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Isabelle Moutoussamy
Laurent Veron

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Source Type Positive Solutions of Nonlinear Parabolic Inequalities

ISABELLE MOUTOUSSAMY - LAURENT VERON

0. - Introduction

In a recent paper Richard and Veron [20] noticed that, if h is a continuous nondecreasing function such that

$$\int_0^1 h(r^{2-N})\ r^{N-1}\ \mathrm{d}r < +\infty \tag{0.1}$$

for some integer $N \geq 3$ and $u \in C^2(B_1(0)\backslash\{0\})$ is a nonnegative function such that

$$\Delta u \leq h(u) \tag{0.2}$$

in $B_1(0)\backslash\{0\}$, where $B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, then

(i) *either* $r^{N-2}\ u(r,\cdot)$ *converges in measure on* S^{N-1} *to some* $\alpha \geq 0$ *as* r *tends to* 0, *or*

(ii) $$\lim_{x \to 0}\ |x|^{N-2}\ u(x) = +\infty.$$

Their proof was based upon an elegant result due to Brézis and Lions [7] on isolated singularities of linear elliptic inequalities. The consequence of Richard and Veron's result was a unification of the description of isolated singularities of nonnegative solutions of

$$\Delta u = \pm\ u^{\nu} \tag{0.3}$$

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when $1 < \nu < \frac{N}{N-2}$. The aim of this article is to give the parabolic version of Richard and Veron's work and to give applications to source type positive solutions of semilinear heat equations.

Let us consider a continuous nondecreasing function g defined on $\mathbb{R}^+$, such that $g(0) = 0$ and

$$g(E) \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^+), \tag{0.4}$$

where $E(x,t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right)$ and N is a positive integer. Assume $u \in C^{2,1}(\overline{Q}\backslash\{(0,0)\})$, where $Q = B_1(0) \times (0,T)$, $T > 0$, satisfies $u(x,0) = 0$ in $B_1(0)\backslash\{0\}$ and

$$-u_t + \Delta u \leq g(u), \tag{0.5}$$

in $\overline{Q}\backslash\{(0,0)\}$ and let $\tilde{u}$ be the extension of u by 0 outside $\overline{Q}$. Then our generic isotropy result is the following

(i) *either there exists* $\gamma \geq 0$ *such that* $k^{N/2}\ \tilde{u}(\sqrt{k}x, kt)$ *converges to* $\gamma E(x,t)$ *locally in measure in* $\mathbb{R}^N \times \mathbb{R}^+$ *when* k *tends to* 0, *or*

$$\lim_{t\to 0}\ t^{N/2}\ u(x,t) = +\infty, \tag{ii}$$

uniformly on any set $\mathbf{E}_\alpha \cap Q$, *where* $\mathbf{E}_\alpha = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : |x| \leq \alpha\sqrt{t}\}$, $\alpha > 0$.

We first apply this result to semilinear heat equations with absorption of the following type

$$u_t - \Delta u + g(u) = 0, \tag{0.6}$$

where g is as above.

Assume $u \in C^{2,1}(Q\backslash\{(0,0)\})$ *is a nonnegative solution of* (0.6) *in* $\overline{Q}\backslash\{(0,0)\}$ *vanishing on* $\overline{B}_1(0) \times \{0\}\backslash\{(0,0)\}$. *Then either* (ii) *holds, or*

(iii) *there exists* $\gamma \geq 0$ *such that* $t^{N/2}[u(x,t) - \gamma E(x,t)]$ *converges to* 0 *uniformly on any set* $\mathbf{E}^c_\alpha \cap Q$, *when* t *tends to* 0,

$$\left[\mathbf{E}^c_\alpha = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : |x| \geq \alpha\ \sqrt{t}\}, \quad \alpha > 0\right].$$

When $g(u) = u^\nu$, with $1 < \nu < \frac{N+2}{N}$, we derive a new proof of Oswald classification's result [19].

In the other sign case, that is

$$u_t - \Delta u = g(u), \tag{0.7}$$

we first apply our "basic isotropy result" when g is just a continuous nonnegative function vanishing at 0 (we do not assume monotonicity or (0.4)) and we prove the following.

Let $u \in C^{2,1}(\overline{Q}\backslash\{(0,0)\})$ be a nonnegative solution of (0.7) in $\overline{Q}\backslash\{(0,0)\}$ vanishing on $\overline{B}_1(0) \times \{0\}\backslash\{(0,0)\}$, then there exists $\gamma \geq 0$ such that $k^{N/2}\ \tilde{u}(\sqrt{k}x, kt) - \gamma\ E(x,t)$ converges to 0 in $L^\infty_{\text{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^N))$ when k tends to 0. Moreover $g(u) \in L^1(Q)$ and u solves (0.7) with $\gamma\delta(x)$ as an initial data ($\delta(x)$ = Dirac mass at 0).

When g is convex (for simplicity) and satisfies

$$g(\phi) \in L^1(Q) \text{ implies } \int_0^T \int_{B_1(0)} [g'(\phi)\ dx]^{q/p}\ dt < +\infty \tag{0.8}$$

for $p, q > 1$ and $\frac{N}{2p} + \frac{1}{q} < 1$, then we can apply Aronson-Serrin's theory on quasilinear parabolic equations [1]; the previous convergence result is improved and we get

$$\lim_{t\to 0}\ t^{N/2}[u(x,t) - \gamma E(x,t)] = 0\ \cdot \tag{0.9}$$

uniformly on any set $\mathbf{E}^c_\alpha \cap Q$; and u is bounded if $\gamma = 0$.

In the particular case of the following equation

$$u_t - \Delta u = u^\nu \tag{0.10}$$

with $1 \leq \nu < \frac{N+2}{N}$, we prove that

$$\lim_{t\to 0}\ \|u(\cdot,t) - \gamma E(\cdot,t)\|_{L^\infty(B_1(0))} = 0. \tag{0.11}$$

We end our paper with an appendix where we present the construction and some properties of the solutions of

$$\begin{cases} u_t - \Delta u \pm g(u) = 0 & \text{in } Q \\ u(x,0) = \gamma\delta(x), \quad u = 0 & \text{on } \partial B_1(0) \times (0,T), \end{cases} \tag{0.12}$$

where g is nondecreasing and satisfies (0.4).

Our paper is organised as follows:

1. - Isotropic singularities of parabolic inequalities
2. - Source type solutions of semilinear heat equations
3. - Appendix.

1. - Isotropic singularities of parabolic inequalities

Throughout the paper we assume that $N \geq 1$ and we shall use the following notations

$$B_R(a) = \{x \in \mathbb{R}^N : |x-a| < R\},$$
$$Q = B_1(0) \times (0,T),$$
$$E(x,t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right),$$
$$\delta(x) = \text{Dirac mass at } (0,0).$$

We first prove the following parabolic Brézis-Lions' type result (see [7] for the elliptic case).

THEOREM 1.1. *Assume* $\Psi \in L^1(Q)$ *and* $\omega \in C^0(\overline{Q}\backslash\{(0,0)\})$ *such that* $\omega_t - \Delta\omega \in L^1_{\text{loc}}(\overline{Q}\backslash\{(0,0)\})$ *and satisfy*

$$\begin{cases} \omega \geq 0 & \text{in } Q, \\ \omega(x,0) = 0 & \text{in } B_1(0)\backslash\{0\}, \\ -\omega_t + \Delta\omega \leq \Psi & \text{a.e. in } Q. \end{cases} \tag{1.1}$$

Then $\omega \in L^\infty[(0,T); L^1(B_1(0))]$ *and there exist* $\beta \geq 0$ *and* $\Phi \in L^1(Q)$ *such that*

$$\begin{cases} -\omega_t + \Delta\omega = \Phi & \text{in } Q, \\ \omega(x,0) = \beta\ \delta(x). \end{cases} \tag{1.2}$$

PROOF. Let ϕ_1 be the first eigenfunction of $-\Delta$ in $H^1_0(B_1(0))$ normalized by $\phi_1(0) = 1$ and λ_1 the corresponding eigenvalue. Then, for $0 < s < t \leq T$, we have by an easy approximation argument

$$\begin{aligned} &-\left[\int_{B_1(0)} \omega(x,\sigma)\phi_1(x)\mathrm{d}x\right]_s^t - \lambda_1 \int_s^t \int_{B_1(0)} \omega\ \phi_1\ \mathrm{d}x\ \mathrm{d}\sigma \\ &-\int_s^t \int_{\partial B_1(0)} \omega\ \frac{\partial\phi_1}{\partial\nu}\ \mathrm{d}S\ \mathrm{d}\sigma \leq \int_s^t \int_{B_1(0)} \Psi\ \phi_1\ \mathrm{d}x\ \mathrm{d}\sigma. \end{aligned} \tag{1.3}$$

If we set $X(t) = \int_{B_1(0)} \omega(x,t)\ \phi_1(x)\ dx$, as $\frac{\partial \phi_1}{\partial \nu} < 0$, we have

$$\frac{d}{dt}\ \left[e^{\lambda_1 t}\ X(t)\right] \geq -e^{\lambda_1 t} \int_{B_1(0)} \Psi(x,t)\ dx$$

in $\mathbf{D}'((0,T])$, which implies that

$$t \mapsto e^{\lambda_1 t} \left[X(t) + \int_0^t \int_{B_1(0)} |\Psi(x,\sigma)|\ dx\ d\sigma\right]$$

is nondecreasing; hence there exists $A \geq 0$ such that

$$A = \lim_{t\downarrow o}\ e^{\lambda_1 t} \left[X(t) + \int_0^t \int_{B_1(0)} |\Psi| dx\ d\sigma\right] = \lim_{t\downarrow 0} \int_{B_1(0)} \omega(x,t)\ \phi_1(x)\ dx \tag{1.4}$$

and $\int_{B_1(0)} \omega(x,t)\ dx$ is bounded for $t \in (0,T]$. Henceforth there exist a positive measure μ and a sequence $\{t_n\}$, $t_n \underset{n\to\infty}{\longrightarrow} 0$ such that $\omega(\cdot,t_n) \rightharpoonup \mu$ in $\mathbf{M}_B(B_1(0))$; μ is concentrated at 0, so

$$\mu = A\ \delta(x) = \lim_{t\downarrow 0}\ \omega(\cdot,t) \tag{1.5}$$

in weak sense. Let h be $-\omega_t + \Delta\omega - \Psi$, then $h \geq 0$ a.e. in Q and let $\phi \in \mathbf{D}(B_1(0))$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ in some neighbourhood of 0. Then, for any $0 < t < T$, we have

$$\int_t^T \int_{B_1(0)} h\ \phi\ dx\ dt = \int_t^T \int_{B_1(0)} \omega(\phi_t + \Delta\phi)\ dx\ dt$$
$$- \int_t^T \int_{B_1(0)} \Psi\ \phi\ dx\ dt - \left[\int_{B_1(0)} \omega(x,\sigma)\ \phi(x)\ dx\right]_t^T.$$

Letting t tend to 0 implies

$$\iint_Q h\ \phi\ dx\ dt = \iint_Q \omega(\phi_t + \Delta\phi)\ dx\ dt$$
$$- \iint_Q \Psi\ \phi\ dx\ dt + A\ \phi(0) - \int_{B_1(0)} \omega(x,T)\ \phi(x)\ dx.$$

As a consequence $-\omega_t + \Delta\omega = \Phi$ and $\Phi \in L^1(Q)$ and we get (1.2).

REMARK 1.1. The result is still true if ω satisfies

$$-\omega_t + \Delta\omega + a\ \omega \leq \Psi \tag{1.6}$$

for some $a \in L^\infty(Q)$. Moreover the initial data $\omega(x,0) = 0$, for $x \neq 0$, can be replaced by the weaker one

$$\omega(x,0) = \varsigma(x), \tag{1.7}$$

for $x \neq 0$, where $\varsigma \in C^0(\overline{B}_1(0))$.

Our main result is the following.

THEOREM 1.2. *Assume g is continuous and nondecreasing on $[0,+\infty)$, $g(0) = 0$ such that*

$$g(E) \in L^1(Q), \tag{1.8}$$

and $u \in C^{2.1}(\overline{Q}\backslash\{(0,0)\})$ satisfies

$$\begin{cases} u \geq 0 & \text{in } Q, \\ u(x,0) = 0 & \text{in } B_1(0)\backslash\{0\} \\ -\ u_t + \Delta u \leq g(u) & \text{in } \overline{Q}\backslash\{(0,0)\}. \end{cases} \tag{1.9}$$

Then we have the following alternatives:

(i) *either there exists $\gamma \geq 0$ such that $k^{N/2}\ \tilde{u}(\sqrt{k}x, kt)$ converges to $\gamma E(x,t)$ locally in measure in $\mathbb{R}^N \times \mathbb{R}^+$, when k tends to 0, and where $\tilde{u}$ is the extension of u by 0 in $\overline{Q}^c$, or*

(ii) $$\lim_{t\downarrow 0} t^{N/2}\ u(x,t) = +\infty$$

uniformly on $\mathbf{E}_\alpha \cap Q$, where $\mathbf{E}_\alpha = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : |x| \leq \alpha\sqrt{t}\}$, $\alpha > 0$.

PROOF. For any $\lambda \geq 0$, let v_λ be the solution of

$$\begin{cases} v_{\lambda t} - \Delta\ v_\lambda + g(v_\lambda) = 0 & \text{in } Q, \\ v_\lambda(x,0) = \lambda\ \delta(x),\ v_\lambda(x,t) = 0 & \text{in } \partial B_1(0) \times [0,T], \end{cases} \tag{1.10}$$

(see Appendix) and for $\delta > 0$, we set

$$p_\delta(t) = p(t) = \begin{cases} |t| - \dfrac{\delta}{2} & \text{if } |t| \geq \delta, \\ \dfrac{t^2}{2\delta} & \text{if } |t| \leq \delta. \end{cases}$$

We define $\omega_\delta = \frac{1}{2}\ [u + v_\lambda - p(u - v_\lambda)]$ and $\omega^\lambda = \inf(u, v_\lambda)$.

Hence $\omega^\lambda = \frac{1}{2}\ (u + v_\lambda - |u - v_\lambda|)$ and

$$0 \le \omega^\lambda \le \omega_\delta \le \omega^\lambda + \frac{\delta}{4}.$$

Step 1. There exist $\beta = \beta(\lambda) \ge 0$ and $\Phi \in L^1(Q)$ such that

$$\begin{cases} \omega_{\delta t} - \Delta\ \omega_\delta = \Phi & \text{in } Q, \\ \omega_\delta(x, 0) = \beta\ \delta(x). \end{cases} \tag{1.11}$$

We have immediately

$$\begin{aligned} -\omega_{\delta t} + \Delta\omega_\delta = &-\frac{1}{2}\ [u_t + v_{\lambda t} - p'(u - v_\lambda)(u_t - v_{\lambda t})] + \frac{1}{2}\ \Delta(u + v_\lambda) \\ &- \frac{1}{2}\ p'(u - v_\lambda)\ \Delta(u - v_\lambda) - \frac{1}{2}\ p''(u - v_\lambda)\ |\nabla(u - v_\lambda)|^2 \end{aligned}$$

and from convexity

$$\begin{aligned} -\omega_{\delta t} + \Delta\omega_\delta \le &-\frac{1}{2}(u_t + v_{\lambda t}) + \frac{1}{2}\ \Delta(u + v_\lambda) \\ &+ \frac{1}{2}\ p'(u - v_\lambda)[u_t - v_{\lambda t} - \Delta(u - v_\lambda)] = F. \end{aligned} \tag{1.12}$$

We now define Q_i $(i = 1, 2, 3)$ by

$$\begin{aligned} Q_1 &= \{(x, t) \in \overline{Q}\backslash\{(0, 0)\} : (u - v_\lambda)(x, t) > \delta\}, \\ Q_2 &= \{(x, t) \in \overline{Q}\backslash\{(0, 0)\} : (u - v_\lambda)(x, t) < -\delta\}, \\ Q_3 &= \{(x, t) \in \overline{Q}\backslash\{(0, 0)\} : |u - v_\lambda|(x, t) \le \delta\}. \end{aligned}$$

On Q_1 we have $p'(u - v_\lambda) = 1$ and

$$F = -v_{\lambda t} + \Delta v_\lambda = g(v_\lambda) \le g(v_\lambda + \delta) = g\left(\omega_\delta + \frac{3}{4}\ \delta\right).$$

On Q_2 we have $p'(u - v_\lambda) = -1$ and

$$F = -u_t + \Delta u \le g(u) \le g\left(\omega_\delta + \frac{3}{4}\ \delta\right).$$

On Q_3, $p'(u-v) = \dfrac{u-v_\lambda}{\delta}$ and

$$\begin{aligned} F = &-\frac{1}{2}\left[u_t\left(1-\frac{u-v_\lambda}{\delta}\right)+v_{\lambda t}\left(1+\frac{u-v_\lambda}{\delta}\right)\right] \\ &+\frac{1}{2}\left(1-\frac{u-v_\lambda}{\delta}\right)\Delta u+\frac{1}{2}\left(1+\frac{u-v_\lambda}{\delta}\right)\Delta v \\ &=\frac{1}{2}\left(1-\frac{u-v_\lambda}{\delta}\right)(\Delta u-u_t)+\frac{1}{2}\left(1+\frac{u-v_\lambda}{\delta}\right)(\Delta v_\lambda-v_{\lambda t}) \\ &\leq\frac{1}{2}\left(1-\frac{u-v_\lambda}{\delta}\right)g(u)+\frac{1}{2}\left(1+\frac{u-v_\lambda}{\delta}\right)g(v_\lambda). \end{aligned}$$

By continuity of g and the mean value theorem, there exists $\theta=\theta(x,t)\in[0,1]$ such that

$$-\omega_{\delta t}+\Delta\omega_\delta \leq F \leq g[\theta u+(1-\theta)\ v_\lambda] \tag{1.13}$$

and clearly $\theta u+(1-\theta)\ v_\lambda \leq \omega_\delta+\delta$. As a consequence we get

$$-\omega_{\delta t}+\Delta\omega_\delta \leq g(\omega_\delta+\delta) \tag{1.14}$$

in $\overline{Q}\backslash\{(0,0)\}$. As $v_\lambda \leq \lambda E$ and $g(cE+d)\in L^1(Q)$, for any c and $d\geq 0$ (from (1.8)), $g(\omega_\delta+\delta)\in L^1(Q)$ which implies (1.11).

Step 2. If $\tilde{\omega}_\lambda$ is the extension of ω^λ by 0 in $\overline{Q}^c$, then we prove that

$$\lim_{k\downarrow 0} k^{N/2}\ \tilde{\omega}^\lambda(\sqrt{k}x,kt)=\beta(\lambda)\ E(x,t) \tag{1.15}$$

in $L^\infty_{\mathrm{loc}}[0,+\infty;L^1(\mathbb{R}^N)]$.

As $0\leq\omega^\lambda\leq\omega_\delta\leq\omega^\lambda+\frac{\delta}{4}$ and supp. $\tilde{\omega}_\lambda(\cdot,t)\subset B_1(0)$, it is sufficient to prove (1.15) with ω^λ replaced by ω_δ. Let $E^*(x,t)$ be the solution of

$$\begin{cases} E^*_t-\Delta E^*=0 & \text{in } Q \\ E^*(x,0)=\delta(x), E^*(x,t)=0 & \text{on } \partial B_1\times[0,T], \end{cases} \tag{1.16}$$

then one has immediately

$$E^*(x,t)\leq E(x,t)\leq E^*(x,t)+\max_{0<\tau\leq t}\left(\frac{1}{4\pi\tau}\right)^{N/2} e^{-1/4\tau} \tag{1.17}$$

in $B_1(0)\times(0,+\infty)$. Let ψ_δ be $\omega_\delta-\beta(\lambda)E^*$; it satisfies

$$\begin{cases} \psi_{\delta t}-\Delta\psi_\delta=\Phi & \text{in } Q, \\ \psi_\delta(x,0)=0,\ \ \psi_\delta(x,t)=0 & \text{on } \partial B_1(0)\times[0,T]. \end{cases} \tag{1.18}$$

As $\Phi \in L^1(Q)$, we have classically

$$\lim_{t \downarrow 0} \int_{B_1(0)} |\psi_\delta(x,t)| \; dx = 0. \tag{1.19}$$

If we set $x = \sqrt{k}y$, $t = k\tau$ in (1.19), we get

$$\lim_{k \to 0} \int_{B_{1/k}(0)} k^{N/2} \; |\psi_\delta(\sqrt{k}y, k\tau)| \; dy = 0. \tag{1.20}$$

As

$$k^{N/2} \; \omega_\delta(\sqrt{k}x, kt) = \beta \; k^{N/2} \; \tilde{E}^*(\sqrt{k}x, kt) + k^{N/2} \; \psi_\delta(\sqrt{k}y, kt)$$

we deduce (using the same extensions $\tilde{\omega}_\delta$, $\tilde{E}^*$ for ω_δ and E^*)

$$\lim_{k \to 0} \max_{0<\tau<t} \|k^{N/2} \; \omega_\delta(\sqrt{k} \; \cdot \; , k\tau) - \beta \; E(\; \cdot \; , \tau)\|_{L^1(R^N)} = 0 \tag{1.21}$$

from (1.17), (1.20). This implies (1.15).

Step 3. End of the proof. Let $\tilde{v}_\lambda$ be the extension of v_λ by 0 outside $\overline{Q}$, then from Proposition 3.1 (see Appendix) we know that

$$\lim_{k \to 0} k^{N/2} \; \tilde{v}_\lambda(\sqrt{k}x, kt) = \lambda E(x,t) \tag{1.22}$$

in $L^\infty_{\text{loc}}[0, +\infty; L^1(R^N)]$ and uniformly on $\mathbf{E}^c_\alpha \cap \mathbb{R}^N \times [0, S]$, $\alpha, S > 0$. Moreover $\lambda \mapsto v_\lambda$ is nondecreasing and it is the same with $\lambda \mapsto \omega^\lambda$ and $\lambda \mapsto \beta(\lambda)$. We shall distinguish two cases.

Case 1: $\lim_{\lambda \to +\infty} \beta(\lambda) = \gamma \in [0 + \infty)$. We choose $\lambda > \gamma$. Let $\{k_n\}$ be any sequence converging to 0. From Step 2, we know that there exists a subsequence $\{k_{n_j}\}$ converging to 0 such that

$$\lim_{n_j \to \infty} k_{n_j}^{N/2} \; \tilde{\omega}_\lambda \; (\sqrt{k_{n_j}}x, k_{n_j}t) = \beta(\lambda) \; E(x,t) \tag{1.23}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^+$. Moreover $\beta(\lambda) \leq \gamma < \lambda$. Henceforth we deduce from (1.22) that

$$\lim_{n_j \to \infty} k_{n_j}^{N/2} \; \tilde{u} \; (\sqrt{k_{n_j}}x, k_{n_j}t) = \beta(\lambda) \; E(x,t) \tag{1.24}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^+$. In order to prove that $\beta(\lambda)$ is independent of $\lambda \in (\gamma, +\infty)$, we take $\lambda' \in (\gamma, +\infty) \backslash \{\lambda\}$ and there exists a subsequence $\{k_{n_{j_\ell}}\}$ of $\{k_{n_j}\}$ converging to 0 and such that

$$\lim_{n_{j_\ell} \to \infty} k_{n_{j_\ell}}^{N/2} \; \tilde{\omega}_{\lambda'} \; (\sqrt{k_{n_{j_\ell}} t}) = \beta(\lambda') \; E(x,t) \tag{1.25}$$

and, using again (1.22),

$$\lim_{n_{j_\ell}\to+\infty} k_{n_{j_\ell}}^{N/2}\ \tilde{u}\ (\sqrt{k_{n_{j_\ell}}}x, k_{n_{j_\ell}}t) = \beta(\lambda')\ E(x,t) \tag{1.26}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^+$, which implies that $\beta(\lambda)$ is constant on $(\gamma, +\infty)$ (and in fact on $[\gamma, +\infty))$ with value γ. As a consequence we get (i).

Case 2. $\lim_{\lambda\to\infty} \beta(\lambda) = +\infty$. We fix $\mu > 0$ and let $\lambda > 0$ such that $\beta(\lambda) \geq \mu$; for $\sigma > 0$, we define

$$q(r) = \min\left(1, \frac{r^+}{\sigma}\right) \qquad \text{and} \qquad j(r) = \int_0^r q(t)\ dt.$$

We deduce from (1.10), (1.14) that

$$\left[\int_{B_1(0)} j(v_\mu - \omega_\delta) dx\right]_{t'}^{t} + \int_{t'}^{t}\int_{B_1(0)} q'(v_\mu - \omega_\delta)\ |\nabla(v_\mu - \omega_\delta)|^2\ dx\ dt$$
$$+ \int_{t'}^{t}\int_{B_1(0)} q(v_\mu - \omega_\delta)[g(v_\mu) - g(\omega_\delta + \delta)]\ dx\ dt \leq 0,$$

for $0 < t' < t \leq T$. Letting δ go to 0 and using the monotonicity of g imply that

$$t \mapsto \int_{B_1(0)} j(v_\mu - \omega^\lambda)(x,t)\ dx$$

is nonincreasing and it is the same with

$$t \mapsto \int_{B_1(0)} (v_\mu - \omega^\lambda)^+(x,t)\ dx.$$

From Step 2 there exists a sequence $\{k_n\}$ converging to 0 such that

$$\lim_{n\to+\infty} k_n^{N/2}\ (\tilde{v}_\mu - \tilde{\omega}^\lambda)(\sqrt{k_n}x, k_n t) = [\mu - \beta(\lambda)]\ E(x,t) \leq 0$$

a.e. in $\mathbb{R}^N$ ($t > 0$ fixed). As

$$k_n^{N/2}\ (\tilde{v}_\mu - \tilde{\omega}^\lambda)^+(\sqrt{k_n}x, k_n t) \leq \mu\ E(x,t),$$

we deduce, by Lebesgue's theorem, that

$$\begin{aligned}\lim_{n\to\infty} k_n^{N/2} \int_{\mathbb{R}^N} (\tilde{v}_\mu - \tilde{\omega}^\lambda)^+(\sqrt{k_n}x, k_n t)\ dx = 0 \\ = \lim_{n\to\infty} \int_{B_1(0)} (v_\mu - \omega^\lambda)^+(x, k_n t)\ dx.\end{aligned} \tag{1.27}$$

As a consequence

$$\lim_{t\to 0} \int (v_\mu - \omega^\lambda)^+(x,t)\ dx = 0$$

and $v_\mu \leq \omega^\lambda$ in Q. If we let μ go to $+\infty$, we deduce (ii) from Proposition 3.2 (see Appendix).

REMARK 1.2. In the case $g(r) = r^\nu$, $\nu > 0$, then (1.8) means that $\nu < \frac{N+2}{N}$.

2. - Source type solutions of semilinear heat equations

Our first result deals with the following heat equation with absorption

$$u_t - \Delta u + g(u) = 0, \tag{2.1}$$

where we assume that g is a continuous nondecreasing real valued function vanishing at 0 and satisfying (1.8).

THEOREM 2.1. *Let $u \in C^{2,1}(\overline{Q}\backslash\{(0,0)\})$ be a nonnegative solution of* (2.1) *in $\overline{Q}\backslash\{(0,0)\}$ vanishing on $\overline{B}_1(0) \times \{0\}\backslash\{(0,0)\}$. Then*

(i) *either $\lim_{t\to 0} t^{N/2}\ u(x,t) = +\infty$, uniformly on any set $\mathbf{E}_\alpha \cap Q$, $\alpha > 0$, or*

(ii) *there exists $\gamma \geq 0$ such that $t^{N/2}[u(x,t) - \gamma E(x,t)]$ converges to 0 uniformly on any set $\mathbf{E}_\alpha^c \cap Q$, when t tends to 0. Moreover $g(u) \in L^1(Q)$ and u satisfies* (2.1) *in Q with initial data $\gamma\delta(x)$.*

PROOF. From Theorem 1.2. we may assume that $k^{N/2}\ \tilde{u}(\sqrt{k}x, kt)$ converges to $\gamma E(x,t)$ locally in measure in $\mathbb{R}^N \times \mathbb{R}^+$, when k tends to 0, γ being some nonnegative real number. Let v_γ be the solution of (1.10) with λ replaced by γ. From the proof of Theorem 1.2 (Step 3, Case 2), we have

$$v_\gamma(x,t) \leq u(x,t) \tag{2.2}$$

in $\overline{Q}\backslash\{(0,0)\}$. For $t \in (0,T]$, let $\ell(t)$ be $\int_{B_1(0)} u(x,t)\ dx$. As we have

$$\ell(t) = \ell(T) + \int_t^T \int_{B_1(0)} g(u)\ dx\ d\sigma - \int_t^T \int_{\partial B_1(0)} \frac{\partial u}{\partial \nu}\ dS\ d\sigma, \tag{2.3}$$

we deduce, from the positivity of $g(u)$ and the continuity of $\frac{\partial u}{\partial \nu}$ on $\partial B_1(0) \times [0,T]$, that $\ell = \lim_{t\to 0} \ell(t)$ exists in $[0,+\infty]$.

Step 1. We claim that $\ell = \gamma$. From (2.2) and the fact that

$$\lim_{t\to 0} \int_{B_1(0)} v_\gamma(x,t)\ dx = \gamma, \tag{2.4}$$

it is clear that $\ell \geq \gamma$. Let us assume that $\ell > \gamma$ and take $\tilde{\gamma} \in (\gamma, \ell)$.
From Theorem 1.2 (Step 2) we have

$$\lim_{t\to 0} \int_{B_1(0)} \inf\ [u(x,t), v_{\tilde{\gamma}}(x,t)]\ dx = \beta(\tilde{\gamma}) \tag{2.5}$$

and $\beta(\tilde{\gamma}) = \gamma$. As $\lim_{t\to 0} v_{\tilde{\gamma}}(x,t) = 0$ for $x \in B_1(0)\backslash\{0\}$, we get for any $t > 0$

$$\lim_{\tau\to 0} \int_{B_1(0)} \inf\ [u(x,t), v_{\tilde{\gamma}}(x,\tau)]\ dx = \ell(t). \tag{2.6}$$

Without any restriction we may suppose that the following inequalities hold for $0 < t \leq T$

$$\ell(t) > \tilde{\gamma} \tag{2.7}$$

$$\int_{B_1(0)} \inf\ [u(x,t), v_{\tilde{\gamma}}(x,t)]\ dx < \tilde{\gamma} \tag{2.8}$$

and, by continuity, there exists a continuous function η defined on $(0,T]$ such that $0 < \eta(t) < t$ and

$$\int_{B_1(0)} \inf\ [u(x,t), v_{\tilde{\gamma}}(x,\eta(t))]\ dx = \tilde{\gamma} \tag{2.9}$$

for $0 < t \leq T$. For $n \geq 2$ we set $\varepsilon_n = \frac{T}{n}$, and w_n the solution of

$$\begin{cases} w_{nt} - \Delta\ w_n + g(w_n) = 0 & \text{in } Q, \\ w_n(x,t) = 0 & \text{on } \partial B_1(0) \times (0,T), \\ w_n(x,0) = \inf[u(x,\varepsilon_n),\ v_{\tilde{\gamma}}(x,\eta(\varepsilon_n))] & \text{in } \overline{B}_1(0). \end{cases} \tag{2.10}$$

From the maximum principle we have

$$\begin{cases} u(x,t+\varepsilon_n) \geq w_n(x,t) \\ v_{\tilde{\gamma}}(x,t+\varepsilon_n) \geq w_n(x,t) \end{cases} \tag{2.11}$$

on $[0, T-\varepsilon_n]$ and

$$0 \leq g[w_n(x,t)] \leq g[v_{\tilde{\gamma}}(x,t+\varepsilon_n)]. \tag{2.12}$$

As $g(v_{\tilde{\gamma}}) \in L^1(Q)$, we deduce from Dunford-Pettis theorem that $\{g(w_n)\}$ is weakly relatively compact in $L^1(Q)$. Moreover, from standard parabolic estimates, $\{w_n\}$ is relatively compact in $L^1(Q)$ and in $C^{1,0}_{\text{loc}}[\overline{Q}\backslash\{(0,0)\}]$. Henceforth there exist a subsequence $\{w_{n_k}\}$ and a function

$$w \in L^1(Q) \cap C^{1,0}_{\text{loc}}[\overline{Q}\backslash\{(0,0)\}]$$

such that

$$\begin{cases} \lim\limits_{n_k\to\infty} w_{n_k} = w & \text{in } L^1(Q) \cap C^{1,0}_{\text{loc}}[\overline{Q}\backslash\{(0,0)\}], \\ \lim\limits_{n_k\to\infty} g(w_{n_k}) = g(w) & \text{weakly in } L^1(Q). \end{cases} \tag{2.13}$$

Let $\phi \in C^{2,1}[\overline{B}_1(0) \times [0,+\infty)]$ with compact support in $\overline{B}_1(0) \times [0,T)$, then we have

$$\iint_Q \{w_n(-\phi_t - \Delta\phi) + \phi\ g(w_n)\}\ dx\ dt = \int_{B_1(0)} w_n(x,0)\ \phi(x,0)\ dx. \tag{2.14}$$

As

$$\int_{B_1(0)} w_n(x,0)\ dx = \tilde{\gamma}$$

and $w_n(x,0) \underset{n\to+\infty}{\longrightarrow} 0$ for any $x \neq 0$, we deduce from (2.13) and (2.14) that

$$\iint_Q \{w(-\phi_t - \Delta\phi) + \phi\ g(w)\}\ dx\ dt = \tilde{\gamma}\ \phi(0,0) \tag{2.15}$$

and $w = v_{\tilde{\gamma}}$ from uniqueness. Henceforth (2.11) implies

$$u(x,t) \geq v_{\tilde{\gamma}}(x,t). \tag{2.16}$$

As

$$\lim_{k\to 0} k^{N/2}\ \tilde{v}_{\tilde{\gamma}}(x\sqrt{k}, kt) = \tilde{\gamma}\ E(x,t)$$

locally in measure in $\mathbb{R}^N \times \mathbb{R}^+$ and

$$\lim_{k\to 0} k^{N/2}\ \tilde{u}(\sqrt{k}x, kt) = \gamma\ E(x,t),$$

this contradicts the fact that $\gamma < \tilde{\gamma}$, and finally $\ell = \gamma$.

Step 2. End of the proof. From (2.3) and $\ell = \gamma$, we deduce that

$$\iint_Q g(u)\ \mathrm{d}x\ \mathrm{d}t < +\infty, \tag{2.17}$$

and

$$\lim_{t\to 0} u(x,t) = \gamma\delta(x), \tag{2.18}$$

in $\mathbf{M}_B(B_1(0))$. Hence u solves (2.1) with $\gamma\delta(x)$ as an initial condition.

Let ρ be the supremum of $u(x,t)$ on $\partial B_1(0) \times [0,T]$ and $v^*_\gamma = v_\gamma + \rho$. Then

$$v^*_{\gamma_t} - \Delta\ v^*_\gamma + g(v^*_\gamma) \geq 0 \tag{2.19}$$

and the function

$$t \mapsto \psi(t) = \int_{B_1(0)} [u(x,t) - v^*_\gamma(x,t)]^+\ \mathrm{d}x$$

is nonincreasing.

As

$$\psi(kt) = \int_{\mathbb{R}^N} k^{N/2}\ [\tilde{u}(\sqrt{k}x, kt) - v^*_\gamma(\sqrt{k}x, kt)]^+\ \mathrm{d}x,$$

$$k^{N/2}[\tilde{u}(\sqrt{k}x, kt) - \tilde{v}^*_\gamma(\sqrt{k}x, kt)]^+ \leq \gamma\ E(x,t)$$

and the existence of $\{k_n\} \to 0$ such that

$$k^{N/2}(\tilde{u} - \tilde{v}^*_\gamma)^+(\sqrt{k_n}x, k_n t) \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{a.e.},$$

we deduce that $\lim_{n'\to\infty} \psi(k_{n'}t) = 0$ for almost all t, and $\psi \equiv 0$. As a consequence $v^*_\gamma \geq u$ in Q and finally

$$v_\gamma(x,t) \leq u(x,t) \leq v_\gamma(x,t) + \rho. \tag{2.20}$$

From Proposition 3.1 and the scaling invariance of E

$$\lim_{k\to 0} k^{N/2}\ [\tilde{u}(\sqrt{k}x, kt) - \gamma E(\sqrt{k}x, kt)] = 0 \tag{2.21}$$

uniformly on any set $\mathbf{E}_\alpha^c \cap \mathbb{R}^N \times [0, L]$, $L > 0$. If we take $t = 1$, and set $k = t$ and $\sqrt{k}x = y$, then $|y| \geq \gamma\sqrt{t}$ and (2.21) reads as

$$\lim_{t \to 0} t^{N/2} \left[u(y,t) - \gamma E(y,t)\right] = 0 \tag{2.22}$$

uniformly on $\mathbf{E}_\alpha^c \cap Q$.

As an application we give a new proof of Oswald classification result [19] when $N \geq 2$.

COROLLARY 2.1. *Assume $g(r) = r^\nu$ with $1 < \nu < \frac{N+2}{N}$ and*

$$u \in C^{2,1}[\mathbb{R}^N \times \mathbb{R}^+ \backslash \{(0,0)\}]$$

is a nonnegative solution of (2.1) *in $\mathbb{R}^N \times (0, +\infty)$ vanishing on $\mathbb{R}^N \times \{0\} \backslash \{(0,0)\}$. Then*

(i) *either $u(x,t) = t^{-1/(\nu-1)}\, f\left(\frac{|x|}{\sqrt{t}}\right)$, where f is the unique positive solution of*

$$\begin{cases} -f'' - \left(\dfrac{N-1}{\eta} + \dfrac{\eta}{2}\right) f' - \dfrac{1}{\nu - 1}\, f + f^\nu = 0 & \text{on } (0, +\infty), \\ f'(0) = 0, \quad \lim\limits_{\eta \to \infty} \eta^{2/(\nu-1)}\, f(\eta) = 0, \end{cases} \tag{2.23}$$

(ii) *or there exists $\gamma \geq 0$ such that u is the unique solution of*

$$\begin{cases} u_t - \Delta u + u^\nu = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(x,0) = \gamma\, \delta(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.24}$$

Before proving the result notice that the existence and the uniqueness of f is due to Brézis, Peletier and Terman [8] and the existence and uniqueness of the solution of (2.24) is due to Brézis and Friedman [6].

PROOF. From Theorem 2.1 we are left with the case where $\lim\limits_{t \to +\infty} t^{N/2} u(x,t) = +\infty$ and $u(x,t) \geq v_n(x,t)$, for any $n \geq 0$, where v_n satisfies

$$\begin{cases} v_{nt} - \Delta\, v_n + v_n^\nu = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ v_n(x,0) = n\, \delta(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.25}$$

From [13],

$$\lim_{n \to +\infty} v_n(x,t) = t^{-1/(\nu-1)}\, f\left(\frac{|x|}{\sqrt{t}}\right) \geq u(x,t).$$

The construction of the upper bound for u is an adaptation of [14]. For any $\varepsilon > 0$, let $U_\varepsilon(x,t)$ be the solution of the following Cauchy-Dirichlet problem

$$(2.26)\quad \begin{cases} U_{\varepsilon t} - \Delta\ U_\varepsilon + U_\varepsilon^\nu = 0 & \text{in } \mathbb{R}^N \backslash B_\varepsilon(0) \times (0,+\infty), \\ U_\varepsilon(x,t) = \left(\frac{1}{\nu-1}\right)^{1/(\nu-1)}\ t^{-1/(\nu-1)} & \text{on } \partial B_\varepsilon(0) \times (0,+\infty), \\ U_\varepsilon(x,0) = 0 & \text{for } |x| > \varepsilon. \end{cases}$$

The function U_ε is obtained as the increasing limit of U_ε^δ $(\delta \to 0)$, where U_ε^δ satisfies

$$(2.27)\quad \begin{cases} U_{\varepsilon t}^\delta - \Delta\ U_\varepsilon^\delta + (U_\varepsilon^\delta)^\nu = 0 & \text{in } \mathbb{R}^N \backslash B_\varepsilon(0) \times (0,+\infty), \\ U_\varepsilon^\delta(x,t) = \left(\frac{1}{\nu-1}\right)^{1/(\nu-1)}\ (t+\delta)^{-1/(\nu-1)} & \text{on } \partial B_\varepsilon(0) \times (0,+\infty), \\ U_\varepsilon^\delta(x,0) = 0 & \text{for } |x| > \varepsilon. \end{cases}$$

As

$$U_\varepsilon^\delta(x,t) \le \left[\frac{1}{(\nu-1)t}\right]^{1/(\nu-1)}$$

the standard parabolic theory [15] asserts the existence of U_ε. Moreover, as in [14] we have

$$(2.28)\qquad u(x,t) \le U_{\varepsilon_1}(x,t) \le U_{\varepsilon_2}(x,t)$$

for $\varepsilon_1 < \varepsilon_2$, $|x| \ge \varepsilon_2$ and $t > 0$ and

$$(2.29)\qquad U_1(x,t) = \varepsilon^{1/(\nu-1)}\ U_\varepsilon(\sqrt{\varepsilon}x, \varepsilon t) = (\sigma\varepsilon)^{1/(\nu-1)}\ U_{\sigma\varepsilon}(\sqrt{\sigma\varepsilon}x, \sigma\varepsilon t)$$

for σ, $\varepsilon > 0$, which implies

$$(2.30)\qquad U_\varepsilon(y,\tau) = \sigma^{1/(\nu-1)}\ U_{\sigma\varepsilon}(\sqrt{\sigma}y, \sigma\tau).$$

If we set $U(y,\tau) = \lim_{\varepsilon\to 0}\ U_\varepsilon(y,\tau)$, then $U(\cdot,\tau)$ is radial in y and satisfies

$$(2.31)\qquad \begin{cases} U_t - \Delta U + U^\nu = 0 & \text{in } \mathbb{R}^N \backslash \{0\} \times (0,+\infty) \\ U(x,0) = 0 & \text{for } x \neq 0 \end{cases}$$

(we have used the a priori estimate of [6]) and

$$(2.32)\qquad U(y,\tau) = \sigma^{1/(\nu-1)}\ U(\sqrt{\sigma}y, \sigma\tau)$$

for $y \neq 0$, $\sigma, \tau > 0$, which implies

$$(2.33)\qquad U(x,t) = t^{-1/(\nu-1)}\ U\left(\frac{x}{\sqrt{t}}, 1\right) = t^{-1/(\nu-1)}\ F(\eta)$$

with $\eta = \frac{|x|}{\sqrt{t}}$. As $N \geq 2$, $\{0\} \times (0, +\infty)$ is a removable singular set for U [4]. As a consequence F satisfies the same equation as f with the same limit conditions; hence $f = F$. From (2.28), u is majorized by U, which implies

$$u(x,t) = t^{-1/(\nu-1)}\ f\left(\frac{|x|}{\sqrt{t}}\right). \tag{2.34}$$

REMARK 2.1. When $N = 1$, $\{0\} \times (0, +\infty)$ is not necessarily a removable singularity for U. As a consequence we just have $F'(0) \leq 0$. Henceforth, if $F(0) > f(0)$ and $F'(0) < 0$, the strict maximum principle implies $F > f$ on $(0, +\infty)$ and a careful (but rather simple) analysis of the proof of the uniqueness of f in [8] shows that this situation is impossible. As a consequence $f = F$.

Let us now consider the following semilinear heat equation with nonlinear forcing term

$$u_t - \Delta u = g(u), \tag{2.35}$$

where g is a continuous real valued function vanishing at 0 and nonnegative on $\mathbb{R}^+$; it is important to notice that we do not make any assumption of monotonicity on g neither integrability condition (1.8) in the following theorem:

THEOREM 2.2. *Assume $u \in C^{2,1}[\overline{Q}\backslash\{(0,0)\}]$ is a nonnegative solution of* (2.35) *in $\overline{Q}\backslash\{(0,0)\}$ vanishing on $\overline{B}_1(0)\times\{0\}\backslash\{(0,0)\}$, and let $\tilde{u}$ be its extension by* 0 *outside Q. Then there exists $\gamma \geq 0$ such that*

$$\lim_{k\to 0} k^{N/2}\ \tilde{u}\ (\sqrt{k}x, kt) = \gamma\ E(x,t), \tag{2.36}$$

in $L^\infty_{\text{loc}}\left[[0,+\infty); L^1(\mathbb{R}^N)\right]$. Moreover $g(u) \in L^1(Q)$ and u solves (2.35) *in $\mathbf{D}'(\overline{Q})$ with initial data $\gamma\ \delta(x)$.*

PROOF. As $-u_t + \Delta u \leq 0$, we deduce from Theorem 1.1 that

$$u \in L^\infty\left[[0,T]; L^1(B_1(0))\right]$$

and that there exist $\gamma \geq 0$ and $\Phi \in L^1(Q)$ such that

$$\begin{cases} -\ u_t + \Delta u = \Phi & \text{in } Q, \\ u(x,0) = \gamma\ \delta(x) & \text{in } B_1(0). \end{cases} \tag{2.37}$$

Hence $\Phi = -g(u)$. Let $E_\gamma(x,t)$ be the solution of

$$\begin{cases} E_{\gamma t} - \Delta\ E_\gamma = 0 & \text{in } Q, \\ E_\gamma(x,t) = u(x,t) & \text{in } \partial B_1(0) \times [0,T], \\ E_\gamma(x,0) = \gamma\ \delta(x) & \text{in } B_1(0). \end{cases} \tag{2.38}$$

Then

$$\gamma\ E^*(x,t) \leq E_\gamma(x,t) \leq \gamma\ E^*(x,t) + \sup_{0\leq\tau\leq t} \|u(\cdot,\tau)\|_{L^\infty(\partial B_1(0))} \tag{2.39}$$

(E^* being defined in 1.16). Using (1.17) we deduce

$$\lim_{k\to 0} k^{N/2}\ \tilde{E}_\gamma(\sqrt{k}x, kt) = \gamma\ E(x,t) \tag{2.40}$$

(where $\tilde{E}_\gamma = 0$ outside $\overline{Q}$) in $L^\infty_{\mathrm{loc}}\left[[0,+\infty); L^1(\mathbb{R}^N)\right]$. If we set

$$w(x,t) = u(x,t) - E_\gamma(x,t), \tag{2.41}$$

then

$$\lim_{t\to 0} \int_{B_1(0)} |w(x,t)|\ dx = 0$$

which implies, with (2.40),

$$\lim_{k\to 0} k^{N/2}\ \tilde{u}\ (\sqrt{k}x, kt) = \gamma\ E(x,t), \tag{2.42}$$

in $L^\infty_{\mathrm{loc}}\left[[0,+\infty); L^1(\mathbb{R}^N)\right]$.

REMARK 2.2. If we know that $u(\cdot,t)$ is radial with respect to x and radially decreasing in $|x|$, for any $t > 0$, we get a much more accurate result as in Proposition 3.1 and Theorem 2.1 and we have

$$\lim_{t\to 0} t^{N/2}\ [u(x,t) - \gamma E(x,t)] = 0, \tag{2.43}$$

uniformly on any set $\mathbf{E}^c_\alpha \cap Q$, for $\alpha > 0$.

REMARK 2.3. If g satisfies

$$\int\int_Q g(E(x,t))\ dx\ dt = +\infty; \tag{2.44}$$

then $\gamma = 0$ as $u \geq E_\gamma$ and $g(u) \in L^1(Q)$; this is in particular the case if

$$\liminf_{\rho\to+\infty} g(\rho)\ \rho^{-(N+2)/N} > 0. \tag{2.45}$$

THEOREM 2.3. *Assume g is a continuous everywhere-differentiable nondecreasing real valued function vanishing at 0 satisfying, for some $p > 1$, $q > 1$ such that*

$$\frac{N}{2p} + \frac{1}{q} < 1, \tag{2.46}$$

the following relation

$$\int_0^T \left\{ \int_{B_1(0)} \{\sup[g'(\phi), g'(\psi)]\}^p \, dx \right\}^{q/p} dt < +\infty, \tag{2.47}$$

for any $\phi, \psi \geq 0$ in Q such that $g(\phi)$ and $g(\psi)$ are integrable in Q. Assume also $u \in C^{2,1}[\overline{Q}\backslash(0,0)\}]$ is a nonnegative solution of (2.35) *in $\overline{Q}\backslash\{(0,0)\}$ vanishing on $\overline{B}_1(0) \times \{0\}\backslash\{(0,0)\}$. Then there exists $\gamma \geq 0$ such that*

$$\lim_{t \to 0} t^{N/2} \; [u(x,t) - \gamma E(x,t)] = 0 \tag{2.48}$$

holds uniformly in any set $\mathbf{E}_\alpha^c \cap Q$. Moreover if $\gamma = 0$, u is bounded in $\overline{Q}$.

PROOF. The assumptions (2.46), (2.47) may look rather strange but in fact they are exactly what we need to apply Aronson and Serrin's results [1]. Let γ be the real number obtained in Theorem 2.2.

Case 1. Assume $\gamma > 0$. Using Theorem 3.2, let u be the solution obtained by the iterative scheme (3.26) (with γ) of

$$\begin{cases} u_{\gamma t} - \Delta \; u_\gamma = g(u_\gamma) & \text{in } Q, \\ u_\gamma(x,t) = 0 & \text{on } \partial B_1(0) \times [0,T], \\ u_\gamma(x,0) = \gamma \; \delta(x) & \text{in } \mathbf{D}'(B_1(0)). \end{cases} \tag{2.49}$$

The function $u_\gamma(\cdot,t)$ is radial in x, radially decreasing with respect to $|x|$ for any $t > 0$ and

$$\gamma \; E^*(x,t) \leq u_\gamma(x,t) \leq u(x,t) \tag{2.50}$$

in $\overline{Q}\backslash\{(0,0)\}$. Let w be $u - u_\gamma$ and

$$d = \frac{g(u) - g(u_\gamma)}{u - u_\gamma}$$

then w satisfies

$$\iint_Q \{w(-\phi_t - \Delta\phi) - d \; w \; \phi) \; dx \; dt = 0 \tag{2.51}$$

for any $\phi \in C^{2,1}\left[\overline{B}_1(0) \times [0,T]\right]$ with compact support in $B_1(0) \times [0,T)$. Let now $\tilde{w}$ and $\tilde{d}$ be the extensions of w and d by 0 in $\overline{B}_1(0) \times [-T,0)$. If we consider $\psi \in C_0^{2,1}[B_1(0) \times (-T,T)]$ and $\varsigma_n(t) = \min(1, nt)$, $n > 0$, then

$$\int_{-T}^{T} \int_{B_1(0)} \{\tilde{w}[-(\psi\varsigma_n)_t - \varsigma_n \; \Delta\psi] - \tilde{d} \; \tilde{w} \; \varsigma_n\} \; dx \; dt = 0, \tag{2.52}$$

but the left-hand side of (2.52) can be written as

$$\iint_Q \zeta_n\ \tilde{w}(-\psi_t - \Delta\psi - d\psi)\ dx\ dt - n \int_0^{1/n} \int_{B_1(0)} w\ dx\ dt.$$

As $\tilde{w}(-\psi_t - \Delta\psi - d\psi) \in L^1(Q)$ and $\lim_{t \to 0} \|w(\cdot,t)\|_{L^1(B_1(0))} = 0$, we deduce that $\tilde{w}$ satisfies

$$\tilde{w}_t - \Delta\ \tilde{w} - \tilde{d}\ \tilde{w} = 0 \tag{2.53}$$

in $\mathbf{D}'[B_1(0) \times (-T,T)]$. As for the coefficient d it is equal to $g'(\xi(x,t))$ where $\xi(x,t) \in [u_\gamma(x,t), u(x,t)]$. As g' satisfies the mean value property, we have

$$g'[\xi(x,t)] \in [\min\{g'[u(x,t)], g'[u_\gamma(x,t)]\}, \max\{g'[u(x,t)], g'[u_\gamma(x,t)]\}] \tag{2.54}$$

and

$$\int_{-T}^{T} \left[\int_{B_1(0)} d^p(x,t)\ dx \right]^{q/p} dt < +\infty.$$

As $\tilde{w}$ is bounded in a neighbourhood of the boundary of $B_1(0) \times (-T,T)$, we deduce from [1, Theorem 1] that $\tilde{w} \in L^\infty[B_1(0) \times (-T,T)]$. We then obtain (2.48) from Remark 3.2.

Case 2. Assume $\gamma = 0$. In that case we write (2.35) as

$$u_t - \Delta u - \frac{g(u)}{u}\ u = 0 \tag{2.55}$$

and we extend $\tilde{u}$ into $B_1(0) \times (-T,T)$ by 0 for $t < 0$. As

$$\frac{g(u)}{u} = g'[\xi(x,t)]$$

we then deduce from (2.47) and [1] that $\tilde{u}$ is bounded in $B_1(0) \times (-T,T)$.

In the power case we obtain a better result:

COROLLARY 2.2. *Assume* $1 < \nu < \frac{N+2}{N}$ *and* $u \in C^{2,1}[\overline{Q}\backslash\{(0,0)\}]$ *is a nonnegative solution of*

$$u_t - \Delta u = u^\nu \tag{2.56}$$

in $\overline{Q}\backslash\{(0,0)\}$ *vanishing on* $\overline{B}_1(0) \times \{0\}\backslash\{(0,0)\}$. *Then there exists* $\gamma \geq 0$ *such that*

$$\lim_{t \to 0} [u(x,t) - \gamma\ E(x,t)] = 0 \tag{2.57}$$

uniformly in $\overline{B}_1(0)$.

PROOF. Let γ be defined by (2.36). From Theorem 2.2, $u^\nu \in L^1(Q)$ and $u(x,0) = \gamma\ \delta(x)$. In order to apply Theorem 2.3, we first notice that if $v^\nu \in L^1(Q)$ then sup $(u^\nu, v^\nu) = [\text{sup } (u,v)]^\nu \in L^1(Q)$ and

$$(2.58)\quad \iint_Q [\sup(u^{\nu-1}, v^{\nu-1})]^{\nu/(\nu-1)}\ dx\ dt = \iint_Q \sup(u^\nu, v^\nu)\ dx\ dt < +\infty.$$

If we take $p = q = \frac{\nu}{\nu-1}$, then $\frac{N}{2p} + \frac{1}{q} = \frac{\nu-1}{\nu} \left(\frac{N+2}{2}\right) < 1$ as $1 \le \nu < \frac{N+2}{2}$, and (2.46) holds. As a consequence u satisfies (2.48). In order to improve this estimate, we define

$$(2.59)\quad \phi(x,t) = (\gamma+1)\ E^*(x,t) + \int_0^t S^*(t-s)\,[(2\gamma+2)\ E^*(\cdot,s)]^\nu\,(x)\ dx.$$

Then $\phi^\nu \in L^1(Q)$ and from Remark 3.4 there exists $T^* \in (0,T]$ such that

$$(2.60)\quad \begin{cases} \phi(x,t) \ge (\gamma+1)\ E^*(x,t) & \text{in } \overline{B}_1(0) \times [0,T^*]\backslash\{(0,0)\}, \\ \phi_t - \Delta\phi \ge \phi^\nu & \text{in } \mathbf{D}'[B_1(0) \times (0,T^*)]. \end{cases}$$

Henceforth if u_γ is the solution of

$$(2.61)\quad \begin{cases} u_{\gamma t} - \Delta u_\gamma = u_\gamma^\nu & \text{in } B_1(0) \times (0,T^*), \\ u_\gamma(x,t) = 0 & \text{on } \partial B_1(0) \times (0,T^*), \\ u_\gamma(x,0) = \gamma\ \delta(x) & \text{in } B_1(0), \end{cases}$$

defined by the iterative scheme

$$(2.62)\quad \begin{cases} u_0 = \gamma\ E^* \\ u_n(x,t) = \gamma\ E^*(x,t) + \displaystyle\int_0^t S^*(t-s)\ u_{n-1}^\nu(\cdot,s)(x)\ ds, \end{cases}$$

it satisfies (see [8-Appendix])

$$(2.63)\quad u_\gamma(x,t) \le \phi(x,t) \le C\ E^*(x,t),$$

where C depends on T^*, N, ν and γ. As a consequence $(\sigma > 0)$

$$(2.64)\quad 0 \le u_\gamma(x,t) - \gamma\ E^*(x,t) \le C'\ \gamma^\nu\ t^\sigma\ E^*(x,t)$$

holds in $\overline{B}_1(0) \times [0, T^*]$. From (2.58) with $v = u_\gamma$ and Theorem 2.3, we deduce that $u - u_\gamma$ is bounded and more precisely [1, Theorem 1]

$$0 \leq (u - u_\gamma)(x,t) \leq K \max_{0 \leq \tau \leq t} \|u(\cdot, \tau)\|_{L^\infty(\partial B_1(0))}. \tag{2.65}$$

As the right-hand side of (2.65) tends to 0, when t tends to 0, we deduce (2.57) from (2.64) and (2.65).

REMARK 2.4. The study of equation (2.56) in the case $\nu \geq \frac{N+2}{N}$ appears as a very deep challenge. In that case there should exist a second critical value $\nu = \frac{N+2}{N-2}$ (if $N \geq 3$) as in the elliptic case [16], [10], [2]. We conjecture that when $\frac{N+2}{N} < \nu < \frac{N+2}{N-2}$ then $u(x,t)$ behaves like $t^{1/(\nu-1)} f\left(\frac{|x|}{\sqrt{t}}\right)$ where $f > 0$ satisfies

$$\begin{cases} f'' + \left(\dfrac{N-1}{\eta} + \dfrac{\eta}{2}\right) f' + \dfrac{1}{\eta - 1} f + f^\nu = 0 & \text{on } (0, +\infty), \\ f'(0) = 0, \ \lim\limits_{n \to \infty} \eta^{2/(\nu-1)} f(\eta) = 0. \end{cases} \tag{2.66}$$

(See [12], [21], [22]) for some similar equations; this would be the perfect analogy with the elliptic case. However two points are missing to apply an energy method as in [9], [11] or [18]: the uniqueness of f and some a priori estimate of the form

$$u(x,t) \leq C\, t^{-1/(\nu-1)} f\left(\frac{|x|}{\sqrt{t}}\right), \tag{2.67}$$

for $|x|$ small).

3. - Appendix

We first consider the following heat equation with absorption

$$\begin{cases} u_t - \Delta u + g(u) = 0 & \text{in } B_1(0) \times (0, +\infty), \\ u(x,t) = 0 & \text{in } \partial B_1(0) \times (0, +\infty), \\ u(x,0) = \lambda \delta(x) & \text{in } \mathbf{D}'(B_1(0)). \end{cases} \tag{3.1}$$

We assume that λ is nonnegative and g is a continuous nondecreasing real valued function, vanishing at 0 for simplicity, and satisfying

$$\int_0^1 \int_{B_1(0)} g[E(x,t)]\, dx\, dt < +\infty. \tag{3.2}$$

We say that u satisfies (3.1) if u is continuous in $\overline{B}_1(0) \times [0, +\infty)\backslash\{(0,0)\}$ and vanishes on $\partial B_1(0) \times [0, +\infty)$, $g(u) \in L^1_{\text{loc}}[\overline{B}_1(0) \times [0, +\infty)]$ and if

$$\int\int [u(-\phi_t - \Delta\phi) + g(u)\phi]\mathrm{d}x \ \mathrm{d}t = \lambda \ \phi(0,0) \tag{3.3}$$

for any $\phi \in C^{2,1}[\overline{B}_1(0) \times [0, +\infty)]$ with compact support. Such a u is $W^{2,1}_p$-regular locally in $\overline{B}_1(0) \times [0, +\infty)\backslash\{(0,0)\}$ for any $p \in [1, +\infty)$ and can be expressed by

$$u(x,t) = \lambda E^*(x,t) - \int_0^t S^*(t-s) \ g[u(\cdot, s)](x) \ \mathrm{d}s, \tag{3.4}$$

where E^* is defined in (1.16) and S^* is the heat semigroup in $B_1(0)$ with Dirichlet boundary conditions [and in fact $E^*(x,t)$ is just $S^*(t)\delta(\cdot)(x)$].

THEOREM 3.1. *For any $\lambda \geq 0$ there exists a unique solution u of* (3.1). *Moreover $\lambda \mapsto u$ is nondecreasing.*

PROOF. The uniqueness is classical (see [6] for example). For the existence we shall use a double approximation method. It is first clear that (3.2) implies

$$\int_0^1 \int_{B_1(0)} g[\rho \ E(x,t) + \rho']\mathrm{d}x \ \mathrm{d}t + \infty \tag{3.5}$$

for any ρ, $\rho' \geq 0$. We define $g_k(r) = \min[g(r), k]$ and $u_n = u_{n,k}$ such that

$$\begin{cases} u_{nt} - \Delta u_n + g_k(u_n) = 0 & \text{in } B_1(0) \times (0, +\infty), \\ u_n(x, 0) = \lambda E^*\left(x, \frac{1}{n}\right) & \\ u_n(x,t) = 0 & \text{on } \partial B_1(0) \times [0, +\infty). \end{cases} \tag{3.6}$$

Henceforth u_n satisfies

$$u_n(x,t) = \lambda E^*\left(x, t + \frac{1}{n}\right) - \int_0^t S^*(t-s) \ g_k[u_n(\cdot, s)](x)\mathrm{d}s. \tag{3.7}$$

It is clear that $0 \leq u_n(x,t) \leq \lambda E^*\left(x, t + \frac{1}{n}\right)$, which implies that, for any $T > 0$, $\{g_k(u_n)\}$ is bounded in $L^\infty[(0,T) \times B_1(0)]$ independently of n. From [3] we know that

$$\phi \mapsto K\phi, \qquad \text{with} \qquad K\phi(t) = \int_0^t S^*(t-s) \ \phi(s) \ \mathrm{d}s,$$

is compact from $L^\infty[[0,T];L^\infty(B_1(0))]$ into $C[[0,T];\ L^\infty(B_1(0))]$. As a consequence there exist a sequence $\{n_\ell\}$ tending to ∞ and a function $h \in L^\infty[(0,+\infty);L^\infty(B_1(0))]$ such that $\{Kg_k(u_{n_\ell})\}$ converges to h in $L^\infty_{\mathrm{loc}}[(0,+\infty);\ L^\infty(B_1(0))]$, and if we set

$$u^k(x,t) = \lambda E^*(x,t) - Kh(t)(x), \tag{3.8}$$

then $h(x,t) = g_k[u^k(x,t)]$ a.e. and u^k is the solution of (3.1) with g replaced by g_k. We also have classically the following two inequalities

$$0 \le u^k \le u^{k'} \le \lambda E^* \tag{3.9}$$

$$0 \le g_k(u^k) \le g(\lambda E^*) \le g(\lambda E) \tag{3.10}$$

for $0 < k' < k$. Henceforth, when k tends to $+\infty$, $\{u^k\}$ converges in $L^1_{\mathrm{loc}}\left[[0,+\infty);\ L^1(B_1(0))\right]$ to some u satisfying (3.3); u is continuous in $\overline{B}_1(0)\times[0,+\infty)\backslash\{(0,0)\}$ and is the solution of (3.1). As for the monotonicity of $\lambda \mapsto u$, it is obvious from the construction.

REMARK 3.1. The above method can be adapted to prove the existence (the uniqueness being a consequence of [6, Lemma 3]) of solutions of the more general equation

$$\begin{cases} u_t - \Delta u + g(u) = 0 & \text{in } B_1(0)\times(0,+\infty) \\ u(x,t) = 0 & \text{on } \partial B_1(0)\times(0,+\infty) \\ u(x,0) = \mu(x) & \text{in } B_1(0), \end{cases} \tag{3.11}$$

where μ is a nonnegative (for simplicity) bounded measure in $B_1(0)$. In that case (3.2) has to be replaced by

$$\int_0^1 \int_{B_1(0)} g[S^*(t)\ \mu(\cdot)(x)]\ dxdt < +\infty. \tag{3.12}$$

PROPOSITION 3.1. *Let u be the solution of* (3.1) *and $\tilde u$ its extension by* 0 *outside $B_1(0)\times[0,+\infty)$. Then we have*

$$\lim_{k\to 0} k^{N/2}\ \tilde u(\sqrt{k}x,kt) = \lambda E(x,t) \tag{3.13}$$

in $L^\infty_{\mathrm{loc}}\left[[0,+\infty);L^1(\mathbb{R}^N)\right]$ and uniformly in $\{(x,t): 0\le t\le T,\ |x|\ge \varepsilon\sqrt{t}\}$ for any ε and $T>0$.

PROOF. As $g(u) \in L^1_{\mathrm{loc}}\left[[0,+\infty)\times \overline{B}_1(0)\right]$, the convergence in $L^\infty_{\mathrm{loc}}\left[[0,+\infty);\ L^1(\mathbb{R}^N)\right]$ has already been proved in Theorem 1.2, Step 2. For the second assertion we first notice that both $\lambda E(\cdot,t)$ and $u(\cdot,t)$ are radial functions with respect to x for any $t > 0$. Moreover, they are decreasing with respect to $|x|$. If we set $w = \lambda E - u$, then

$$\text{(3.14)} \qquad \begin{cases} w_t - \Delta w = g(u) & \text{in } B_1(0) \times (0, +\infty), \\ w(x,t) > 0 & \text{in } \partial B_1(0) \times (0, +\infty), \\ w(x,0) = 0, \quad w > 0 & \text{in } B_1(0) \times (0, +\infty). \end{cases}$$

As $g[u(\cdot, t)]$ is radial for any $t > 0$ and radially decreasing with respect to $|x|$, it is the same for $w(\cdot, t)$. Henceforth

$$\text{(3.15)} \qquad \begin{aligned} &\int [\lambda E(x,t) - k^{N/2}\tilde{u}(\sqrt{k}x, kt)]\mathrm{d}x \\ &\geq \int_{|x| \leq \varepsilon\sqrt{t}} [\lambda E(x,t) - k^{N/2}\tilde{u}(\sqrt{k}x, kt)]\mathrm{d}x \\ &\geq \text{meas. } [B_{\varepsilon\sqrt{t}}(0)]\ [\lambda E(\varepsilon\sqrt{t}, t) - k^{N/2}\tilde{u}(\sqrt{kt}\ \varepsilon, kt)] \end{aligned}$$

(we have used the fact that E is invariant with respect to the scaling transformation). But the right-hand side of (3.15) is just

$$\text{meas. } \left[B_{\varepsilon\sqrt{t}}(0)\right] \max_{|x| \geq \varepsilon\sqrt{t}} \left[\lambda E(x,t) - k^{N/2}\tilde{u}(\sqrt{k}x, kt)\right],$$

which is positive, as for the left-hand side, and it converges to 0 uniformly in $(0, T)$ when k tends to 0.

REMARK 3.2. For a general g we do not know whether

$$\text{(3.16)} \qquad \lim_{k \to 0} k^{N/2}\ \tilde{u}(0, kt) = \lambda E(0, t)$$

holds or not. However if $g(r) \leq C\ r^\nu$, $1 < \nu < \frac{N+2}{N}$, it is proved in [8] that

$$\text{(3.17)} \qquad 0 \leq \lambda E(x,t) - u(x,t) \leq K\ \lambda^\nu\ t^\sigma\ E(x,t),$$

where K and σ are positive constants depending only on N and ν. As a consequence we get (3.16).

REMARK 3.3. An interesting goal would be the study of the possible limit u_∞ of $u = u_\lambda$ solutin of (3.1). In fact at least four phenomena should occur:

(i) $u_\infty(x,t) = +\infty$ for all $(x,t) \in B_1(0) \times [0, +\infty)$,

(ii) $u_\infty(x,0) = +\infty$ for all $x \in B_1(0)$ but $u_\infty(x,t) < +\infty$ for $t > 0$,

(iii) $u_\infty(x,t) < +\infty$ for all $(x,t) \in \overline{B}_1(0) \times [0, +\infty) \backslash \{(0,0)\}$ but

$$\lim_{t\to 0} \int_{|x|<\varepsilon} u_\infty(x,t)\,dx = +\infty, \tag{3.18}$$

(iv) $u_\infty(0,t) = +\infty$, for all $t > 0$ but $u_\infty(x,t) < +\infty$ for each $x \in B_1(0)\setminus\{0\}$ and $t > 0$.

These phenomena should be linked with the nature of the two following integrals

$$\int^{\infty} \frac{ds}{g(s)} \quad \text{and} \quad \int^{\infty} ds\Big/\sqrt{\int^{s} g(\sigma)d\sigma}. \tag{3.19}$$

The most interesting case is case (iii) and a solution corresponding to this case is called a very singular solution [8]. In any case we have the following result.

PROPOSITION 3.2. *Let u_∞ be the limit of $u = u_\lambda$ when λ tends to $+\infty$; then*

$$\lim_{t\to 0} t^{N/2}\, u_\infty(x,t) = +\infty \tag{3.20}$$

uniformly on the sets $\mathbf{E}_\alpha$, $\alpha > 0$.

PROOF. From Proposition 3.1, we have

$$\lim_{t\to 0} \int_{|x|<c\sqrt{t}} [\lambda E(x,t) - u_\lambda(x,t)]\,dx = 0 \tag{3.21}$$

for $0 < c$. As a consequence

$$\lim_{t\to 0} \int_0^{c\sqrt{t}} u_\lambda(r,t)\, r^{N-1}\,dr = \frac{\lambda}{(4\pi)^{N/2}} \int_0^c e^{-\rho^2/4}\,\rho^{N-1}\,d\rho. \tag{3.22}$$

Henceforth

$$\liminf_{t\to 0}\, t^{N/2} u_\lambda(c\sqrt{t},t) \geq \lambda(4\pi)^{-N/2}\, N\, C^{-N} \int_0^c e^{-\rho^2/4}\,\rho^{N-1}\,d\rho, \tag{3.23}$$

which implies the result, as $u_\lambda(\cdot,t)$ (and u_∞, when it exists) is radial in x and radially decreasing with respect to $|x|$, for any $t > 0$.

To end this Section, we consider the following heat equation with forcing nonlinearity

$$\begin{cases} u_t - \Delta u = g(u) & \text{in } B_1(0)\times(0,T) \\ u(x,t) = 0 & \text{in } \partial B_1(0)\times[0,T) \\ u(x,0) = \lambda\,\delta(x) & \text{in } \mathbf{D}'(B_1(0)). \end{cases} \tag{3.24}$$

We assume $\lambda \geq 0$, we make on g the same assumptions of monotonicity as those of Theorem 3.1 and we prove

THEOREM 3.2. *Assume there exist* $T > 0$, $\Lambda > 0$ *and a nonnegative function* $\phi \in C^0[\overline{B}_1(0) \times [0,T]\backslash\{(0,0)\}]$ *satisfying*

$$(3.25) \qquad \begin{cases} g(\phi) \in L^1[B_1(0) \times [0,T]], & \\ \phi(x,t) \geq \Lambda \; E^*(x,t) & \text{in } \overline{B}_1(0) \times [0,T]\backslash\{(0,0)\}, \\ \phi_t - \Delta\phi \geq g(\phi) & \text{in } \mathbf{D}'(B_1(0) \times (0,T)). \end{cases}$$

Then for any $\lambda \in [0,\Lambda]$ *there exists at least one nonnegative function* $u \in C^0[\overline{B}_1(0) \times [0,T]\backslash\{(0,0)\}]$, *such that* $g(u)$ *is integrable in* $B_1(0) \times (0,T)$, *satisfying* (3.24).

PROOF. We consider the following sequence $\{u_n\}$, $n \geq 0$,

$$(3.26) \qquad \begin{cases} u_0 = \lambda \; E^* \\ u_n(x,t) = \lambda \; E^*(x,t) + \int_0^t S^*(t-s) \; g[u_{n-1}(\cdot,s)](x) \; ds \end{cases}$$

and we claim that

$$(3.27) \qquad u_0 \leq u_1 \leq \ldots \leq u_n \leq \phi, \qquad \text{for every } n \in N,$$

in $\overline{B}_1(0) \times \{(0,0)\}$. From (3.26) it is clear that $u_0 \leq u_1$, as g is nonnegative on $[0,+\infty)$. As $g(\phi) \geq g(u_0)$ and

$$t \mapsto \int_{B_1(0)} (u_1 - \phi)^+(x,t)dx$$

is nonincreasing, from (3.25) and (3.26), we get $u_1 \leq \phi$. If we assume now that (3.27) is true to the order n, then from the definition of u_{n+1} we have $u_{n+1} \geq u_n$. Moreover $g(\phi) \geq g(u_n)$ and

$$t \mapsto \int_{B_1(0)} (u_{n+1} - \phi)^+(x,t)dx$$

is nonincreasing, which implies $u_{n+1} \leq \phi$ and (3.27). Set $u = \lim_{n \to +\infty} u_n$, then $g(u_n)$ converges to $g(u)$ in $L^1[B_1(0) \times (0,T)]$ and everywhere in $\overline{B}_1(0) \times (0,T]$. As a consequence, u satisfies

$$(3.28) \qquad u(x,t) = \lambda \; E^*(x,t) + \int_0^t S^*(t-s) \; g[u(\cdot,s)](x) \; ds,$$

for $0 \leq t \leq T$, which ends the proof.

REMARK 3.4. The conditions on g which insure the existence of ϕ satisfying (3.25) are not known except in the power case

$$g(r) \leq C\ r^{\nu}, \qquad r \geq 0, \tag{3.29}$$

$1 < \nu < \frac{N+2}{N}$. In that case, a suitable adaptation of [8, Appendix] shows that

$$\left| \int_0^t S^*(t-s)\ g[kE^*(\cdot,s)](x)\mathrm{d}s \right| \leq D\ k^{\nu}\ t^{\sigma}\ E^*(x,t), \tag{3.30}$$

and, as a consequence, for any $\lambda > 0$, the function ϕ defined by

$$\phi(x,t) = \Lambda\ E^*(x,t) + \int_0^t \tilde{S}(t-s)\ g[2\Lambda\ E^*(\cdot,s)](x)\ \mathrm{d}s, \tag{3.31}$$

satisfies (3.25) on some small interval $[0,T]$. If we assume moreover that g is Lipschitz continuous and

$$g'(r) \leq C'\ r^{\nu-1}, \tag{3.32}$$

with $1 \leq \nu < \frac{N+2}{N}$, *then u is unique* (see the proof of Theorem 2.3).

REMARK 3.5. If u is a nonnegative solution of (3.24), radial with respect to x and radially decreasing in $|x|$ (as the one obtained in Theorem 3.2), it is clear that the convergence results of Proposition 3.1 still hold; in the proof, $\gamma E - u$ has to be replaced by $u - \gamma E^*$.

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Département de Mathématiques
Faculté des Sciences et Techniques
Parc de Grandmont
37200 Tours, FRANCE