - \frac{b}{b_2} W_2, \mathcal{G}_0\right] = 0\\
&\Rightarrow a = 0.
\end{aligned}$$

This proves the assertion, from which it follows that $\operatorname{Ind}_{G_2}^G \sigma_{\omega_s}$ is irreducible for any $s \in \mathbb{R}$. Moreover, these representations are pairwise inequivalent, for if $g \cdot \omega_{s_1} \cong \omega_{s_2}$ for some $g \in G$, then $g \in G_\theta = (G_2)_\theta$. It follows that for distinct s, the functionals $\varphi_s = \varphi + s\alpha$ lie in distinct G-orbits. Moreover,

$$\pi = \operatorname{Ind}_{G_0}^G \gamma_\theta = \operatorname{Ind}_{G_2}^G \int_{\mathbb{R}}^{\oplus} \sigma_{\omega_s} ds = \int_{\mathbb{R}}^{\oplus} \operatorname{Ind}_{G_2}^G \sigma_{\omega_s} ds = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi_s} ds.$$

Clearly the orbits $G \cdot \varphi_s$ are those that meet $p^{-1}(G_1 \cdot \psi)$. Moreover, we have

$$G \cdot \varphi_s \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi_s.$$

Indeed the right side is obviously contained in the left; and conversely, if $\varphi' = g \cdot \varphi_s \in p^{-1}(G_1 \cdot \psi)$, then $g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi$ for some $g_1 \in G_1$. Then $g_1 \cdot \varphi' = \varphi_{s_1}$ for some s_1, which says $g_1 g \cdot \varphi_s = \varphi_{s_1}$. Since the functionals φ_s, φ_{s_1} lie in distinct orbits unless $s = s_1$, we have $\varphi' \in G_1 \cdot \varphi_s$. Thus in this third case we have verified that the spectrum and multiplicity portions of the orbital spectrum formula are valid. As indicated above, we handle the measures at the end.

(iv) $G_\gamma = G_1$. The stabilizers of θ in $\mathcal{G}_1$ and $\mathcal{G}_2$ reverse roles – we have $(\mathcal{G}_2)_\theta = (\mathcal{G}_0)_\theta$ and $(\mathcal{G}_0)_\theta \subsetneq (\mathcal{G}_1)_\theta = \mathcal{G}_\theta$. In this and the next case, unlike the previous three, ν_ψ is not induced from γ_θ. Therefore π is not induced from G_0. We shall analyze it by computing its restriction to G_2 via the Mackey Subgroup Theorem. Indeed

$$\pi|_{G_2} = (\operatorname{Ind}_{G_1}^{G} \nu_\psi)|_{G_2} = \operatorname{Ind}_{G_0}^{G_2}(\nu_\psi|_{G_0}),$$

where we have used the facts that $G_1 G_2 = G$ and $G_1 \cap G_2 = G_0$. But the inequality $(\mathcal{G}_1)_\theta \neq (\mathcal{G}_0)_\theta$ insures (by Theorem 3.1) that

$$\nu_\psi|_{G_0} = \gamma_\theta.$$

Hence

$$\pi|_{G_2} = \operatorname{Ind}_{G_0}^{G_2} \gamma_\theta.$$

Since $(\mathcal{G}_2)_\theta = (\mathcal{G}_0)_\theta$, the latter is irreducible – it must be σ_ω. Hence π itself must be irreducible. In fact $\pi = \pi_\varphi$. To see this we reason as follows. Since π is irreducible and lies over σ_ω, it must be that $\pi = \pi_{\varphi + t\beta}$ for some $t \in \mathbb{R}$. We show $t = 0$. Let $\mathcal{B}$ be a real polarization for ψ which satisfies the Pukanszky condition – so $(\mathcal{G}_1)_\psi \subset \mathcal{B} \subset \mathcal{G}_1$, $\psi[\mathcal{B}, \mathcal{B}] = 0$, $\dim \mathcal{G}_1/\mathcal{B} = \dim \mathcal{B}/(\mathcal{G}_1)_\psi$ and $B \cdot \psi = \psi + \mathcal{B}^\perp(\mathcal{G}_1)$. I claim that $\mathcal{B}$ is also a real polarization for φ satisfying the Pukanszky condition. By the irreducibility facts already established, we have

$$\dim \mathcal{G} \cdot \varphi = \dim \mathcal{G}_2 \cdot \omega = \dim \mathcal{G}_0 \cdot \theta + 2 = \dim \mathcal{G}_1 \cdot \psi + 2.$$

We also have $\mathcal{G}_\varphi \subset \mathcal{G}_\theta = (\mathcal{G}_1)_\theta$, so $\mathcal{G}_\varphi \subset (\mathcal{G}_1)_\psi$. Since $\dim \mathcal{G}/\mathcal{G}_1 = 1$, it must be that $\dim(\mathcal{G}_1)_\psi/\mathcal{G}_\varphi = 1$ also. Since $\varphi|_{\mathcal{G}_1} = \psi$ (not so for φ^t), we obtain that $\mathcal{B}$ is a real polarization for φ. It also satisfies Pukanszky. In fact we know $B \cdot \varphi \subset \varphi + \mathcal{B}^\perp(\mathcal{G})$ [2, p. 68]. On the other hand, if $\varsigma \in \mathcal{B}^\perp(\mathcal{G})$, then $\varsigma_1 = \varsigma|_{\mathcal{G}_1} \in \mathcal{B}^\perp(\mathcal{G}_1)$. Hence there is $b \in B$ so that

$$b \cdot \psi = \psi + \varsigma_1.$$

Suppose

$$b \cdot \varphi = \varphi + \varsigma'.$$

Then $\varsigma - \varsigma' = s_1\alpha$ for some $s_1 \in \mathbb{R}$. Set $\varphi_1 = b \cdot \varphi$, $\psi_1 = \varphi_1|_{\mathcal{G}_1}$, $\theta_1 = \varphi_1|_{\mathcal{G}_0}$. We have $(G_0)_{\theta_1} \cdot \varphi_1 = \varphi_1 + \mathcal{G}_1^\perp$ (by Lemma 2.4). Also, applying Theorem 3.1 to $\mathcal{G}_0 \triangleleft \mathcal{G}_1$, we have $(\mathcal{G}_1)_{\psi_1} = (\mathcal{G}_1)_{\theta_1}$. So $(G_0)_{\theta_1} \subset (G_1)_{\psi_1} \subset B$ (since $\mathcal{B}$ is also a real polarization for ψ_1 by [2, p. 69]). Therefore there is $g_1 \in (G_1)_{\psi_1} \subset B$ so that $g_1 \cdot \varphi_1 = \varphi_1 + s_1\alpha$. Hence

$$g_1 b \cdot \varphi = g_1 \cdot \varphi_1 = \varphi_1 + s_1\alpha = \varphi + \varsigma' + s_1\alpha = \varphi + \varsigma.$$

Thus $\pi = \mathrm{Ind}_{G_1}^G \nu_\psi = \mathrm{Ind}_{G_1}^G \; \mathrm{Ind}_B^{G_1} \chi_\psi = \mathrm{Ind}_B^G \chi_\varphi$ must be equivalent to π_φ.

Now we attend to the orbital spectrum and multiplicities. Suppose $\varphi' \in p^{-1}(G_1 \cdot \psi)$. Then $g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi$ for some $g_1 \in G_1$, so $g_1 \cdot \varphi' = \varphi + s\alpha$ for some s. But we already saw that $(G_0)_\theta \cdot \varphi = \varphi + \mathcal{G}_1^\perp$, therefore φ and $\varphi + s\alpha$ are in the same G-orbit. Hence $G \cdot \varphi$ is the only orbit meeting $p^{-1}(G_1 \cdot \psi)$. It remains to show

$$G \cdot \varphi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi.$$

In fact,

$$\begin{aligned} \varphi' = g \cdot \varphi \in p^{-1}(G_1 \cdot \psi) \;&\Rightarrow g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi, &&\text{for some } g_1 \in G_1\\ &\Rightarrow g_1 \cdot \varphi' = \varphi + s\alpha, &&\text{for some } s \in \mathbb{R}\\ &\Rightarrow g_1 \cdot \varphi' = g_0 \cdot \varphi, &&\text{for some } g_0 \in (G_0)_\theta\\ &\Rightarrow \varphi' \in G_1 \cdot \varphi. \end{aligned}$$

As with the previous cases, the discussion of the spectral measure is postponed.

(v) $G_\gamma = G$. This is the final case. Herein the stabilizers satisfy

$$(\mathcal{G}_1)_\theta \not\supseteq (\mathcal{G}_0)_\theta, \quad (\mathcal{G}_2)_\theta \not\supseteq (\mathcal{G}_0)_\theta$$
$$\dim \mathcal{G}_\theta/(\mathcal{G}_0)_\theta = 2.$$

As in the previous case

$$\pi|_{G_2} = \mathrm{Ind}_{G_0}^{G_2} \gamma_\theta,$$

but this time the latter is reducible. In fact, since $(\mathcal{G}_2)_\theta \not\supseteq (\mathcal{G}_0)_\theta$, it must decompose

$$\pi|_{G_2} = \mathrm{Ind}_{G_0}^{G_2} \gamma_\theta = \int_{\mathbb{R}}^{\oplus} \sigma_{\omega + s\alpha} ds. \tag{3.2}$$

Now, since $(\mathcal{G}_1)_\theta \not\supseteq (\mathcal{G}_0)_\theta$, there exist a non-zero $a_1 \in \mathbb{R}$ and $W_1 \in \mathcal{G}_0$ such that

$$\theta[a_1 X + W_1, \mathcal{G}_0] = 0.$$

We will also use that $(\mathcal{G}_2)_\theta = (\mathcal{G}_2)_{\omega_s}$, $\omega_s = \omega + s\alpha$ (which follows from Theorem 3.1 in the case $(\mathcal{G}_2)_\theta \not\supseteq (\mathcal{G}_0)_\theta$). Now write

$$[a_1 X + W_1, Y] = a_1 Y + U_1, \qquad \text{for some } U_1 \in \mathcal{G}_0.$$

Then I claim:

$$\mathcal{G}_{\omega_s} = (\mathcal{G}_2)_{\omega_s} \quad \text{iff} \quad s \neq -\frac{\theta(U_1)}{a_1}.$$

First

$$\mathcal{G}_{\omega_s} \subset \mathcal{G}_\theta = (\mathcal{G}_1)_\theta + (\mathcal{G}_2)_\theta$$
$$= (\mathcal{G}_1)_\theta + (\mathcal{G}_2)_{\omega_s}.$$

So it is enough to show

$$(\mathcal{G}_1)_\theta \cap \mathcal{G}_{\omega_s} \subset (\mathcal{G}_2)_{\omega_s} \quad \text{iff} \quad s \neq -\frac{\theta(U_1)}{a_1}.$$

Since $(\mathcal{G}_0)_\theta \subset (\mathcal{G}_2)_\theta = (\mathcal{G}_2)_{\omega_s}$, the inclusion is completely controlled by the element $a_1X + W_1 \in (\mathcal{G}_1)_\theta$. If $s \neq -\frac{\theta(U_1)}{a_1}$, then

$$\omega_s[a_1X + W_1, \mathcal{G}_2] \supset \omega_s[a_1X + W_1, Y]$$
$$= \omega_s(a_1Y + U_1)$$
$$= a_1s + \theta(U_1) \neq 0.$$

If $s = -\frac{\theta(U_1)}{a_1}$, then $\omega_s[a_1X + W_1, \mathcal{G}_2] = 0$ since $\mathcal{G}_2 = \mathbb{R}Y + \mathcal{G}_0$. But $\mathcal{G} = \mathbb{R}(a_1X + W_1) + \mathcal{G}_2$, so $\mathcal{G}_{\omega_s} \not\subset (\mathcal{G}_2)_{\omega_s}$ if $s = -\frac{\theta(U_1)}{a_1}$.

Thus it follows from the claim that

$$\mathrm{Ind}_{G_2}^G \sigma_{\omega_s} \text{ is irreducible} \iff s \neq -\frac{\theta(U_1)}{a_1}.$$

Next we must decide which of these are equivalent. Since

$$G = \exp \ \mathbb{R}(a_1X + W_1)G_2,$$

it is enough to examine the action of $\exp\ \lambda(a_1X + W_1)$, $\lambda \in \mathbb{R}$, on σ_{ω_s}. We have

$$\exp \ \lambda(a_1X + W_1) \cdot \omega_s|_{\mathcal{G}_0} = \omega_s|_{\mathcal{G}_0} = \theta.$$

Also

$$\exp[-\lambda(a_1X + W_1)] \cdot \omega_s(Y) = \omega_s(Y + \lambda(a_1Y + U_1) + \ldots)$$
$$= e^{\lambda a_1}s + \frac{\theta(U_1)}{a_1}(e^{\lambda a_1} - 1).$$

Hence there are two G-orbits on the set of distinct irreducible representations $\left\{\sigma_{\omega_s} : s \neq -\frac{\theta(U_1)}{a_1}\right\}$, namely

$$\mathcal{O}^\pm = \left\{\sigma_{\omega_s} : s + \frac{\theta(U_1)}{a_1} \gtrless 0\right\}.$$

It follows from (3.2) and [7] that

$$\pi = \pi^+ \oplus \ \pi^-$$

where

$$\pi^{\pm} = \mathrm{Ind}_{G_2}^{G} \sigma_{\omega_s}, \text{ and } s \text{ is any real number satisfying } s + \frac{\theta(U_1)}{a_1} \gtrless 0 \text{ (resp.)}.$$

We must reconcile these Mackey parameters with the orbital parameters. To achieve that we start as usual with the observation that all G-orbits meeting $p^{-1}(G_1 \cdot \psi)$ must pass through the functionals $\varphi + s\alpha$, $s \in \mathbb{R}$. But the preceding shows that only three distinct orbits occur:

$$G \cdot (\varphi + s_0\alpha),\ G \cdot (\varphi + s_1\alpha),\ G \cdot (\varphi + s_2\alpha);$$
$$s_0 = -\frac{\theta(U_1)}{a_1},\ s_1 < -\frac{\theta(U_1)}{a_1},\ s_2 > -\frac{\theta(U_1)}{a_1}.$$

Write $\varphi^0 = \varphi + s_0\alpha$, $\varphi^- = \varphi + s_1\alpha$, $\varphi^+ = \varphi + s_2\alpha$. Then only the latter two are generic, since $\dim G \cdot \varphi^{\pm} = \dim G_0 \cdot \theta + 2$, while $\dim G \cdot \varphi^0 = \dim G_0 \cdot \theta$. Finally we show

$$G \cdot \varphi^{\pm} \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \varphi^{\pm}.$$

Indeed, if $\varphi' = g \cdot \varphi^+ \in p^{-1}(G_1 \cdot \psi)$, then $g_1 \cdot \varphi'|_{\mathcal{G}_1} = \psi$ for some $g_1 \in G_1$. Then $g_1 \cdot \varphi' = \varphi + s\alpha$ for some $s \in \mathbb{R}$. But $\varphi^+ = \varphi + s_2\alpha$, $s_2 > -\frac{\theta(U_1)}{a_1}$. The only way φ' and φ^+ can be in the same orbit is if $s > s_0$ also. But the above computation (of the orbit of exp $\mathbb{R}(a_1X + W_1)$) shows that one can get from φ_s to φ_{s_2} via an element in $(G_1)_\theta \subset G_1$. Hence $\varphi' \in G_1 \cdot \varphi^+$. A similar argument works for φ^-.

To complete the proof of Theorem 3.3 we must prove the equality of the spectral measure classes obtained in cases (i)-(v) – that is, the point mass, 2-point measure or Lebesgue measure on the line – with the orbital measure (class) μ^{ψ}_{G,G_1}. Let us examine the latter more carefully in the co-dimension one situation. The orbit $G_1 \cdot \psi$ has its canonical G_1-invariant measure [2, pp. 19-20]. The manifold $p^{-1}(G_1 \cdot \psi)$ can be identified as a measure space to $G_1 \cdot \psi \times \mathcal{G}_1^{\perp}(\mathcal{G})$ via the map

$$\varphi' \to (\varphi'|_{\mathcal{G}_1}, \varphi'(Y)\alpha).$$

Its canonical measure (class) is the direct product of the canonical measure (class) on $G_1 \cdot \psi$ with Lebesgue measure. In case $p^{-1}(G_1 \cdot \psi)/G_1$ is (generically) discrete – i.e., cases (i), (iv) or (v) – it is clear that the canonical measure μ^{ψ}_{G,G_1}, being the push-forward of the canonical measure on $p^{-1}(G_1 \cdot \psi)$, gives a discrete measure concentrated on the generic orbit classes. What about the continuous measures in (ii) or (iii)? In case (iii), it is absolutely obvious that the parameter $s\alpha$ parametrizes the G_1-orbits on $p^{-1}(G_1 \cdot \psi)$, and so the push-forward is the Lebesgue class ds. As for case (ii), we know that $G \cdot \varphi^t$, $\varphi^t = \varphi + t\beta$, $t \in \mathbb{R}$, parametrize the orbits that lie over $G_1 \cdot \psi$. But we also saw that $G \cdot \varphi^t = G \cdot (\varphi + t\beta) = G \cdot (\varphi + t(x_0/y_0)\alpha)$ also parametrizes the orbits. Hence the push-forward of the canonical measure class from the manifold $p^{-1}(G_1 \cdot \psi)$, under the action of G_1, again yields Lebesgue measure (class) dt. This completes our argument.

We conclude the section with the remark that every one of the five cases described in Theorem 3.3 actually occurs. Examples may be found in section 6. (See (2a) for (i), then (3f) for (ii), (2b) for (iii), (3f) for (iv) and (1) for (v).)

4. - Monomial representations

In this section we prove our main result, Theorem 2.2, in the case that the inducing representation ν is a character. Let us formulate the notation. Suppose G is completely solvable, $H \subset G$ is closed connected and $\chi \in \hat{H}$ is a unitary character. Then $\mathcal{O}_\chi$ in $\mathcal{H}^*$ is the singleton $\{\eta\}$, $\eta = -\mathrm{i}d\chi$. We shall write

$$\mathcal{H}_\chi^\perp = \mathcal{H}_\chi^\perp(\mathcal{G}) = p_{\mathcal{G},\mathcal{H}}^{-1}(\mathcal{O}_\chi) = \{\varphi \in \mathcal{G}^* : \varphi|_{\mathcal{H}} = \eta\}.$$

Theorem 2.2 becomes the statement

THEOREM 4.1. *We have*

$$\mathrm{Ind}_H^G \chi = \int_{\mathcal{H}_\chi^\perp(\mathcal{G})/H}^{\oplus} \pi_\varphi \mathrm{d}\mu_{G,H}^\chi(\varphi). \tag{4.1}$$

PROOF. The proof of formula (4.1) is by induction on $\dim G/H$. It follows from Theorem 3.1 (or Theorem 2.3) and Theorem 3.3 that it is true if

$$\dim G/H = 1.$$

Now let $\dim G/H$ be larger than one and assume by induction that formula (4.1) is true for lower co-dimension. Since G is completely solvable, we can find a closed connected subgroup G_1 such that

$$H \subset G_1 \subset G \quad \text{and} \quad \dim G/G_1 = 1.$$

The induction assumption applies and the orbital spectrum formula is valid for $\mathrm{Ind}_H^{G_1}\chi$,

$$\mathrm{Ind}_H^{G_1} \chi = \int_{\mathcal{H}_\chi^\perp(\mathcal{G}_1)/H}^{\oplus} \nu_\psi \mathrm{d}\mu_{G_1,H}^\chi(\psi).$$

If G_1 is normal in G, a direct application of [10, Thm. 2.2] yields that $\mathrm{Ind}_H^G\chi$ satisfies the orbital spectrum formula. That is, formula 4.1 is true. Therefore, we may assume that G_1 is *not* normal (i.e., H is not strongly subnormal in the sense of [10]).

The method from here on is based upon that of [9]. The argument is much more complicated since more possibilities ensue for the structure of co-dimension one induced representations. There is also one very new feature, not at all present in the nilpotent case, which we will explain during the course of the proof. As in [9] we start by invoking: induction in stages, the induction hypothesis, and commutation of direct integrals and induced representations. We obtain

$$\mathrm{Ind}_H^G \chi = \mathrm{Ind}_{G_1}^G \ \mathrm{Ind}_H^{G_1} \chi = \mathrm{Ind}_{G_1}^G \int_{\mathcal{H}_\chi^\perp(\mathcal{G}_1)/H}^{\oplus} \nu_\psi \mathrm{d}\mu_{G_1,H}^\chi(\psi)$$
$$= \int_{\mathcal{H}_\chi^\perp(\mathcal{G}_1)/H}^{\oplus} \mathrm{Ind}_{G_1}^G \nu_\psi \mathrm{d}\mu_{G_1,H}^\chi(\psi).$$

Therefore we must prove

$$\int_{\mathcal{H}_\chi^\perp(\mathcal{G}_1)/H}^{\oplus} \mathrm{Ind}_{G_1}^G \nu_\psi \mathrm{d}\mu_{G_1,H}^\chi(\psi) = \int_{\mathcal{H}_\chi^\perp(\mathcal{G})/H}^{\oplus} \pi_\varphi \mathrm{d}\mu_{G,H}^\chi(\varphi). \tag{4.2}$$

To prove equation (4.2) we must show the two direct integrals are equivalent, and as usual, that means in spectrum, multiplicity, and spectral measure. What exactly does that require? To be precise we recall the formulation in [9, paragraph after Thm. 1.5]. Let $\int_X^{\oplus} \pi_x \mathrm{d}\mu(x)$ be a direct integral of irreducible unitary representations of a type I group G. Then $x \to \pi_x$, $X \to \mathrm{Irr}(G)$, is a Borel injection. Writing $x_1 \sim x_2$ to mean $\pi_{x_1} \cong \pi_{x_2}$, setting $\overline{X} = X/{\sim} \subset \hat{G}$ and $\overline{\mu}$ the push-forward of μ under $X \to \overline{X}$, we can rewrite the direct integral as

$$\int_{\overline{X}}^{\oplus} n_{\overline{x}} \pi_{\overline{x}} \mathrm{d}\overline{\mu}(\overline{x}),$$

where $n_{\overline{x}} = \#\{y \in X : y \sim x\}$. To show that two direct integrals

$$\int_X^{\oplus} \pi_x \mathrm{d}\mu(x) \quad \text{and} \quad \int_Y^{\oplus} \sigma_y \mathrm{d}\nu(y)$$

are equivalent, we must show equality of spectra $\overline{X} = \overline{Y}$, multiplicity $n_{\overline{x}} = n_{\overline{y}}$ and measure $\overline{\mu} = \overline{\nu}$ – at least up to null sets. If the parameter spaces X and Y come equipped with a Borel surjection $p : X \to Y$ which satisfies

$$\sigma_{p(x)} \cong \pi_x \quad \text{and} \quad p_*\mu \equiv \nu,$$

then equality of spectrum and spectral measure follows automatically. Equality of multiplicity still requires separate proof. Indeed, in our proof of equality in (4.2) we shall be able to handle the spectrum and spectral measure relatively easily. The proof of equality of multiplicity is much more difficult and occupies the major portion of the argument.

Spectrum. By Theorem 3.3 we have that, regardless of the structure of $\mathrm{Ind}_{G_1}^G \nu_\psi$, the spectrum of that induced representation is

$$G \cdot p_{\mathcal{G},\mathcal{G}_1}^{-1}(\psi).$$

Hence to deduce equality of spectrum in (4.2) we only require

$$G \cdot p_{\mathcal{G},\mathcal{G}_1}^{-1}(\mathcal{H}_\chi^\perp(\mathcal{G}_1)) = G \cdot \mathcal{H}_\chi^\perp(\mathcal{G}).$$

This is completely trivial since

$$p_{\mathcal{G},\mathcal{G}_1}^{-1}(\mathcal{H}_\chi^\perp(\mathcal{G}_1)) = \mathcal{H}_\chi^\perp(\mathcal{G}).$$

Multiplicity. This is the heart of the theorem – the proof of equal multiplicity in (4.2) is long and complicated. To begin we have already observed that the multiplicity in the right side of equation (4.2) is

$$n_\varphi^\chi = \#H\text{-orbits on } G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}).$$

This is because the map $H \cdot \varphi \to G \cdot \varphi$, $\mathcal{H}_\chi^\perp(\mathcal{G})/H \to G \cdot \mathcal{H}_\chi^\perp(\mathcal{G})/G$ is surjective with n_φ^χ elements in the fiber over $G \cdot \varphi$ (see the discussion following Thm. 1.5 in [9]). Now in order to evaluate n_φ^χ and relate it to the multiplicity on the left side of (4.2), we need two auxiliary results. The first is the analog of [9, Prop. 1.7] (although generic does not mean Zariski open here – see section 2).

LEMMA 4.2. *Let $H \subset N \triangleleft G$ be simply connected exponential solvable Lie groups, N normal. Fix a character $\chi \in \hat{H}$. Then generically on $\mathcal{H}_\chi^\perp(\mathcal{N})$, we have*

$$G \cdot \theta \cap \mathcal{H}_\chi^\perp(\mathcal{N}) \text{ has the same dimension as } \mathcal{G} \cdot \theta \cap \mathcal{H}^\perp(\mathcal{N}).$$

PROOF. We know the action of G on $\mathcal{N}^*$ is exponential solvable (i.e., has non-purely imaginary weights). Hence there is a local smooth cross section. In fact there is an open co-null set $\mathcal{U} \subset \mathcal{N}^*$ such that for every $\theta \in \mathcal{U}$, there is a G-invariant neighbourhood $\mathcal{O}_\theta$ such that $\mathcal{O}_\theta \cap \mathcal{H}_\chi^\perp(\mathcal{N})$ is open in $\mathcal{H}_\chi^\perp(\mathcal{N})$, and a non-singular bi-analytic map

$$\Phi : \mathcal{O}_\theta \to \Sigma \times V$$

where V is Euclidean space, Σ is an open ball in Euclidean space and

$$\Phi^{-1}\{(\sigma, v) : v \in V\} \text{ constitutes a } G\text{-orbit for each fixed } \sigma \in \Sigma,$$

$$\Phi^{-1}(\sigma, v) \text{ and } \Phi^{-1}(\sigma', v') \text{ lie in the same orbit } \Longleftrightarrow \sigma = \sigma'.$$

The remainder of the proof is virtually identical to [9, Prop. 1.7]. Consider

$$f : \mathcal{O}_\theta \cap \mathcal{H}_\chi^\perp(\mathcal{N}) \to \Sigma$$

defined by

$$f = p_\Sigma \circ \Phi|_{\mathcal{O}_\theta \cap \mathcal{H}_\chi^\perp(\mathcal{N})},$$

and restrict attention to the dense open subset on which f has maximum rank. f determines a foliation there on which the leaves are the orbit intersections $G \circ \theta \cap \mathcal{H}_\chi^\perp(\mathcal{N})$. The proof that

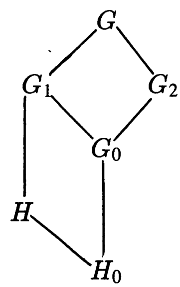
$$\dim G \circ \theta \cap \mathcal{H}_\chi^\perp(\mathcal{N}) = \dim(\mathrm{d}f)_\theta = \dim \mathcal{G} \cdot \theta \cap \mathcal{H}^\perp(\mathcal{N})$$

is then absolutely identical to the end of the proof of Proposition 1.7 in [9].

It follows from Lemma 4.2 that the multiplicity on the right side of (4.2) is generically given by

$$\dim G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = \dim \mathcal{G} \cdot \varphi \cap \mathcal{H}^\perp(\mathcal{G}).$$

To relate it to the left side we need our second auxiliary result. We invoke Proposition 3.2 and set $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$. $\mathcal{H}_0$ is an ideal in $\mathcal{H}$. If $\mathcal{H}_0 = \mathcal{H}$, then $\mathcal{H} \subset \mathcal{G}_0 \subset \mathcal{G}_2$ which is a co-dimension one ideal in $\mathcal{G}$. This contradicts our assumption that $\mathcal{H}$ is not contained in any such ideal. Hence $\mathcal{G}_1 = \mathcal{H} + \mathcal{G}_0$ and $\dim \mathcal{H}/\mathcal{H}_0 = 1$. Moreover, it is no loss of generality to assume $X \in \mathcal{H}$, so that $\mathcal{H} = \mathbb{R}X + \mathcal{H}_0$. Schematically we depict



where each slanted line connotes co-dimension one. (Of course $H_0 = \exp \mathcal{H}_0$.)

Now for any $\varphi \in \mathcal{G}^*$ we set $\psi = \varphi|_{\mathcal{G}_1}$, $\theta = \psi|_{\mathcal{G}_0}$, $\gamma = \gamma_\theta \in \hat{G}_0$ and $\mathcal{G}_\gamma = \mathcal{G}_\theta + \mathcal{G}_0$. The possibilities for $\mathcal{G}_\gamma$ are outlined in Theorem 3.3, i.e. $\mathcal{G}_\gamma$ is one of: (i) $\mathcal{G}_0$; (ii) a co-dimension one subalgebra not equal to $\mathcal{G}_1$ or $\mathcal{G}_2$; (iii) $\mathcal{G}_2$; (iv) $\mathcal{G}_1$; (v) $\mathcal{G}$. Using $p = p_{\mathcal{G},\mathcal{G}_1}$, we set

$$\begin{aligned} \mathcal{U} &= \{\varphi \in \mathcal{G}^* : \mathcal{G}_\gamma = \mathcal{G}\} & \mathcal{U}^1 &= p(\mathcal{U}) \\ \mathcal{U}_2 &= \{\varphi \in \mathcal{G}^* : \mathcal{G}_\gamma = \mathcal{G}_2\} & \mathcal{U}_2^1 &= p(\mathcal{U}_2) \\ \mathcal{U}_0 &= \{\varphi \in \mathcal{G}^* : \mathcal{G}_\gamma = \mathcal{G}_0\} & \mathcal{U}_0^1 &= p(\mathcal{U}_0) \\ \mathcal{U}_n &= \{\varphi \in \mathcal{G}^* : \mathcal{G}_\gamma \text{ is not an ideal}\} & \mathcal{U}_n^1 &= p(\mathcal{U}_n). \end{aligned}$$

We observe that the family $\mathcal{U}, \mathcal{U}_2, \mathcal{U}_0, \mathcal{U}_n$ constitutes a G-invariant partition of $\mathcal{G}^*$, and their projections form a G_1-invariant partition of $\mathcal{G}_1^*$. We have a further partition

$$\mathcal{U}_n = \mathcal{U}_{n1} \cup \mathcal{U}_{nn} \qquad \mathcal{U}_n^1 = \mathcal{U}_{n1}^1 \cup \mathcal{U}_{nn}^1$$

where

$$\begin{aligned} \mathcal{U}_{n1} &= \{\varphi \in \mathcal{G}^* : \mathcal{G}_\gamma = \mathcal{G}_1\} \qquad & \mathcal{U}_{n1}^1 &= p(\mathcal{U}_{n1}) \\ \mathcal{U}_{nn} &= \{\varphi \in \mathcal{U}_n : \mathcal{G}_\gamma \neq \mathcal{G}_1\} \qquad & \mathcal{U}_{nn}^1 &= p(\mathcal{U}_{nn}). \end{aligned}$$

We note the latter partition is *only* G_1*-invariant.* We also have

$$\left.\begin{aligned} p^{-1}(\mathcal{V}^1) &= \mathcal{V} \\ p^{-1}(\mathcal{V}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)) &= \mathcal{V} \cap \mathcal{H}_\chi^\perp(\mathcal{G}) \end{aligned}\right\} \text{ for } \mathcal{V} \text{ (resp. } \mathcal{V}^1) = \mathcal{U}, \mathcal{U}_2, \mathcal{U}_0, \mathcal{U}_{n1}, \mathcal{U}_{nn} \text{ (resp. } \mathcal{U}^1, \mathcal{U}_2^1, \mathcal{U}_0^1, \mathcal{U}_{n1}^1, \mathcal{U}_{nn}^1).$$

It is the need to deal with spectra that are not G-invariant in one side of (4.2) that is completely absent from the nilpotent situation. We deal with it in part (iv) of the next result.

LEMMA 4.3. (i) *Set* $\chi_0 = \chi|_{H_0}$. *Then the map*

$$\psi \to \theta = \psi|_{\mathcal{G}_0}, \quad \mathcal{H}_\chi^\perp(\mathcal{G}_1) \to (\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$$

is an H*-equivariant bijection.*

(ii) *Equivalence of the induced representations* $\operatorname{Ind}_{G_1}^G \nu_{\psi_1} \cong \operatorname{Ind}_{G_1}^G \nu_{\psi_2}$ *implies* ψ_1 *and* ψ_2 *lie in the same cell, i.e., either* $\mathcal{U}^1, \mathcal{U}_2^1, \mathcal{U}_0^1, \mathcal{U}_{n1}^1, \mathcal{U}_{nn}^1$.

(iii) *For* ψ_j, $j = 1, 2$, *lying in* $\mathcal{U}^1, \mathcal{U}_2^1, \mathcal{U}_0^1$, *the representations* $\operatorname{Ind}_{G_1}^G \nu_{\psi_j}$, $j = 1, 2$ *are equivalent* $\iff \theta_1 = \psi_1|_{\mathcal{G}_0}$ *and* $\theta_2 = \psi_2|_{\mathcal{G}_0}$ *lie in the same* G*-orbit.*

(iv) *Suppose* $\mathcal{U}_n \cap \mathcal{H}_\chi^\perp(\mathcal{G})$ *contains an open set in* $\mathcal{H}_\chi^\perp(\mathcal{G})$ *(equivalently* $\mathcal{U}_n^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$ *contains an open set in* $\mathcal{H}_\chi^\perp(\mathcal{G}_1)$*). Then either* $\mathcal{U}_{n1}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$ *is of lower dimension (equivalently* $\mathcal{U}_{nn}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$ *accounts for everything in* $\mathcal{U}_n^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$ *measure-theoretically) or* $\mathcal{U}_{n1}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) = \mathcal{U}_n^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$. *In the former case, for* ψ_j, $j = 1, 2$, *lying in* $\mathcal{U}_{nn}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$, *the representations* $\operatorname{Ind}_{G_1}^G \nu_{\psi_j}$, $j = 1, 2$, *are equivalent* $\iff \theta_1$ *and* θ_2 *lie in the same* G*-orbit. In the latter case, the representations* $\operatorname{Ind}_{G_1}^G \nu_{\psi_j}$, $j = 1, 2$, *are equivalent* $\iff \psi_1$ *and* ψ_2 *lie in the same* G_1*-orbit.*

PROOF. (i) The restriction $\psi \to \theta = \psi|_{\mathcal{G}_0}$ is clearly surjective and H-equivariant. It is injective since $\mathcal{G}_1 = \mathcal{H} + \mathcal{G}_0$.

(ii) Each of the representations $\mathrm{Ind}_{G_1}^G \nu_{\psi_j}$ takes one of the five forms enumerated in Theorem 3.3. The assertion is that unitary equivalence cannot occur between different forms. In fact, from Theorem 3.3 we see that equivalence could only occur between cases (i) and (iv), or between (ii) and (iii). If both $\mathrm{Ind}_{G_1}^G \nu_{\psi_j}$, $j = 1, 2$, are irreducible and equivalent, then they must lie over the same G-orbit in $\mathcal{G}_0^*$. In particular the stabilizers $\mathcal{G}_{\gamma_1}$ and $\mathcal{G}_{\gamma_2}$ would have to be G-conjugate. This is impossible if $\mathcal{G}_{\gamma_1} = \mathcal{G}_0$ and $\mathcal{G}_{\gamma_2} = \mathcal{G}_1$. In the second pair, if $\mathcal{G}_{\gamma_1} = \mathcal{G}_2$ and $\mathcal{G}_{\gamma_2}$ is a co-dimension one sub-algebra other than $\mathcal{G}_1$ or $\mathcal{G}_2$, we know (from Theorem 3.3) that both representations $\mathrm{Ind}_{G_1}^G \nu_{\psi_j}$, $j = 1, 2$, are induced from G_0. So once again they must lie over the same G-orbit in $\mathcal{G}_0^*$. The stabilizers $\mathcal{G}_{\gamma_1}, \mathcal{G}_{\gamma_2}$ must be conjugate, which is impossible since $\mathcal{G}_2$ is an ideal.

(iii) If for $j = 1, 2$, $\psi_j \in \mathcal{U}_0^1$ or $\psi_j \in \mathcal{U}_2^1$, then

$$\mathrm{Ind}_{G_1}^G \nu_{\psi_j} = \mathrm{Ind}_{G_0}^G \gamma_{\theta_j}, \quad \theta_j = \psi_j|_{\mathcal{G}_0}.$$

Hence the two representations are equivalent iff the functionals θ_j, $j = 1, 2$, are in the same G-orbit. Next, let $\psi \in \mathcal{U}^1$, so that $\mathrm{Ind}_{G_1}^G \nu_\psi = \pi^+ \oplus \pi^-$ as in Theorem 3.3. Using the material from Theorem 3.3 part (v), we also have

$$\mathrm{Ind}_{G_0}^G \gamma_\theta = \mathrm{Ind}_{G_2}^G \ \mathrm{Ind}_{G_0}^{G_2} \gamma_\theta = \mathrm{Ind}_{G_2}^G \int^{\oplus} \sigma_{\omega_s} ds = \int^{\oplus} \mathrm{Ind}_{G_2}^G \sigma_{\omega_s} ds$$
$$= \int_{-\infty}^{s_0 \oplus} \pi_{\varphi_s} ds \oplus \int_{s_0}^{\infty \oplus} \pi_{\varphi_s} ds = \infty(\pi^+ \oplus \pi^-).$$

Thus $\mathrm{Ind}_{G_1}^G \nu_\psi$ and $\mathrm{Ind}_{G_0}^G \gamma_\theta$ are quasi-equivalent. But that is enough to conclude again that for $\psi_j \in \mathcal{U}^1$, $\mathrm{Ind}_{G_1}^G \nu_{\psi_1} \cong \mathrm{Ind}_{G_1}^G \nu_{\psi_2}$ iff $G \cdot \theta_1 = G \cdot \theta_2$.

(iv) For $\varphi \in \mathcal{U}_n$, $\theta = \varphi|_{\mathcal{G}_0}$, $\mathcal{G}_\gamma = \mathcal{G}_\theta + \mathcal{G}_0$, we have $\mathcal{G}_\gamma$ is of co-dimension one in $\mathcal{G}$ and not an ideal. If we set

$$\mathcal{G}_1^s = \exp\ sY \cdot \mathcal{G}_1 = \mathbb{R}(X - sY) + \mathcal{G}_0, \tag{4.3}$$

then all such stabilizers are accounted for as s varies in $\mathbb{R}$. Each of the subsets $\mathcal{U}_{n1}$ and $\mathcal{U}_{nn}$ is G_1-invariant, and it is obvious that

$$\dim \mathcal{U}_{n1} < \dim \mathcal{U}_{nn}, \quad \dim \mathcal{U}_{n1}^1 < \dim \mathcal{U}_{nn}^1$$

(unless they are all zero). It follows from the fact that we have local cross sections (see the structure theory in Lemma 4.2) that either

$$\dim \mathcal{U}_{n1}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1) < \dim \mathcal{U}_{nn}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1)$$

or

$$\mathcal{U}_{nn}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1) = \emptyset.$$

(Alternatively, the map $\psi \to s(\psi)$, $\mathcal{G}_{\psi|\mathcal{G}_0} = \mathcal{G}_0 + \mathbb{R}(X - s(\psi)Y) + \mathcal{G}_0$, is smooth on the variety $\mathcal{U}_n^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1)$, and $\mathcal{U}_{n1}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1)$ is the pre-image of zero). Now when $\psi \in \mathcal{U}_{nn}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1)$, we find ourselves in case (ii) so that

$$\mathrm{Ind}_{G_1}^G \nu_\psi = \mathrm{Ind}_{G_0}^G \gamma_\theta, \quad \theta = \varphi|_{\mathcal{G}_0}.$$

Therefore the equivalence assertion in the lemma is immediate. When

$$\psi \in \mathcal{U}_{n1}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1),$$

then (by Theorem 3.3) we know $\mathrm{Ind}_{G_1}^G \nu_\psi$ is irreducible. If ψ_1 and ψ_2 are in the same G_1-orbit, then

$$\nu_{\psi_1} \cong \nu_{\psi_2} \Rightarrow \mathrm{Ind}_{G_1}^G \nu_{\psi_1} \cong \mathrm{Ind}_{G_1}^G \nu_{\psi_2}.$$

Conversely, if $\psi_1, \psi_2 \in \mathcal{U}_{n1}^1 \cap \mathcal{H}_X^\perp(\mathcal{G}_1)$ and $\mathrm{Ind}_{G_1}^G \nu_{\psi_1} \cong \mathrm{Ind}_{G_1}^G \nu_{\psi_2}$, then we can say the following. By the Mackey Machine, the representations must lie over the same G- orbit in $\mathcal{G}_0^*$. Hence if $\theta_j = \psi_j|_{\mathcal{G}_0}$, there is $g \in G$ so that $g \cdot \theta_1 = \theta_2$. But then

$$g \cdot G_{\gamma_1} = G_{\gamma_2}.$$

Since both $G_{\gamma_{\theta_j}} = G_1$, and the normalizer of G_1 in G is G_1 itself, it must be that $g \in G_1$. Thus

$$\mathrm{Ind}_{G_1}^G \nu_{\psi_2} \cong \mathrm{Ind}_{G_1}^G \nu_{\psi_1} \cong \mathrm{Ind}_{G_1}^G g \cdot \nu_{\psi_1} \cong \mathrm{Ind}_{G_1}^G \nu_{g \cdot \psi_1}.$$

But, again by the Mackey Machine, these can be equivalent only if ν_{ψ_2} and $\nu_{g\cdot\psi_1}$ are equivalent representations of G_1. Hence ψ_2 and $g \cdot \psi_1$ lie in the same G_1-orbit. Since $g \in G_1$, the same is true of ψ_1 and ψ_2. This concludes the proof of Lemma 4.3.

Next we consider a partition of $\mathcal{H}_X^\perp(\mathcal{G})$ according to generic dimensions of the G- and H-orbits passing through it. We distinguish two mutually exclusive cases. Let $\varphi \in \mathcal{H}_X^\perp(\mathcal{G})$. Either

(4.4)
(a) $\dim G \cdot \varphi \cap \mathcal{H}_X^\perp(\mathcal{G}) > \dim H \cdot \varphi$ or
(b) $\dim G \cdot \varphi \cap \mathcal{H}_X^\perp(\mathcal{G}) = \dim H \cdot \varphi$.

Write $\varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})_a$, $\mathcal{H}_\chi^\perp(\mathcal{G})_b$ accordingly. It is standard (see Lemma 4.4 (iii) below, or [9, §1]) that

$$(4.5)\qquad \begin{aligned}&\text{(a)}\ \dim \mathcal{G}\cdot\varphi \cap \mathcal{H}^\perp(\mathcal{G}) > \dim \mathcal{H}\cdot\varphi \iff \dim \mathcal{G}\cdot\varphi > 2\ \dim \mathcal{H}\cdot\varphi\\ &\text{(b)}\ \dim \mathcal{G}\cdot\varphi \cap \mathcal{H}^\perp(\mathcal{G}) = \dim \mathcal{H}\cdot\varphi \iff \dim \mathcal{G}\cdot\varphi = 2\ \dim \mathcal{H}\cdot\varphi.\end{aligned}$$

Combining with Lemma 4.2, we see *generically*

$$(4.6)\qquad \begin{aligned}&\text{(a)}\ \varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})_a \iff \dim \mathcal{G}\cdot\varphi > 2\ \dim \mathcal{H}\cdot\varphi\\ &\text{(b)}\ \varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})_b \iff \dim \mathcal{G}\cdot\varphi = 2\ \dim \mathcal{H}\cdot\varphi.\end{aligned}$$

REMARK. In the nilpotent situation, case (b) characterizes finite multiplicity in equation (4.1). This is not so for exponential solvable groups – see [9, Expl. 8(ii)], although I do not have a corresponding example for completely solvable groups. Also in the nilpotent situation, only one of conditions (a) or (b) can occur (measure-theoretically), since generic means Zariski open. For completely solvable groups, I imagine both conditions might constitute sets of positive measure, although again I do not have an example. We shall prove equal multiplicity in equation (4.2) by examining each of the eight pairs obtained by intersecting the four sets $\mathcal{U}, \mathcal{U}_0, \mathcal{U}_2, \mathcal{U}_n$ with cases (a) and (b). In every example it may be that no more than one of these has positive measure, but I do not have any such general result. So I treat all of them independently. (Also $\mathcal{U}_n$ splits into $\mathcal{U}_{n1}$ and $\mathcal{U}_{nn}$ so actually there are ten pairs in total.)

We need one more lemma before starting the case-by-case analysis.

LEMMA 4.4. (i) *For any* $\varphi \in \mathcal{G}^*$, $\theta = \varphi|_{\mathcal{G}_0}$, *we have* $\mathcal{G}\cdot\theta = (\mathcal{G}_0)_\varphi^\perp(\mathcal{G}_0)$.

(ii) $\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^\perp(\mathcal{G}_0) = \dim \mathcal{G}\cdot\theta - \dim \mathcal{H}_0\cdot\varphi$.

(iii) $\dim \mathcal{G}\cdot\varphi \cap \mathcal{H}^\perp(\mathcal{G}) = \dim \mathcal{G}\cdot\varphi - \dim \mathcal{H}\cdot\varphi$.

PROOF. (i) We have $\mathcal{G}\cdot\theta((\mathcal{G}_0)_\varphi) = \theta[\mathcal{G},(\mathcal{G}_0)_\varphi] \subset \varphi[\mathcal{G},(\mathcal{G}_0)_\varphi] = 0$. Hence $\mathcal{G}\cdot\theta \subset (\mathcal{G}_0)_\varphi^\perp(\mathcal{G}_0)$. In fact the dimensions are equal. Indeed by Lemma 2.4 we have

$$(\mathcal{G}_0)_\theta\cdot\varphi = \mathcal{G}_\gamma^\perp.$$

Moreover,

$$\begin{aligned}\dim \mathcal{G}_\gamma^\perp = \dim \mathcal{G}/(\mathcal{G}_\theta + \mathcal{G}_0) &= \dim(\mathcal{G}/\mathcal{G}_\theta)/((\mathcal{G}_\theta+\mathcal{G}_0)/\mathcal{G}_\theta)\\ &= \dim(\mathcal{G}/\mathcal{G}_\theta)/(\mathcal{G}_0/(\mathcal{G}_0)_\theta).\end{aligned}$$

Therefore

$$\begin{aligned}\dim(\mathcal{G}_0)_\theta - \dim(\mathcal{G}_0)_\varphi &= \dim(\mathcal{G}_0)_\theta\cdot\varphi\\ &= \dim \mathcal{G}/\mathcal{G}_\theta - \dim \mathcal{G}_0/(\mathcal{G}_0)_\theta\\ &= \dim \mathcal{G}\cdot\theta - \dim \mathcal{G}_0 + \dim(\mathcal{G}_0)_\theta.\end{aligned}$$

That is

$$\dim \mathcal{G}\cdot\theta = \dim \mathcal{G}_0 - \dim(\mathcal{G}_0)_\varphi = \dim(\mathcal{G}_0)^{\perp}_{\varphi}(\mathcal{G}_0).$$

(ii) Using part (i) we compute

$$\begin{aligned}\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^{\perp}(\mathcal{G}_0) &= \dim[(\mathcal{G}_0)^{\perp}_{\varphi}(\mathcal{G}_0)\cap \mathcal{H}_0^{\perp}(\mathcal{G}_0)]\\ &= \dim[(\mathcal{G}_0)_\varphi + \mathcal{H}_0]^{\perp}(\mathcal{G}_0)\\ &= \dim \mathcal{G}_0/((\mathcal{G}_0)_\varphi + \mathcal{H}_0)\\ &= \dim \mathcal{G}_0 - \dim(\mathcal{G}_0)_\varphi - \dim \mathcal{H}_0 + \dim((\mathcal{G}_0)_\varphi \cap \mathcal{H}_0)\\ &= \dim(\mathcal{G}_0)^{\perp}_{\varphi}(\mathcal{G}_0) - (\dim \mathcal{H}_0 - \dim(\mathcal{H}_0)_\varphi)\\ &= \dim \mathcal{G}\cdot\theta - \dim \mathcal{H}_0\cdot\varphi.\end{aligned}$$

(iii) This is similar to (ii),

$$\begin{aligned}\dim \mathcal{G}\cdot\varphi \cap \mathcal{H}^{\perp}(\mathcal{G}) &= \dim \mathcal{G}^{\perp}_{\varphi}\cap\mathcal{H} = \dim(\mathcal{G}_\varphi + \mathcal{H})^{\perp}\\ &= \dim \mathcal{G} - \dim(\mathcal{G}_\varphi + \mathcal{H})\\ &= \dim \mathcal{G} - \dim \mathcal{G}_\varphi - \dim \mathcal{H} + \dim \mathcal{H}_\varphi\\ &= \dim \mathcal{G}\cdot\varphi - \dim \mathcal{H}\cdot\varphi.\end{aligned}$$

We turn at last to the proof of equal multiplicity in equation (4.2) for the various intersections indicated above. We combine the preceding lemmas and observe that: in cases $\mathcal{U}, \mathcal{U}_0, \mathcal{U}_2$ and $\mathcal{U}_{nn}$ we must show that generically

$$\#H\text{-orbits on } G\cdot\varphi \cap \mathcal{H}^{\perp}_{\chi}(\mathcal{G}) = \#H\text{-orbits on } G\cdot\theta\cap(\mathcal{H}_0)^{\perp}_{\chi_0}(\mathcal{G}_0);$$

while in case $\mathcal{U}_{n1}$ we must show that generically

$$\#H\text{-orbits on } G\cdot\varphi \cap \mathcal{H}^{\perp}_{\chi}(\mathcal{G}) = \#H\text{-orbits on } G_1\cdot\psi\cap\mathcal{H}^{\perp}_{\chi}(\mathcal{G}_1).$$

Furthermore, in these varieties we also have generically that

$$\begin{aligned}\dim G\cdot\varphi\cap\mathcal{H}^{\perp}_{\chi}(\mathcal{G}) &= \dim \mathcal{G}\cdot\varphi\cap\mathcal{H}^{\perp}(\mathcal{G})\\ \dim G_1\cdot\psi\cap\mathcal{H}^{\perp}_{\chi}(\mathcal{G}_1) &= \dim \mathcal{G}_1\cdot\psi\cap\mathcal{H}^{\perp}(\mathcal{G}_1)\\ \dim G\cdot\theta\cap(\mathcal{H}_0)^{\perp}_{\chi_0}(\mathcal{G}_0) &= \dim \mathcal{G}\cdot\theta\cap\mathcal{H}_0^{\perp}(\mathcal{G}_0).\end{aligned}$$

Finally we keep in mind at all times both equations (4.4)-(4.6) and the fact that all co-adjoint orbits are even-dimensional.

($\mathcal{U}_0a$) Here we have $\dim \mathcal{G}\cdot\varphi > 2\ \dim \mathcal{H}\cdot\varphi$. Then the multiplicity is infinite on the right side of equation (4.2). To show infinite multiplicity on the left, it is enough to prove

$$\dim \mathcal{G}\cdot\theta\cap\mathcal{H}_0^{\perp}(\mathcal{G}_0) > \dim \mathcal{H}\cdot\theta.$$

But in case $\mathcal{U}_0$ we have the following facts:

$$\mathcal{G}_\theta = (\mathcal{G}_0)_\theta \Rightarrow \mathcal{G}_\varphi = (\mathcal{G}_\theta)_\varphi = ((\mathcal{G}_0)_\theta)_\varphi = (\mathcal{G}_0)_\varphi;$$

$$\dim \mathcal{G}_\theta/\mathcal{G}_\varphi = \dim(\mathcal{G}_0)_\theta/(\mathcal{G}_0)_\varphi = \dim(\mathcal{G}_0)_\theta \cdot \varphi = \dim \mathcal{G}_\gamma^\perp = 2;$$

$$\begin{aligned}\dim \mathcal{G}\cdot\theta &= \dim \mathcal{G} - \dim \mathcal{G}_\theta = \dim \mathcal{G} - \dim \mathcal{G}_\varphi - 2\\ &= \dim \mathcal{G}\cdot\varphi - 2 \geq 2 \dim \mathcal{H}\cdot\varphi;\end{aligned}$$

$$\begin{aligned}\mathcal{H}_\theta &= \mathcal{H} \cap (\mathcal{G}_1)_\theta = \mathcal{H} \cap (\mathcal{G}_0)_\theta = (\mathcal{H}_0)_\theta \Rightarrow \mathcal{H}_\varphi\\ &= (\mathcal{H}_0)_\varphi \Rightarrow \dim \mathcal{H}\cdot\varphi > \dim \mathcal{H}_0\cdot\varphi.\end{aligned}$$

Combining with Lemma 4.4 we get

$$\begin{aligned}\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^\perp(\mathcal{G}_0) &= \dim \mathcal{G}\cdot\theta - \dim \mathcal{H}_0\cdot\varphi\\ &\geq 2 \dim \mathcal{H}\cdot\varphi - \dim \mathcal{H}_0\cdot\varphi\\ &> \dim \mathcal{H}\cdot\varphi\\ &\geq \dim \mathcal{H}\cdot\theta.\end{aligned}$$

($\mathcal{U}_0 b$) Now we have $\dim \mathcal{G}\cdot\varphi = 2 \dim \mathcal{H}\cdot\varphi$. In fact we can deduce from this that

$$\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^\perp(\mathcal{G}_0) = \dim \mathcal{H}\cdot\theta.$$

But unlike the nilpotent case, neither equality guarantees finite multiplicity (apparently – see the Remark after equation (4.6)). So there is no necessity to derive the latter. Instead, to prove equal multiplicity in equation (4.2), we proceed directly to show that the sets of

$$H\text{-orbits in } G\cdot\varphi \cap \mathcal{H}_\chi^\perp(\mathcal{H}) \quad \text{and} \quad H\text{-orbits in } G\cdot\theta \cap (\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$$

are in bijective correspondence. In fact the map

$$\varphi' \to \varphi'|_{\mathcal{G}_0}$$

is clearly a surjective map $G\cdot\varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) \to G\cdot\theta \cap (\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$. Moreover it is H-equivariant. Thus it takes H-orbits to H-orbits. It remains only to prove that distinct H-orbits on $G\cdot\varphi\cap\mathcal{H}_\chi^\perp(\mathcal{G})$ restrict to distinct H-orbits on $G\cdot\theta\cap(\mathcal{H}_0)_\chi^\perp(\mathcal{G}_0)$. So take $\varphi \in \mathcal{U}_0 \cap \mathcal{H}_\chi^\perp(\mathcal{G})$, $g\cdot\varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})$, $g\cdot\theta = h\cdot\theta$ for some $h \in H$. We must produce $h' \in H$ so that $g\cdot\varphi = h'\cdot\varphi$. We have

$$h^{-1}g \in G_\theta = (G_0)_\theta \text{ and } (G_0)_\theta\cdot\varphi = \varphi + \mathcal{G}_\gamma^\perp = \varphi + \mathcal{G}_0^\perp.$$

Also

$$h^{-1}g\cdot\varphi|_{\mathcal{H}} = g\cdot\varphi|_{\mathcal{H}} \in \mathcal{H}_\chi^\perp(\mathcal{G})$$

since χ is a character. Therefore

$$h^{-1}g\cdot\varphi\in(\varphi+\mathcal{G}_0^\perp)\cap\mathcal{H}_\chi^\perp(\mathcal{G})=\varphi+\mathcal{G}_1^\perp.$$

Now

$$H_\theta\cdot\varphi=(H_0)_\theta\cdot\varphi\in(G_0)_\theta\cdot\varphi\subset\varphi+\mathcal{G}_0^\perp.$$

Also

$$H_\theta\cdot\varphi|_{\mathcal{H}}=\varphi|_{\mathcal{H}}\ \text{so}\ H_\theta\cdot\varphi\subset\varphi+\mathcal{G}_1^\perp.$$

It suffices to prove

$$H_\theta\cdot\varphi=\varphi+\mathcal{G}_1^\perp.$$

If so, then $h^{-1}g\cdot\varphi=h'\cdot\varphi$ for some $h'\in H_\theta$, and we are done. Well, the fact that

$$\psi\to\theta,\ \mathcal{H}_\chi^\perp(\mathcal{G}_1)\to(\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$$

is an H-equivariant bijection implies $\mathcal{H}_\psi=\mathcal{H}_\theta$. But

$$\begin{aligned}2\ \dim\mathcal{H}\cdot\psi\le\dim\mathcal{G}_1\cdot\psi&=\dim\mathcal{G}_0\cdot\theta+2=\dim\mathcal{G}_2\cdot\omega\\&=\dim\mathcal{G}\cdot\varphi-2=2(\dim\mathcal{H}\cdot\varphi-1).\end{aligned}$$

Strict inequality is impossible, so

$$\dim\mathcal{H}\cdot\psi=\dim\mathcal{H}\cdot\varphi-1\Rightarrow\dim\mathcal{H}_\varphi+1=\dim\mathcal{H}_\psi=\dim\mathcal{H}_\theta.$$

Therefore $H_\theta\cdot\varphi$ is an open and connected subset of $\varphi+\mathcal{G}_1^\perp$. The same is true of any other H_θ-orbit in $\varphi+\mathcal{G}_1^\perp$, hence $H_\theta\cdot\varphi=\varphi+\mathcal{G}_1^\perp$.

($\mathcal{U}_2a$) As in ($\mathcal{U}_0a$) we assume $\dim\mathcal{G}\cdot\varphi>2\ \dim\mathcal{H}\cdot\varphi$ and deduce that $\dim\mathcal{G}\cdot\theta\cap\mathcal{H}_0^\perp(\mathcal{G}_0)>\dim\mathcal{H}\cdot\theta$ (thereby obtaining infinite multiplicity on both sides of equation (4.2)). The salient facts this time are:

$$\begin{aligned}\dim\mathcal{G}_1\cdot\psi&=\dim\mathcal{G}_0\cdot\theta+2=\dim\mathcal{G}_2\cdot\omega+2=\dim\mathcal{G}\cdot\varphi\\&\Rightarrow\dim\frac{\mathcal{G}}{\mathcal{G}_\varphi}=\dim\frac{\mathcal{G}_0}{(\mathcal{G}_0)_\theta}+2.\end{aligned}$$

Also

$$(G_0)_\theta\cdot\varphi=\varphi+\mathcal{G}_\gamma^\perp=\varphi+\mathcal{G}_2^\perp\Rightarrow\dim\frac{(\mathcal{G}_0)_\theta}{(\mathcal{G}_0)_\varphi}=1.$$

Thus, by Lemma 4.4, we have

$$\begin{aligned}\dim\mathcal{G}\cdot\theta&=\dim\frac{\mathcal{G}_0}{(\mathcal{G}_0)_\varphi}=\dim\frac{\mathcal{G}_0}{(\mathcal{G}_0)_\theta}+1\\&=\dim\frac{\mathcal{G}}{\mathcal{G}_\varphi}-1=\dim\mathcal{G}\cdot\varphi-1\\&>2\ \dim\mathcal{H}\cdot\varphi.\end{aligned}$$

Again by Lemma 4.4 we obtain

$$\begin{aligned}\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^{\perp}(\mathcal{G}_0) &= \dim \mathcal{G}\cdot\theta - \dim \mathcal{H}_0\cdot\varphi \\ &> 2\ \dim \mathcal{H}\cdot\varphi - \dim \mathcal{H}_0\cdot\varphi \\ &\geq \dim \mathcal{H}\cdot\varphi \\ &\geq \dim \mathcal{H}\cdot\theta.\end{aligned}$$

($\mathcal{U}_2 b$) As in ($\mathcal{U}_0 b$) we must show that the H-equivariant restriction

$$G\cdot\varphi \cap \mathcal{H}_\chi^{\perp}(\mathcal{G}) \to G\cdot\theta \cap (\mathcal{H}_0)_{\chi_0}^{\perp}(\mathcal{G}_0)$$

sets up a bijection of H-orbits. In fact in this case we can show that the map itself is a bijection. To prove that we only need to demonstrate injectivity. Let $\varphi \in \mathcal{U}_2 \cap \mathcal{H}_\chi^{\perp}(\mathcal{G})$, $g\cdot\varphi \in \mathcal{H}_\chi^{\perp}(\mathcal{G})$, $g\cdot\theta = \theta$. We have to show $g \in G_\varphi$. Well, $g\cdot\theta = \theta$, and both φ and $g\cdot\varphi$ in $\mathcal{H}_\chi^{\perp}(\mathcal{G})$ says that

$$(g\cdot\varphi)|_{\mathcal{G}_1} = \varphi|_{\mathcal{G}_1}.$$

Hence $g\cdot\varphi = \varphi_s = \varphi + s\alpha$ for some $s \in \mathbb{R}$. But in case $\mathcal{U}_2$, the functionals φ_s, for distinct s, lie in distinct G-orbits. Thus $s = 0$, that is $g \in G_\varphi$.

($\mathcal{U}a$) The technique remains unchanged. Assume $\dim \mathcal{G}\cdot\varphi > 2\ \dim \mathcal{H}\cdot\varphi$; we prove $\dim \mathcal{G}\cdot\theta \cap \mathcal{H}_0^{\perp}(\mathcal{G}_0) > \dim \mathcal{H}\cdot\theta$. We recite the pertinent facts:

$$\dim \mathcal{G}_1\cdot\psi = \dim \mathcal{G}_0\cdot\theta = \dim \mathcal{G}_2\cdot\omega = \dim \mathcal{G}\cdot\varphi - 2;$$

$$(G_0)_\theta\cdot\varphi = \varphi + \mathcal{G}_\gamma^{\perp} = \varphi,\ \text{since}\ \mathcal{G}_\gamma = \mathcal{G} \Rightarrow (\mathcal{G}_0)_\theta = (\mathcal{G}_0)_\varphi.$$

Next

$$\dim \frac{\mathcal{G}}{\mathcal{G}_\varphi} - 2 = \dim \frac{\mathcal{G}_0}{(\mathcal{G}_0)_\theta} = \dim \frac{\mathcal{G}_0}{(\mathcal{G}_0)_\varphi} = \dim \frac{\mathcal{G}}{(\mathcal{G}_0)_\varphi} - 2 \Rightarrow \mathcal{G}_\varphi = (\mathcal{G}_0)_\varphi.$$

Therefore by Lemma 4.4, we have

$$\begin{aligned}\dim \mathcal{G}\cdot\theta &= \dim \frac{\mathcal{G}_0}{(\mathcal{G}_0)_\varphi} \\ &= \dim \frac{\mathcal{G}}{(\mathcal{G}_0)_\varphi} - 2 \\ &= \dim \frac{\mathcal{G}}{\mathcal{G}_\varphi} - 2 \\ &\geq 2\ \dim \mathcal{H}\cdot\varphi.\end{aligned}$$

Also

$$\mathcal{H}_\varphi = \mathcal{H} \cap \mathcal{G}_\varphi = \mathcal{H} \cap (\mathcal{G}_0)_\varphi = (\mathcal{H}_0)_\varphi$$
$$\Rightarrow \dim \mathcal{H} \cdot \varphi = \dim \frac{\mathcal{H}}{\mathcal{H}_\varphi} = \dim \frac{\mathcal{H}}{(\mathcal{H}_0)_\varphi}$$
$$> \dim \frac{\mathcal{H}_0}{(\mathcal{H}_0)_{\varphi'}} = \dim \mathcal{H}_0 \cdot \varphi.$$

Therefore applying Lemma 4.4 again, we get

$$\begin{aligned}\dim \mathcal{G} \cdot \theta \cap \mathcal{H}_0^\perp(\mathcal{G}_0) &= \dim \mathcal{G} \cdot \theta - \dim \mathcal{H}_0 \cdot \varphi \\ &\geq 2 \dim \mathcal{H} \cdot \varphi - \dim \mathcal{H}_0 \cdot \varphi \\ &> \dim \mathcal{H} \cdot \varphi \\ &\geq \dim \mathcal{H} \cdot \theta.\end{aligned}$$

($\mathcal{U}b$) In this case we establish a bijective relationship between the H-orbits on $G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G})$ and on $G \cdot \theta \cap (\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$. As in ($\mathcal{U}_0 b$), we take $\varphi \in \mathcal{U} \cap \mathcal{H}_\chi^\perp(\mathcal{G})$, $g \cdot \varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})$, $g \cdot \theta = h \cdot \theta$ for some $h \in H$. We produce $h' \in H$ so that $g \cdot \varphi = h' \cdot \varphi$. But as in the cited case

$$h^{-1} g \in G_\theta \quad \text{and} \quad h^{-1} g \cdot \varphi|_{\mathcal{H}} = \varphi|_{\mathcal{H}} = -\mathrm{id}_\chi,$$

so

$$h^{-1} g \cdot \varphi = \varphi_s = \varphi + s\alpha \quad \text{for some } s \in \mathbb{R}.$$

Then it must be that $(s_0 - s)s_0 > 0$ – that is, s and 0 lie on the same side of s_0 – since that is the only way $\varphi = \varphi_0$ and φ_s can be in the same orbit. But now consider $H_\theta \cdot \varphi$. We have

$$2 \dim \mathcal{H} \cdot \psi \leq \dim \mathcal{G}_1 \cdot \psi = \dim \mathcal{G} \cdot \varphi - 2 = 2(\dim \mathcal{H} \cdot \varphi - 1).$$

Therefore

$$\dim \mathcal{H} \cdot \psi = \dim \mathcal{H} \cdot \varphi - 1 \Rightarrow \dim \mathcal{H}_\psi = \dim \mathcal{H}_\varphi + 1.$$

As in ($\mathcal{U}_0 b$) we have (by the fact that $\mathcal{H}_\chi^\perp(\mathcal{G}) \to (\mathcal{H}_0)_{\chi_0}^\perp(\mathcal{G}_0)$ is an H-equivariant bijection) that $\mathcal{H}_\psi = \mathcal{H}_\theta$. Hence

$$\dim \mathcal{H}_\theta = \dim \mathcal{H}_\varphi + 1.$$

Thus

$$\dim H_\theta \cdot \varphi = 1,$$

and so $H_\theta \cdot \varphi$ is an open and connected subset of $\varphi + \mathcal{G}_1^\perp$. But the same is true of any H_θ-orbit $H_\theta \cdot \varphi_s$, $(s_0 - s)s_0 > 0$. Hence

$$H_\theta \cdot \varphi = \{\varphi_s : (s_0 - s)s_0 > 0\},$$

and so

$$h^{-1}g\cdot\varphi = h'\cdot\varphi \qquad \text{for some } h'\in H_\theta,$$

i.e.

$$g\cdot\varphi = hh'\cdot\varphi.$$

We pass now to the subvariety $\mathcal{U}_n$, which requires more delicate arguments than the previous three. The right side of equation (4.2) yields the same multiplicity formula for *any* orbit meeting $\mathcal{H}_\chi^\perp(\mathcal{G})$. But in counting multiplicities on the left we must distinguish between $\mathcal{U}_{n1}$ and $\mathcal{U}_{nn}$. Lemma 4.3 (iv) helps us greatly since it insures that both cases cannot occur simultaneously. Either the family $\mathcal{U}_{n1}^1\cap\mathcal{H}_\chi^\perp(\mathcal{G}_1)$ accounts for all of $\mathcal{U}_n^1\cap\mathcal{H}_\chi^\perp(\mathcal{G}_1)$, or measure-theoretically it accounts for nothing.

($\mathcal{U}_{nn}a$) We assume here and in the next case that we are in the first of the two situations of Lemma 4.3 (iv). First we take dim $\mathcal{G}\cdot\varphi > 2$ dim $\mathcal{H}\cdot\varphi$, forcing infinite multiplicity on the right side of (4.2). To show infinite multiplicity on the left, we must prove (as usual) that dim $\mathcal{G}\cdot\theta\cap\mathcal{H}_0^\perp(\mathcal{G}) >$ dim $\mathcal{H}\cdot\theta$. The relevant facts are as follows:

$$(G_0)_\theta\cdot\varphi = \varphi + \mathcal{G}_\gamma^\perp \Rightarrow \dim\frac{(\mathcal{G}_0)_\theta}{(\mathcal{G}_0)_\varphi} = 1;$$

$$\dim\mathcal{G}_1\cdot\psi = \dim\mathcal{G}_0\cdot\theta + 2 = \dim\mathcal{G}_2\cdot\omega = \dim\mathcal{G}\cdot\varphi.$$

Combining these with Lemma 4.4, we obtain

$$\dim\mathcal{G}\cdot\theta = \dim\frac{\mathcal{G}_0}{(\mathcal{G}_0)_\varphi} = \dim\frac{\mathcal{G}_0}{(\mathcal{G}_0)_\theta} + 1 = \dim\mathcal{G}\cdot\varphi - 1 > 2\ \dim\mathcal{H}\cdot\varphi.$$

Applying Lemma 4.4 again, we get

$$\begin{aligned}\dim\mathcal{G}\cdot\theta\cap\mathcal{H}_0^\perp(\mathcal{G}_0) &= \dim\mathcal{G}\cdot\theta - \dim,\mathcal{H}_0\cdot\varphi\\ &> 2\ \dim\mathcal{H}\cdot\varphi - \dim\mathcal{H}_0\cdot\varphi\\ &\geq \dim\mathcal{H}\cdot\varphi\\ &\geq \dim\mathcal{H}\cdot\theta.\end{aligned}$$

($\mathcal{U}_{nn}b$) Now we have dim $\mathcal{G}\cdot\varphi = 2$ dim $\mathcal{H}\cdot\varphi$. We will show (as in ($\mathcal{U}_2b$)) that the restriction

$$G\cdot\varphi\cap\mathcal{H}_\chi^\perp(\mathcal{G}) \to G\cdot\theta\cap(\mathcal{H}_0)_{\chi_0}^\perp(g_0)$$

is actually a bijection. So let $\varphi\in\mathcal{U}_{nn}\cap\mathcal{H}_\chi^\perp(\mathcal{G})$, $g\cdot\varphi\in\mathcal{H}_\chi^\perp(\mathcal{G})$, $g\cdot\theta=\theta$. We shall prove that $g\in G_\varphi$. The argument is almost identical to that of ($\mathcal{U}_2b$). Since $g\cdot\theta=\theta$, and both φ and $g\cdot\varphi$ are in $\mathcal{H}_\chi^\perp(\mathcal{G})$, it must be that $g\cdot\varphi=\varphi_s=\varphi+s\alpha$ for some $s\in\mathbb{R}$. But when $\varphi\in\mathcal{U}_{nn}$, the functionals φ_s, for distinct s, lie in distinct G-orbits (see Theorem 3.3). Thus $s=0$ and $g\cdot\varphi=\varphi$.

($U_{n1}a$) Now we pass to the second of the situations in Lemma 4.3 (iv) – the set $\mathcal{U}_{nn}^1$ fails to meet $\mathcal{H}_\chi^\perp(\mathcal{G}_1)$, but $\mathcal{U}_{n1}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) = \mathcal{U}_n^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$ contains an open set in $\mathcal{H}_\chi^\perp(\mathcal{G}_1)$. First we take $\dim \mathcal{G} \cdot \varphi > 2 \ \dim \mathcal{H} \cdot \varphi$. This gives infinite multiplicity on the right side of (4.2), but this time, to match it on the left, we must verify that

$$\dim \mathcal{G}_1 \cdot \psi > 2 \ \dim \mathcal{H} \cdot \psi.$$

In fact, in this case we prove directly that

$$\dim G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) > \dim H \cdot \psi.$$

To achieve that set $2n = \dim \mathcal{G} \cdot \varphi$, $m = \dim \mathcal{H} \cdot \varphi$, so that $2n > 2m$. Then

$$\begin{aligned} \dim G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) &= \dim \mathcal{G} \cdot \varphi \cap \mathcal{H}^\perp(\mathcal{G}) \\ &= \dim \mathcal{G} \cdot \varphi - \dim \mathcal{H} \cdot \varphi \\ &= 2n - m. \end{aligned}$$

Now I claim that

$$G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = G_1 \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}).$$

Indeed, when $\mathcal{U}_{n1}^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) = \mathcal{U}_n^1 \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$, we have

$$\varphi \in \mathcal{U}_{n1} \cap \mathcal{H}_\chi^\perp(\mathcal{G}) \text{ and } g \cdot \varphi \in \mathcal{H}_\chi^\perp(\mathcal{G}) \Rightarrow g \in G_1,$$

because of equation (4.3). But then the surjective projection

$$G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = G_1 \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) \to G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$$

has fiber of dimension at most 1. Therefore

$$\dim G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) \geq 2n - m - 1.$$

On the other hand

$$\dim H \cdot \psi \leq \dim H \cdot \varphi = m.$$

Therefore

$$\dim G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1) > \dim H \cdot \psi.$$

($\mathcal{U}_{n1}b$) We conclude by assuming $\dim \mathcal{G} \cdot \varphi = 2 \ \dim \mathcal{H} \cdot \varphi$, and observing that in this case there is a bijection between the

H-orbits in $G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G})$ and the H-orbits in $G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$.

The critical points already appear in the previous case. The proof that

$$G \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = G_1 \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G})$$

still applies. Hence the restriction map

$$G_1 \cdot \varphi \cap \mathcal{H}_\chi^\perp(\mathcal{G}) \to G_1 \cdot \psi \cap \mathcal{H}_\chi^\perp(\mathcal{G}_1)$$

is an H-equivariant surjection. In fact, it is a bijection of H-orbits. To show this, we take $\varphi \in \mathcal{U}_{n1} \cap \mathcal{H}_\chi^\perp(\mathcal{G})$, $g_1 \cdot \varphi \in \mathcal{H}_\chi^\perp(\mathcal{G})$, $g_1 \cdot \psi = h \cdot \psi$ for some $h \in H$. We produce $h' \in H$ so that $g \cdot \varphi = h' \cdot \varphi$. We have $h^{-1} g \in G_\psi$. So

$$h^{-1} g \cdot \varphi = \varphi_s \quad \text{for some } s \in \mathbb{R}.$$

We finish by showing (as in ($\mathcal{U}b$)) that

$$H_\theta \cdot \varphi = \varphi + \mathcal{G}_1^\perp.$$

In fact

$$2 \ \dim \mathcal{H} \cdot \psi \leq \dim \mathcal{G}_1 \cdot \psi = \dim \mathcal{G} \cdot \varphi - 2 = 2(\dim \mathcal{H} \cdot \varphi - 1)$$
$$\Rightarrow \dim \mathcal{H} \cdot \psi = \dim \mathcal{H} \cdot \varphi - 1.$$

As usual, we have $\mathcal{H}_\psi = \mathcal{H}_\theta$, so $\dim \mathcal{H}_\theta / h_\varphi = 1$. Therefore $H_\theta \cdot \varphi$ is an open connected subset of $\varphi + \mathcal{G}_1^\perp$. The same is true of any other φ_s, hence $H_\theta \cdot \varphi = \varphi + \mathcal{G}_1^\perp$.

This concludes the proof of equal multiplicity in equation (4.2). Since we have already shown equal spectrum, it remains only to prove the equivalence of the

Spectral Measures. Although the proof of equality of measure classes in equation (4.2) is not nearly as difficult as that of the multiplicity, it still requires that we look at the cases (i)-(v) of Theorem 3.3 separately. On the other hand, we can ignore the (a)-(b) dichotomy. Now the measures under consideration are the push-forwards – under the action of H – of Lebesgue measure on the affine spaces $\mathcal{H}_\chi^\perp(\mathcal{G})$ and $\mathcal{H}_\chi^\perp(\mathcal{G}_1)$. But we have a canonical H-equivariant projection

$$\mathcal{H}_\chi^\perp(\mathcal{G}) \to \mathcal{H}_\chi^\perp(\mathcal{G}_1)$$

which factors to a Borel surjection

$$\mathcal{H}_\chi^\perp(\mathcal{G})/H \to \mathcal{H}_\chi^\perp(\mathcal{G}_1)/H$$

that carries the (class of the) push-forward of Lebesgue measure on the first to the (class of the) push-forward of Lebesgue measure on the second. Therefore in cases (i) and (iv) – that is $\mathcal{U}_0$ and $\mathcal{U}_{n1}$ – where $\mathrm{Ind}_{G_1}^G \nu_\psi$ is irreducible, the equality of the measure classes in (4.2) follows immediately from the remarks in the paragraph after equation (4.2). In fact, the same conclusion is true for case (v) (that is $\mathcal{U}$) as well. There we have $\mathrm{Ind}_{G_1}^G \nu_\psi = \pi^+ \oplus \pi^-$, and we augment the projection to

$$\varphi \to (\psi, \ \mathrm{sgn}(\varphi)), \ \psi = \varphi|_{\mathcal{G}_1},$$

where

$$\operatorname{sgn}(\varphi) = \begin{cases} + & \varphi(Y) > s_0 \\ & \varphi(Y) < s_0. \end{cases}$$

The variety $\{\varphi \in \mathcal{H}_\chi^\perp(\mathcal{G}) \cap \mathcal{U}_{n1} : \varphi(Y) = s_0\}$ is of lower dimension and may be ignored. Clearly the projection factors to a Borel surjection of the parameter spaces in equation (4.2) which moves the measure class of $\mathrm{d}\mu^\chi_{G,H}$ to $\delta\mathrm{d}\mu^\chi_{G_1,H}$, where δ is the discrete measure class concentrated on the 2-point fiber over $\psi = \varphi|_{\mathcal{G}_1} \in \mathcal{H}_\chi(\mathcal{G}_1) \cap \mathcal{U}^1_{n1}$.

The remaining cases (ii) and (iii) (that is $\mathcal{U}_{n1}$ and $\mathcal{U}_2$) require more substantial reasoning. First take $\mathcal{U}_2$. Then

$$\operatorname{Ind}_{G_1}^G \nu_\psi = \int^\oplus \pi_{\varphi+s\alpha} \mathrm{d}s.$$

Now suppose $\mathcal{C}$ is a Borel cross-section for the action of H on $\mathcal{H}_\chi^\perp(\mathcal{G}_1)$. For $\psi \in \mathcal{H}_\chi^\perp(\mathcal{G}_1)$, we define $\varphi_\psi \in \mathcal{H}_\chi^\perp(\mathcal{G})$ by $\varphi_\psi|_{\mathcal{G}_1} = \psi$ and $\varphi_\psi(Y) = 0$. Then I claim

$$\mathcal{Q} = \{\varphi_\psi + s\alpha : \psi \in \mathcal{C},\ \ s \in \mathbb{R}\}$$

is a Borel cross-section for the action of H on $\mathcal{H}_\chi^\perp(\mathcal{G})$. Indeed, for any $\psi \in \mathcal{H}_\chi^\perp(\mathcal{G}_1)$, $h \in H$, there is $s_{h,\psi} \in \mathbb{R}$ such that

$$h \cdot \varphi_\psi = \varphi_{h\cdot\psi} + s_{h\cdot\psi}\alpha.$$

Next for any $h \in H$, we have a real number λ_h so that $h \cdot \alpha = \lambda_h \alpha$. Hence if $\varphi \in \mathcal{H}_\chi^\perp(\mathcal{G}), \psi = \varphi|_{\mathcal{G}_1}$ and $h \cdot \psi \in \mathcal{C}$, then

$$\begin{aligned} h \cdot \varphi &= h \cdot (\varphi_\psi + \varphi(Y)\alpha) \\ &= \varphi_{h\cdot\psi} + s_{h,\psi}\alpha + \varphi(Y)\lambda_h\alpha \in \mathcal{Q}. \end{aligned}$$

On the other hand if $\varphi \in \mathcal{Q}$ and $h \cdot \varphi \in \mathcal{Q}$, then

$$\psi = \varphi|_{\mathcal{G}_1} \in \mathcal{C}, \quad h \cdot \psi = h \cdot (\varphi|_{\mathcal{G}_1}) = (h \cdot \varphi)|_{\mathcal{G}_1} \in \mathcal{C} \quad \Rightarrow \quad h \in H_\psi.$$

But in case (iii) we have

$$H_\psi \subset H_\theta \subset G_\theta \subset G_2,$$

which implies $h \cdot \alpha = \alpha$. Also in this case φ_ψ and $\varphi_\psi + s_{h,\psi}\alpha$ cannot be in the same orbit unless $s_{h,\psi} = 0$. Therefore

$$\begin{aligned} h \cdot \varphi = h \cdot (\varphi_\psi + \varphi(Y)\alpha) &= \varphi_{h\cdot\psi} + s_{h,\psi}\alpha + \varphi(Y)\alpha \\ &= \varphi_\psi + \varphi(Y)\alpha = \varphi. \end{aligned}$$

Hence the restriction

$$\varphi_\psi + s\alpha \to \psi, \ \ \mathcal{Q} \to \mathcal{C}$$

is a Borel surjection which takes the measure class of $d\mu^{\chi}_{G,H}$ to $dsd\mu^{\chi}_{G_1,H}$.

In case (ii), the argument is exactly the same until the point where we concluded $h \in H_\psi$. In this case we do not know $H_\psi \subset G_2$, so perhaps $\lambda_h \neq 1$. But we still have that φ_ψ and $\varphi_\psi + s_{h,\psi}\alpha$ cannot be in the same orbit unless $s_{h,\psi} = 0$. Therefore

$$h \cdot \varphi = h \cdot (\varphi_\psi + s_{h,\psi}\alpha) = \varphi_\psi + \varphi(Y)\lambda_h\alpha.$$

Once again the fact that the orbits $(\varphi_\psi)_s$ are distinct forces $\varphi(Y)\lambda_h = \varphi(Y)$. So either $\varphi(Y) = 0$ or $\lambda_h = 1$, either of which implies $h \cdot \varphi = \varphi$. This concludes the proof of equality of the measure classes in equation (4.2), and with it the proof of Theorem 4.1.

5. - The general case

In this section we extend Theorem 4.1 to the full generality asserted in Theorem 2.2. Namely we allow the inducing representation to be an arbitrary irreducible instead of just a character. It turns out that, unlike in sections 3 and 4, the ideas involved in the corresponding step, in the nilpotent case, suffice to handle completely solvable groups (see [4, §6], [9, §3]).

THEOREM 5.1. *We have*

$$\text{(5.1)} \qquad \operatorname{Ind}_H^G \nu_\psi = \int^{\oplus}_{p^{-1}(H\cdot\psi)/H} \pi_\varphi d\mu^{\nu}_{G,H}(\varphi).$$

PROOF. (The reader is referred to section 2 for a review of the notation if necessary.) We utilize the fact that any irreducible representation of a completely solvable group is monomial. In fact, given $\nu_\psi \in \hat{H}$, there exists a real polarization $\mathcal{F}$ for ψ satisfying the Pukanszky condition, that is a subalgebra $\mathcal{F}$ of $\mathcal{H}$ such that

$$\mathcal{H}_\psi \subset \mathcal{F} \subset \mathcal{H} \qquad \psi[\mathcal{F}, \mathcal{F}] = 0$$

$$\dim \mathcal{H}/\mathcal{F} = \dim \mathcal{F}/\mathcal{H}_\psi \qquad K \cdot \psi = \psi + \mathcal{F}^{\perp}(\mathcal{H}) \qquad K = \exp(\mathcal{F}).$$

Then $\nu_\psi = \operatorname{Ind}_K^H \chi_\psi$. Therefore

$$\operatorname{Ind}_H^G \nu_\psi = \operatorname{Ind}_H^G \ \operatorname{Ind}_K^H \chi_\psi = \operatorname{Ind}_K^G \chi_\psi.$$

If we invoke Theorem 4.1, we see that, to prove formula (5.1), we must demonstrate

$$\int_{p^{-1}(H\cdot\psi)/H}^{\oplus} \pi_\varphi \mathrm{d}\mu^{\nu}_{G,H}(\varphi) = \int_{\mathcal{F}_\chi^\perp(\mathcal{G})/K}^{\oplus} \pi_\varphi \mathrm{d}\mu^{\chi}_{G,K}(\varphi), \quad \chi = \chi_\psi. \tag{5.2}$$

Exactly as in the proof of equation (4.2), we must demonstrate equality of spectrum and multiplicity, and equivalence of the spectral measures in (5.2).

Spectrum. We must prove that

$$G \cdot p^{-1}(H \cdot \psi) = G \cdot \mathcal{F}_\chi^\perp(\mathcal{G}). \tag{5.3}$$

One inclusion is obvious. An element from the left side of (5.3) is of the form $g \cdot \varphi$, $\varphi|_{\mathcal{H}} = \psi$. But then $\varphi|_{\mathcal{F}} = \psi|_{\mathcal{F}} = \mathrm{id}\chi$. That is, $g \cdot \varphi$ is also in the right side of (5.3). Conversely, suppose $\varphi \in \mathcal{F}_\chi^\perp(\mathcal{G})$. Then $\varphi|_{\mathcal{F}} = \psi|_{\mathcal{F}}$. But the Pukanszky condition insures the existence of an element $k \in K$ such that $\varphi|_{\mathcal{H}} = k \cdot \psi$. Hence $k^{-1} \cdot \varphi$ is in the left side of (5.3).

Multiplicity. The multiplicity on the left side of (5.3) is

$$n^{\nu}_{\varphi} = \#H\text{-orbits in } G \cdot \varphi \cap p^{-1}(H \cdot \psi).$$

The multiplicity on the right side is

$$n^{\chi}_{\varphi} = \#K\text{-orbits in } G \cdot \varphi \cap \mathcal{F}_\chi^\perp(\mathcal{G}).$$

The equality of n^{ν}_{φ} and n^{χ}_{φ} is a consequence of

PROPOSITION 5.2. *The mapping $K \cdot \varphi \to H \cdot \varphi$ is a bijection of the K-orbits in $G \cdot \varphi \cap \mathcal{F}_\chi^\perp(\mathcal{G})$ onto the H-orbits in $G \cdot \varphi \cap p^{-1}(H \cdot \psi)$.*

This is precisely [4, Prop. 5] or [9, Prop. 3.2]. Both of these are proven under the assumption that G is nilpotent. In fact, nilpotence is unnecessary – the argument is valid in the context of arbitrary exponential solvable groups. Here is a drastically simpler proof than that of [4].

PROOF OF PROPOSITION 5.2. By the observations made above in the equality of spectrum argument, a G-orbit meets $p^{-1}(H \cdot \psi)$ iff it meets $\mathcal{F}_\chi^\perp(\mathcal{G})$. So the map is well-defined. It is surjective as follows. If

$$\varphi' \in G \cdot \varphi \cap p^{-1}(H \cdot \psi),$$

then $h \cdot \varphi'|_{\mathcal{H}} = \psi$ for some $h \in H$. That says

$$h \cdot \varphi'|_{\mathcal{F}} = \psi|_{\mathcal{F}} = -\mathrm{id}\chi.$$

Hence the H-orbit $H \cdot \varphi'$ is the image of the K-orbit $K \cdot (h \cdot \varphi')$. Finally the mapping is injective because of the Pukanszky condition. Indeed, if for $\varphi_1, \varphi_2 \in G \cdot \varphi \cap \mathcal{F}_\chi^\perp(\mathcal{G})$, the orbits $K \cdot \varphi_1$, $K \cdot \varphi_2$ map to the same H-orbit, then there is $h \in H$ so that $h \cdot \varphi_1 = \varphi_2$. In particular $h \cdot \varphi_1|_{\mathcal{H}} = \varphi_2|_{\mathcal{H}}$. If we write $\psi_1 = \varphi_1|_{\mathcal{H}}$, $\psi_2 = \varphi_2|_{\mathcal{H}}$, then since $\varphi_1, \varphi_2 \in \mathcal{F}_\chi^\perp(\mathcal{G})$, we have $\psi_1, \psi_2 \in \mathcal{F}_\chi^\perp(\mathcal{H})$. By the Pukanszky condition, there is an element $k \in K$ so that $k \cdot \psi_1 = \psi_2$. Therefore $k^{-1}h \in H_{\psi_1}$. But by [2, p. 69] $\mathcal{F}$ is also a real polarization for ψ_1. In particular $H_{\psi_1} \subset K$, and so $h \in K$. That is φ_1 and φ_2 are in the same K-orbit.

We complete the proof of Theorem 5.1 by examining the

Spectral Measures. The basic idea is presented in [9, Prop. 3.2]. We first observe that $h \cdot \mathcal{F}_\chi^\perp(\mathcal{G}) = p^{-1}(H \cdot \psi)$. The latter's canonical measure class is determined as a fiber space

$$\begin{array}{c} \mathcal{H}^\perp(g) \to p^{-1}(H \cdot \psi) \\ \downarrow \\ H \cdot \psi, \end{array}$$

with the H-invariant measure on the base and Lebesgue measure on the fiber. But $H \cdot \mathcal{F}_\chi^\perp(\mathcal{G})$ can also be realized as a fiber space

$$\begin{array}{c} \mathcal{F}_\chi^\perp(\mathcal{G}) \to H \cdot \mathcal{F}_\chi^\perp(\mathcal{G}) \\ \downarrow \\ K\backslash H, \end{array}$$

where the projection sends $\varphi \to Kh$ if $h \cdot \varphi|_{\mathcal{F}} = -\mathrm{id}x$. Again K/H has H-invariant measure and the affine fiber has Lebesgue measure. But these two fiber measures are equivalent. Indeed the first fiber space is naturally identified (measure-theoretically) to

$$\mathcal{G}/\mathcal{H} \times \mathcal{H}/\mathcal{H}_\psi;$$

the latter to

$$\mathcal{G}/\mathcal{F} \times \mathcal{H}/\mathcal{F}.$$

Moreover the two invariant measure classes agree with the Lebesgue measure upon identification. But we have a canonical duality between $\mathcal{H}/\mathcal{F}$ and $\mathcal{F}/\mathcal{H}_\psi$, and it follows that the Lebesgue measure classes are the same. We observe then that the map (composition of injection and quotient by H)

$$\mathcal{F}_\chi^\perp(\mathcal{G}) \to H \cdot \mathcal{F}_\chi^\perp(\mathcal{H}) \to H \cdot \mathcal{F}_\chi^\perp(\mathcal{G})/H$$

factors to a bijection

$$\mathcal{F}_\chi^\perp(\mathcal{G})/K \to H \cdot \mathcal{F}_\chi^\perp(\mathcal{G})/H$$

(using the same argument as in the injectivity part of Proposition 5.2). Thus an application of [10, Prop. 4.2] gives precisely the equivalence of the respective push-forwards.

6. - Examples

We give here several examples to illustrate Theorems 3.3, 4.1 and 5.1. For each completely solvable Lie algebra $\mathcal{G}$, we list generators and non-zero bracket relations. We also list co-adjoint orbits Ω and parameters for cross-sections.

(1) $(ax+b)$-algebra. $\mathcal{G} = \mathrm{sp}\{A, X\}, [A, X] = X,\ \varphi = \varphi_{\alpha,\xi} = \alpha A^* + \xi X^* \in \mathcal{G}^*$

$$\Omega_\alpha = \alpha A^*,\ \ \alpha \in \mathbb{R},$$
$$\Omega^+ = \{\alpha A^* + \xi X^* : \alpha \in \mathbb{R},\ \ \xi > 0\},$$
$$\Omega^- = \{\alpha A^* + \xi X^* : \alpha \in \mathbb{R},\ \ \xi < 0\},$$
$$\mathcal{G}_1 = \mathrm{sp}\{A\},\ \ \mathcal{G}_0 = \{0\},\ \ \psi = \alpha A^*,\ \ \chi = \chi_\alpha = e^{i\alpha A^*},\ \ \theta = 0,\ \ \mathcal{G}_\theta = \mathcal{G},$$
$$\mathrm{Ind}_{G_1}^G \chi_\alpha = \pi_{\Omega^+} \oplus \pi_{\Omega^-},\ \text{ for any }\ \alpha \in \mathbb{R},$$

since

$$p^{-1}(\alpha A^*) = \{\alpha A^* + \xi X^* : \xi \in \mathbb{R}\}\ \text{ and}$$
$$G \cdot \varphi_{0,\pm 1} \cap p^{-1}(\alpha A^*) = G_1 \cdot \varphi_{0,\pm 1}.$$

(2) $\mathcal{G} = \mathrm{sp}\{A, X, Y, Z\},\ \ [X, Y] = Z, [A, X] = X, [A, Y] = Y, [A, Z] = 2Z,$

$$\varphi = \varphi_{\alpha,\xi,\eta,\varsigma} = \alpha A^* + \xi X^* + \eta Y^* + \varsigma Z^* \in \mathcal{G}^*,$$
$$(\exp aA\ \ \exp xX\ \ \exp yY\ \ \exp zZ)^{-1} \cdot \varphi_{\alpha,\xi,\eta,\varsigma}$$
$$= \varphi_{\alpha-\xi e^a x-\eta e^a y-\varsigma(e^{2a}xy+e^a z),\ e^a(\xi-y\varsigma),\ e^a(\eta+x\varsigma),\ e^{2a}\varsigma},$$
$$\Omega_\alpha = \alpha A^*,\ \ \alpha \in \mathbb{R},$$
$$\Omega_{\xi,\eta} = \{\alpha A^* + r(\xi X^* + \eta Y^*) : \alpha \in \mathbb{R},\ \ r > 0\},\ \ \xi^2 + \eta^2 = 1,$$
$$\Omega_\pm = \{\alpha A^* + \xi X^* + \eta Y^* + \varepsilon r Z^* : \alpha, \xi, \eta \in \mathbb{R},\ \ r > 0\},\ \ \varepsilon = \pm 1,$$
$$\mathcal{G}_1 = \mathrm{sp}\{A, X, Z\},\ \ \mathcal{G}_0 = \mathrm{sp}\{X, Z\},$$

(a) $\psi_\varsigma = \varsigma Z^*,\ \ \varsigma \neq 0,\ \ \theta = \varsigma Z^*,\ \ \mathcal{G}_\theta = \mathcal{G}_0,$
$\mathrm{Ind}_{G_1}^G \nu_{\psi_\varsigma} = \mathrm{Ind}_{G_0}^G \gamma_\theta$ is irreducible and equivalent to $\pi_{\Omega_{\mathrm{sgn}(\varsigma)}}$,

(b) $\psi_\xi = \xi X^*,\ \ \xi \neq 0,\ \ \theta = \xi X^*,\ \ \mathcal{G}_\theta = \mathcal{G}_0 + \mathbb{R}Y = \mathcal{G}_2$
$\mathrm{Ind}_{G_1}^G \nu_{\psi_\xi} = \mathrm{Ind}_{G_0}^G \gamma_\theta = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi_{\xi,s}} \mathrm{d}s.$

Note: For fixed ξ, the spectrum "sees only half" of the 2-dimensional orbits.

(3) $\mathcal{G} = \mathrm{sp}\{A, X, Y, Z\},\ \ [X, Y] = Z,\ \ [A, X] = X, [A, Y] = -Y,$

$$\varphi = \varphi_{\alpha,\xi,\eta,\varsigma} = \alpha A^* + \xi X^* + \eta Y^* + \varsigma Z^* \in \mathcal{G}^*,$$

$$(\exp aA \exp xX \exp yY \exp zZ)^{-1} \cdot \varphi_{\alpha,\xi,\eta,\varsigma}$$

$$= \varphi_{\alpha-\xi e^a x+\eta e^{-a} y+\varsigma xy, \xi e^a-\varsigma y, \eta e^{-a}+\varsigma x, \varsigma},$$

$$\Omega_\alpha = \alpha A^*, \quad \alpha \in \mathbb{R},$$

$$\Omega_{\xi,\eta} = \{\alpha A^* + r\xi X^* + r^{-1}\eta Y^* : \alpha \in \mathbb{R},\ r > 0\}, \quad (\xi,\eta) \in \mathcal{C},$$

a Borel cross-section for the action of $\mathbb{R}$ on $\mathbb{R}^2$ by $a \cdot (\xi,\eta) = (e^a\xi, e^{-a}\eta)$,

$$\Omega(\alpha,\xi) = G \cdot \varphi_{\alpha,0,0,\varsigma}, \quad \alpha \in \mathbb{R}, \quad \varsigma \neq 0$$

the generic orbits.

(a) $\mathcal{H} = \{0\}, \ \chi = 1$,

$\operatorname{Ind}_H^G \chi$ is the regular representation $= \int_{\mathcal{G}^*}^{\oplus} \pi_\varphi d\varphi = \infty \int^{\oplus} \pi_{\Omega(\alpha,\varsigma)} d\alpha d\varsigma$,

since the non-generic orbits account for measure zero on $\mathcal{G}^*$, and the H-orbits are 0-dimensional on the 2-dimensional variety $\Omega(\alpha,\varsigma)$.

(b) $\mathcal{H} = \operatorname{sp}\{X\}, \ \chi = \chi_\xi = e^{i\xi X^*}$,

$\operatorname{Ind}_H^G \chi_\xi = \int^{\oplus} \pi_{\Omega(\alpha,\varsigma)} d\alpha d\varsigma$,

since the generic orbits lie over $\xi X^* \in \mathcal{H}^*$ and

$$\Omega(\alpha,\varsigma) \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = \{\varphi_{\alpha+x\xi,\xi,x\varsigma,\varsigma} : x \in \mathbb{R}\} = H \cdot \varphi_{\alpha,\xi,0,\varsigma}.$$

(c) $\mathcal{H} = \operatorname{sp}\{A\}, \ \chi = \chi_\alpha = e^{i\alpha A^*}$,

$\operatorname{Ind}_H^G \chi_\alpha = 2\int^{\oplus} \pi_{\Omega(\alpha',\varsigma)} d\alpha' d\varsigma$,

since the generic orbits lie over $\alpha A^* \in \mathcal{H}^*$, and

$$\Omega(\alpha',\varsigma) \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = \{\varphi_{\alpha,-\varsigma y,\varsigma x,\varsigma} : xy = (\alpha-\alpha')\varsigma^{-1}, x, y \in \mathbb{R}\},$$

on which H has two open orbits – except when $\alpha = \alpha'$ wherein H has four open orbits. The latter is measure zero so does not figure in the direct integral.

(d) $\mathcal{H} = \operatorname{sp}\{Z\}, \ \chi = \chi_\varsigma = e^{i\varsigma Z^*}$,

$$\operatorname{Ind}_H^G \chi_\varsigma = \begin{cases} \infty \int_{\mathcal{C}}^{\oplus} \pi_{\xi,\eta} dc & \varsigma = 0, \quad dc = \text{canonical measure on } \mathcal{C} \\ \infty \int^{\oplus} \pi_{\Omega(\alpha,\varsigma)} d\alpha & \varsigma \neq 0, \end{cases}$$

since in the first case we get the regular representation of G/Z; and in the latter case the generic orbits $\Omega(\alpha,\varsigma), \alpha \in \mathbb{R}$, cover $\varsigma Z^* \in \mathcal{H}^*$, while the H-orbits are of dimension 0 on the 2-dimensional variety $\Omega(\alpha, \varsigma) \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = \Omega(\alpha, \varsigma)$.

(e) $\mathcal{H} = \mathrm{sp}\{A, X\}, \ \chi = \chi_\alpha = e^{i\alpha A}$,

$\mathrm{Ind}_H^G \chi_\alpha = \int^{\oplus} \pi_{\Omega(\alpha,\varsigma)} \mathrm{d}\varsigma$,

since only the generic orbits $\Omega(\alpha, \varsigma)$, $\varsigma \neq 0$, lie over $\alpha A^* \in \mathcal{H}^*$, and $\Omega(\alpha, \varsigma) \cap \mathcal{H}_\chi^\perp(\mathcal{G}) = \{\varphi_{(\alpha,0,\eta,\varsigma)} : \eta \in \mathbb{R}\} = H \cdot \varphi_{(\alpha,0,0,\varsigma)}$.

(f) Now set $\mathcal{G}_1 = \mathrm{sp}\{A, X, Z\}$, $\mathcal{G}_0 = \mathrm{sp}\{X, Z\}$,

$\psi = \varphi|_{\mathcal{G}_1} = \alpha A^* + \xi X^* + \varsigma Z^*$, $\theta = \varphi|_{\mathcal{G}_0} = \xi X^* + \varsigma Z^*$, $\mathcal{G}_\gamma = \mathcal{G}_0 + \mathcal{G}_\theta$.

Then

$$\mathcal{G}_\gamma = \begin{cases} \mathcal{G} & \xi = \varsigma = 0 \\ \mathcal{G}_2 & \xi \neq 0, \ \varsigma = 0 \\ \mathcal{G}_1 & \xi = 0, \ \varsigma \neq 0 \\ \mathcal{G}_0 + \mathbb{R}(\varsigma A + \xi X) & \xi\varsigma \neq 0. \end{cases}$$

Therefore

$$\mathrm{Ind}_{G_1}^G \nu_\psi = \begin{cases} \pi_{\Omega_{0,1}} \oplus \pi_{\Omega_{0,-1}} & \xi = \varsigma = 0 \\ \int^{\oplus} \pi_{\Omega_{\xi,\eta}} \mathrm{d}\eta & \xi \neq 0, \ \varsigma = 0 \\ \pi_{\Omega(\alpha,\varsigma)} & \xi = 0, \ \varsigma \neq 0 \\ \int^{\oplus} \pi_{\Omega(\alpha',\varsigma)} \mathrm{d}\alpha' & \xi\varsigma \neq 0. \end{cases}$$

Final Remark. One can construct examples of mixed finite multiplicity by combining [4, Expl. 4] with [10, Thm. 5.1] – so e.g. $G = AN$, $H = AM$, N/M as in [4, Expl. 4], $A \cong \mathbb{R}$ acting semisimply on N preserving M so that $A_\gamma = \{1\}$, $\gamma \in \hat{N}_M$ (see [10]). I suspect, but do not know for sure, that – as in the nilpotent case [5] – the parity of the multiplicity in Theorem 5.1 must always be constant.

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