ANNALI DELLA
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Classe di Scienze

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Cartesian currents and variational problems for mappings into spheres

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 16, nº 3 (1989), p. 393-485
<http://www.numdam.org/item?id=ASNSP_1989_4_16_3_393_0>

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1. - Introduction

In a previous paper [33] we have discussed the problem of the existence of generalized *equilibrium deformations* in nonlinear hyperelasticity, i.e. of mappings u from a bounded domain Ω of $\mathbb{R}^n$, $n \geq 2$, that are one to one and preserve the orientation, and which minimize physically resonable energies associated to a *perfectly* hyperelastic material. The simple key idea, which

Pervenuto alla Redazione il 28 Marzo 1989 e in forma definitiva il 15 Maggio 1989.

made possible to prove existence theorems by the direct methods of Calculus of Variations, was the following. We observe that, in the context of nonlinear eleasticity it is very natural (compare section 6 and [33]) to look at the problem in the product space $\mathbb{R}^n_x \times \mathbb{R}^n_y$ and to regard the deformation u as a *graph* or more precisely as the n-dimensional current integration of n-forms in $\mathbb{R}^n_x \times \mathbb{R}^n_y$ over the graph of $u : G_u$. We therefore work in the setting of rectifiable currents of Federer and Fleming, and we consider the weak sequential closure of the class of graphs associated to diffeomorphisms, for which suitable L^p and L^q-norms respectively of u and u^{-1} and of the minors of their Jacobian matrices are equibounded; we denote such a "norm" by $\|\cdot\|_{\mathrm{Dif}^{p,q}}$. It turns out that coercivity of the energy with respect to $\|\cdot\|_{\mathrm{Dif}^{p,q}}$ is equivalent to the physical requirement that the energy become large for large stretchings and large compressions. And this allows in conclusion to minimize physically reasonable energies in classes of deformations with various boundary conditions.

This paper is strongly related to [33] and aims to show that the same simple idea gives a natural way, and in a sense the right way, to approach *variational problems with constraints for vector valued mappings,* for instance for mappings into a non-flat Riemannian manifold such as a sphere.

Consider for example the problem of minimizing the Dirichlet integral

$$\mathcal{D}(u) := \frac{1}{2}\int_{B^3} |\mathrm{D}u|^2 \mathrm{d}x$$

among mappings from the unit ball B^3 of $\mathbb{R}^3$ into the sphere $S^2 \subset \mathbb{R}^3$, with say prescribed value u_0 on ∂B^3. The usual approach is the following. One considers $\mathcal{D}(u)$ as defined in the Sobolev space $H^{1,2}(B^3, S^2)$, thus by direct methods one concludes at once with the existence of a minimizer in $H^{1,2}(B^3, S^2) \cap \{u : u = u_0 \text{ on } \partial B^3\}$. We believe that this is one of the possible approaches and it is not the most suited for the Dirichlet problem. Let us explain this claim. The class of smooth mappings $C^1(B^3, S^2)$ is not dense in $H^{1,2}(B^3, S^2)$ and even empty if we restrict ourselves to functions with boundary data $u_0 : S^2 \to S^2$ with non-zero degree; moreover, even for zero degree boundary values we have, see [37],

$$\begin{aligned}&\inf\{\mathcal{D}(u) \mid u \in H^{1,2}(B^3, S^2), u = u_0 \text{ on } \partial B^3\}\\ &< \inf\{\mathcal{D}(u) \mid u \in H^{1,2}(B^3, S^2) \cap C^0(\bar{B}^3, S^2),\ u = u_0 \text{ on } \partial B^3\}.\end{aligned}$$

Actually if we regard u as the associated rectifiable current in $B^3 \times S^2$ which is roughly the current integration over the graph of u, T_u (compare section 2), one sees that in general T_u has a boundary in $B^3 \times S^2$ or, equivalently, the graph of $u \in H^{1,2}(B^3, S^2)$ has holes. Thus, defining $\mathcal{D}(u)$ in $H^{1,2}(B^3, S^2)$ in a 'pointwise way' is like choosing zero as value of

$$\int_{-1}^{1} |\dot{u}|$$

for u =sign x and, when we minimize in $H^{1,2}$, we in fact allow the minimizer to create new boundaries in the interior of B^3, thus decreasing the energy.

We propose to regard smooth functions as graphs or, more precisely, as *cartesian currents* and to work on the class of weak limits T of sequences of smooth currents with equibounded energy. We define the energy on this class by means of the classical Lebesgue extension formula, very roughly

$$\mathcal{D}(T) := \inf\left\{ \liminf_{k\to+\infty} \mathcal{D}(u_k) \mid u_k \text{ smooth}, \ \sup_k \mathcal{D}(u_k) < +\infty, \right.$$
$$\left. \text{graph of } u_k \rightharpoonup T \right\}.$$

Actually the class of weak limits of sequences of smooth graphs with equibounded energy seems a priori not closed in general, thus we consider the smallest sequentially closed set of currents T which contains all smooth graphs and we define $\mathcal{D}(T)$ as the *relaxed functional* associated to $\mathcal{D}$.

Since the smallest sequentially closed set containing the family of smooth graphs can be obtained by successive closures (in a transfinite way), we conclude at once that their elements have *no interior boundaries.*

The previous proposal can be carried on in a reasonable way if our functional $\mathcal{E}$, defined on smooth functions, "*controls*" the graph of u. Since, as it is well known, a good control of T_u is given by the *mass of* T_u, i.e. the area of G_u, this means that $\mathcal{E}$ *ought* to be coercive with respect to the area of the graph of u. By the isoperimetric inequality for parallelograms, we have

$$\frac{1}{2}|\mathrm{D}u|^2 \geq |M_2(\mathrm{D}u)| \ , \quad u : B^3 \to S^2,$$

$M_2(\mathrm{D}u)$ standing for the second order minors of the jacobian matrix $\mathrm{D}u$; thus the Dirichlet integral is coercive with respect to the area. In this respect we are led to distinguish *regular* functionals, the ones which are coercive with respect to the area, from the others. For instance, while the Dirichlet integral is coercive, hence regular, for mappings from B^3 into S^2, it is not coercive, hence not regular, for mappings from B^3 into S^3 or $\mathbb{R}^3$; actually, in these last cases, one easily sees that there is lost of control on graphs with equibounded Dirichlet's integral.

The aim of this paper is to develop this idea, which in many respects can be considered as classical, mainly in specific significant examples. We shall see that the resulting problems will have minimizers which have in general completely different features from the ones obtained as minimizers in Sobolev classes. For example, in the case considered above of the Dirichlet integral for mappings $u : B^3 \to S^2$, which arises as a simplified model in the theory of liquid crystals, we shall see that our minimizers have in general "*line singularities*" instead of point singularities of one degree, and point singularities can occur only with zero degree, compare [16], [14].

In our context, minimizing in $H^{1,2}$ is in general like a problem with a free boundary more than a boundary value problem, in the sense already mentioned that the minimizers are free to produce new boundaries, in that way lowering their energy. On the other hand, such kind of problems are natural and important both from the mathematical and the physical point of view. In the last section we shall briefly discuss some of them and, for instance, we shall see that they might be useful in order to give a mathematical model for describing the fractures of an elastic body.

The paper is organized as follows.

In section 2 we shall discuss several classes of cartesian currents and the problem of the convergence of determinants on the basis of the results in [33], and we shall make a few relevant remarks. In particular we shall discuss relationships among boundaries, traces and weak convergence. Results concerning the weak convergence of determinants in the context of Sobolev spaces have been obtained by Reshetnyak [54], [55], and Ball [4]; recently Müller [50] has given a simpler proof of the convergence result in [33]. Here we shall show that actually this proof fits into a more general context and in fact gives a more general result.

In section 3 the notion of *degree for cartesian currents* is discussed. As in [30], the definition of degree is based on the constancy theorem; we shall prove that all classical properties of the degree remain valid. This will allow us to describe easily weak diffeomorphisms in terms of degree, extending in this way some results in [5], [53], [61].

In section 4 we define the *polyconvex extension* of a general integrand, roughly, as the largest polyconvex integrand which lies below the given one and the *parametric integrand* associated to such an extension; then we shall compute these extensions in many specific cases. In fact we shall not work with the Lebesgue extension of a given functional, but actually with its parametric extension, which has an explicit integral representation, and in some specific cases we shall prove that it coincides with the Lebesgue extension. But, in general, we can only conjecture that for regular functionals the two extensions coincide. This will be used in order to discuss the problem of the existence of energy minimizing maps with prescribed degree from an n-dimensional Riemannian manifold into S^n. In fact we prove existence of a minimizer for regular functionals, for instance we prove existence of a minimizer of the Dirichlet integral among maps from the sphere S^2 or the torus T^2 into S^2 with prescribed degree.

In section 5 we discuss several problems in which one looks for minimizers of the Dirichlet integral, or of the more general functional of liquid crystals, among mappings from a domain of $\mathbb{R}^3$ into S^2 satisfying suitable "boundary conditions", and we shall prove existence.

Finally, in section 6 we formulate a few variational problems for graphs with holes, giving conditions under which they can be solved; in particular we formulate a setting which can be useful for a possible static model of fractures in the nonlinear theory of hyperelastic materials.

This work was written while the third author was visiting the Istituto di Matematica Applicata dell'Università di Firenze under the support of CNR and it has been partially supported by the Ministero della Pubblica Istruzione.

2. - Cartesian currents, boundaries and weak convergence of determinants

In this section we shall discuss relationships among mappings, their graphs and the associated current integration of forms over graphs. As a result we shall introduce a few classes of cartesian currents and of weak diffeomorphisms, already considered in [33], and we shall make some relevant remarks on them. In particular we shall clarify in which sense these classes have to be considered as the natural extension of the class of smooth mappings and of the class of smooth diffeomorphisms.

NOTATIONS. We denote the standard basis of $\mathbb{R}^{n+N} = \mathbb{R}^n_x \times \mathbb{R}^N_y$ by $(e_1, \ldots, e_n, \epsilon_1, \ldots, \epsilon_N)$ and the coordinates relative to this basis by $(x_1, \ldots, x_n, y_1, \ldots, y_N) = (x, y)$. The dual basis is denoted by $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}y_1, \ldots, \mathrm{d}y_N)$. We use the standard notations for multiindices

$$I(p,n) = \{\alpha = (\alpha_1, \ldots, \alpha_p) : \alpha_i \text{ integers}, \ 1 \leq \alpha_1 < \ldots < \alpha_p \leq n\}$$

and for convenience we set

$$I(0,n) = \{0\} \ , \ \mathrm{d}x^0 = \mathrm{d}y^0 = 1,$$

and $|\alpha| = p$ for $\alpha \in I(p,n)$. If $\alpha \in I(p,n)$, $p = 0, 1, \ldots, n$, then $\bar{\alpha} \in I(n-p,n)$ denotes the complement of α in $\{1, 2, \ldots, n\}$ in the natural order; we have $\bar{\bar{\alpha}} = \alpha$, $\bar{0} = (1, \ldots, n)$. Moreover, for $\alpha \in I(p,n)$ and $\beta \in I(q,n)$ with $p + q \leq n$, α and β disjoint, $\sigma(\alpha, \beta)$ denotes the sign of the permutation which reorders naturally (α, β). We set

$$\mathrm{d}x^\alpha = \mathrm{d}x_{\alpha_1} \wedge \ldots \wedge \mathrm{d}x_{\alpha_p} \ , \ e^\alpha = e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_p},$$

in particular

$$dx := dx_1 \wedge \ldots \wedge dx_n = dx^{\bar{0}};$$

if $|\alpha| = n-1$, $|\beta| = 1$, we shall often write i instead of $\bar{\alpha}$, $i = 1, \ldots, n$, and j instead of β, $j = 1, \ldots, n$ and $\bar{i}$ for $(1, \ldots, i-1, i+1, \ldots, n)$; we shall also use the standard notation $\widehat{dx_i}$ for $dx^{\bar{i}}$ and $\hat{e}_i$ for $e_{\bar{i}}$. With the previous notations every r-form in $\mathbb{R}^{n+N}$, $r \leq n+N$, can be written as

$$\omega(x,y) = \sum_{|\alpha|+|\beta|=r} \omega_{\alpha\beta}(x,y)\, dx^\alpha \wedge dy^\beta.$$

Finally, we use in the space of r-forms the inner product induced by the Euclidean inner product in $\mathbb{R}^{n+N}$.

MINORS; TANGENT n-VECTOR AND AREA OF A GRAPH. Let $u(x)$ be a smooth mapping from a bounded domain Ω of $\mathbb{R}^n$ into $\mathbb{R}^N$. For $\alpha, \beta \in I(p,n)$, $1 \le p \le \min(n,N)$, we denote by $M_{\beta\alpha}(\mathrm{D}u(x))$ the determinant of the minor of the Jacobian matrix $\mathrm{D}u(x)$ with rows $\beta = (\beta_1, \ldots, \beta_p)$ and columns $\alpha = (\alpha_1, \ldots, \alpha_p)$, and for convenience we set $M_{00}(\mathrm{D}u(x)) = 1$. From now on we shall refer to $M_{\beta\alpha}(\mathrm{D}u(x)), 0 \le |\alpha| = |\beta| \le \min(n,N)$, as to the *minors* of the Jacobian matrix $\mathrm{D}u(x)$.

The minors $M_{\beta\alpha}(Du(x))$ are related to the Grassmanian coordinates of the tangent plane to the graph,

$$G_u := \{(x,y) \in \Omega \times \mathbb{R}^N \mid y = u(x)\}$$

of u. In fact the vectors $e_j + v_j$, $j = 1, \ldots, n$,

$$v_j := \sum_{i=1}^{N} D_j u^i(x) \epsilon_i$$

yield a basis of the tangent plane to G_u at $(x, u(x))$, thus, if we set

$$M(Du(x)) := (e_1 + v_1) \wedge \ldots \wedge (e_n + v_n)$$

or by a simple computation

$$M(Du(x)) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}}(Du(x))\, e^{\alpha} \wedge \epsilon^{\beta}, \tag{2.1}$$

the *tangent n-vector to G_u at $(x, u(x))$* is given by

$$\xi(x, u(x)) = \frac{M(Du(x))}{|M(Du(x))|} \tag{2.2}$$

and its components relatively to the basis $\{e^{\alpha} \wedge \epsilon^{\beta}\}$ are given by

$$\xi_{\alpha\beta}(x, u(x)) = \frac{\sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}}(Du(x))}{|M(Du(x))|}. \tag{2.3}$$

Observe that we have

$$|M(Du(x))| = \left\{ \sum_{|\alpha|+|\beta|=n} M_{\beta\bar{\alpha}}(Du(x))^2 \right\}^{1/2} = \frac{1}{\xi_{\bar{0}0}}$$

and

$$\xi_{\bar{i}j} = (-1)^{i-1} \xi_{\bar{0}0} D_i u^j,$$

so

$$\frac{\xi_{\alpha\beta}}{\xi_{\bar{0}0}} = \sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}} \left((-1)^{i-1} \frac{\xi_{\bar{i}j}}{\xi_{\bar{0}0}} \right). \tag{2.4}$$

This last relation in fact, characterizes the *simple* n-vectors $\xi \in \wedge_n \mathbb{R}^{n+N}$ with non-zero first component, $\xi_{\bar{0}0} > 0$. This is easily seen since, being ξ simple and $\xi_{\bar{0}0} > 0$, the plane associated to ξ is the graph of a linear map L, so $\xi = (e_1 + Le_1) \wedge \ldots \wedge (e_n + Le_n)$.

Finally we notice that the area of the graph G_u is given by

$$\int_\Omega |M(\mathrm{D}u(x))| \mathrm{d}x.$$

CURRENTS AND INTEGRATION OF FORMS OVER A GRAPH. We recall here some basic facts from the theory of integral currents of Federer and Fleming [31] and we refer for more information to [59] and [30], [38], [48].

We denote by $\mathcal{D}^n(U)$ the space of all infinitely differentiable n-forms with compact support in an open set U of $\mathbb{R}^{n+N}$. Members of the dual space $\mathcal{D}_n(U)$, in the sense of distributions, are called *n-dimensional currents in U*. If $T \epsilon \mathcal{D}_n(U)$ and $V \subset U$ is an open set, the *mass of T in V* is defined by

$$M_V(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^n(U),\ \operatorname{spt} \omega \subset V,\ |\omega(x)| \leq 1 \text{ in } V\}$$

where $|\omega(x)|$ denotes the Euclidean norm of the n-form ω.

A current T with finite mass, $M_U(T) < +\infty$, extends naturally, as a linear and continuous functional, to the space of all compactly supported continuous n-forms with the sup norm. Consequently from the Riesz representation theorem we deduce the existence of a Radon measure $\|T\|$ on U, of a $\|T\|$-measurable function $\vec{T} : U \to \wedge_n \mathbb{R}^{n+N}$ satisfying $|\vec{T}(X)| = 1$, $\|T\| - a.e$, such that

$$T(\omega) = \int < \omega(x), \vec{T}(x) > \mathrm{d}\|T\| ,$$

that is T is representable by integration. Finally, Lebesgue's theorem allows us to extend T to all n-forms with $\|T\|$-summable coefficients (in particular, with Borel bounded coefficients) and to define the *restriction* of T to a Borel subset A of U by

$$T \llcorner A(\omega) = \int_A < \omega, \vec{T} > d\|T\|.$$

We have

$$M_V(T) = M(T \llcorner V) = \|T\|(V), \quad V \subset U, \quad V \text{ open}.$$

For any $T \in \mathcal{D}_n(U)$ the support of $T, \operatorname{spt} T$, is defined in the standard way, the boundary of T is defined by means of Stokes theorem as the $(n-1)$-dimensional current given by

$$\partial T(\omega) = T(d\omega).$$

A sequence of currents $\{T_k\} \subset \mathcal{D}_n(U)$ is said *to converge weakly in* U *to* T if it converges in the sense of distributions, i.e.

$$T_k(\omega) \to T(\omega), \text{ for every } \omega \in \mathcal{D}^n(U).$$

We shall denote the weak convergence in U by $T_k \rightharpoonup T$ in U.

It is easy to prove that the mass is lower semicontinuous with respect to the weak convergence and that from a sequence of currents with equibounded masses we can extract a subsequence converging to a current with finite mass. Conversely Banach-Steinhaus theorem yields that currents with finite masses, weakly converging to a current with finite mass, necessarily have equibounded masses. If we fix the standard basis in $\mathbb{R}^{n+N}$, so that n-forms in $\mathbb{R}^{n+N}$ are written as

$$\omega(x, y) = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}(x, y)\, dx^\alpha \wedge dy^\beta,$$

we can define the *components of the current* $T, T_{\alpha\beta}$, by considering the Schwartz distribution given by

$$\phi \in \mathcal{D}(U) \to T_{\alpha\beta}(\phi) := T(\phi(x, y) dx^\alpha \wedge dy^\beta);$$

then

$$T(\omega) = \sum_{|\alpha|+|\beta|=n} T_{\alpha\beta}(\omega_{\alpha\beta})$$

and clearly T is representable by integration if and only if each $T_{\alpha\beta}$ is a Radon measure.

An important example of current in $\mathbb{R}^{n+N}$ is given by integration of n-forms over an n-dimensional oriented smooth submanifold $\mathcal{M}$ of $\mathbb{R}^{n+N}$ with locally finite area. This current is denoted by $[\![\mathcal{M}]\!]$ and it is given by

$$[\![\mathcal{M}]\!](\omega) = \int_{\mathcal{M}} \omega = \int_{\mathcal{M}} < \omega(z), \xi(z) > d\|\mathcal{M}\|$$

where $\xi(z) = \vec{\mathcal{M}}(z)$ is the n-vector orienting the tangent plane $T_z\mathcal{M}$ to $\mathcal{M}$ at z and $\|\mathcal{M}\|$ is given by the restriction of the $n-$dimensional Hausdorff measure $\mathcal{H}^n$ to $\mathcal{M}$. In this case, by Stokes formula,

$$\partial[\![\mathcal{M}]\!] = [\![\partial\mathcal{M}]\!]$$

and the mass of $[\![\mathcal{M}]\!]$ is the area of $\mathcal{M}$.

In case $\mathcal{M}$ is the graph of a smooth mapping $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, the current $[\![G_u]\!]$ is given by

$$[\![G_u]\!](\omega) = \int < \omega, \xi > \mathrm{d}\mathcal{H}^n \llcorner G_u \tag{2.5}$$

where ξ is the tangent n-vector to G_u in (2.2) or equivalently by

$$[\![G_u]\!](\omega) = \int_\Omega U^{\#}\omega \tag{2.6}$$

where U is the map $U : \Omega \to \mathbb{R}^{n+N}$, $U(x) = (x, u(x))$ and $U^{\#}\omega$ is the pullback of the n-form ω by U. In other words $[\![G_u]\!]$ is the image of the current $[\![\Omega]\!]$ under U

$$[\![G_u]\!](\omega) = U_{\#}[\![\Omega]\!] = [\![\Omega]\!](U^{\#}\omega).$$

A simple computation yields then

$$[\![G_u]\!](\omega) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar{\alpha}) \int_\Omega \omega_{\alpha\beta}(x, u(x)) M_{\beta\bar{\alpha}}(\mathrm{D}u(x))\, \mathrm{d}x \tag{2.7}$$

and

$$[\![G_u]\!]_{\alpha\beta}(\phi) = \sigma(\alpha, \bar{\alpha}) \int_\Omega \phi(x, u(x)) M_{\beta\bar{\alpha}}(Du(x))\, dx \tag{2.8}$$

for its components. So we see that the minors of Du define the components of $[\![G_u]\!]$ and actually $[\![G_u]\!]$.

For a generic current T with finite mass there is no way of defining a 'tangent space' and, even if spt T is a smooth n-manifold, the n-vector $\vec{T}$ associated to T has not to be related in any way to the tangent space to spt T. For this reason, Federer-Fleming [31] introduced the subclass of *integer multiplicity rectifiable currents* $R_n(U)$. This class has good closure properties, and its elements enjoy, in a weak sense, the differential properties of smooth manifolds. Since rectifiable currents are relevant in the sequel, we shall now describe them very briefly.

A subset $\mathcal{M} \subset \mathbb{R}^{n+N}$ is said to be *n-rectifiable* if, except for a $\mathcal{H}^n$-zero set $\mathcal{N}_0$, it is the countable union of $\mathcal{H}^n$-measurable sets $\mathcal{N}_j$ which are subsets of smooth n-dimensional manifolds $\mathcal{M}_j$

$$\mathcal{M} = \mathcal{N}_0 \cup \bigcup_{j=1}^{\infty} \mathcal{N}_j \quad , \quad \mathcal{H}^n(\mathcal{N}_0) = 0 \ , \quad \mathcal{N}_j \subset \mathcal{M}_j. \tag{2.9}$$

For $\mathcal{H}^n$-a.e., z in a n-rectifiable set $\mathcal{M}$, the *approximate tangent space* $\mathrm{Tan}_z \mathcal{M}$ *of* $\mathcal{M}$ *at* z is defined as the tangent space to $\mathcal{M}_j$ at z. Apparently $\mathrm{Tan}_z \mathcal{M}$ seems to depend on the decomposition (2.9), but one can shows that this is not the case. In fact, assuming $\mathcal{M}$ is a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(U \cap K) < +\infty$ for all compact K, one can show (see e.g. [59], pag 60-66)) that $\mathcal{M}$ *is rectifiable if and only if for* $\mathcal{H}^n$ *– a.e. point* $z_0 \in \mathcal{M}$ *there exists an n-dimensional plane* P *such that*

$$\lim_{\lambda \to 0} \int_{\eta_{z_0,\lambda}(\mathcal{M})} f(z)\, d\mathcal{H}^n(z) = \int_P f(z)\, d\mathcal{H}^n(z), \quad \forall f \in C_c^0(\mathbb{R}^{n+N}),$$

where $\eta_{z_0,\lambda}(z) = \lambda^{-1}(z - z_0)$, *or equivalently*

$$\lim_{\lambda\to 0} \lambda^{-n} \int_{\mathcal{M}} f(\lambda^{-1}(z - z_0))d\mathcal{H}^n(z) = \int_P f(y)d\mathcal{H}^n(y).$$

A current $T \in \mathcal{D}_n(U)$ is said to be an integer multiplicity rectifiable current, briefly a *rectifiable current*, if it can be expressed as

$$T(\omega) = \int_{\mathcal{M}} < \omega(z), \xi(z) > \theta(z)d\mathcal{H}^n(z)$$

where $\mathcal{M}$ is an n-rectifiable subset, θ is an $\mathcal{H}^n$-locally summable positive integer valued function, called *multiplicity* of T at z, and ξ an *orientation* on $\mathcal{M}$, that is ξ is an $\mathcal{H}^n$-measurable n-vector field on $\mathcal{M}$ which for $\mathcal{H}^n$-a.e. $z \in \mathcal{M}$ is associated with $\mathrm{Tan}_z\mathcal{M}$ (i.e. $\xi(z)$ can be expressed in the form $\tau_1 \wedge \ldots \wedge \tau_n$, where $\tau_1, \ldots, \tau_n$ form an orthonormal basis for $\mathrm{Tan}_z\mathcal{M}$). A rectifiable current T is denoted by $\tau(\mathcal{M}, \theta, \xi)$. The important closure property of $R_n(U)$ is described by the following theorem due to Federer and Fleming for which we refer to [31], [59], [30] and for a simpler proof to [62].

FEDERER-FLEMING CLOSURE THEOREM. *Let* $\{T_k\} \subset R_n(U), T_k \to T,\ T \in \mathcal{D}_n(U)$. *If for all* $V \subset\subset U$

$$\sup_k\{M_V(T_k) + M_V(\partial T_k)\} < +\infty,$$

then $T \in R_n(U)$.
Consequently, from any sequence of rectifiable currents satisfying the previous sup bound we can extract a subsequence converging weakly to a rectifiable current.

We notice that in general the boundary of a rectifiable current is not rectifiable, and not even of finite mass, but one can show:

BOUNDARY RECTIFIABILITY THEOREM. *If* $T \in R_n(U)$ *and* $M_U(\partial T) < +\infty$, *then* $\partial T \in R_{n-1}(U)$.

Finally we mention the following:

RECTIFICABILITY THEOREM. *Suppose that*

$$T \in \mathcal{D}_n(U) \quad , \quad U \subset \mathbb{R}^{n+N},$$

is such that $M_W(T)+M_W(\partial T) < +\infty$ *for every* $W \subset\subset U$, *and that the measure* $\|T\|$ *has positive upper density for* $\|T\|$ *– a.e.* x *in* U, *i.e.*

$$\theta^{*n}(\|T\|, x) = \limsup_{\rho\to 0} \frac{\|T\|(B_\rho(x))}{\alpha_n \rho^n} > 0, \quad \|T\| - \text{a.e. in } U.$$

Then T *is rectifiable.*

BOUNDARIES: THE CLASSES $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$ AND $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$. We begin by discussing relationships among mappings, their graphs and the associated current integration over graphs for some subclasses of non-smooth functions of the Sobolev spaces $H^{1,p}(\Omega, \mathbb{R}^N)$.

We consider for $p \geq 1$ the family of functions in $H^{1,p}(\Omega, \mathbb{R}^N)$ whose minors are p-summable and we denote this family by $\mathcal{A}^p(\Omega, \mathbb{R}^N)$,

$$\mathcal{A}^p(\Omega, \mathbb{R}^N) := \{u \in L^p(\Omega, \mathbb{R}^N) : M_{\beta\bar{\alpha}}(Du(x)) \in L^p(\Omega, \mathbb{R}^N)\}.$$

In $\mathcal{A}$ we define

$$\|u\|_{\mathcal{A}^p} := \left\{\int_\Omega (|u|^p + |M(Du)|^p)dx\right\}^{1/p}$$

and we say that $\{u_k\} \subset \mathcal{A}^p$ converges weakly in $\mathcal{A}^p$ to $u \in \mathcal{A}^p$, $u_k \overset{\mathcal{A}^p}{\rightharpoonup} u$, if and only if

$$u_k \rightharpoonup u,$$

$$M(Du_k(x) \rightharpoonup M(Du(x)),$$

weakly in L^p. Notice that $\mathcal{A}^p$ is not a linear space and $\|\cdot\|_{\mathcal{A}^p}$ is not a norm.

To $u \in \mathcal{A}^p(\Omega, \mathbb{R}^n)$ we associate the n-dimensional current $T_u \in \mathcal{D}_n(\Omega \times \mathbb{R}^N)$ with components $(T_u)_{\alpha\beta}$ defined for all

$$\phi(x, y) \in C_c^\infty(\Omega \times \mathbb{R}^N)$$

by

$$(T_u)_{\alpha\beta}(\phi(x, y)) := \sigma(\alpha, \bar{\alpha}) \int_\Omega \phi(x, u(x)) M_{\beta\bar{\alpha}}(Du(x))\, dx; \tag{2.10}$$

of course if u is smooth: $T_u = [\![G_u]\!]$.

PROPOSITION 1. *We have*

(i) *if* $u \in \mathcal{A}^p(\Omega, \mathbb{R}^N), p \geq 1$, *then* T_u *is a rectifiable current with bounded mass in* $\Omega \times \mathbb{R}^N$, *and*

$$M(T_u) = \int_\Omega |M(Du)| dx;$$

(ii) *suppose* $p > 1$. *A sequence* $\{u_k\} \subset \mathcal{A}^p(\Omega, \mathbb{R}^N)$ *converges weakly in* $\mathcal{A}^p$ *to some* $u \in \mathcal{A}^p$ *if and only if the currents* T_{u_k} *converge weakly to the current* T_u *and we have* $\sup_k \|u_k\|_{\mathcal{A}^p} < +\infty$.

PROOF. From [44], theor. 3 and 2, there exists a sequence of closed sets $F_k \subset \Omega$ with $\mathcal{H}^n(\Omega \backslash F_k) < \frac{1}{k}$ and a sequence of functions $u_k \in C^1(\Omega, \mathbb{R}^N)$ with

$$u_k = u \quad \text{and} \quad Du_k = Du \quad \text{on } F_k. \tag{2.11}$$

Set $\Omega_0 = \bigcup_{k=1}^{\infty} F_k$ and by induction define

$$H_k := \bigcup_{i=1}^{k} F_k \setminus \bigcup_{i=1}^{k-1} F_k,$$

$$\mathcal{N}_k := G_{u_k} \cap \pi^{-1}(H_k),$$

where $\pi : \mathbb{R}^{n+N} \to \mathbb{R}^n$ is the linear projection $(x, y) \to x$. Clearly $\mathcal{H}^n(\Omega \setminus \Omega_0) = 0$, $\mathcal{M} := \bigcup_{k=1}^{\infty} \mathcal{N}_k$ is covered by a countable family of measurable subsets of C^1-submanifolds, and by the area formula

$$\mathcal{H}^n(\mathcal{M}) = \sum_k \mathcal{H}^n(\mathcal{N}_k) = \sum_k \int_{H_k} |M(Du_k)|dx = \int_{\Omega_0} |M(Du(x))|dx.$$

Using (2.11) we then easily conclude that T_u is rectifiable

$$T_u = \tau(\mathcal{M}, 1, \xi), \quad \xi = \text{Tan } G_{u_k} \quad \text{on } \mathcal{N}_k,$$

moreover

$$M(T_u) = \mathcal{H}^n(\mathcal{M}) = \int_{\Omega_0} |M(Du(x)|dx < +\infty$$

and (i) is proved. Notice that the previous argument gives a way of regarding T_u as the current "integration over the graph of u".
Let us prove (ii), compare [33] theor 3 of sec. 3. If $\{u_k\}$ converges weakly in $\mathcal{A}^p$ to $u \in \mathcal{A}^p$, then $u_k \to u$ strongly in L^p and $\phi(x, u_k) \to \phi(x, u)$ strongly in all L^q and for all $\phi \in C_c^\infty(\Omega \times \mathbb{R}^N)$. Writing (2.10) for u_k and passing to the limit, one sees at once that $T_{u_k} \rightharpoonup T_u$. Conversely, suppose that $T_{u_k} \rightharpoonup T_u$ and $\sup_k \|u_k\|_{\mathcal{A}_p} < +\infty$. Passing to a subsequence, u_k converge strongly in L^p (actually in $L^q, q < p*$, $p*$ being the Sobolev exponent of p) to some v and $\text{meas}\{x : |u_k(x)| > t\} \to 0$, as $t \to +\infty$, uniformly in k. From

$$(T_{u_k})_{\bar{0}0}(\phi(x, y)) = \int_\Omega \phi(x, u_k)dx \to (T_u)_{\bar{0}0}(\phi(x, y))$$
$$= \int_\Omega \phi(x, u(x))dx,$$

we then deduce that $v = u$ and that $u_k \to u$ in $L^p(\Omega, \mathbb{R}^N)$. Analogously, from $(T_{u_k})_{\alpha\beta} \rightharpoonup (T_u)_{\alpha\beta}$, one deduces, since $p > 1$, that $M_{\beta\bar{\alpha}}(Du_k) \rightharpoonup M(Du)$ in $L^p(\Omega)$.

q.e.d.

Let u be a smooth mapping from Ω into $\mathbb{R}^N$, say $u \in C^1(\Omega, \mathbb{R}^N)$. Clearly the submanifold G_u has no topological boundary in $\Omega \times \mathbb{R}^N$, in fact its topological boundary lies in $\partial\Omega \times \mathbb{R}^N$. The same is true also for the measure theoretic boundary of the current $[\![G_u]\!]$, $[\![\partial G_u]\!] = 0$; in fact, by Stokes theorem, for every $(n-1)$-form η with compact support in $\Omega \times \mathbb{R}^N$, we have $[\![G_u]\!](d\eta) = 0$. Now, if $u \in \mathcal{A}^p(\Omega, \mathbb{R})$, $p \geq 1$, and $N = 1$, i.e. u is a scalar function, using the standard Gauss-Green formula, one easily sees that also $\partial T_u = 0$ in $\Omega \times \mathbb{R}$. The situation changes in the vector valued case, in fact the elements of $\mathcal{A}^p(\Omega, \mathbb{R}^N), N > 1$, in general have boundary in $\Omega \times \mathbb{R}^N$. A simple example, compare [33] section 3, is given by the mapping $u_0(x) = x/|x|$ from the unit ball $B(0,1) \subset \mathbb{R}^2$ into $\mathbb{R}^2$, which belongs to $\mathcal{A}^p(B(0,1), \mathbb{R}^2)$ for all $p < 2$, but for which we have

$$\partial T_{u_0} = -[\![\{0\} \times \partial\{y \in \mathbb{R}^2 : |y| \leq 1\}]\!].$$

In other words the graph of u_0 has a hole like the function $x/|x|$ from $\mathbb{R}$ in $\mathbb{R}$ but, while in dimension 1 the summability of the gradient prevents the formation of such holes, if $n, N \geq 2$, even the summability of all minors does not exclude holes.

EXAMPLE 1. In general, consider any smooth mapping (in fact it suffices a Lipschitz mapping) $\varphi : S^{n-1} \subset \mathbb{R}^n \to \mathbb{R}^N$ and its homogeneous extension $u : B(0,1) \subset \mathbb{R}^n \to \mathbb{R}^N$,

$$u(x) = \varphi\left(\frac{x}{|x|}\right).$$

It is easily seen that $u(x) \in \mathcal{A}^p(B(0,1), \mathbb{R}^N)$ for all $p < \frac{n}{n-1}$. Proceeding as in [33] example 1 sec. 3, it is not difficult to see that ∂T_u lies in $\{0\} \times \mathbb{R}^N$ and is given by

$$\partial T_u = -[\![\{0\}]\!] \times \varphi_\# [\![S^{n-1}]\!].$$

This means that ∂T_u is the integration over the manifold $\varphi(S^{n-1})$ with its multiplicity in $\{0\} \times \mathbb{R}^N$.
However, one can find functions u which have essentially the same singularity of $\frac{x}{|x|}$ at zero, but with $\partial T_u = 0$: for example the homogeneous extension of

$$\varphi(\theta) = (\cos(2|\pi - \theta|), \sin(2|\pi - \theta|)), \ 0 \leq \theta \leq 2\pi.$$

But $\partial T_u = 0$ if $u \in H^{1,p}(\Omega, \mathbb{R}^N)$ and $p \geq \min(n, N)$; in fact we have

PROPOSITION 2. *If* $u \in H^{1,\bar{n}}(\Omega, \mathbb{R}^N)$, *where* $\bar{n} = \min(n, N)$, *then* $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$ *and* $\partial T_u = 0$

PROOF. Obviously $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$. Let $u_k \in C^1(\Omega, \mathbb{R}^N) \cap H^{1,\bar{n}}(\Omega, \mathbb{R}^N)$ be a sequence converging strongly in $H^{1,\bar{n}}$ to u. It suffices to show that $T_{u_k} = [\![G_{u_k}]\!] \rightharpoonup T_u$; in fact T_u has then no boundary in $\Omega \times \mathbb{R}^N$ as weak limit of the boundaryless currents $[\![G_{u_k}]\!]$. As in the proof of proposition 1, we get

$$(T_{u_k})_{\alpha\beta}(\phi) \to (T_u)_{\alpha\beta}(\phi)$$

for all α, β with $|\alpha| + |\beta| = n$ and $|\beta| < \bar{n}$. For the last components $|\beta| = \bar{n}$, we have

$$(T_{u_k})_{\alpha\beta}(\phi) = \sigma(\alpha, \bar{\alpha}) \int_\Omega \phi(x, u_k) M_{\beta\bar{\alpha}}(Du_k)\, dx$$

and $M_{\beta\bar{\alpha}}(Du_k)$ converges strongly in L^1 to $M_{\beta\bar{\alpha}}(Du)$, so $M_{\beta\bar{\alpha}}(Du_k)$ is equiabsolutely continuous. This easily yields that $(T_{u_k})_{\alpha\beta}(\phi) \to (T_u)_{\alpha\beta}(\phi)$.

q.e.d.

The previous discussion shows that there are elements in $\mathcal{A}^p$ which cannot be approximated weakly in $\mathcal{A}^p$ by smooth functions, a necessary conditon for that being that $\partial T_u = 0$. For many reasons, and especially in connection with the Calculus of Variation where it is natural to work in classes of weak limits of smooth functions, it is convenient to introduce the following class of cartesian currents.

DEFINITION 1. $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$ denotes the smallest set in $\mathcal{A}^p(\Omega, \mathbb{R}^N)$ containing $C^1(\Omega, \mathbb{R}^N) \cap \mathcal{A}^p(\Omega, \mathbb{R}^N)$ and which is closed with respect to the weak convergence of sequences in $\mathcal{A}^p$.

DEFINITION 2. $\mathrm{cart}^p(\Omega, \mathbb{R}^N) = \{ u \in \mathcal{A}^p(\Omega, \mathbb{R}^N) \mid \partial T_u = 0\}$.

From proposition 1, obviously $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$ is sequentially weakly closed, thus

$$\mathrm{Cart}^p(\Omega, \mathbb{R}^N) \subset \mathrm{cart}^p(\Omega, \mathbb{R}^N).$$

It is reasonable to conjecture that

$$\begin{aligned}\mathrm{Cart}^p(\Omega, \mathbb{R}^N) &= \mathrm{cart}^p(\Omega, \mathbb{R}^N)\\ &= \{ u \in \mathcal{A}^p(\Omega, \mathbb{R}^N) \mid \exists u_k \in C^1(\Omega, \mathbb{R}^N) \cap \mathcal{A}^p(\Omega, \mathbb{R}^N) \quad \text{with } u_k \overset{\mathcal{A}^p}{\rightharpoonup} u \},\end{aligned}$$

but we are not able to prove or disprove such a conjecture.

The following compactness theorem, which is valid in both spaces, makes these spaces, besides being natural, useful in the Calculus of Variations.

THEOREM 1. *Let $\{u_k\}$ be a sequence in $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$ (respectively in $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$), $p > 1$. If $\sup_k \|u_k\|_{\mathcal{A}_p} < +\infty$ and $u_k \rightharpoonup u$ weakly in L^1, then u belongs to $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$ (resp. $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$) and $u_k \overset{\mathcal{A}^p}{\rightharpoonup} u$, i.e.*

$$u_k \rightharpoonup u,$$

$$M_{\beta\bar{\alpha}}(Du_k) \rightharpoonup M_{\beta\bar{\alpha}}(Du),$$

weakly in L^p for all α, β, with $|\alpha| + |\beta| = n$.

The proof of this theorem is given in a slightly different context in [33] theor. 1 of sec. 4. Here we sketch briefly the main steps for future purposes.

LEMMA 1. *Given* $u \in H^{1,1}(\Omega, \mathbb{R}^N)$ *and* $v_{\beta\bar{\alpha}} \in L^1(\Omega)$, $|\alpha|+|\beta| = n$, $|\beta| \geq 2$, *consider the current* S *in* $\Omega \times \mathbb{R}^N$ *with components*

$$\tag{2.12} \begin{aligned} &S_{\bar{0}0}(\phi) = \int_\Omega \phi(x,u)dx \ , \ \ S_{\bar{i}j}(\phi) = (-1)^{i-1}\int_\Omega \phi(x,u)D_i u^j dx, \\ &S_{\alpha\beta}(\phi) = \sigma(\alpha,\bar{\alpha})\int_\Omega \phi(x,u)v_{\beta\bar{\alpha}}dx, \quad |\alpha|+|\beta| = n, \ \ |\beta| \geq 2. \end{aligned}$$

S is a rectifiable current if and only if $v_{\beta\bar{\alpha}}(x) = M_{\beta\bar{\alpha}}(Du(x))$ *for a.e.* x.

PROOF. If $v_{\beta\bar{\alpha}} = M_{\beta\bar{\alpha}}(Du)$, then $S = T_u$ and proposition 1 (i) says that S is rectifiable. Suppose now S rectifiable, then $S = \tau(\mathcal{M}, \theta, \xi)$. Define

$$\mathcal{M}_+ := \{z \in \mathcal{M} : \xi_{\bar{0}0} > 0\}$$

and denote by π the linear projection $(x,y) \to x$.
Then, compare [33] theorems 1, 2 and remark 1 sec. 3, from the area formula and the expression of $S_{\bar{0}0}$ it follows that $\pi(\mathcal{M} - \mathcal{M}_+) = 0$, $\theta = 1$, $\mathcal{H}^n - a.e.$ on $\mathcal{M}_+$, and for a.e., $x \in \pi(\mathcal{M}_+)$ there is a unique $\tilde{u}(x)$ such that $(x, \tilde{u}(x)) \in \mathcal{M}_+$ and moreover $\tilde{u}(x) = u(x)$, $\mathcal{H}^n - a.e.$ in Ω; while from the absolute continuity of $\pi_{\#}S_{\alpha\beta}$ with respect to Lebesgue's measure, we get $\mathcal{M} = \mathcal{M}_+$, $\mathcal{H}^n - a.e.$. Using again the area formula and the expression of all components but $S_{\bar{0}0}$, we then get a.e. in Ω

$$\frac{\xi_{\bar{i}j}}{\xi_{\bar{0}0}} = (-1)^{i-1}D_i u^j,$$

$$\frac{\xi_{\alpha\beta}}{\xi_{\bar{0}0}} = \sigma(\alpha,\bar{\alpha})v_{\beta\bar{\alpha}}.$$

Since S is rectifiable, ξ is simple and as $\xi_{\bar{0}0}$ is positive, (2.4) reads $v_{\beta\bar{\alpha}} = M_{\beta\bar{\alpha}}(Du)$.

q.e.d.

PROOF OF THEOREM 1. Passing to a subsequence

$$\begin{aligned} u_k &\to u \quad \text{in } L^p, \\ Du_k &\rightharpoonup Du \quad \text{weakly in } L^p, \\ M_{\beta\bar{\alpha}}(Du_k) &\rightharpoonup v_{\beta\bar{\alpha}} \quad \text{weakly in } L^p, \end{aligned}$$

for some $v_{\beta\bar{\alpha}} \in L^p$, for $|\alpha|+|\beta| = n$, $|\beta| \geq 2$. Defining S as in the lemma, we have

$$T_{u_k} \rightharpoonup S.$$

Since T_{u_k} is rectifiable, $\partial T_{u_k} = 0$, and the masses of the T_{u_k}'s are equibounded, Federer-Fleming closure theorem yields that S is rectifiable and lemma 1 gives

$u \in \mathcal{A}^p$ and $S = T_u$. This concludes the proof of the theorem for $\text{cart}^p(\Omega, \mathbb{R}^N)$.

q.e.d.

Finally we notice that the classes $\text{Cart}^p(\Omega, \mathbb{R}^N)$ and $\text{cart}^p(\Omega, \mathbb{R}^N)$ are neither linear nor convex subclasses of $H^{1,p}(\Omega, \mathbb{R}^N)$ and that they 'coincide' with the classes denoted by the same symbols introduced in [33].

BOUNDARIES AND TRACES. Suppose Ω has a smooth boundary. As

$$\text{cart}^p(\Omega, \mathbb{R}^N) \subset H^{1,p}(\Omega, \mathbb{R}^N),$$

for each $u \in \text{cart}^p(\Omega, \mathbb{R}^N)$ the trace of u on $\partial\Omega$ is well defined in the sense of Sobolev spaces. The current T_u has no boundary in $\Omega \times \mathbb{R}^N$; but, since the mass of T_u in $\Omega \times \mathbb{R}^N$ is finite, T_u can be seen as a current in $\mathbb{R}^n \times \mathbb{R}^N$, simply by setting

$$T_u(\omega) = T_u(\chi_\Omega \omega), \quad \forall \omega \in \mathcal{D}^n(\mathbb{R}^n \times \mathbb{R}^N),$$

χ_Ω being the characteristic function of Ω. When seen as a current in $\mathbb{R}^n \times \mathbb{R}^N$, T_u has of course boundary ∂T_u, and a natural question is whether the trace of u determines the boundary of T_u, that is if $u, v \in \text{cart}^p(\Omega, \mathbb{R}^N)$ and $u = v$ on $\partial\Omega$, is it true that $\partial T_u = \partial T_v$? The answer is in general negative, as shown by the example below, and it is positive in case $u, v \in \text{cart}^p(\Omega, \mathbb{R}^N) \cap H^{1,\bar{n}}(\Omega, \mathbb{R}^N)$, $\bar{n} = \min(n, N)$.

EXAMPLE 2. As in example 1, consider a smooth function $\varphi : S^1 \subset \mathbb{R}^2 \to \mathbb{R}^N$ and its homogeneous extension to $B(0,1) \subset \mathbb{R}^2, u(x) = \varphi(x/|x|)$. Denote by $v(x)$ the restriction of $u(x)$ on $B_+(0,1) = B(0,1) \cap \{x_2 > 0\}$. Since v is regular in $B_+(0,1), T_v$ has no boundary in $B_+(0,1)$, and obviously v belongs to $\text{Cart}^p(B^+(0,1), \mathbb{R}^N)$ for all $p < 2$. Regarding T_v as current on $\mathbb{R}^2 \times \mathbb{R}^N$, as before one sees that the boundary of T_v lies in $\partial B_+(0,1) \times \mathbb{R}^N$ and

$$\begin{aligned} \partial T_v \llcorner \pi^{-1}\{x_2 = 0\} = [\![[(-1,0),(0,0)]]\!] \times [\![\{\varphi(P_1)\}]\!] + \\ + [\![[(0,0),(0,1)]]\!] \times [\![\{\varphi(P_2)\}]\!] \; - \; \delta_0 \times \varphi_\# [\![S^1_+]\!] \end{aligned}$$

where $S^1_+ = S^1 \cap \{x_2 > 0\}$, and P_1, P_2 are the points of S^1 with cartesian coordinates $(1,0),(0,1)$ or, equivalently, polar coordinates 0 and π. If we choose $N = 2$ and φ in polar coordinates, $\varphi(\theta) = (\frac{1}{2}\sin 2\theta, \frac{1}{2} - \frac{1}{2}\cos 2\theta)$, i.e. $v(x) = (x_1x_2/|x|^2, x_2^2/|x|^2)$, the trace of v in $\{x_2 = 0\}, \gamma v$, is zero, $\varphi(P_1) = \varphi(P_2) = (0,0)$, while

$$\partial T_v \llcorner \pi^{-1}\{x_2 = 0\} = \; [\![G_{\gamma v}]\!] - [\![\{0\} \times \partial B((0,1/2),1/2)]\!].$$

Of course the boundary in $\{x_2 = 0\}$ of the current associated to the function $(0,0)$ is given by $[\![G_{\gamma v}]\!]$.

Essentially a similar situation occurs for the mapping $w = (z/|z|)^{2k}$ in complex coordinates. The associated current has boundary in $\{0\} \times \mathbb{R}^2$ given by $2k[\![S^1]\!]$; the boundary of the current associated to the restriction of $(z/|z|)^{2k}$ to Im$z > 0$ is given by

$$[\![G_w \text{ on Im} z = 0]\!] - k[\![\{0\} \times S^1]\!].$$

Notice that $2(x_1x_2/|x|^2, x_2^2/|x|^2)$ can be obtained from $z^2/|z|^2$ by a rotation R of 45 degrees in the z plane plus a translation in the y-plane

$$2v(x) = w(Rz) + (0,1) \ , \qquad z = x_1 + ix_2.$$

THEOREM 2. *Let Ω be a bounded domain with smooth boundary, and let u and v be functions in $H^{1,\bar{n}}(\Omega, \mathbb{R}^N)$, $N \geq 1$, $\bar{n} = \min(n, N)$, with the same trace on $\partial\Omega$, $u - v \in H_0^{1,\bar{n}}(\Omega, \mathbb{R}^N)$.*
Then ∂T_u and ∂T_v lie in $\partial\Omega \times \mathbb{R}^N$ and $\partial T_u = \partial T_v$.

PROOF. We fix some open set $\tilde{\Omega} \supset\supset \Omega$ and extend u and v as functions in $H^{1,\bar{n}}(\tilde{\Omega})$ with $u = v$ on $\tilde{\Omega}\backslash\Omega$. Denote by u_ϵ, v_ϵ the standard mollifiers of u, v and let $\Omega_\epsilon = \{x : \text{dist}(x, \Omega) < \epsilon\}$. Obviously

$$T_{u_\epsilon} \llcorner \pi^{-1}\Omega \rightharpoonup T_u,$$
$$T_{v_\epsilon} \llcorner \pi^{-1}\Omega \rightharpoonup T_v,$$
$$\partial(T_{u_\epsilon} \llcorner \pi^{-1}\Omega_\epsilon) = \partial(T_{v_\epsilon} \llcorner \pi^{-1}\Omega_\epsilon),$$

the last equality being true, since u_ϵ, v_ϵ are regular and $v_\epsilon = u_\epsilon$ on $\partial\Omega_\epsilon$. Moreover, since the minors of Du_ϵ and Dv_ϵ are equiabsolutely continuous, we have

$$M(T_{u_\epsilon} \llcorner \pi^{-1}(\Omega_\epsilon\backslash\Omega)), \ M(T_{v_\epsilon} \llcorner \pi^{-1}(\Omega_\epsilon\backslash\Omega)) \to 0, \quad \text{as } \epsilon \to 0.$$

Therefore, for all $(n-1)$, form ω with compact support in $\mathbb{R}^n \times \mathbb{R}^N$, we have

$$|\partial(T_{u_\epsilon} \llcorner \pi^{-1}\Omega)(\omega) - \partial(T_{v_\epsilon} \llcorner \pi^{-1}\Omega)(\omega)| = |(T_{u_\epsilon} - T_{v_\epsilon}) \llcorner \pi^{-1}(\Omega_\epsilon\backslash\Omega)(\omega)|$$
$$\leq \left[M(T_{u_\epsilon} \llcorner \pi^{-1}(\Omega_\epsilon\backslash\Omega)) + M(T_{v_\epsilon} \llcorner \pi^{-1}(\Omega_\epsilon\backslash\Omega)\right] \sup|d\omega| \to 0, \quad \text{as } \epsilon \to 0$$

and this concludes the proof since

$$\partial(T_{u_\epsilon} \llcorner \pi^{-1}\Omega) \rightharpoonup \partial T \ , \ \partial(T_{v_\epsilon} \llcorner \pi^{-1}\Omega) \rightharpoonup \partial T \text{ in } \mathbb{R}^n \times \mathbb{R}^N$$

q.e.d.

WEAK CONVERGENCE OF MINORS. As we have seen in theorem 1, bounded sequences in Cart$^p(\Omega, \mathbb{R}^N)$ or cart$^p(\Omega, \mathbb{R}^N)$ which converge weakly in L^1 have minors converging weakly to the minors of the L^p-limits; moreover the graphs have no holes. Both these properties are relevant in the Calculus of Variations, but on the other hand one can ask in general whether the minors of a bounded sequence of mappings in $\mathcal{A}^p$ converge to the minors of the limit function in

L^p. The answer to this question is in general negative. For instance, in [7], counterexample 7.4, one can find a sequence of functions $\{u_k\} \subset H^{1,p}(\Omega, \mathbb{R}^n)$ for all $p < n$, which is equibounded in $\mathcal{A}^p$ for $q < \frac{n}{n-1}$, is weakly converging in $H^{1,p}$ for all $p < n$ to a function u, but such that the minors $M(Du_k)$ do not converge to $M(Du)$. According to our next theorem, the lack of convergence of the minors is related here to the fact that the masses of the boundaries of the associated currents T_{u_k} diverge to plus infinity,

$$M(\partial T_{u_k}) \to +\infty.$$

In fact a sufficient condition for the weak compactness of bounded sequences in the $\mathcal{A}^p$ is the equiboundedness of the masses of the ∂T_{u_k}, as stated by the following theorem which has exactly the same proof of theorem 1.

THEOREM 3. *Let $\{u_k\}$ be a sequence in $\mathcal{A}^p(\Omega, \mathbb{R}^N)$, $p > 1$. If u_k converges weakly to u in $L^1(\Omega, \mathbb{R}^N)$ and*

$$\sup_k \|u_k\|_{\mathcal{A}^p} < +\infty \quad , \quad \sup_k M(\partial T_{u_k}) < +\infty,$$

then

$$u_k \rightharpoonup u, \quad M(Du_k) \rightharpoonup M(Du),$$

weakly in L^p.

Actually theorem 3 on account of lemma 1, more than on Federer-Fleming closure theorem, relies on the rectifiability theorem. In fact, consider a sequence $\{u_k\} \subset \mathcal{A}^p$ such that

$$u_k \rightharpoonup u, \quad Du_k \rightharpoonup Du, \quad M_{\beta\bar{\alpha}}(Du_k) \rightharpoonup v_{\beta\bar{\alpha}}$$

hold weakly in L^p with $|\alpha| + |\beta| = n, |\beta| \geq 2$, and as in lemma 1 define the current S with components given by (2.15). As in the proof of proposition 1, we can find a sequence of disjoint Borel sets H_k and a sequence of functions $w_k \in C^1(\Omega, \mathbb{R}^N)$ with

$$w_k = u \ , \ Dw_k = Du \quad \text{on } H_k$$

and we can also assume that the $v_{\beta\bar{\alpha}}(x)$ are in H_k restrictions of continuous functions. Set $\Omega_0 = \bigcup_k H_k$, $\mathcal{N}_k = G_{w_k} \cap \pi^{-1}(H_k)$, $\mathcal{M} = \bigcup_k \mathcal{N}_k$. Then one easily sees that $T_{w_k} \rightharpoonup S$ defined in (2.12) and that

$$S(\omega) = \int < \omega(z), \vec{S}(z) > \mathrm{d}\|S\|, \qquad z = (x, y), \tag{2.14}$$

where

$$\|S\| = \theta(z)\mathcal{H}^n \llcorner \mathcal{M} \ ,$$

$$\theta(z) = |M(Dw_k)|^{-1}(1 + |Dw_k|^2 + \Sigma v^2_{\beta\bar{\alpha}})^{1/2} \ ,$$

$$z \in \mathcal{N}_k \text{ so } w_k(z) = u(z),$$

(2.15)
$$\vec{S}(z) = \sum_{|\alpha|+|\beta|=n} \varsigma_{\alpha\beta} \ e^\alpha \wedge \epsilon^\beta,$$

$$\varsigma_{\bar{0}\bar{0}}(z) = (1 + |Du(x)|^2 + \Sigma v^2_{\alpha\beta}(x))^{-1/2},$$

$$\varsigma_{\bar{i}j}(z) = (-1)^{i-1} D_i u^j(z) \varsigma_{\bar{0}\bar{0}}(z) \ , \ \varsigma_{\alpha\beta} = \sigma(\alpha, \bar{\alpha}) v_{\beta\bar{\alpha}}(z) \varsigma_{\bar{0}\bar{0}}(z).$$

Obviously $\theta^{*n}(\|S\|, z) > 0$ $\|S\|$-a.e., so if we assume $\sup_k \|\partial T_{u_k}\| < +\infty$, which gives $M(\partial S) < +\infty$, the rectifiability theorem yields at once that S is rectifiable and lemma 1 that $v_{\beta\bar{\alpha}} = M_{\beta\bar{\alpha}}(Du)$.

Using the special structure of our currents T_{u_k} and S, and essentially repeating the simplest part of the proof of the rectifiability theorem, as pointed out by S. Müller [50], we can actually give a much weaker condition ensuring the weak convergence of minors.

THEOREM 4. *Let u_k be a bounded sequence in $\mathcal{A}^p$ converging weakly in L^p to some function u. Denote by $\mathcal{F}$ the class of linear combinations of $(n-1)$-forms in $\mathbb{R}^{n+N}$ of the type*

(2.16)
$$\phi(x, y^i)\, \mathrm{d}x^\alpha \wedge \mathrm{d}y^\beta, \qquad |\alpha| + |\beta| = n - 1, \ i \in \bar{\beta}.$$

Suppose that

(2.17)
$$\sup_k \sup \{\partial T_{u_k}(\omega) \mid \omega \in \mathcal{F}, |\omega| \leq 1\} < +\infty.$$

Then $M(Du_k) \rightharpoonup M(Du)$ weakly in L^p.

This theorem is an immediate consequence of the following proposition.

PROPOSITION 3. *Let S be the current defined in* (2.12) *or equivalently in* (2.14), (2.15). *Suppose that*

(2.18)
$$\sup\{\partial S(\omega) \mid \omega \in \mathcal{F}, |\omega| \leq 1\} < +\infty$$

where $\mathcal{F}$ is the family of forms in theorem 4. *Then S is rectifiable and in particular we have $v_{\beta\bar{\alpha}} = M_{\beta\bar{\alpha}}(Du)$ for all α, β with $|\alpha| + |\beta| = n$, $\beta \geq 2$.*

PROOF. Using the previous notations, on account of lemma 1, it suffices to show that, for all k and all $z_0 \in \mathcal{N}_k$, $\vec{S}(z_0)$ is a simple n-vector orientating $\mathrm{Tan}_{z_0} \mathcal{N}_k$.
Following [59], pag 186-187, we use a blow up argument around z_0. Writing $\eta_\lambda(z) = \lambda^{-1}(z - z_0)$,

$$\theta^{*n}(\|S\|, \mathcal{M} - \mathcal{N}_k, z_0) = 0,$$

one easily sees, compare [59], that for $\lambda \to 0$

$$\eta_{\lambda\#}S(\omega) \to \theta(z_0)\int_P <\varsigma(z_0),\omega(z)> \mathrm{d}\mathcal{H}^n$$

where P is the tangent space $T_{z_0}\mathcal{N}_k$ of $\mathcal{N}_k$ at z_0, i.e. of G_{u_k} at z_0, while

$$\partial\eta_{\lambda\#}S(\omega) \to 0, \quad \forall\omega \in \mathcal{F}.$$

Finally, since the masses of $\eta_{\lambda\#}S$ are locally equibounded, passing possibly to a subsequence $\lambda_\ell \downarrow 0$, we conclude that

$$\eta_{\lambda_\ell\#}S \rightharpoonup S_\infty, \ \partial S_\infty(\omega) = 0, \quad \forall\omega \in \mathcal{F},$$

where S_∞ is given by

$$S_\infty(\omega) = \theta(z_0)\int_P <\varsigma(z_0),\omega(z)> \ d\mathcal{H}^n(z),$$

$$P = \text{the plane } y = u(x_0) + \pi\cdot(x - x_0)\ ,$$

$$\pi = Du(x_0) \text{ with the natural orientation.}$$

We shall now prove that $\varsigma(z_0)$ orients P. Denote by R the linear transformation

$$\bar{x} = x\ ,\ \bar{y} = y - u(x_0) - \pi\cdot(x - x_0).$$

Observing that the forms of the type

$$\phi(x, y^i - u(x_0) - \pi\cdot(x - x_0))\,\mathrm{d}x^\alpha \wedge \mathrm{d}(y - u(x_0) - \pi\cdot(x - x_0))^\beta,$$

with $|\alpha| + |\beta| = n - 1$ and $i \in \bar{\beta}$, belong to $\mathcal{F}$, we get

$$\partial R_\# S(\eta) = 0,$$

for all y of the type

$$\Phi(\bar{x}, \bar{y}^i)d\bar{x}^\alpha \wedge d\bar{y}^\beta, \quad |\alpha| + |\beta| = n - 1, \quad i \in \bar{\beta}.$$

This, compare [59] p. 187, easily gives that $R_\#\varsigma$ is the orienting n-vector of the plane $\mathbb{R}^n \times \{0\}$ and therefore ς is the simple n-vector orienting P.

q.e.d.

A condition implying of course (2.17) is

$$\partial T_{u_k}(\omega) = 0 \quad \forall\omega \in \mathcal{F}. \tag{2.19}$$

It is easily seen that (2.19) can be written more explicitly as the family of "Green-formulas": *for all* α, β, i, *with* $|\alpha| + |\beta| = n - 1$, $i \in \bar{\beta}$,

$$\int_\Omega \Big[\sum_{j \in \bar{\alpha}} \sigma(j, \bar{\alpha} - j) \phi_{x_j}(x, u_k^i) M_{\beta(\bar{\alpha}-j)}(Du_k) \qquad (2.20)$$

$$+ \sigma(i, \beta) \phi_{y^i}(x, u_k^i) M_{(\beta+i)\bar{\alpha}}(Du_k) \Big] dx = 0$$

where $\beta + i$ denotes $\beta \cup \{i\}$ in the natural order so that $\sigma(i, \beta)$ is the sign of the permutation which reorders the indices in the normal order, i.e. $dx_i \wedge dx^\beta = \sigma(i, \beta) dx^{\beta+i}$. Thus, again on account of proposition 3,

PROPOSITION 4. *Let* $\{u_k\}$ *be a sequence in* $L^1(\Omega, \mathbb{R}^N)$ *with minors in* $L^1(\Omega)$. *Suppose that* $u_k \to u$, $M_{\beta\bar{\alpha}}(Du_k) \to v_{\beta\bar{\alpha}}$ *strongly in* L^1 *and that* (2.20) *holds for all elements* u_k. *Then*

$$v_{\beta\bar{\alpha}} = M_{\beta\bar{\alpha}}(Du).$$

We remark that proposition 3 trivially implies theorem 1, and actually allows to give a much simpler proof of it. This was indeed pointed out by S. Müller in a recent paper [50], where he proved in a slightly different way proposition 4 under slightly stronger assumptions. We have preferred starting with our original approach because it extends immediately to more general and relevant situations.

Formally, the relations (2.20) can be written as

$$\int_\Omega \sum_{j \in \bar{\alpha}} \sigma(i, \bar{\alpha} - j) \frac{\partial}{\partial x_j} \phi(x, u_k^i) M_{\beta(\bar{\alpha}-j)}(Du_k) dx = 0,$$

since by Laplace formula

$$\sigma(i, \beta) M_{(\beta+1)\bar{\alpha}}(Du_k) = \sum_{j \in \bar{\alpha}} \sigma(j, \bar{\alpha} - j) D_j u_k^i M_{\beta(\bar{\alpha}-j)}(Du_k), \qquad (2.21)$$

and they yield, for $\phi = \phi(x)$, that "all minors have free divergence"

$$\frac{\partial}{\partial x_j} M_{\beta(\bar{\alpha}-j)}(Du_k) = 0. \qquad (2.22)$$

Notice that for smooth functions (2.21), (2.22) hold and are indeed equivalent to (2.20).

As one might expect, condition (2.19), equivalently (2.20), is weaker than $\partial T_{u_k} = 0$. For the reader's convenience we state, omitting its simple proof, the following proposition.

PROPOSITION 5. *Let $u \in H^{1,1}(\Omega, \mathbb{R}^N)$ and $M(Du) \in L^1$. Then $\partial T_u = 0$ if and only if for all α, β, with $|\alpha| + |\beta| = n-1$, and for all ϕ in $\mathcal{D}(\Omega \times \mathbb{R}^N)$, we have*

$$\int_\Omega \Big[\sum_{j \in \bar{\alpha}} \sigma(j, \bar{\alpha} - j) \phi_{x_j}(x, u) M_{\beta(\bar{\alpha}-j)}(Du) \tag{2.23}$$
$$+ \sum_{j \in \bar{\beta}} \sigma(j, \beta) \phi_{y^i}(x, u) M_{(\beta+i)\bar{\alpha}}(Du) \Big] dx = 0.$$

Example 2 below shows a function u in $\mathcal{A}^p$ satisfying (2.20), but not (2.23), i.e. with non-zero ∂T_u. But first let us introduce one more family of functions. We denote by $A^p(\Omega, \mathbb{R}^N)$ the subfamily of mappings u of $\mathcal{A}^p(\Omega, \mathbb{R}^N)$ for which the Green-formulas (2.20) hold with u_k replaced by u

$$A^p(\Omega, \mathbb{R}^N) = \{u \in \mathcal{A}^p(\Omega, \mathbb{R}^N) : \text{(2.20) hold for all } |\alpha| + |\beta| = n-1,\ i \in \bar{\beta}\}.$$

It is easily seen that $x/|x| \in \mathcal{A}^p(B(0,1), \mathbb{R}^N)$ does not belong to $A^p(\Omega, \mathbb{R}^N)$, thus

$$A^p(\Omega, \mathbb{R}^N) \underset{\neq}{\subset} \mathcal{A}^p(\Omega, \mathbb{R}^n).$$

Moreover by the example below

$$\operatorname{cart}^p(\Omega, \mathbb{R}^N) \underset{\neq}{\subset} A^p(\Omega, \mathbb{R}^N).$$

However sequences in $A^p(\Omega, \mathbb{R}^N), p > 1$, which are equibounded in $\mathcal{A}^p$, are weakly compact, i.e. the minors converge weakly in L^p to the minors of the limit function; and this makes it possible to discuss variational problems in $A^p(\Omega, \mathbb{R}^N)$.

EXAMPLE 2. As in example 1, consider the Lipschitz-mapping $\varphi : S^1 \to \mathbb{R}^3$ whose components in polar coordinates, $x = (r, \theta)$, are given by:

$$\varphi_1(\theta) = \cos 4\theta, \quad \varphi_2(\theta) = \sin 4\theta, \quad \varphi_3(\theta) = 0,$$

for $0 \leq \theta \leq \pi/2$;

$$\varphi_1(\theta) = 1, \quad \varphi_2(\theta) = 0, \quad \varphi_3(\theta) = \theta - \pi/2,$$

for $\pi/2 \leq \theta \leq \pi$;

$$\varphi_1(\theta) = \cos 4(\pi - \theta), \quad \varphi_2(\theta) = \sin 4(\pi - \theta), \quad \varphi_3(\theta) = \frac{\pi}{2},$$

for $\pi \leq \theta \leq \frac{3}{2}\pi$;

$$\varphi_1(\theta) = 1, \quad \varphi_2(\theta) = 0, \quad \varphi_3(\theta) = 2\pi - \theta,$$

for $\frac{3}{2} \leq \theta \leq 2\pi$; and consider its homogeneous extension $u(x) = \varphi\left(\frac{x}{|x|}\right)$. As we have seen $u \in \mathcal{A}^p(B(0,1), \mathbb{R}^3)$ and, using our special φ,

$$\begin{aligned}\partial T_u = [\![0]\!] \times \varphi_{\#}[\![S^1]\!] &= [\![0]\!] \times [\![\{y : |y|^2 = 1, y_3 = 0]\!] \\ &+ [\![0]\!] \times [\![\{(1,0,y_3) : 0 \leq y_3 \leq \pi/2]\!] \\ &- [\![0]\!] \times [\![\{y : y_1^2 + y_2^2 = 1, y_3 = \pi/2]\!] \\ &- [\![0]\!] \times [\![\{1,0,y_3\} : 0 \leq y_3 \leq \pi/2]\!] \\ = [\![0]\!] \times [\![\{y : y_1^2 + y_2^2 = 1, y_3 = 0\}]\!] &- [\![0]\!] \times [\![\{y : y_1^2 + y_2^2 = 1, \ y_3 = \pi/2\}]\!],\end{aligned}$$

thus $\partial T_u \neq 0$. Observe now that if P is one of the three 2-dimensional coordinate planes in $\mathbb{R}^n_y$ and if π_p is the orthogonal projection of $\mathbb{R}^2 \times \mathbb{R}^3$ on $\mathbb{R}^2 \times P$, then obviously $\partial\pi_{p\#}T_u = \pi_{p\#}\partial T_u = 0$. Thus one easily sees that $\partial T_u = 0$ on all $(n-1)$-forms, $n = 2$, of the type

$$(2.24) \qquad \omega = \sum_{\substack{|\alpha|+|\beta|=n-1 \\ i \in \bar{\beta}}} \varphi_{\alpha\beta i}(x, y^{\beta_1}, y^{\beta_2}, \ldots, y^{\beta_{|\beta|}}, y^i)\, dx^\alpha \wedge dy^\beta ,$$

which clearly contain the $(n-1)$-form of type $\mathcal{F}$ in theorem 4, so $\partial T_u(\omega), \forall \omega \in \mathcal{F}$. We notice that the forms in (2.24) are exactly the ones used in [50] and that $\partial T_u(\omega) = 0$, for all ω of type (2.24), is equivalent to the relations (2.20) where ϕ is allowed to depend on $u^{\beta_1}, \ldots, u^{\beta_{|\beta|}}$.

Of course the example above extends to all dimensions, for instance a similar example from $\mathbb{R}^3 \to \mathbb{R}^3$ is given by the function $\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2)$.

We observe that, if $\{u_k\}$ is any sequence of smooth functions converging in $L^1(B(0,1), \mathbb{R}^3)$ to $u(x) = \varphi\left(\frac{x}{|x|}\right)$ defined above, then

$$\int_\Omega |M(Du_k)|^2 dx \to +\infty.$$

Otherwise, passing to a subsequence, $T_{u_k} \rightharpoonup T_u$, $u \in \mathrm{cart}^p(B(0,1), \mathbb{R}^3)$, so in particular $\partial T_u = 0$ contradicting the fact that $\partial T_u \neq 0$.

Finally, we observe that condition (2.17) or (2.19), (2.20) are not invariant by orthogonal transformation in $\mathbb{R}^N_y$, while the conclusions in theorem 4 or proposition 4 are invariant.

VARIATIONAL PROBLEMS. Consider a variational integral of the type

$$\mathcal{F}(u, \Omega) = \int_\Omega F(x, u(x), M(Du(x)))\, dx$$

where $M(Du(x))$ stands for all minors of the Jacobian matrix of a mapping $u : \Omega \to \mathbb{R}^N$, and suppose that $F(x, u, M)$ is convex in M and, for simplicity, satisfies

$$|M|^p \leq F(x,u,M) \leq c(1+|M|^p) \ , \ p>1.$$

Of course we can extend the functional $\mathcal{F}$, naturally defined on the class of smooth mappings, to $\mathcal{A}^p(\Omega,\mathbb{R}^N)$ in the trivial way, for $u \in \mathcal{A}^p(\Omega,\mathbb{R}^N)$

$$\mathcal{F}(u,\Omega) = \int_\Omega F(x,u(x),M(Du(x)))dx,$$

so $\mathcal{F}$ is defined in each of the spaces

$$\mathrm{Cart}^p(\Omega,\mathbb{R}^N) \subset \mathrm{cart}^p(\Omega,\mathbb{R}^N) \underset{\neq}{\subset} A^p(\Omega,\mathbb{R}^N) \underset{\neq}{\subset} \mathcal{A}^p(\Omega,\mathbb{R}^N).$$

In view of the weak compactness theorems we have proved, and of a classical theorem of semicontinuity, we can conclude, compare e.g. [33], the existence of a minimizer with prescribed Dirichlet's boundary in each of the classes $\mathrm{Cart}^p(\Omega,\mathbb{R}^N), \mathrm{cart}^p(\Omega,\mathbb{R}^N), A^p(\Omega,\mathbb{R}^N)$. But a few remarks are necessary.

Let us start by discussing the 'Dirichlet boundary conditions'. For the sake of simplicity, let us fix a smooth mapping in $\mathbb{R}^n$ and its restriction u_0 in Ω. We have several possibilities of prescribing the "Dirichlet boundary condition u_0" for elements of (subfamilies of) $\mathcal{A}^p(\Omega,\mathbb{R}^N)$.

(a) As $\mathcal{A}^p(\Omega,\mathbb{R}^N) \subset H^{1,p}(\Omega,\mathbb{R}^N)$, we can require that the trace of $u \in \mathcal{A}^p(\Omega,\mathbb{R}^N)$ and of u_0 on $\partial\Omega$ be equal, i.e. $u - u_0 \in H_0^{1,p}(\Omega,\mathbb{R}^N)$.

(b) We can require that $\partial T_u \llcorner (\partial\Omega \times \mathbb{R}^N) = \partial T_{u_0} \llcorner (\partial\Omega \times \mathbb{R}^N)$.

(c) More generally, we can impose that $(\partial T_u - \partial T_{u_0})(\omega) = 0$ for all $(n-1)$-forms in some family containing the forms of the type

$$\phi(x)\widehat{dx_i},$$

so that $u - u_0 \in H_0^1(\Omega,\mathbb{R}^N)$.
For all corresponding 'Dirichlet's problems' we can easily prove existence, but the geometrical or physical situation described might be completely different. In a sense, only problem (b), which gives the maximum of prescriptions, should be considered as the 'Dirichlet problem.'

Secondly, and this is probably more important for the sequel, in $A^p(\Omega,\mathbb{R}^N)$ we should expect a Lavrentiev phenomenon, i.e. that

$$\inf_{A^p(\Omega,\mathbb{R}^N)} \mathcal{F}(u,\Omega) \ < \ \inf_{\mathrm{Cart}^p} \mathcal{F}(u,\Omega).$$

In fact the example 2 above shows that $\mathcal{F}(u,\Omega)$ in $A^p(\Omega,\mathbb{R}^N)$ is not the 'Lebesgue extension' of $\mathcal{F}$ on $A^p(\Omega,\mathbb{R}^N)$,

$$\tilde{\mathcal{F}}(u,\Omega) := \inf\left\{\liminf_{k\to\infty} \mathcal{F}(u_k,\Omega) : \{u_k\} \text{ smooth, } u_k \rightharpoonup u \text{ in } L^1\right\},$$

since if $\partial[\![G_u]\!] \neq 0$, $\tilde{\mathcal{F}}(u,\Omega) = +\infty$ and the possible occurrence of boundaries in $\Omega \times \mathbb{R}^N$ of ∂T_u, $u \in A^p(\Omega, \mathbb{R}^N)$, makes the problem, as it will be clarified in the sequel, more like a boundary problem with free 'boundaries' in the interior of Ω than like a real Dirichlet problem. In some situations we shall instead see that $\mathcal{F}$ on $\text{Cart}^p(\Omega, \mathbb{R}^N)$ is the Lebesgue extension of $\mathcal{F}$ on smooth mappings, and we are tempted to conjecture this fact in general, but we have no proof of it.

Lavrentiev phenomenon, related to the choice of the space and the extension of the functional there, occurs in many contexts, for example in elasticity in connection with cavitation (see e.g. [6] and for a discussion [33], see also [45]) and it seems typical when considering mappings on manifolds, see e.g. [37], [14]. In the following sections, we shall explain how this may depend on the choice of Sobolev spaces instead of a space of 'cartesian currents', in the same way as for A^p and cart^p.

The rest of this section will be dedicated to defining briefly general cartesian currents and discussing some of their relevant properties. We shall refer to [33] for the proofs. The general setting, which at first sight might appear exaggerated, is justified by the relevance of problems where one asks to minimize integrals, for example, in dimension $n = 2$, of the type

$$\int_\Omega \left[\frac{1}{2}|Du|^2 + |\det Du|\right] \, dx, \qquad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R},$$

where one has a quadratic behaviour in $|Du|$ and only a linear growth with respect to the minors (or some of the minors).

CARTESIAN CURRENTS AND THE PROJECTION FORMULA. Let Ω be a bounded domain in $\mathbb{R}^n_x$. From now on we shall denote by U the cylinder $\Omega \times \mathbb{R}^N_y$ in $\mathbb{R}^{n+N} = \mathbb{R}^n_x \times \mathbb{R}^N_y$. We define the cartesian norm of a current $T \epsilon \mathcal{D}_n(U)$ by setting

$$\|T\|_C = M_U(T) + \|T\|_{L^1(\Omega)}$$

where

$$\|T\|_{L^1(\Omega)} := \sup\{T_{\bar{0}0}(|y|\phi(x,y)) : \phi \in \mathcal{D}(U) \ , \|\phi\|_{\infty . U} \leq 1\},$$

$T_{\bar{0}0}$ being the first component of T with respect to the fixed basis of $\mathbb{R}^n_y \times \mathbb{R}^N_y$. Finally we denote by C the Banach space of currents with finite cartesian norm

$$C := \{T \in \mathcal{D}_n(U) : \|T\|_C < +\infty\}.$$

The Banach space $\mathcal{C}$ is the dual of a separable Banach space and the weak* convergence in $\mathcal{C}$ amount to weak convergence in the sense of currents with equibounded $\mathcal{C}$ norm, compare [33].

DEFINITION 3. (compare [33]). The family of cartesian currents $\text{cart}(\Omega, \mathbb{R}^N)$, is defined by

$$\text{cart}(\Omega, \mathbb{R}^N) := \{T \in C : T \in R_n(U), \partial T = 0 \text{ in } \Omega \times \mathbb{R}^N, \ \pi_\# T = [\![\Omega]\!], T_{\bar{0}0} \geq 0\}$$

i.e. as the family of rectifiable currents with finite cartesian norm which have no boundary in $\Omega \times \mathbb{R}^N$, project into $[\![\Omega]\!]$ with multiplicity 1, and carry the orientation of $\mathbb{R}^n_x$.

DEFINITION 4. We define $\mathrm{Cart}(\Omega, \mathbb{R}^N)$ as the smallest set in $\mathcal{C}$ containing the current integrations over C^1 graphs and which is sequentially closed under the weak* convergence.

The currents in $\mathrm{cart}(\Omega, \mathbb{R}^N)$ are roughly integration over graphs which might have 'vertical pieces', compare [33]. We shall now make that statement precise.

Let $T = \tau(\mathcal{M}, \theta, \xi)$ be an n-dimensional rectifiable current in $\mathbb{R}^n_x \times \mathbb{R}^N_y$ with finite mass and finite L^1-norm, and let π be the linear projection $(x, y) \to x$. By Lebesgue's theorem, T is well defined on the pullback $\pi^{\#}\omega$ of n-forms ω with compact support in $\mathbb{R}^n_x$. Thus the projection $\pi_{\#}T$ of T is defined as the current in $\mathcal{D}^n(\mathbb{R}^n_x)$ given by

$$\pi_{\#}T(\omega) := T(\pi^{\#}\omega) = \int < \pi^{\#}\omega, \xi > \theta \mathrm{d}\mathcal{H}^n \llcorner \mathcal{M}.$$

We shall now give an explicit formula for $\pi_{\#}T$ ([59] 27.2).
For $\mathcal{H}^n \llcorner \mathcal{M}$ *a.e.* z, π defines the linear map

$$\mathrm{d}\pi_z : \mathrm{Tan}_z \mathcal{M} \to \Lambda_1 \mathbb{R}^n,$$

given by $\tau = (\tau_1, \dots, \tau_{n+N}) \in \mathrm{Tan}_z \mathcal{M} \to (\tau_1, \dots, \tau_n)$. The Jacobian of π is given by

$$J_n\pi(z) = | \Lambda_n \, d\pi_z(\xi(z))|$$

and clearly $J_n\pi(z) = 0$ if and only if $\mathrm{Tan}_z \mathcal{M}$ contains vertical vectors. The area formula (cfr. [30] 3.2.19, 3.2.20, [59] 12) states that, given an n-rectifiable set W in $\mathbb{R}^{n+N}$, then $\pi(W)$ is n-rectifiable in $\mathbb{R}^n_x$, the function $x \to \mathcal{H}^0(W \cap \pi^{-1}(x))$ is $\mathcal{H}^n \llcorner \pi(W)$- measurable and finite, and

$$\int J_n\pi \, d\mathcal{H}^n \llcorner W = \int \mathcal{H}^0(W \cap \pi^{-1}(x)) \, d\mathcal{H}^n \llcorner \pi(W)(x). \tag{2.25}$$

Let $\mathcal{M}_+$ be the set of points of $\mathcal{M}$ where $d\pi$ has maximal rank

$$\mathcal{M}_+ = \{z \in \mathcal{M} : J_n\pi(z) > 0\}.$$

Applying the area formula, then one gets

$$\pi_{\#}T(\omega) = \int < \omega(x), \frac{\xi(x)}{|\xi(x)|} > |\xi(x)| d\mathcal{H}^n \llcorner \pi(\mathcal{M}_+) \tag{2.26}$$

where

$$\xi(x) = \sum_{z \in \pi^{-1}(x)} \theta(z) \frac{\Lambda_n d\pi_z(\xi(z))}{| \Lambda_n \, d\pi_z(\xi(z))|}.$$

Notice that $\frac{\wedge_n d\pi_z(\xi(z))}{|\wedge_n d\pi(\xi(z))|} = \pm e_1 \wedge \ldots \wedge e_n$, and that, if $T \in \mathrm{cart}(\Omega, \mathbb{R}^N)$, then

$$J_n \pi(z) = \xi_{\bar{0}0}(z)$$

so that

$$\mathcal{M}_+ = \{z \in \mathcal{M} : \xi_{\bar{0}0}(z) > 0\}.$$

Let $T \in \mathrm{cart}(\Omega, \mathbb{R}^N)$. Since the constants are $T_{\alpha\beta}$-summable and $|y|$ is $T_{\bar{0}0}$-summable, by Lebesgue's dominated convergence theorem, we deduce

$$\begin{aligned} T_{\bar{0}0}(\phi(x,y)y^j) &= \int_{\mathcal{M}} \phi(x,y) y^j \xi_{\bar{0}0}(x,y)\theta(x,y)\, d\mathcal{H}^n(x,y), \\ T_{\alpha\beta}(\phi(x,y)) &= \int_{\mathcal{M}} \phi(x,y)\xi_{\alpha\beta}(x,y)\theta(x,y)\, d\mathcal{H}^n(x,y), \end{aligned}$$

for all bounded and Borel functions ϕ in $\Omega \times \mathbb{R}^N$; in particular we can consider the measures in Ω

$$\begin{aligned} &\mathrm{d}u^j(T) := \pi_\#(T_{\bar{0}0} \llcorner y^j) : \phi \to T_{\bar{0}0}(y^j \phi(x)), \qquad \phi \in \mathcal{D}(\Omega), \\ &M_{\beta\bar{\alpha}}(T) := \sigma(\alpha, \bar{\alpha})\pi_\# T_{\alpha\beta} : \phi \to \sigma(\alpha,\bar{\alpha}) T_{\alpha,\beta}(\phi(x)), \quad \phi \in \mathcal{D}(\Omega), \end{aligned}$$

for $|\alpha| + |\beta| = n$. Moreover, we denote by

$$M_{\beta\bar{\alpha}}(T) = M_{\beta\bar{\alpha}}(T)^a dx + M_{\beta\bar{\alpha}}(T)^s$$

the Lebesgue decomposition of the measure $M_{\beta\bar{\alpha}}(T)$. Using the projection formula and lemma 1, one can show:

THEOREM 5. *Let* $T \in \mathrm{cart}(\Omega, \mathbb{R}^N)$.

(i) *The measures* $\mathrm{d}u^j(T)$ *are absolutely continuous with respect to Lebesgue measure, and if we denote by* $u^j_T(x)$ *their densities we have* $u^j_T \in BV(\Omega, \mathbb{R}^N)$ *and*

$$\mathrm{D}_i u^j_T = M_{j\bar{i}}(T).$$

(ii) $\mathcal{H}^n(\Omega \backslash \pi(\mathcal{M}_+)) = 0$, $\theta(z) = 1$, $\mathcal{H}^n \llcorner \mathcal{M}_+$*-a.e.*, and $\mathcal{H}^n \llcorner \mathcal{M}_+$-a.e. *we have*

$$\mathcal{M}_+ = \{z = (x,y) : x \in \pi(\mathcal{M}_+), y = u_T(x)\}.$$

(iii) *For all* $\alpha, \beta, |\alpha| + |\beta| = n, |\beta| > 0$,

$$T_{\alpha\beta} \llcorner \mathcal{M}_+(\phi(x,y)) = \sigma(\alpha,\bar{\alpha}) \int_\Omega \phi(x, u_T(x)) M_{\beta\bar{\alpha}}(T)^a dx$$

and

$$T_{\bar{0}0}(\phi(x,y)) = \int_\Omega \phi(x, u_T(x)) dx.$$

Moreover,

$$M_{\beta\bar{\alpha}}(T)^a(x) = M_{\beta\bar{\alpha}}((Du_T)^a(x)).$$

(iv) *If* $(\pi_{\#}|T_{\alpha\beta}|)^s = 0$, *then* $|T_{\alpha\beta}| \llcorner (\mathcal{M}\setminus\mathcal{M}_+) = 0$, *hence*

$$T_{\alpha\beta}(\phi(x,y)) = \sigma(\alpha,\bar{\alpha}) \int_\Omega \phi(x, u_T(x)) M_{\beta\bar{\alpha}}((Du_T)^a(x))\, dx.$$

In particular, if $(\pi_{\#}|T_{\alpha\beta}|)^s = 0$ *for all* $\alpha, \beta, |\alpha| + |\beta| = n$, *then* $\|T\|(\mathcal{M}\setminus\mathcal{M}_+) = 0$, *i.e.* $\mathcal{M} = \mathcal{M}_+$, $\mathcal{H}^n \llcorner \mathcal{M}$*a.e.. This implies* $u_T \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, *and*

$$T_{\alpha\beta}(\phi(x,y)) = \sigma(\alpha,\bar{\alpha}) \int_\Omega \phi(x, u_T(x)) M_{\beta\bar{\alpha}}(Du_T(x))dx$$

i.e. $T = T_{u_T}$.

Furthermore if $\{T_k\} \subset \operatorname{cart}(\Omega, \mathbb{R}^N)$, $\sup_k \|T_k\|_C < +\infty$, *and* $T_k \rightharpoonup T$, *then* $u_{T_k} \rightharpoonup u_T$ *in* $BV(\Omega, \mathbb{R}^N)$, *and for all* $\alpha, \beta, M_{\beta\bar{\alpha}}(T_k) \rightharpoonup M_{\beta\bar{\alpha}}(T)$ *in the sense of measures if either the supports of the* T_k *are bounded or the* $\pi_{\#}|T_{\alpha\beta}|$ *are equiabsolutely continuous with respect to the Lebesgue measure. In particular* $\operatorname{cart}^p(\Omega, \mathbb{R}^N)$ *is weak* sequentially closed in* C.

As shown in [33], the measures $M_{\beta\alpha}(T)$ do not fix the current T, in the sense that for different T, S we can have

$$\pi_{\#}(T_{\alpha\beta}) = \pi_{\#}(S_{\alpha\beta}) \qquad \forall \alpha, \beta \text{ with } |\alpha| + |\beta| = n;$$

actually we can also have

$$\pi_{\#}(|T_{\alpha\beta}|) = \pi_{\#}(|S_{\alpha\beta}|),$$

as shown by the currents in $\mathbb{R}_x \times \mathbb{R}^2_{(y^1, y^2)}$

$$T_0 = [\![\mathbb{R}_x \times (0,0)]\!] + [\![0 \times \{y : y_1^2 + (y_2 - 1)^2 = 1\}]\!]$$
$$S_0 = [\![\mathbb{R}_x \times (0,0)]\!] + 2[\![0 \times \{y : y_1^2 + (y_2 - 1/2)^2 = 1/4\}]\!];$$

as in [33] sec. 3, one can see that $T_0, S_0 \in \operatorname{Cart}(\mathbb{R}, \mathbb{R}^2)$. While any current T in $\operatorname{cart}(\Omega, \mathbb{R}^N)$ has density 1 in $\mathcal{M}_+$, the density of its "vertical part" $T \llcorner (\mathcal{M}\setminus\mathcal{M}_+)$ is essentially uncontrollable; for example, the vertical part of the current

$$T_{(z/|z|)^k} + k[\![\{0\} \times B(0,1)]\!] \in \operatorname{Cart}(B(0,1), \mathbb{R}^2), \quad z \in \mathrm{C}, \ k \in \mathrm{N},$$

obviously given by $k[\![\{0\} \times B(0,1)]\!]$, has density k.

Finally we recall that using the closure theorem of Federer-Fleming, one shows the following compactness theorem (see [33]).

THEOREM 6. *Let* $\{T_k\} \subset \operatorname{cart}(\Omega, \mathbb{R}^N)$ *be an equibounded sequence in* C, $\sup_k \|T_k\|_C < +\infty$. *Then there exists a subsequence which converges weakly to a current* T *in* $\operatorname{cart}(\Omega, \mathbb{R}^N)$. *Moreover the same result holds in* $\operatorname{Cart}(\Omega, \mathbb{R}^N)$.

PROOF OF THEOREM 5. As $T \in \text{cart}(\Omega, \mathbb{R}^N)$, we have

$$T_{\bar{0}0}(\phi(x)) = \int_{\mathcal{M}_+} \phi(x)\xi_{\bar{0}0}(x,y)\theta(x,y)\, d\mathcal{H}^n(x,y) = \int_\Omega \phi(x)\, dx.$$

On the other hand, by the projection formula we also have

$$\int_{\mathcal{M}_+} \phi(x)\xi_{\bar{0}0}(x,y)\theta(x,y)\, d\mathcal{H}^n = \int_{\pi(\mathcal{M}_+)} \phi(x) \sum_{y\in\pi^{-1}(x)} \theta(x,y)\, dx \quad ,$$

hence we conclude that $\mathcal{H}^n(\Omega \setminus \pi(\mathcal{M}_+)) = 0$ and

$$\sum_{y\in\pi^{-1}(x)} \theta(x,y) = 1 \qquad \text{for a.e. } x \in \Omega. \tag{2.27}$$

For future use, we denote by $\mathcal{N}$ a Borel set contained in $\pi(\mathcal{M}_+)$ with $\mathcal{H}^n(\pi(\mathcal{M}_+) \setminus \mathcal{N}) = 0$ and such that (2.27) holds for $\mathcal{N}$. For each $x \in \mathcal{N}$ there exists a unique $y = \tilde{u}(x)$ such that $(x, \tilde{u}(x)) \in \mathcal{M}_+$ and by definition $\mathcal{H}^n(\mathcal{M}_+ \setminus \pi^{-1}(\mathcal{N})) = 0$, and $\theta(z) = 1$ a.e. in $\mathcal{M}_+$.

From

$$\int_\Omega \phi(x)\, du^j(T) = T_{\bar{0}0}(\phi(x)) = \int_\Omega \phi(x)\tilde{u}^j(x)\, dx,$$

where in the last inequality we have again used the area formula, we conclude that $du^j(T)$ is absolutely continuous with respect to the Lebesgue measure with density u_T which agrees a.e. in Ω with $\tilde{u}^j$. This proves (ii) and part of (i). The second part follows easily by considering the $(n-1)$- form

$$\omega(x,y) = y^j\phi(x)\ \widehat{dx_i}, \qquad\qquad \phi(x) \in C^1(\Omega, \mathbb{R}^N).$$

We have in fact

$$d\omega = y^j D_i\phi(x)\, dx_i \wedge \widehat{dx_i} + \phi(x)\, dy^j \wedge \widehat{dx_i} \quad ,$$

thus

$$0 = \partial T(\omega) = T(d\omega) = (-1)^{i-1}\left\{\int D_i\phi\, du^j(T) + \int \phi(x)\, dM_{ji}(T)\right\}. \tag{2.28}$$

Let us prove (iii). By the projection formula and (ii) we have

$$\begin{aligned}(T_{\alpha\beta} \llcorner \mathcal{M}_+) &= \int_{\mathcal{M}_+} \phi(x)\theta(x,y)\xi_{\alpha\beta}(x,y)\ \ d\mathcal{H}^n(x,y)\\ &= \int_\Omega \phi(x)\frac{\xi_{\alpha\beta}(x,u_T(x))}{\xi_{\bar{0}0}(x,u_T(x))}\ dx \quad ;\end{aligned}$$

in particular we deduce that $\frac{\xi_{\alpha\beta}(x, u_T(x))}{\xi_{\bar{0}0}(x, u_T(x))}$ is a summable function, and $\pi_\#(T_{\alpha\beta} \llcorner \mathcal{M}_+)$ is absolutely continuous with respect to Lebesgue's measure in Ω. As $\pi_\#(T_{\alpha\beta} \llcorner (\mathcal{M} \setminus \mathcal{M}_+))$ is concentrated over $\Omega \setminus \mathcal{N}$, we therefore conclude by the uniqueness of Lebesgue's decomposition that $\pi_\#(T_{\alpha\beta} \llcorner \mathcal{M}_+) = \sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}}(T)^a \, dx$. The first two formulas of (iii) then follows easily, while the third one follows from (2.4). Since $(\pi_\# |T_{\alpha\beta}|)^s = \pi_\#(|T_{\alpha\beta}| \llcorner \mathcal{M} \setminus \mathcal{M}_+)$, (iv) follows easily. Let us finally prove the last part of the theorem. The sequence $\{u_{T_k}\}$ is equibounded in $BV(\Omega, \mathbb{R}^N)$ and in particular in $L^{1^*}(\Omega, \mathbb{R}^N)$, 1^* being the Sobolev exponent associated to 1; consequently $\{u_{T_k}\}$ is equiabsolutly continuous in $L^1(\Omega, \mathbb{R}^N)$. From

$$\int_\Omega \phi(x, u_{T_k}) u_{T_k} \, dx \to \int_\Omega \phi(x, u_T) u_T \, dx,$$

valid for all ϕ with compact support, it then easily follows that u_{T_k} converges weakly (hence strongly by the equiboundedness in BV) to u_T. Similarly one proceeds for the minors.

q.e.d.

For future purposes, it is convenient to introduce a more general class of cartesian currents. Set $\bar{n} = \min(n, N)$ and denote now by p a multiindex $p = (p_0, p_1, \ldots, p_{\bar{n}})$ where $p_i \in \mathbb{R}$, $p_i \geq 1$, $i = 0, 1, \ldots \bar{n}$. For $T \in \mathcal{D}_n(U)$ we set

$$\|T\|_{L^{p_0}} := \sup \{ T(|y|^{p_0} \phi(x, y) \, dx) \mid \phi \in \mathcal{D}(U), \ \sup_U |\phi| \leq 1 \};$$

for an n-form ω in U

$$\omega = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}(x, y) \, dx^\alpha \wedge dy^\beta,$$

we set

$$\|\omega\|_{M^{p_k}} := \left\{ \int_\Omega \sup_y \left\{ \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} |\omega_{\alpha\beta}(x, y)|^2 \right\}^{p'_k/2} dx \right\}^{1/p'_k},$$

p'_k being the dual exponent of p_k, $p'_k = \frac{p_k}{p_k - 1}$, and

$$\|\omega\|_{M^p} := \max_{k=0,\ldots,\bar{n}} \|\omega\|_{M^{p_k}}.$$

Finally, for $T \in \mathcal{D}_n(U)$ we define

$$\|T\|_{M^{p_k}} := \sup \left\{ \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} T_{\alpha\beta}(\omega_{\alpha\beta}) \mid \omega \in \mathcal{D}^n(U), \ \|\omega\|_{M^{p_k}} \leq 1 \right\}$$

and

$$\|T\|_{M^p} := \max_{k=0,\dots,\bar{n}} \|T\|_{M^{p_k}} \quad , \quad \|T\|_{\mathrm{cart}^p} := \|T\|_{L^{p_0}} + \|T\|_{M^p}$$

and we consider the Banach space

$$\mathcal{C}^p = \{T \in \mathcal{D}^n(U) : \|T\|_{\mathrm{cart}^p} < +\infty\}.$$

Observe that $\mathcal{C}^p$ is the dual space of the separable Banach space$\{\ \omega \mid \|\omega\|_{M^p} < +\infty\}$ and that the weak* convergence in $\mathcal{C}^p$ is equivalent to the weak convergence of currents with equibounded cart^p-norm. Notice that if $u \in C^1(\Omega, \mathbb{R}^N)$, $T = [\![G_u]\!]$ and $M_k(Du)$ denote all minors of order k, then

$$\|T\|_{L^{p_0}} = \|u\|_{L^{p_0}(\Omega.\mathbb{R}^N)} \ , \ \|T\|_{M^p} = \max_{k=0,1,\dots,\bar{n}} \left\{ \int_\Omega |M_k(Du)|^{p_k} dx \right\}^{1/p_k} ,$$

and in general $\|T\|_{M^{p_k}}$ is the total variation of the vector valued measure $(T_{\alpha\beta})_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}}$, if $p_k = 1$, while for $p_k > 1$, $\|T\|_{M^{p_k}}$ is the L^{p_k}- norm of the vector valued Radon-Nykodym derivative of $(\pi_{\#}|T_{\alpha\beta}|)_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}}$ with respect to Lebesgue's n-dimensional measure.

DEFINITION 5. We define

$$\mathrm{cart}^p(\Omega, \mathbb{R}^N) := \mathcal{C}^p \cap \ \mathrm{cart}(\Omega, \mathbb{R}^N)$$

and we define $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$ as the smallest set in $\mathcal{C}^p$ containing the currents integrations over C^1 graphs and sequentially closed with respect to the weak* convergence in $\mathcal{C}^p$.

Of course $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$ is sequentially weak* closed,

$$\mathrm{Cart}^p(\Omega, \mathbb{R}^N) \subset \mathrm{cart}^p(\Omega, \mathbb{R}^N) \subset \mathrm{cart}(\Omega, \mathbb{R}^N)$$

and all elements of $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$ enjoy the properties stated in theorem 5. For $p = (1, \dots, 1)$, $\mathrm{cart}^p(\Omega, \mathbb{R}^N) = \mathrm{cart}(\Omega, \mathbb{R}^N)$; notice however that $\|T\|_{C^{(1,1,\dots,1)}}$ and $\|T\|_C$ are equivalent but not equal; for $p = (p, \dots, p)$, $p > 1$, one easily sees that the spaces of currents and of functions, both denoted $\mathrm{Cart}^p(\Omega, \mathbb{R}^N)$, coincide. Finally we explicitly remark that the compactness result in theorem 6 holds also in $\mathrm{cart}^p(\Omega, \mathbb{R}^N)$, $p = (p_i)_{i=0,\dots,\bar{n}}$, $p_i \geq 1$.

WEAK DIFFEOMORPHISMS. For future purposes we shall now recall the definition of a few classes of weak diffeomorphisms, introduced in [33]. Let $\Omega, \hat{\Omega}$ be respectively bounded domains in $\mathbb{R}^n_x$ and $\mathbb{R}^n_y$.

We denote by $\pi : \mathbb{R}^n_x \times \mathbb{R}^n_y \to \mathbb{R}^n_x$ and $\hat{\pi} : \mathbb{R}^n_x \times \mathbb{R}^n_y \to \mathbb{R}^n_y$ the standard linear projection operators and by i the map defined by

$$i(x, y) := (y, x).$$

Given a current T we denote by $\widehat{T}$ the current $i_{\#}(T)$. If T is integration over the graph of a smooth diffeomorphism u with inverse $\hat{u}$, we have

$$[\![G_u]\!] = \widehat{[\![G_{\hat{u}}]\!]}.$$

DEFINITION 6. The class of weak diffeomorphisms $\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})$, with $p, q > 1$, is defined by

$$\mathrm{dif}^{p,q}(\Omega,\hat{\Omega}) := \left\{ T \in \mathrm{cart}^p(\Omega,\mathbb{R}^n) \mid \widehat{T} \in \mathrm{cart}^q(\hat{\Omega},\mathbb{R}^n) \right\}.$$

We say that $\{T_k\}$ converges weakly to T in $\mathrm{dif}^{p,q}$, if $T_k \rightharpoonup T$ in the sense of currents and

$$\|T_k\|_{\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})} := \|T\|_{\mathrm{cart}^p(\Omega,\mathbb{R}^n)} + \|\widehat{T}\|_{\mathrm{cart}^q(\hat{\Omega},\mathbb{R}^n)}$$

are equibounded.

DEFINITION 7. We define $\mathrm{Dif}^{p,q}(\Omega,\hat{\Omega})$, $p, q > 1$, as the smallest set in $\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})$ which contains the class of C^1-diffeomorphisms between Ω and $\hat{\Omega}$, denoted by $\mathrm{Diff}(\Omega,\hat{\Omega})$, and which is sequentially closed with respect to the weak convergence in $\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})$.

It is not difficult to see that $\mathrm{Dif}^{p,q}(\Omega,\hat{\Omega})$ coincides with the family introduced in [33] and that both families $\mathrm{dif}^{p,q}$ and $\mathrm{Dif}^{p,q}$ are closed with respect to the weak sequential convergence with equibounded $\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})$ norms, compare [33].
Also, theorem 2 sec.4 of [33] is valid for the elements of $\mathrm{dif}^{p,q}(\Omega,\hat{\Omega})$, in particular:

THEOREM 7. *We have*

$$\mathrm{dif}^{p,q}(\Omega,\hat{\Omega}) = \Big\{ T \in \mathcal{D}_n(\mathbb{R}^{2n}) \mid \exists u \in \mathrm{cart}^p(\Omega,\mathbb{R}^n), \exists \hat{u} \in \mathrm{cart}^q(\hat{\Omega},\mathbb{R}^n) \text{ such that } T = T_u = \widehat{T_{\hat{u}}} \Big\}.$$

Moreover we have

$$\hat{u}(u(x)) = x, \quad \mathcal{H}^n\text{-a.e. } x \in \Omega, \qquad u(\hat{u}(y)) = y, \quad \mathcal{H}^n\text{-a.e. } y \in \hat{\Omega},$$

$$\int_{\hat{\Omega}} \varphi(y)\,dy = \int_{\Omega} \varphi(u(x)) \det Du(x)\,dx\ ;$$

also almost everywhere

$$\det Du > 0\ , \quad \det D\hat{u} > 0$$

and

$$M_{\alpha\bar{\beta}}(D\hat{u}(y)) = \sigma(\alpha,\bar{\alpha})\sigma(\beta,\bar{\beta})\frac{M_{\beta\bar{\alpha}}(Du(\hat{u}(y)))}{\det Du(\hat{u}(y))}.$$

We remark that theorem 7 says that in a weak sense u and $\hat{u}$ are each the inverse of the other.

Similar definitions can be given in the case $p, q = 1$, or p, q multiindices; moreover, several other classes of diffeomorphisms u from Ω into $u(\Omega)$ can be defined. But, since we are not going to use those classes in the sequel, we simply refer to [33].

Finally we mention that in $\mathrm{dif}^{p,q}$ or $\mathrm{Dif}^{p,q}$ a compactness theorem is valid, thus variational integrals which are coercive with respect to both $\mathrm{dif}^{p,q}$ norms can be easily minimized. This kind of functionals are relevant in nonlinear elasticity, compare [33].

3. - The degree of cartesian currents

In this section we shall discuss the notion of degree for cartesian currents. As in [30] 4.1.26, and [1] 1.7, our definition is based on the following

CONSTANCY THEOREM (see [30] 4.1.7, [59] 26.27). *If Ω is a connected open set in $\mathbb{R}^n$ and T is an n-dimensional current in Ω, without boundary in Ω, then there exists a constant $m \in \mathbb{R}$ such that*

$$T = m[\![\Omega]\!];$$

if moreover T is rectifiable, then m is an integer.

In the second part of this section we shall discuss relationships between local and global weak diffeomorphisms in terms of degree. And, finally, we shall make a few remarks on the degree of generalized mappings between manifolds.

DEGREE. Let $T \in \mathrm{cart}(\Omega, \mathbb{R}^n_y)$ where Ω is a bounded domain of $\mathbb{R}^n_x$. Denote by $\pi, \hat{\pi}$ respectively the linear projection of $\mathbb{R}^n_x \times \mathbb{R}^n_y$ into $\mathbb{R}^n_x$ and $\mathbb{R}^n_y$. For any Borel set $A \subset \Omega$, we consider the rectifiable current $T_A = T \llcorner \pi^{-1}(A)$ in $\mathbb{R}^n_x \times \mathbb{R}^n_y$, and its projection on $\mathbb{R}^n_y$, $\hat{\pi}_\#(T \llcorner \pi^{-1}(A))$. Finally set for simplicity $\Gamma_{T,A} := \mathbb{R}^n_y \backslash \mathrm{spt}\partial \hat{\pi}_\#(T_A)$. For any $y \in \Gamma_{T,A}$, we consider its connected component C_y in $\Gamma_{T,A}$ and we notice that the current $\hat{\pi}_\# T_A \llcorner C_y$ has no boundary in C_y, thus by the constancy theorem

$$\hat{\pi}_\# T_A \llcorner C_y = m[\![C_y]\!]. \tag{3.1}$$

DEFINITION 1. The degree of T with respect to A at y, $\deg(T, A, y)$, is defined as the number m in (3.1).

By definition $\deg(T, A, y)$ is an integer, is constant on each connected component of $\Gamma_{T,A}$, is zero on connected components with infinite measure, and if ω is an n-form on $\mathbb{R}^n_y$ with $\mathrm{spt}\, \omega \subset C_y$ and $\int \omega dy = 1$, then (compare e.g. [52])

$$\deg(T, A, y) = T_A(\hat{\pi}^\# \omega);$$

obviously we also have

$$\hat{\pi}_{\#} T_A \llcorner \Gamma_{T,A} = \sum_i \deg(T, A, y_i) [\![C_i]\!]$$

where $\{C_i\}$ is the family of connected components of $\Gamma_{T,A}$ and y_i is a point in C_i.

Using the projection formula in section 2, with π replaced by $\hat{\pi}$, we shall now give a pointwise expression of $\deg(T, A, y)$. As T is rectifiable, $T = \tau(\mathcal{M}, \theta, \xi)$ and

$$\xi(z) = \sum_{|\alpha|+|\beta|=n} \xi_{\alpha\beta}(z)\, e^{\alpha} \wedge \epsilon^{\beta}, \quad \mathcal{H}^n - \text{a.e. } z \in \mathcal{M}.$$

Set now

$$\mathcal{M}^+ := \{ z \in \mathcal{M} \mid |\xi_{0\bar{0}}(z)| > 0 \}$$

(3.2)
$$\Phi(y, A) := \sum_{z \in \hat{\pi}^{-1}(y) \cap \mathcal{M}^+ \cap \pi^{-1}(A)} \theta(z) \frac{\xi_{0\bar{0}}(z)}{|\xi_{0\bar{0}}(z)|};$$

recall that $\xi_{0\bar{0}}(z) e^0 \wedge \epsilon^{\bar{0}} = \wedge_n d\hat{\pi}_z(\xi(z))$. From the projection formula we get that $\phi(y, A)$ is finite for $\mathcal{H}^n$-a.e. $y \in \hat{\pi}(\mathcal{M}^+ \cap \pi^{-1}(A))$ and that for every n-form ω with support in C_y

$$\deg(T, A, y) \int \omega = \deg(T, A, y) \int_{C_y} < \omega, \epsilon^{\bar{0}} > \, d\mathcal{H}^n = \hat{\pi}_{\#} T_A(\omega)$$
$$= \int_{\hat{\pi}(\mathcal{M}^+ \cap \pi^{-1}(A))} < \omega, \epsilon^{\bar{0}} > \, \phi(y, A)\, d\mathcal{H}^n.$$

Thus we conclude at once

PROPOSITION 1. *Let* $T \in \text{cart}(\Omega, \mathbb{R}^n_y)$. *Then for almost every* y *in* $\Gamma_{T,A}$

(3.3)
$$\deg(T, A, y) = \Phi(y, A).$$

Moreover if C_y *is the connected component of* y *in* $\Gamma_{T,A}$, *and if* $\deg(T, A, y) \neq 0$, *then* a.e.

$$C_y \subset \hat{\pi}(\mathcal{M}^+ \cap \pi^{-1}(A));$$

in particular if C_y *is not contained in the image of* $\mathcal{M}^+ \cap \pi^{-1}(A)$, *then* $\deg(T, A, y) = 0$.

If $T \in \text{cart}^p(\Omega, \mathbb{R}^N)$, $p > 1$, then $\theta(z) = 1$, $\mathcal{H}^n$ -a.e. on $\mathcal{M}$, and

$$\xi_{0\bar{0}}(z) = \xi_{\bar{0}0} \; \det Du_T(x), \quad \text{a.e. } x \in \Omega \;, \; z = (x, u_T(x)),$$

u_T being the function in $\mathcal{A}^p$ so that $T_{u_T} = T$; thus (3.3) reads as

$$\text{(3.4)} \qquad \deg(T, A, y) = \sum_{x \in u_T^{-1}(y) \cap A} \text{sign} \det Du_T(x), \quad \text{a.e. } y \in \Gamma_{T,A}.$$

The next theorem shows that the degree defined above enjoys all properties of the classical degree for smooth mappings (see e.g. [52]).

THEOREM 1. *Let $T \in \text{cart}(\Omega, \mathbb{R}^n_y)$. We have*
(i) **(excision)** *Let A, B be two Borel sets in $\mathbb{R}^n_x$ with $A \subset B$. Suppose that $y \in \Gamma_{T,A} \cap \Gamma_{T,B}$ and that the connected component of y in $\Gamma_{T,A} \cap \Gamma_{T,B}$ has an empty intersection with $\hat{\pi}(\mathcal{M}_+ \cap (B \setminus A))$, then*

$$\text{(3.5)} \qquad \deg(T, A, y) = \deg(T, B, y).$$

More generally, if $\{A_i\}$ is a numerable family of disjoint Borel sets, $y \in \bigcap_{i=1}^{\infty} \Gamma_{T,A_i} \bigcap \Gamma_{T, \bigcup_1^\infty A_i}$ and $\mathcal{H}^n \left(\Gamma_{T, \bigcup_1^\infty A_i} \bigcap \bigcap_{i=1}^{\infty} \Gamma_{T,A_i} \right) > 0$, then

$$\text{(3.6)} \qquad \deg(T, \bigcup_{i=1}^{\infty} A_i, y) = \sum_{i=1}^{\infty} \deg(T, A_i, y).$$

(ii) **(Homology invariance)** *Let $T, S \in \text{cart}(\Omega, \mathbb{R}^n_y)$ and let $A \subset \Omega$ be a Borel set. If*

$$\partial T_A = \partial S_A$$

then $\Gamma_{T,A} = \Gamma_{S,A}$ and

$$\deg(T, A, y) = \deg(S, A, y), \quad \forall y \in \Gamma_{T,A}.$$

(iii) **(Homotopy invariance)** *Let $T_t \in \text{cart}(\Omega, \mathbb{R}^n_y)$, $t \in [0,1]$, and let $A \subset \Omega$ be a Borel set. Suppose that $T_{tA} = T_t \llcorner \pi^{-1}(A)$ is a continuous map from $[0,1]$ into $\mathcal{D}_n(\mathbb{R}^n_x \times \mathbb{R}^n_y)$, i.e. $t \to T_{tA}(\omega)$ is continuous for all $\omega \in \mathcal{D}^n(\mathbb{R}^n)$. Suppose finally that y_0 be an interior point of $\bigcap_{t \in [0,1]} \Gamma_{T_t,A}$, then $\deg(T_t, A, y_0)$ is independent of t.*
(iv) **(Leray product theorem)** *Let $\phi : \mathbb{R}^n_y \to \mathbb{R}^n_y$ be a Lipschitz map and denote by $\tilde{\phi}$ the map $(x, y) \to (x, \phi(y))$. If C_i are the connected components of $\Gamma_{T,A}$, A a Borel subset of Ω and y_i are points in C_i, then for any $y \notin \overline{\phi(\mathbb{R}^n \setminus \Gamma_{T,A})}$ we have*

$$\deg(\tilde{\phi}_\# T, A, y) = \sum_i \deg(T, A, y_i) \deg(\phi, C_i, y).$$

PROOF. (i) For a.e. $y \in \Gamma_{T, \bigcup_i^\infty A_i} \bigcap \bigcap_i^\infty \Gamma_{T,A_i}$ we have

$$\deg(T, \bigcup_i^\infty A_i, y) = \phi(y, \bigcup_i^\infty A_i) = \sum_i \phi(y, A_i) = \sum_i \deg(T, A_i, y),$$

so (3.6) follows and, as (3.5) is an immediate consequence of (3.6), (i) is proved.

(ii) Since $\partial T_A = \partial S_A$, we have $\partial\hat{\pi}_\# T_A = \partial\hat{\pi}_\# S_A$, hence $\Gamma_{T,A} = \Gamma_{S,A}$. Let $y \in \Gamma_{T,A}$ and let B be a ball around y and contained in the connected component of y in $\Gamma_{T,A}$. If $\omega \in \mathcal{D}^n(B)$ with $\int \omega = 1$, from the definition of degree we have

$$\deg(T, A, y) - \deg(S, A, y) = \hat{\pi}_\# (T_A - S_A)(\omega).$$

Now, since $\partial(T_A - S_A) = 0$, one easily sees that there is an $(n+1)$-dimensional current Σ such that $T_A - S_A = \partial\Sigma$ (compare e.g. [30] 4.1.11); thus

$$\deg(T, A, y) - \deg(S, A, y) = \hat{\pi}_\# \partial\Sigma(\omega) = \Sigma(\hat{\pi}^\# d\omega) = 0,$$

being $d\omega \in \mathcal{D}^{n+1}(\mathbb{R}^n_y)$.

(iii) Let $\omega \in \mathcal{D}^n(\mathbb{R}^n_y)$ with sptω contained in some ball $B(y_0, r) \subset \bigcap_{t\in[0,1]} \Gamma_{T_t,A}$. We have

$$\deg(T_t, A, y_0) = T_{tA}(\hat{\pi}^\# \omega) = T_{tA}(\eta\hat{\pi}^\# \omega)$$

where $\eta \in C^\infty_0(\mathbb{R}^n_x)$, $\eta = 1$ on Ω. Since $\deg(T_t, A, y_0)$ is an integer and is continuous in t, it must be constant for all t.

(iv) Since $y \notin \overline{\phi(\mathbb{R}^n \backslash \Gamma_{T,A})}$ we can find a ball B around y, $B \subset \mathbb{R}^n \backslash \phi(\mathbb{R}^n \backslash \Gamma_{T,A})$; moreover since $\phi(\mathbb{R}^n \backslash \Gamma_{T,A}) \supset \mathrm{spt}\Phi_\# \hat{\pi}_\# \partial T_A$, we can also assume that B is contained in the connected component of y in $\Gamma_{\tilde{\phi}_\# T,A}$. Also, we obviously have $\phi^{-1}(B) \subset \Gamma_{T,A}$ and $B \subset \mathbb{R}^n \backslash \phi(\partial C_i)$ for all i. Hence we can write

$$\deg(\tilde{\Phi}_\# T, A, y)[\![B]\!] = \hat{\pi}_\#(\tilde{\phi}_\# T)_A \llcorner B = \Phi_\#(\hat{\pi}_\# T) \llcorner B = \Phi_\#(\hat{\pi}_\# T_A \llcorner \phi^{-1}(B));$$

on the other hand

$$\hat{\pi}_\# T_A \llcorner \phi^{-1}(B) = \sum_i \deg(T, A, y_i)\, [\![C_i \llcorner \phi^{-1}(B)]\!]$$

so

$$\begin{aligned}\Phi_\#(\hat{\pi}_\# T_A \llcorner \phi^{-1}(B)) &= \sum_i \deg(T, A, y_i)\, \Phi_\# [\![C_i \llcorner \phi^{-1}(B)]\!] \\ &= \sum_i \deg(T, A, y_i)\, \deg(\phi, C_i, y)[\![B]\!].\end{aligned}$$

q.e.d.

The degree defined above is stable with respect to the weak convergence of currents with equibounded masses. This is stated in the two following propositions, the first of which is a simple rewriting of the homology invariance of the degree.

PROPOSITION 2. *Let* $\Omega \subset\subset \tilde{\Omega}$ *be two bounded domains in* $\mathbb{R}^n$ *and let*

$$T, T_k \in \operatorname{cart}(\Omega, \mathbb{R}^N)$$

be such that

$$\sup_k M(T_k) < +\infty \ , \ T_k \rightharpoonup T.$$

Suppose that

$$(T_k - T_h) \llcorner \pi^{-1}(\tilde{\Omega} \backslash \bar{\Omega}) = 0, \quad \forall h, k,$$

then for all $y \in \Gamma_{T, \bar{\Omega}}$

$$\deg(T_k, \bar{\Omega}, y) = \deg(T, \bar{\Omega}, y).$$

PROOF. In fact for all k

$$T_k \llcorner \pi^{-1}(\tilde{\Omega} \backslash \bar{\Omega}) = T \llcorner \pi^{-1}(\tilde{\Omega} \backslash \bar{\Omega})$$

hence

$$\partial(T_k \llcorner \pi^{-1}(\bar{\Omega})) = \partial(T_k \llcorner \pi^{-1}(\tilde{\Omega} \backslash \bar{\Omega})) - \partial(T \llcorner \pi^{-1}(\tilde{\Omega} \backslash \bar{\Omega})) = \partial(T \llcorner \pi^{-1}(\bar{\Omega}))$$

and the result follows at once from theorem 1 (ii).

q.e.d.

PROPOSITION 3. *Let* $T, T_k \in \operatorname{cart}^p(\Omega, \mathbb{R}^n), p > 1,$ *and let* A *be a Borel set in* Ω. *Suppose that*

$$\sup_k \|T_k\|_{\operatorname{cart}^p} < +\infty, \quad T_k \rightharpoonup T.$$

Then

$$\deg(T_k, A, y) \to \deg(T, A, y)$$

provided y *is an interior point of* $\Gamma_{T,A} \bigcap \bigcap_k \Gamma_{T_k,A}$.

PROOF. Let $\omega = \phi(y)dy$ be an n-form with compact support in the connected component of y in $\Gamma_{T,A} \bigcap \bigcap_k \Gamma_{T_k,A}$ with $\int \omega = 1$. We have

$$\begin{aligned}
&\deg(T_k, A, y) = \hat{\pi}_{\#}(T_k \llcorner \pi^{-1}(A))(\phi(y)dy) \\
&= \int_A \phi(u_{T_k}(x)) \det Du_{T_k}(x)dx \to \int_A \phi(u_T(x)) Du_T(x)dx = \deg(T, A, y).
\end{aligned}$$

q.e.d.

EXAMPLE 1. Consider the current $T \in \operatorname{Cart}(B_1(0), \mathbb{R}^n)$

$$T := [\![G_{x/|x|}]\!] + [\![\{0\} \times \hat{B}_1]\!]$$

where $\hat{B}_1$ is the unit ball around zero in $\mathbb{R}^n_y$. As we have seen in [33], T is the weak limit of a sequence of smooth diffeomorphisms which preserve the orientation from B_1 into the unit ball $\hat{B}_1$ in $\mathbb{R}^n_y$ with the same boundary (and obviously with degree 1) and with equibounded masses; thus, with the notations of [33], $T \in \mathrm{Dif}(B_1, \hat{B}_1)$. It is easily seen that

$$\Gamma_{T,B_1} = \mathbb{R}^n_y \backslash \partial \hat{B}_1$$

$$\deg(T, B_1, y) = \begin{cases} 1 & \text{for } y \in \hat{B}_1 \backslash \partial \hat{B}_1 \\ 0 & \text{for } y \notin \hat{B}_1 \cup \partial \hat{B}_1. \end{cases}$$

DEGREE AND WEAK DIFFEOMORPHISMS. Consider a cartesian current

$$T = \tau(\mathcal{M}, \theta, \xi) \in \mathrm{cart}(\Omega, \mathbb{R}^n)$$

(or a current $T = T_u \in \mathrm{cart}^p(\Omega, \mathbb{R}^n), p > 1$). We say that T is a *(weak) local diffeomorphism from* Ω *into* $\hat{\Omega}$ *if*

(i) $\xi_{0\bar{0}}(z) \geq 0$, a.e. $z = (x, y) \in \mathcal{M}$ *(or* $\det Du(x) \geq 0$, a.e. $x \in \Omega$*),*

(ii) $\mathrm{spt}\partial T \subset \partial\Omega \times \partial\hat{\Omega}$.

We observe that $\mathbb{R}^n \backslash \overline{\hat{\Omega}}$ is contained in a connected component of infinite measure of $\Gamma_{T,\Omega}$, since $\partial T \subset \partial\Omega \times \partial\hat{\Omega}$, thus

$$\deg(T, \Omega, y) = 0, \qquad \forall y \notin \hat{\Omega} \cup \partial\hat{\Omega},$$

and

$$\mathrm{spt}\hat{\pi}_{\#}T \subset \overline{\hat{\Omega}};$$

in particular $\hat{\pi}(\mathcal{M}) \subset \overline{\hat{\Omega}}$. From (3.2), (3.4) we immediately get

THEOREM 2. *Let* $T \in \mathrm{cart}(\Omega, \mathbb{R}^n_y)$ *be a local diffeomorphism into* $\hat{\Omega}$, *let* C *be a connected component of* $\Gamma_{T,\Omega}$ *and let* $y \in C$.

i) *If* $\deg(T, \Omega, y) = 0$, *then* $\hat{\pi}(\mathcal{M}) \cap C = \emptyset$.

ii) *If* $\deg(T, \Omega, y) = m > 0$, *there for almost every* $y \in C$ *then is at least one and no more then* m *points* $(x, y) \in \mathcal{M}$, *and* $\hat{\pi}(\mathcal{M}) \supset C$. *Moreover, if* $T = T_u \in \mathrm{cart}^p(\Omega, \mathbb{R}^n_y), p > 1$, *then for a.e.* $y \in C$ *there exist exactly* m *distinct points* $x_1, \ldots, x_m$ *in* Ω *such that*

$$\{x \mid u(x) = y\} = \{x_1, \ldots, x_m\}$$

and $u(\Omega) \supset C$.

EXAMPLE 2. Identify $\mathbb{R}^2_x \simeq \mathbb{R}^2_y$ with the complex plane C and denote by B_1 and $\hat{B}_1$ respectively the unit ball in $\mathbb{R}^2_x$ and $\mathbb{R}^2_y$. Consider the map

$$u(z) : B_1 \to \hat{B}_1, \;\; z \to z^m,$$

and the current $[\![G_{z^m}]\!] \in \text{Cart}^p(B_1, \mathbb{R}^2)$, $\forall p$, obviously

$$\deg([\![G_{z^m}]\!], B_1, y) = \begin{cases} m, & y \in \hat{B}_1, \\ 0, & y \notin \hat{B}_1 \cup \partial\hat{B}_1, \end{cases}$$

and each non-zero point in $\hat{B}_1$ is the image of exactly m distinct points in B_1. Consider now the current in $\text{Cart}(B_1, \mathbb{R}^2_y)$ associated to $z^m/|z|^m$,

$$T = T_{z^m/|z|^m} + m[\![\{0\} \times \hat{B}_1]\!].$$

It is easily seen that

$$\deg(T, B, y) = \begin{cases} m, & y \in \hat{B}_1, \\ 0, & y \notin \hat{B}_1 \cup \partial\hat{B}_1, \end{cases}$$

but every point in $\hat{B}_1$ is the image of only one point in B_1 while the vertical piece has multiplicity m.

In particular for $m = 1$, and taking into account theorem 5 (iv) sec. 2, we deduce at once

THEOREM 3. *Let $T \in \text{cart}(\Omega, \mathbb{R}^n_y)$ be a local diffeomorphism with*

$$\deg(T, \Omega, y) = 1$$

for a.e. *y in $\hat{\Omega}$. Then $\hat{T} \in \text{cart}(\hat{\Omega}, \mathbb{R}^n)$, i.e. $T \in R_n(\mathbb{R}^n_x \times \hat{\Omega})$, $M(T) < +\infty$, $\partial T = 0$ in $\mathbb{R}^n_x \times \hat{\Omega}$, $\hat{\pi}_\# T = [\![\hat{\Omega}]\!]$, and $\xi_{0\bar{0}}(z) \geq 0$ on $\mathcal{M}$. If $\xi_{0\bar{0}}(z) > 0$ on $\mathcal{M}$, then there exists a map $\hat{u} : \hat{\Omega} \to \mathbb{R}^n_x$ in $\mathcal{A}(\hat{\Omega}, \mathbb{R}^n_y)$ such that $T = \widehat{T_{\hat{u}}}$, i.e.*

$$T_{\alpha\beta}(\phi) = \sigma(\beta, \bar{\beta}) \int_{\hat{\Omega}} \phi(\hat{u}(y), y)\, M_{\alpha\bar{\beta}}(D\hat{u}(y))\, dy\ .$$

Moreover if $T \in \text{cart}^p(\Omega, \mathbb{R}^n_y)$, $p > 1$, then $\hat{u}$ enjoys all properties of the "inverse function", of u and in particular the properties stated in theorem 7 sec. 2.

An immediate consequence of theorem 3 is the following corollary which gives a slight variant of results in [5], [61].

COROLLARY 1. *Let Ω be a bounded domain and let $T_u \in \text{cart}^p(\Omega, \mathbb{R}^n_y)$ be a local diffeomorphism from Ω into $\hat{\Omega}$ with*

$$\det Du > 0.$$

If $\partial T_u = \partial S$ where $S \in \text{dif}^{p,q}(\Omega, \hat{\Omega})$, $q > 1$, then $\widehat{T_u} \in \text{cart}(\hat{\Omega}, \mathbb{R}^n_x)$.

Moreover, if

$$\int_\Omega \frac{|M(Du)|^q}{(\det Du)^{q-1}} dx < +\infty. \tag{3.7}$$

then $T_u \in \text{dif}^{p,q}(\Omega, \hat{\Omega})$.

PROOF. Since $\deg(S, \Omega, y) = 1$ for all $y \in \hat{\Omega}$, we conclude by the homology invariance that $\deg(T_u, \Omega, y) = 1$ a.e. in $\hat{\Omega}$ and thus the first part of the theorem follows from theorem 3. The second part is obvious, compare e.g. [33].

q.e.d.

COROLLARY 2. *Let* Ω *be a bounded domain with Lipschitz boundary and let*

$$u \in H^{1,p}(\Omega, \mathbb{R}^n), \;\; p \geq n.$$

Suppose that T_u *is a local weak diffeomorphism from* Ω *into* $\hat{\Omega}$ *with* $\det Du(x) > 0$ a.e. *in* Ω, *and that* $u = u_0$ *on* $\partial\Omega$ *in the sense of the traces in* $H^{1,p}(\Omega, \mathbb{R}^n)$, *where* $u_0 \in H^{1,p}(\Omega, \mathbb{R}^n)$ *is an homeomorphism from* Ω *into* $\hat{\Omega}$. *Then* u *is one to one from* Ω *into* $\hat{\Omega}$, *in particular there exists* $\hat{u} : \hat{\Omega} \to \Omega$ *in* $\mathcal{A}^1$ *such that* $T_u = \widetilde{T_{\hat{u}}}$. *If moreover* (3.7) *holds for some* $q > 1$, *then* $\hat{u} \in \mathcal{A}^q(\Omega, \mathbb{R}^n_x)$ *and actually* $T \in \text{dif}^{p,q}(\Omega, \hat{\Omega})$.

REMARK 1. As it is clear from the proof of corollary 1, the almost everywhere injectivity of u follows from the condition $\deg(T_u, \Omega, y) = 1$ in $\hat{\Omega}$. This can be also written for weak local diffeomorphisms T_u with $\det Du > 0$ a.e., as

$$\int_\Omega \det Du(x)dx \;\leq\; \mathcal{H}^n(u(\Omega))$$

by the area formula (compare [18], [53]).

We conclude this section with a few remarks which will be useful in the sequel. Since the definition of degree depends on the constancy theorem which also holds on oriented manifolds without boundary [30] 4.1.31, we can extend our definition of degree to currents $T \in \text{cart}(\Omega, \mathbb{R}^k)$ with $\text{spt}T$ contained in an n-dimensional oriented and properly immersed submanifold $\mathcal{N}$ of $\mathbb{R}^k$. One also sees that all properties of the degree remain true, with the exception of the homological property which depends on the homology of $\Omega \times \mathcal{N}$. In particular the degree is defined for mappings $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $u(x) \in S^2$, or for currents $T \in \text{cart}(\Omega, \mathbb{R}^3)$ with $\text{spt}T \subset \Omega \times S^2$, simply $T \in \text{cart}(\Omega, S^2)$.

4. - The Dirichlet integral for mappings into S^2 and the parametric extension of variational integrals

In this section we shall discuss the problem of minimizing the Dirichlet integral among maps from a domain $\Omega \subset \mathbb{R}^2$ into S^2 with prescribed degree. This will lead us to a natural extension of the Dirichlet integral, as a parametric integral, to a class of cartesian currents.

For simplicity we choose as Ω the unit ball in $\mathbb{R}^2$ and we think of S^2 as the unit sphere in $\mathbb{R}^3$

$$S^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}.$$

Also we fix a non-constant smooth map $\gamma : \partial\Omega \to S^2$. It is well known that the Dirichlet integral

$$\mathcal{D}(u) = \frac{1}{2}\int_\Omega |Du|^2 dx$$

has a minimizer $\underline{u}$ (not necessarily unique) in the class

$$E := \{u \in H^{1,2}(\Omega, \mathbb{R}^3),\ u(x) \in S^2 \text{ a.e. on } \Omega,\ u = \gamma \text{ on } \partial\Omega\} .$$

Moreover $\underline{u}$ is a regular harmonic function, i.e. satisfies

$$\text{(4.1)} \qquad \begin{cases} -\Delta u = u|Du|^2 & \text{on } \Omega \\ u(x) \in S^2 & \text{on } \Omega \\ u(x) = \gamma(x) & \text{on } \partial\Omega. \end{cases}$$

In [15], [39], see also [13], the problem of finding a harmonic map $\bar{u}$ topologically different from $\underline{u}$ is studied. One splits E into connected components E_m by means of the degree theory and one seeks a minimizer on each E_m. More precisely to each map u in E, in particular to each smooth map, we associate the map

$$U : \mathbb{R}^2 \to S^2$$

defined by gluing u in Ω and $\underline{u}$ in $\mathbb{R}^n \setminus \Omega$

$$U(x) := \begin{cases} u(x) & \text{if } x_1^2 + x_2^2 < 1 \\ \underline{u}\left(\dfrac{x_1}{x_1^2 + x_2^2}, \dfrac{x_2}{x_1^2 + x_2^2}\right) & \text{if } x_1^2 + x_2^2 > 1 \end{cases}$$

and if π_S denotes the stereographic projection from S^2 into $\mathbb{R}^2_x$, we consider the map

$$U \circ \pi_S : S^2 \to S^2.$$

The degree of $U \circ \pi_S$ is well defined (compare [52], and [15] sec. 3) and is given for all y in S^2 by

$$\deg(U \circ \pi_S, S^2, y) = \frac{1}{4\pi}\int_{S^2} (U \circ \pi_S)^\# \omega$$

where ω is the volume 2-form on S^2

$$\omega = (-1)^{i-1} y_i \widehat{dy_i};$$

and by a simple computation

$$\deg(U \circ \pi_S, S^2, y) = \frac{1}{4\pi}\int_{\mathbb{R}^2} U \cdot U_{x_1} \wedge U_{x_2} dx = Q(u) - Q(\underline{u})$$

where

$$Q(w) = \frac{1}{4\pi} \int_\Omega w \cdot w_{x_1} \wedge w_{x_2}.$$

We notice that, if we think of u as a map from Ω into S^2, and $y \notin \overline{\underline{u}(\partial\Omega)}$, then

$$\begin{aligned} &\deg(U \circ \pi_S, S^2, y) = \deg(U \circ \pi_S, \pi_S^{-1}(\Omega), y) + \deg(U \circ \pi_S, \pi_S^{-1}(\mathbb{R}^2 \backslash \Omega), y) \\ &= \deg(u, \Omega, y) + \deg\left(\underline{u}\left(\frac{x}{|x|^2}\right), \mathbb{R}^2 \backslash \Omega, y\right) \\ &= \deg(u, \Omega, y) - \deg(\underline{u}, \Omega, y) \end{aligned}$$

where $\deg(u, \Omega, y)$ is defined for $u : \Omega \to S^2$ as in sec. 3.

Define

$$E_m := \{u \in E : Q(u) - Q(\underline{u}) = m\}$$

or equivalently

$$E_m := \{u \in E : \deg(u, \Omega, y) = \deg(\underline{u}, \Omega, y) + m, \quad \forall y \notin \overline{\underline{u}(\partial\Omega)}\},$$

then clearly

$$E = \bigcup_{m \in Z} E_m.$$

Now we look for a minimizer of $\mathcal{D}(u)$ in E_m

$$\text{(4.2)} \qquad \begin{cases} \mathcal{D}(u) \to \min \\ u \in E_m. \end{cases}$$

Notice that any smooth solution of (4.2) solves also (4.1) since the degree is a null Lagrangian.

The main difficulty in trying to carry out this program is the following. Suppose $\{u_k\}$ is a minimizing sequence for (4.2). Clearly, we may assume that $u_k \rightharpoonup \bar{u}$ weakly in $H^{1,2}$ and thus, by semicontinuity, we have $\mathcal{D}(\bar{u}) \leq \inf\{\mathcal{D}(u) : u \in E_m\}$. However in general $\bar{u}$ does not belong to E_m since the sets E_m are not closed under weak convergence, the degree, i.e. $Q(u)$, being not continuous with respect to the weak convergence in $H^{1,2}$. In fact (see [15]) consider the family of mappings

$$v_\epsilon : \Omega \backslash \text{ the annulus } B(0, 2\epsilon) \backslash B(0, \epsilon) \to S^2$$

given by

$$v_\epsilon(x) = \underline{u}(x) \quad \text{in} \quad \Omega \backslash B(0, 2\epsilon)$$

and in polar coordinates (r, θ) by

$$\text{(4.3)} \qquad v_\epsilon = \frac{2\epsilon^{m+1}}{\epsilon^{2m+2} + r^{2m}} \begin{pmatrix} r^m \cos m\theta \\ -r^m \sin m\theta \\ -\epsilon^{m+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{in } B(0, \epsilon).$$

One can define v_ϵ in $B(0,2\epsilon)\setminus B(0,\epsilon)$ in such a way that v_ϵ is continuous in Ω and

$$\int_{B(0,2\epsilon)\setminus B(0,\epsilon)} |Dv_\epsilon|^2 dx = O(\epsilon^2)$$

thus one easily sees that for ϵ small enough $v_\epsilon \in E_m$, i.e. $E_m \neq \emptyset$; but

$$v_\epsilon \rightharpoonup \underline{u} \in E_0$$

and

$$\mathcal{D}(v_\epsilon) \to \mathcal{D}(\underline{u}) + 4\pi|m|.$$

Our basic observation is now that by the isoperimetric inequality for parallelograms:

$$\frac{1}{2}|Du|^2 \geq |D_1 u \wedge D_2 u| =: J_2 Du;$$

thus the equiboundedness of the sequence $\{v_\epsilon\}$ in $H^{1,2}$ implies the equiboundedness of the areas of the graphs of the v_ϵ

$$\sup_{0<\epsilon<\frac{1}{2}} \mathcal{H}^2(G_{v_\epsilon}) = \sup_{0<\epsilon<\frac{1}{2}} \int_\Omega \sqrt{1+|Dv_\epsilon|^2 + (J_2 Dv_\epsilon)^2}\, dx < +\infty.$$

Therefore the currents $[\![G_{v_\epsilon}]\!]$ converge in the sense of $\text{cart}(\Omega,\mathbb{R}^3)$ to a current T and one easily sees that T is given by

$$T = [\![G_{\underline{u}}]\!] + m[\![\{0\}\times S^2]\!].$$

Notice that $\deg(T, S^2, y) = m$. This simple remark suggests that the natural space associated to problem (4.1) is a suitable space of currents rather than $H^{1,2}$.

THE CLASS $\text{Cart}^{2,1}(\Omega, S^2)$. Let Ω be a bounded domain of $\mathbb{R}^2$ and p the multiindex (2,2,1). We define $\text{cart}^{2,1}(\Omega,S^2)$ as the class of currents in $\text{cart}^p(\Omega,\mathbb{R}^3)$ with $\text{spt}T \subset \Omega\times S^2$ (compare sec. 2).

PROPOSITION 1. *Let* $T = \tau(\mathcal{M},\theta,\xi) \in \text{cart}^{2,1}(\Omega,S^2)$.

i) *The function* u_T*, associated to* T *by theorem 5 sec. 2, belongs to* $H^{1,2}(\Omega,S^2)$ *and by the isoperimetric inequality* $u_T \in \mathcal{A}^1(\Omega,\mathbb{R}^3)$*; moreover the current integration over the graph of* u_T, T_{u_T} *belongs to* $\text{Cart}^{2,1}(\Omega,S^2)$*, more precisely there exists a sequence of* C^∞*-maps* $\{u_k\}$ *with values in* S^2 *such that*

$$[\![G_{u_k}]\!] \rightharpoonup T_{u_T}.$$

ii) *The singular part of* $T, S := T - T_{u_T}$*, is vertical,* i.e.

$$S_{\bar{0}0} = 0\ ,\ S_{ij} = 0\ ,\ S_{0\bar{i}} = T_{0\bar{i}} \llcorner (\mathcal{M}\setminus\mathcal{M}_+)$$

and without boundary in $\Omega\times\mathbb{R}^3$.

PROOF. Since the $\text{cart}^p(\Omega, \mathbb{R}^3)$-norm, $p = (p_0, p_1, p_2), p_0 = p_1 = 2, p_2 = 1$, is finite, $M_{j\bar{i}}(T)$ is absolutely continuous with respect to Lebesgue's measure and by theorem 5 sec. 2

$$u_T(x) \in S^2 \text{ a.e. }, \qquad T_{\bar{0}0}(\phi(x,y)) = \int_\Omega \phi(x, u_T(x))dx,$$

$$T_{ij}(\phi(x,y)) = (-1)^{i-1} \int_\Omega \phi(x, u_T) D_i u_T^j \, dx, \quad \|T\|_{M_1^p} = \left(\int_\Omega |Du_T|^2 dx \right)^{1/2},$$

thus $u_T \in H^{1,2}(\Omega, S^2)$. By the isoperimetric inequality, $u_T \in \mathcal{A}^1(\Omega, \mathbb{R}^3)$ and we can consider the current T_{u_T}. By the density theorem in [58] we can approximate strongly in $H^{1,2}(\Omega, S^2)$ u_T by a sequence of C^∞-maps $\{u_k\}$ with values in S^2; as in the proof of proposition 2 sec. 2, one then sees that

$$[\![G_{u_k}]\!] \rightharpoonup T_{u_T},$$

thus $T_{u_T} \in \text{Cart}^{2,1}(\Omega, S^2)$. This concludes the proof of (i). Taking into account the definition of T_{u_T} and theorem 5 (iii), we finally get (ii).

q.e.d.

The following theorem gives the structure of the vertical part of the currents in $\text{cart}^{2,1}(\Omega, S^2)$.

THEOREM 1. *Let $T \in \text{cart}^{2,1}(\Omega, S^2)$. Then there exists a finite number of points $x_i, i = 1, \ldots, k$ in Ω and k integers d_i such that*

$$T = T_{u_T} + \sum_{i=1}^{k} d_i [\![\{x_i\} \times S^2]\!]. \tag{4.4}$$

PROOF. Let ω_S be the volume 2-form on S^2. It is well known that every 2-form on S^2, $\phi(y)\omega_{S^2}$, can be decomposed as

$$\phi(y)\omega_{S^2} = \left(\frac{1}{|S^2|} \int_{S^2} \phi(y) \mathrm{d}\mathcal{H}^2 \right) \omega_{S^2} + \mathrm{d}\eta$$

(see e.g. [49] chap. 7, [12] theor. 6.17); more generally, one sees that locally in x, for every 2-form $\phi(x,y)\omega$ in $\Omega \times S^2$, $\omega = \hat{\pi}^\# \omega_{S^2}$, there exists a smooth 1- form $\eta(x,y)$ such that

$$\phi(x,y)\omega = \bar{\phi}(x)\, \omega + \mathrm{d}_y \eta(x,y) \ , \quad \bar{\phi}(x) = \frac{1}{|S^2|} \int_{S^2} \phi(x,y) \, \mathrm{d}\mathcal{H}^2(y) \tag{4.5}$$

where d_y is the exterior differentiation operator with respect to y. If d_x is the exterior differentiation operator with respect to x, we then have

$$\phi(x,y)\omega = \bar{\phi}(x)\, \omega + d\eta - \mathrm{d}_x \eta.$$

Let $S := T - T_{u_T}$. Since S is vertical and boundaryless, we have

$$S(d_x \eta) = 0 \ , \ S(d\eta) = 0,$$

thus

$$S(\phi(x,y)\omega) = S(\bar{\phi}(x)\omega). \tag{4.6}$$

In particular we see that $S \llcorner \pi^{-1}(x)$ is boundaryless , therefore, by the constancy theorem, it is an integer multiple of $[\![\{x\} \times S^2]\!]$ for a.e. $x \in \Omega$. Denote now by μ the measure $S \llcorner \omega$, i.e.

$$\phi(x,y) \to S(\phi(x,y)\omega) = \int_{\mathcal{M}} \phi(x,y) < \omega, \xi > \theta d\mathcal{H}^2$$

and observe that $< \omega, \xi >= \pm 1$, then

$$\mu = \theta < \omega, \xi > \mathcal{H}^2 \llcorner \mathcal{M} \tag{4.7}$$

and by (4.6)

$$\mu = \pi_{\#}\mu \times \frac{1}{4\pi}\mathcal{H}^2 \llcorner S^2; \tag{4.8}$$

this yields at once that $\pi_{\#}\mu$ is absolutely continuous with respect to $\mathcal{H}^0$, so

$$\pi_{\#}\mu = \mathrm{d}(x)\,\mathcal{H}^0 \quad \text{in } \Omega.$$

Computing the value of η on $\pi^{-1}(x)$ by means of (4.7), (4.8), we get

$$\mathrm{d}(x) = 4\pi\,\theta(x,y) < \omega, \xi(x,y) > \qquad \forall y \in \pi^{-1}(x),$$

i.e. $\mathrm{d}(x)/4\pi$ is integer valued; the total variation of μ being finite, (4.4) follows at once.

q.e.d.

Our next theorem deals with the approximation property of elements in $\mathrm{cart}^{2,1}(\Omega, S^2)$.

THEOREM 2. *We have*

$$\mathrm{cart}^{2,1}(\Omega, S^2) = \mathrm{Cart}^{2,1}(\Omega, S^2),$$

more precisely, for every $T \in \mathrm{cart}^{2,1}(\Omega, S^2)$ there exists a sequence of smooth functions u_h from Ω into S^2 with equibounded $H^{1,2}$ and $\mathrm{cart}^{2,1}$-norms such that

$$[\![G_{u_h}]\!] \rightharpoonup T;$$

moreover if

$$T = T_{u_T} + \sum_{i=1}^{k} d_i [\![\{x_i\} \times S^2]\!] \tag{4.9}$$

then

$$\frac{1}{2}\int_\Omega |Du_h|^2 dx \to \frac{1}{2}\int_\Omega |Du_T|^2 dx + 4\pi \sum_{1=1}^{k} |d_i|. \tag{4.10}$$

PROOF. Suppose first that

$$T = T_{u_T} + m[\![\{0\} \times S^2]\!].$$

Let u_ϵ be a family of smooth functions converging for $\epsilon \to 0$ in $H^{1,2}(\Omega, S^2)$ to u_T. Obviously

$$\int_{B_{2\epsilon}} |Du_\epsilon|^2 dx \to 0.$$

Thus we can find, for each ϵ, a radius r_ϵ with $\epsilon \leq r_\epsilon \leq 2\epsilon$ such that

$$\epsilon^{-1} \int_{\partial B_{r_\epsilon}} \left| \frac{\partial u_\epsilon}{\partial \theta} \right|^2 d\mathcal{H}^1(\theta) \to 0. \tag{4.11}$$

Define now, as in [15],

$$w_\epsilon := \begin{cases} v_\epsilon & \text{in } B_\epsilon \\ u_\epsilon & \text{in } \Omega \backslash B_{r_\epsilon} \\ \tilde{w}_\epsilon & \text{in } B_{r_\epsilon} \backslash B_\epsilon, \end{cases}$$

where v_ϵ is the mapping given in (4.3) and $\tilde{w}_\epsilon$ is the linear transition map

$$\tilde{w}_\epsilon := \begin{cases} A_1 r + B_1 \\ A_2 r + B_2 \\ \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2} \end{cases}$$

where A_1, A_2, B_1, B_2 depend only on θ, ϵ, and are determined in such a way to make w_ϵ continuous in Ω. Because of (4.11) one sees that

$$\int_{B_{r_\epsilon} \backslash B_\epsilon} |Dw_\epsilon|^2 dx \to 0, \quad \text{as } \epsilon \to 0,$$

also

$$\int_{B_\epsilon} |Dw_\epsilon|^2 dx \to 4\pi |m|,$$

so we conclude that

$$\frac{1}{2}\int_{\Omega} |Dw_\epsilon|^2 dx \to \frac{1}{2}\int_{\Omega} |Du_T|^2 dx + 4\pi|m|,$$

and obviously

$$[\![G_{w_\epsilon}]\!] \rightharpoonup T_{u_T} + m[\![\{0\} \times S^2]\!].$$

Repeating the same construction around every point $x_i, i = 1, \ldots, k$, one easily gets the conclusion in the general case.

q.e.d.

REMARK 1. While $\mathrm{cart}^{2,1}(\Omega, S^2)$ is a strictly larger class than the graphs of mappings in $H^{1,2}(\Omega, S^2)$, we point out that $\mathrm{cart}^{2,1}(\Omega, \mathbb{R}^2_y)$, coincide with the 'graphs' of $H^{1,2}(\Omega, \mathbb{R}^2_y)$-maps. This is easily seen from the proof of theorem 1, since in this case the vertical part should be, by the constancy theorem, a sum of points times $\mathbb{R}^2_y$ which has infinite mass.

POLYCONVEX AND PARAMETRIC EXTENSIONS OF VARIATIONAL INTEGRALS. We shall now extend variational integrals, and in particular the Dirichlet integral, a priori defined on classes of smooth mappings to cartesian currents. Let G be the matrix associated to a linear transformation G from $\mathbb{R}^n_x$ into $\mathbb{R}^N_y$ endowed with the standard bases $(e_1, \ldots, e_n)$, $(\epsilon_1, \ldots, \epsilon_N)$; set, compare sec. 2,

$$M(G) := \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}}(G)\, e^\alpha \wedge \epsilon^\beta.$$

Clearly M maps the space of $N \times n$-matrices $M_{N,n}$ into $\Lambda_n \mathbb{R}^{n+N}$, and

$$\frac{M(G)}{|M(G)|}$$

is the simple tangent n-vector of the graph of the linear transformation $x \to Gx$. For any n-vector $\xi \in \Lambda_n \mathbb{R}^{n+N}$ we denote by $\xi_{\alpha\beta}$ its coordinates

$$\xi = \sum_{|\alpha|+|\beta|=n} \xi_{\alpha\beta}\; e^\alpha \wedge \epsilon^\beta$$

and by Σ_1 the image of the map M in $\Lambda_n \mathbb{R}^{n+N}$

$$\Sigma_1 := \{\xi \mid \xi = M(G) \text{ for some } G \in M_{N,n}\},$$

i.e. the class of all simple vectors in $\Lambda_n \mathbb{R}^{n+N}$ with $\xi_{\bar{0}0} = 1$. Observe that the map M is injective, Σ_1 is not convex, and obviously $\Sigma_1 \subset \Lambda_1$ where

$$\Lambda_1 := \{\xi \in \Lambda_n \mathbb{R}^{n+N} : \xi_{\bar{0}0} = 1\}.$$

Set also

$$\Lambda_0 := \{\xi \in \wedge_n \mathbb{R}^{n+N} : \xi_{\bar{0}0} = 0\}$$
$$\Lambda_+ := \{\xi \in \wedge_n \mathbb{R}^{n+N} : \xi_{\bar{0}0} > 0\}.$$

For $k = 0, 1, \ldots, \min(n, N)$, denote by V_k the linear subspace of $\wedge_n \mathbb{R}^{n+N}$ given by the linear combinations of vectors of the type $v_1 \wedge \ldots \wedge v_{n-k} \wedge w_1 \wedge \ldots \wedge w_k$ with $v_i \in \mathbb{R}^n$, $w_i \in \mathbb{R}^N$,

$$V_k := \wedge_{n-k} \mathbb{R}^n \wedge \wedge_k \mathbb{R}^N.$$

The spaces V_k are naturally orthogonal and

$$\wedge_n \mathbb{R}^{n+N} = \bigoplus_{k=0}^{\min(n,N)} V_k;$$

in fact $\xi \in \wedge_n \mathbb{R}^{n+N}$, i.e.

$$\xi = \sum_{|\alpha|+|\beta|=n} \xi_{\alpha\beta}\ e^\alpha \wedge \epsilon^\beta$$

then

$$\xi = \sum_{k=0}^{\min(n.N)} \xi_k, \qquad \xi_k := \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \xi_{\alpha\beta}\, e^\alpha \wedge \epsilon^\beta \in V_k \ .$$

If $\xi = M(G)$, $G \in M_{N\times n}$, and P_k is the projection from $\wedge_n \mathbb{R}^{n+N}$ onto V_k, $\xi_k = P_k \xi$, we set

$$M_k(G) = P_k M(G), \quad k = 0, 1, \ldots, \min(n, N),$$

that is

$$M_0(G) = e_1 \wedge \ldots \wedge e_n$$
$$M_1(G) = \sum_{ij} (-1)^{i-1} G_{ij}\, e^{\bar{i}} \wedge \epsilon_j$$
$$M_k(G) = \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \sigma(\alpha, \bar{\alpha}) M_{\beta\bar{\alpha}}(G)\, e^\alpha \wedge \epsilon^\beta.$$

Observe in particular that M_1 gives an isometry between $M_{N\times n}$ and $V_1 = \wedge_{n-1} \mathbb{R}^n \wedge \Lambda_1 \mathbb{R}^N$ which *depends only on the choice of* $e_1 \wedge \ldots \wedge e_n$ since

$$M(G) = \wedge_n (\mathrm{id}_{\mathbb{R}^n} \times L_G)(e_1 \wedge \ldots \wedge e_n)$$

where $(\mathrm{id}_{\mathbb{R}^n} \times L_G) : \mathbb{R}^n \to \mathbb{R}^{n+N}$. This allows us to associate a matrix G_{ξ_1} to each $\xi_1 \in V_1$ by

$$G_{\xi_1} = M_1^{-1}(\xi_1).$$

By means of the previous notations, the necessary and sufficient conditions for the simplicity of an n-vector ξ with $\xi_{\bar{0}0} = 1$ given in (2.4) can be stated as

$$P_0\xi = e_1 \wedge \ldots \wedge e_n$$
$$P_k\xi = M_k(G_{\xi_1}), \qquad \xi_1 = P_1\xi \quad ,$$

in particular the class Σ_1 of simple n-vectors with $\xi_{\bar{0}0} = 1$ is a graph over V_1. We observe that in general $M_k(G), G \in M_{N\times n}$, is not a simple n-vector, and actually it is easy to see that, *for all* k, $\xi \in V_k$ *is simple if and only if* $\xi = M_k(L)$ *"for an orthogonal matrix"* L. Here by "orthogonal matrix" we mean a matrix with the following property: there exist orthogonal bases $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_N)$, respectively in $\mathbb{R}^n$ and $\mathbb{R}^N$, such that

$$Lv_i = 0 \qquad i + 1, \ldots, n - k$$
$$Lv_i = w_{i-n-k} \quad i = n - k + 1, \ldots, n.$$

Consider now a nonnegative smooth integrand f defined on the class of $N \times n$-matrix

$$f : M_{N\times n} \to \mathbb{R}_+,$$

an important model being the Dirichlet integrand

$$\frac{1}{2}|G|^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N} G_{ji}^2 \quad ;$$

identify $M_{N\times n}$ with Σ_1 by means of the map M and regard f as a map $\underset{\sim}{f}$ from Σ_1 into $\mathbb{R}_+$

$$\underset{\sim}{f}(\xi) = f(M^{-1}(\xi)) \qquad \forall \xi \in \Sigma_1.$$

Then, we define for all $\xi \in \Lambda_1$

$$\tilde{f}(\xi) := \sup\{\phi(\xi) \mid \phi : \Lambda_1 \to \mathbb{R} \text{ affine}, \quad \phi(\eta) \le \underset{\sim}{f}(\eta), \ \forall \eta \in \Sigma_1\} \tag{4.12}$$

and we extend $\tilde{f}$ to $\Lambda_+ = \{\xi \in \wedge_n\mathbb{R}^{n+N} \mid \xi_{\bar{0}0} > 0\}$ as the one degree homogeneous function

$$\overline{\tilde{f}}(\xi) = \xi_{\bar{0}0}\,\tilde{f}\left(\frac{\xi}{\xi_{\bar{0}0}}\right) \quad \forall \xi \in \Lambda_+. \tag{4.13}$$

We shall refer to $\tilde{f}$ as to the *polyconvex extension of* f (for related facts see e.g. [17], [4], [19], [41]). Observe that $\tilde{f}$ is the largest convex and lower semicontinuous minorant of the function which agrees with $\underset{\sim}{f}$ on Σ_1 and has values $+\infty$ on $\Lambda_1 \backslash \Sigma_1$, i.e.

$$\tilde{f}(\xi) = \sup\{g(\xi) \mid g : \Lambda_1 \to \mathbb{R} \text{ convex}, \ g(\eta) \le \underset{\sim}{f}(\eta), \ \forall \eta \in \Sigma_1\}.$$

In particular, $\tilde{f}$ ($\overline{\tilde{f}}$) is convex and lower semicontinuous on Λ_1 (Λ_+). We recall that $f : M_{n\times N} \to \mathbb{R}$ is called *polyconvex* (see e.g. [49], [4]) if there exists a convex function $g : \Lambda_1 \to \mathbb{R}$ such that $f(G) = g(M(G))$, $\forall G \in M_{n\times N}$. Thus f *is polyconvex if and only if* $\tilde{f} = \underset{\sim}{f}$ *on* Σ_1.

The function $\overline{\tilde{f}}$ can also be seen as the convex extension of the 1-homogeneous function on the cone Σ_+ over Σ_1 which agrees with $\underset{\sim}{f}$ on Σ_1. More precisely, extend $\underset{\sim}{f}$ to the cone

$$\Sigma_+ := \{\lambda\xi \mid \lambda > 0,\ \xi \in \Sigma_1\} = \{\xi \in \Lambda_n\mathbb{R}^{n+N} \mid \xi_{\bar{0}0} > 0,\ \xi/\xi_{\bar{0}0} \in \Sigma_1\}$$

by

$$\bar{f}(\xi) := \xi_{\bar{0}0}\, \underset{\sim}{f}\left(\frac{\xi}{\xi_{\bar{0}0}}\right) \qquad \forall\ \xi \in \Sigma_+$$

and set

$$\text{(4.14)} \qquad \tilde{\bar{f}}(\xi) := \{\phi(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \mathbb{R} \text{ linear},\ \phi(\eta) \leq \tilde{f}(\eta),\ \forall \eta \in \Sigma_+\}.$$

Then it is easily seen that $\overline{\tilde{f}}(\xi) = \tilde{\bar{f}}(\xi)$ for all $\xi \in \Lambda_+$. Observe that $\tilde{\bar{f}}$ is defined for all $\xi \in \Lambda_n\mathbb{R}^{n+N}$ and it is the so called Γ-regularization (see e.g. [27]) of the function

$$\bar{f}_\infty(\xi) = \begin{cases} \bar{f}(\xi) & \xi \in \Sigma_+ \\ +\infty & \xi \in \Lambda_n\mathbb{R}^{n+N}\setminus\Sigma_+, \end{cases}$$

and it is the bipolar of $\bar{f}_\infty$. It is also easily seen that for $\xi \in \Lambda_0$

$$\tilde{\bar{f}}(\xi) = \inf\{\liminf_{k\to\infty} \overline{\tilde{f}}(\xi_k) \mid \xi_k \in \Lambda_+, \quad \xi_k \to \xi\} \qquad .$$

while $\tilde{\bar{f}}(\xi) = +\infty$ for $\xi \in \Lambda_- := \{\xi \in \Lambda_m\mathbb{R}^{n+N} \mid \xi_{\bar{0}0} < 0\}$.

Since in the sequel ξ will be the tangent n-vector to a cartesian current, we finally denote by F the extension of $\underset{\sim}{f}$ to $\Lambda_+ \cup \Lambda_0$ given by $\tilde{\bar{f}}$, or equivalently

$$\text{(4.15)} \qquad F(\xi) = \begin{cases} \overline{\tilde{f}}(\xi) & \text{if } \xi_{\bar{0}0} > 0 \\ \inf\{\liminf_{k\to\infty} \overline{\tilde{f}}(\xi_k) \mid \xi_k \in \Lambda_+,\ \xi_k \to \xi\} & \text{if } \xi_{\bar{0}0} = 0 \end{cases}$$

and, if convenient, we shall think of F as extended to $+\infty$ in Λ_-.

Of course the previous definitions extend naturally to the case in which the integrand f depends also on x, u, just considering x and u as parameters, and to the case in which f is only defined on a proper subset S of $M_{N\times n}$. If $\Sigma_S = M(S)$, then $\overline{\tilde{f}}$ will be finite on the cone over the convex hull of Σ_S. Finally we observe that the function F in (4.15) is lower semicontinuous and convex in $\Lambda_+ \cup \Lambda_0$, and if Γ is the segment in $\Lambda_+ \cup \Lambda_0$ with end points $\xi_0 \in \Lambda_0$ and ξ' in Λ_+, then

$$F(\xi) = \lim_{\substack{\eta \to \xi_0 \\ \eta \in \Gamma}} F(\eta).$$

Given now the variational integral

$$\mathcal{F}(u) := \int_\Omega f(x, u(x), Du(x))\, dx$$

where f is a smooth function from $\Omega \times \mathbb{R}^N \times M_{N\times n}$ into $\mathbb{R}_+$ (or in general from $\Omega \times \mathbb{R}^N \times S$, where S is a subset of $M_{N\times n}$, into $\mathbb{R}_+$), we define *its extension as a parametric integral for* $T \in \mathrm{cart}(\Omega, \mathbb{R}^N_y)$, $T = \tau(\mathcal{M}, \theta, \xi)$, *by*

$$\mathcal{F}(T) := \int_{\mathcal{M}} F(\pi z, \hat{\pi} z, \xi(z))\, \theta(z)\, d\mathcal{H}^n(z) \tag{4.16}$$

where F is given by (4.15), compare [33].

We remark that in order to compute $\mathcal{F}(T)$ we actually do not need to know F in $\Lambda_+ \cup \Lambda_0$; in fact, since the tangent n-vector to a cartesian current is simple, we ought to know F only on the *simple* n-vectors in $\Lambda_+ \cup \Lambda_0$. If f is polyconvex, we trivially know F on the simple n-vectors ξ with $\xi_{\bar{0}0} > 0$

$$F(\pi z, \hat{\pi} z, \xi(x)) = \xi_{\bar{0}0}\, \underset{\sim}{f}(\pi z, \hat{\pi} z, (\xi/\xi_{\bar{0}0})),$$

thus only the values of F on the simple vectors with zero first component are to be found. But in general these cannot be obtained easily from f by semicontinuity, i.e. as

$$\bar{f}(\xi) := \inf\{\liminf_{k\to\infty} \bar{f}(\xi_k) \mid \xi_k \in \Sigma_+,\ \xi_k \to \xi\} \tag{4.17}$$

since in general

$$\bar{f}(\xi) \geq F(\xi), \qquad \xi \in \Lambda_0.$$

A trivial but useful observation is the following: if $g : \Lambda_1 \to \mathbb{R}_+$ is a convex function which agrees with $\underset{\sim}{f}$ on Σ_1, and satisfies, for all $\xi \in \Lambda_0$

$$\bar{g}(\xi) := \inf\{\liminf_{k\to\infty} \bar{g}(\xi_k) \mid \xi_k \in \Sigma_+,\ \xi_k \to \xi\} \geq \bar{f}(\xi),$$

then

$$F(\xi) = \bar{f}(\xi), \qquad \text{for all } \xi \in \Lambda_0. \tag{4.18}$$

Before we discuss some examples, let us recall the following semicontinuity theorem which is a simple consequence of theorem 2 sec. 5 of [33].

THEOREM 3. *Let $F(x,u,\xi)$ be a parametric integrand on $\Omega\times\mathbb{R}^N\times\Lambda_+\cup\Lambda_0\to\mathbb{R}^+$ associated to a function f by* (4.15). *Suppose F is continuous on a convex set $K\subset\Lambda_+\cup\Lambda_0$ such that $\overset{\circ}{K}\subset K\subset\overline{K}$, and $\overset{\circ}{K}=\overline{K}$. Suppose that $\{T_k\}$ is a sequence of rectifiable currents $T_k=\tau(\mathcal{M}_k,\theta_k,\xi_k)$ which are graphs,* i.e. *$\pi_\# T_k=[\![\Omega]\!]$, $\xi_{k\bar{0}0}\geq 0$, with equibounded L^1-norms in Ω and masses in $\Omega\times\mathbb{R}^N$ and which converge weakly in $\mathcal{D}^n(\Omega\times\mathbb{R}^N)$ to some rectifiable cartesian current T_0. Then*

$$\mathcal{F}(T_0)\leq\liminf_{k\to\infty}\mathcal{F}(T_k).$$

We emphasize the fact that the currents T_k,T_0 may have boundaries in $\Omega\times\mathbb{R}^N$.

EXAMPLE 1. Consider the integrand

$$f(Du):=|M(Du)|=\{1+\sum_{|\alpha|+|\beta|=n}|M_{\alpha\beta}(Du)|^2\}^{1/2}\ ,\ u:\mathbb{R}^n\to\mathbb{R}^N,$$

or equivalently

$$\underset{\sim}{f}(\xi)=|\xi|\quad\forall\xi\in\Sigma_1,\ \text{i.e. } \xi \text{ simple and } \xi_{\bar{0}0}=1.$$

It is easily seen that

$$F(\xi)=\|\xi\|\qquad\forall\xi\in\Lambda_+\cup\Lambda_0$$

where $\|\xi\|$ is the *mass* of the n-vector ξ in the sense of Federer-Fleming, cfr. [30],

$$\|\xi\|:=\sup\{\phi(\xi)\mid\phi:\Lambda_n\mathbb{R}^{n+N}\to\mathbb{R}\ \text{linear},\ |\phi(\eta)|\leq|\eta|,\ \forall\eta\ \text{simple}\}.$$

In fact we have for $\xi\in\Lambda_+\cup\Lambda_0$

$$F(\xi)=\sup\{\phi(\xi)\mid\phi:\Lambda_n\mathbb{R}^{n+N}\to\mathbb{R}\ \text{linear},\ |\phi(\eta)|\leq|\eta|,\ \forall\eta\ \text{simple}\ \eta_{\bar{0}0}>0\},$$

and, since any simple n-vector with $\eta_{\bar{0}0}=0$ can be approximated by simple n-vectors in Λ_+ and $|\eta|$ is continuous,

$$F(\xi)=\sup\{\phi(\xi)\mid\phi\in C_1\},$$

$$C_1:=\{\phi\mid\phi:\Lambda_n\mathbb{R}^{n+N}\to\mathbb{R}\ \text{linear},\ |\phi(\eta)|\leq|\eta|,\ \forall\eta\ \text{simple},\ \eta_{\bar{0}0}\geq 0\}.$$

It is now sufficient to observe that the values $\phi(\xi)$, $\xi\in\Lambda_+\cup\Lambda_0$ for $\phi\in C_1$ or $\phi\in C_2:=\{\phi\mid\phi:\ \Lambda_n\mathbb{R}^{n+N}\to\mathbb{R}$ linear, $|\phi(\eta)|\leq|\eta|$, $\forall\eta$ simple$\}$, coincide, because we can realize them with linear functions which are zero on V_0 and positive in Λ_+ if $\xi_{\bar{0}0}>0$, and with linear functions independent of $\eta_{\bar{0}0}$ if $\xi_{\bar{0}0}=0$ and because $-\eta$ is simple if η is simple.

Finally it is easily seen that $\|\xi\| = |\xi|$ if ξ is simple, actually ξ is simple if and only if $\|\xi\| = |\xi|$, so the extension of

$$\int_\Omega \left[1 + \sum_{|\alpha|+|\beta|=n} |M_{\alpha\beta}(Du)|^2\right]^{1/2} dx$$

is given, for example for $T \in \operatorname{cart}(\Omega, \mathbb{R}^N)$, $T = \tau(\mathcal{M}, \theta, \xi)$, by

$$\mathcal{F}(T) = \int_{\mathcal{M}} |\xi| \theta d\mathcal{H}^n = M(T).$$

Consider now the functional

$$\int_\Omega |Du|\, dx, \qquad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N,$$

with integrand $f(G) = |G|$, i.e.

$$\underset{\sim}{f}(\xi) = |\xi_1|, \quad \xi_1 = P_1\xi, \quad \forall\xi \text{ simple}, \ \xi_{\bar{0}0} = 1.$$

We decompose every linear map $\phi : \Lambda_n\mathbb{R}^{n+N} \to \mathbb{R}$ as

$$\phi = \phi_0 + \phi_1 + \ldots + \phi_{\min(n.N)}$$

where the ϕ_i's are linear maps on $V_i := \Lambda_{n-i} \wedge \Lambda_i\mathbb{R}^N$, $\phi_i : V_i \to \mathbb{R}$. Then for all $\xi \in \Lambda_+ \cup \Lambda_0$

$$\begin{aligned} F(\xi) = \sup\{\phi(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \mathbb{R} \text{ linear}, \\ \phi(\eta) \le |P_1\eta|, \ \forall\eta \text{ simple } \eta_{\bar{0}0} > 0\} \\ = \sup\{\phi_1(\xi_1) \mid \phi_1 : V_1 \to \mathbb{R} \text{ linear}, \\ \phi_1(\eta_1) \le |P_1\eta|, \ \forall\eta \text{ simple } \eta_{\bar{0}0} \ge 0\}, \end{aligned}$$

where the second equality follows by homogeneity.
As previously, since

$$\sup\{\phi_1(\xi_1) \mid \phi_1 : V_1 \to \mathbb{R}, \ \phi_1(\eta_1) \le |\eta_1|, \ \forall\eta \text{ simple}\} = \|\xi_1\|,$$

one sees that

$$F(\xi) = \|\xi_1\|, \qquad \xi_1 = P\xi.$$

Therefore we see that the extension of $\int_\Omega |Du| dx$ is finite for currents T with associate function u_T in $BV(\Omega, \mathbb{R}^N)$ and in this case it is given by

$$\mathcal{F}(T) = \int_\Omega |Du_T|$$

where the right hand-side denotes the total variation of the vector valued measure Du_T. We remark that the functional $\int_\Omega |Du_T|$ should be considered, and we shall do it, as a *degenerate* functional, in fact it controls only the first two components of the current T. We notice also that finally the extension of

$$\int_\Omega a(u)|Du|$$

is, for $T = \tau(\mathcal{M}, \theta, \xi)$,

$$\mathcal{F}(T) = \int_{\mathcal{M}} a(\hat{\pi} z)|\xi_1|\, \theta \, d\mathcal{H}^n(z),$$

but giving an 'explicit' expression is not easy (compare [3] for related results). Of course if $N = 1$, or $n = 1$, we have no degeneracy and we can reread $\mathcal{F}(T)$ completely and obtaining for $N = 1$ the representation formula in [20]; the expression being similar for $n = 1, N \geq 1$.

EXAMPLE 2. Consider the Dirichlet integral

$$\frac{1}{2}\int_\Omega |Du|^2 dx$$

for mappings from a domain $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^2$, that is the integrand $f(p) := \frac{1}{2}|p|^2$ for $p \in M_{2\times 2}$. For any 2-vector $\xi \in \wedge_2 \mathbb{R}^4$, i.e.

$$\xi = \xi_{\bar{0}0}\, e_1 \wedge e_2 + \eta_{\bar{i}j}\, e^{\bar{i}} \wedge \epsilon_j + \delta\, \epsilon_1 \wedge \epsilon_2,$$

denote by G_η the matrix $M^{-1}(\xi)$, i.e. the matrix such that $P_1 M(G_\eta) = P_1 \xi, G_\eta =: ((G_\eta)_{ij}), (G_\eta)_{ij} = (-1)^{i-1}\eta_{\bar{i}j}$. Any affine map $\phi : \Lambda_1 \to \mathbb{R}$ has the form

$$\phi(\xi) = a + b \cdot G_\eta + d\delta, \quad \xi \in \Lambda_1,$$

and

$$\phi(M(G)) = a + b \cdot G + d \text{ det} G;$$

therefore for $\xi \in \Lambda_1$

$$\tilde{f}(\xi) = \sup\left\{a + b \cdot G_\eta + d\delta \mid a + b \cdot G + d\text{det}G \leq \frac{1}{2}|G|^2, \quad \forall G \in M_{2\times 2}\right\}.$$

A necessary condition for $\phi(M(G)) \leq \frac{1}{2}|G|^2 \ \forall G$, being that

$$\frac{1}{2}|G|^2 - d \text{ det } G \geq 0, \qquad \forall G \in M_{2\times 2},$$

it follows at once

$$|d| \sup_G \frac{2 \text{ det} G}{|G|^2} \leq 1,$$

hence $|d| \leq 1$, since by the isoperimetric inequality for parallelograms

$$\sup_G \frac{2\det G}{|G|^2} = 1.$$

For $|d| \leq 1$

$$\frac{1}{2}|G|^2 - d \det G$$

is a nonnegative quadratic form, hence a convex function, therefore the maximum of $a + b \cdot G_\eta$, with the constraint $a + b \cdot G \leq \frac{1}{2}|G|^2 - d \det G$, $\forall G$, is obtained for a, b such that

$$a + b \cdot G_\eta = \frac{1}{2}|G_\eta|^2 - d \det G_\eta \ ,$$

so

$$\tilde{f}(\xi) = \sup_{|d|\leq 1} \left\{ \frac{1}{2}|G_\eta|^2 + d(\delta - \det G_\eta) \right\} = \frac{1}{2}|G_\eta|^2 + |\delta - \det G_\eta|, \qquad \forall \xi \in \Lambda_1.$$

A simple computation then gives

$$F(\xi) = \begin{cases} \frac{1}{2}\frac{|G_\eta|^2}{\xi_{\bar{0}0}} + |\delta - \frac{\det G_\eta}{\xi_{\bar{0}0}}| & \text{for } \xi_{\bar{0}0} > 0 \\ |\delta| & \text{for } \xi_{\bar{0}0} = 0, \ \eta = 0 \\ +\infty & \text{for } \xi_{\bar{0}0} = 0, \ \eta \neq 0. \end{cases}$$

Let $T \in \text{cart}(\Omega, \mathbb{R}^2)$, $T = \tau(\mathcal{M}, \theta, \xi)$. Since ξ is simple and $\xi_{\bar{0}0} > 0$ on $\mathcal{M}_+$ we have

$$\mathcal{F}(T, \mathcal{M}_+) = \int_{\mathcal{M}_+} F(\xi)\, d\mathcal{H}^n = \frac{1}{2}\int_\Omega |(Du_T)^a|^2 dx.$$

In particular we deduce that $\mathcal{F}(T) = +\infty$ whenever u_T does not belong to $H^{1,2}(\Omega, \mathbb{R}^2)$. On the other hand we have seen that if $u_T \in H^{1,2}(\Omega, \mathbb{R}^2)$, $T = T_{u_T} \in \text{cart}^{1,1}(\Omega, \mathbb{R}^2)$, so $\mathcal{M} = \mathcal{M}_+$. Therefore we can conclude that the extension of the Dirichlet integral is

$$\mathcal{F}(T) = \begin{cases} \frac{1}{2}\int_\Omega |Du_T|^2 dx & \text{if } u_T \in H^{1,2}(\Omega, \mathbb{R}^2) \\ +\infty & \text{otherwise.} \end{cases}$$

Consider also the Dirichlet integral for mappings from Ω into $S^1 \subset \mathbb{R}^2$, i.e. the integrand f which is defined for each u on the subset of 2×2-matrix G such that $G^T \cdot n = 0$, $n = \frac{u}{|u|}$. A simple computation shows that the extension F associated to $f : \{G \in M_{2\times 2} \mid G^T \cdot n = 0\} \to \mathbb{R}_+$, $f(G) = \frac{1}{2}|G|^2$, is given for all $\xi = (\xi_0, \xi_1, \xi_2) \in \Lambda_+ \cup \Lambda_0$ by

$$F(n, \xi) = \begin{cases} \frac{1}{2}|G_{\xi_1}|^2 & \text{if } G_{\xi_1}^T \cdot n = 0, \ \xi_2 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

that is the parametric extension of the Dirichlet integral, for mappings from $\Omega \subset \mathbb{R}^2$ into S^1, is finite and coincide with the parametric extension of the Dirichlet integral for mappings from Ω into $\mathbb{R}^2$, if and only if $T \in H^{1,2}(\Omega, S^1)$.

If we instead consider the gradient integral

$$\int_\Omega |Du| dx$$

for mappings from $\Omega \subset \mathbb{R}^1$ into S^1, it is not difficult to see that its parametric extension coincides with the restriction of the parametric extension of the gradient integral for currents $T \in \text{cart}^{1,1}(\Omega, \mathbb{R}^2)$. This together with theorem 3 gives at once that the parametric extension $\mathcal{F}$ is lower semicontinuous in $\text{cart}^{1,1}(\Omega, S^1)$ with respect to the weak convergence of currents. As in theorem 2 one also easily see that

$$\text{cart}(\Omega, S^1) = \{T \mid T = T_{u_T} + \sum_{i=1}^k d_i [\![\{x_i\} \times S^1]\!] \quad , \quad u_T \in H^{1,1}(\Omega, S^1)\}$$

and therefore $\mathcal{F}(T)$ is given by

$$\mathcal{F}(T) = \int_\Omega |Du_T| dx + \sum_{i=1}^k |d_i| \mathcal{H}^1(S^1).$$

Before we discuss the extension of the Dirichlet integral in higher dimensions and codimensions, let us make the following remark which shows that our extension procedure is formally the same as the procedure leading to the notions of mass and comass and which at the same time slightly simplifies our notations.

Given a nonnegative real valued function $\underset{\sim}{f}$ on a subset Σ_S of Σ_1, extend $\underset{\sim}{f}$ as the 1- homogeneous function $\bar{f}$ on the cone C_{Σ_S} over Σ_S. For any linear map $\phi : \Lambda_n \mathbb{R}^{n+N} \to \mathbb{R}$, we define the *comass of ϕ respect to f* by

$$\|\phi\|_f := \sup\{\phi(\eta) \mid \eta \in C_{\Sigma_S}, \bar{f}(\eta) \leq 1\}.$$

Observe that for $\Sigma_S = \Sigma_1$, C_{Σ_S} coincides with the simple n-vectors ξ with $\xi_{\bar{0}0} > 0$, so, if also $f(\eta) = |\eta|$, $\|\phi\|_f$ is the comass of ϕ in the sense of Federer-Fleming. We also define the *mass of any n-vector in $\Lambda_n \mathbb{R}^{n+N}$ with respect to* f by

$$\|\xi\|_f := \sup\{\phi(\xi) \mid \phi : \Lambda_n \mathbb{R}^{n+N} \text{ linear}, \ \|\phi\|_f \leq 1\}.$$

One sees at once that $\|\phi\|_f \leq 1$ is equivalent to $\phi(\eta) \leq \bar{f}(\eta)$, $\forall \eta \in C_{\Sigma_S}$, and therefore we have

$$F(\xi) = \|\xi\|_f, \quad \forall \xi \in \Lambda_+ \cup \Lambda_0.$$

EXAMPLE 3. Consider the Dirichlet integral $\frac{1}{2} \int_\Omega |Du|^2 dx$ for mappings

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$$

and denote by $f(p) = \frac{1}{2}|p|^2$, $p \in M_{N\times n}$, its integrand. By definition, the parametric integrand F associated to f is given by

$$F(\xi) = \sup\{\phi(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \mathbb{R} \text{ linear, } \phi(\eta) \le |\eta_1|^2/2\eta_{\bar{0}0},\ \forall\eta \in \Sigma_+\}$$

and of course $\phi(\eta) \le |\eta_1|^2/2\eta_{\bar{0}0}$, $\forall\eta \in \Sigma_+$, is equivalent to

$$\phi(M(G)) \le \frac{1}{2}|G|^2, \quad \forall G \in M_{N\times n}.$$

if we decompose ϕ as the sum $\phi_0+\phi_1+\ldots+\phi_{\min(n.N)}$, where the ϕ_i's are linear maps on V_i, a simple scaling argument gives that, if $\phi(\eta) \le |\eta_1|^2/2\eta_{\bar{0}0}$, $\forall\eta \in \Sigma_+$, then $\phi_3,\ldots,\phi_{\min(n.N)}$ ought to be identically zero, so that

$$F(\xi) = \sup\Big\{\phi_0(\xi_0) + \phi_1(\xi_1) + \phi_2(\xi_2) \mid \phi_0(e_1 \wedge \ldots \wedge e_n) + \phi_1(M_1(G)) + \phi_2(M_2(G)) \le \frac{1}{2}|G|^2,\ \forall G\Big\},$$

and

$$\frac{1}{2}|G|^2 - \phi_2(M_2(G)) \ge 0, \quad \forall G \in M_{N\times n},$$

which is equivalent to $\|\phi_2\|_f \le 1$.
As in example 2, we observe now that $\frac{1}{2}|G|^2 - \phi_2(M_2(G))$ is convex, provided $\|\phi_2\|_f \le 1$, therefore for $\xi_{\bar{0}0} > 0$ the supremum of $\phi_0(\xi_0) + \phi_1(\xi_1)$, under the previous constraint on ϕ, is taken for fixed ϕ_2 on

$$\xi_{\bar{0}0}[\phi_0(e_1 \wedge \ldots \wedge e_n) + \phi_1(\xi_1/\xi_{\bar{0}0})] = \xi_{\bar{0}0}\left[\frac{1}{2}|G_{\xi_1/\xi_{\bar{0}0}}|^2 - \phi_2(M_2(G_{\xi_1/\xi_{\bar{0}0}})\right],$$

hence

$$\begin{aligned} F(\xi) &= \sup_{\|\phi_2\|_f\le 1}\left\{\frac{|\xi_1|^2}{2\xi_{\bar{0}0}} + \phi_2(\xi_2 - \xi_{\bar{0}0}M_2(G_{\xi_1/\xi_{\bar{0}0}}))\right\} \\ &= \frac{1}{2}|G_{\xi_1/\xi_{\bar{0}0}}|^2 + \|\xi_2 - \xi_{\bar{0}0}M_2(G_{\xi_1/\xi_{\bar{0}0}})\|_f, \qquad \text{for } \xi \in \Lambda_+. \end{aligned}$$

Suppose now $\xi_{\bar{0}0} = 0$. Fix $\phi_2 \equiv 0$ and ϕ_1 such that $\phi_1(\xi_1) = k > 0$; clearly we can choose ϕ_0 so that

$$\phi_0(e_1 \wedge \ldots \wedge e_n) + \phi_1(M_1(G)) \le \frac{1}{2}|G|^2, \quad \forall G \in M_{N\times n},$$

thus

$$F(\xi) = +\infty, \quad \text{for } \xi_{\bar{0}0} = 0 \text{ and } \xi_1 \ne 0.$$

Finally, for ξ satisfying $\xi_{\bar{0}0} = 0$, $\xi_1 = 0$, it is easily seen that

$$F(\xi) = \sup\Big\{\phi_2(\xi) \mid \phi : \Lambda_n\mathbb{R}^{n+N} \to \mathbb{R} \text{ linear}, \phi(M(G)) \le \frac{1}{2}|G|^2,\ \forall G \in M_{N\times n}\Big\}$$

$$= \sup\Big\{\phi_2(\xi) \mid \phi_2 : V_2 \to \mathbb{R} \text{ linear}, \phi_2(M_2(G)) \le \frac{1}{2}|G|^2,\ \forall G \in M_{N\times n}\Big\}$$

and, in conclusion we get

$$F(\xi) = \begin{cases} \frac{1}{2}|G_{\xi_1/\xi_{\bar{0}0}}|^2 + \|\xi_2 - \xi_{\bar{0}0} M_2(G_{\xi_1/\xi_{\bar{0}0}})\|_f & \text{if } \xi_{\bar{0}0} > 0 \\ \|\xi_2\|_f & \text{if } \xi_{\bar{0}0} = 0,\ \xi_1 = 0 \\ +\infty & \text{if } \xi_{\bar{0}0} = 0,\ \xi_1 \neq 0. \end{cases}$$

Assume now that either $n = 2$ or $N = 2$. As we have remarked, every simple n-vector in V_2, i.e. every simple n-vector with $\xi_{\bar{0}0} = 0$ and $\xi_1 = 0$, can be written as $\xi = M_2(L)$, where L is an "orthogonal matrix" (compare the biginning of this subsection), thus satisfying

$$\frac{1}{2}|L|^2 = |M_2(L)|.$$

So, if either $n = 2$ or $N = 2$, we have for ξ with $\xi_{\bar{0}0} = 0, \xi_1 = 0$,

$$\|\xi\|_f = \|\xi_2\|_f = \|\xi_2\| = |\xi_2| = |\xi|$$

and we can conclude that the parametric extension of the Dirichlet integral to cart$(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^2$ (respectively to cart$(\Omega, \mathbb{R}^2), \Omega \subset \mathbb{R}^n$) is finite and only if the function u_T associated to $T = \tau(\mathcal{M}, \theta, \xi)$ is in $H^{1,2}(\Omega, \mathbb{R}^N)$, and the singular part of T has finite mass and a completely vertical tangent n-vector, i.e. $\xi_{\bar{0}0} = 0$ and $\xi_1 = 0$ for $\xi \in \mathcal{M}_+$; in this case the parametrix extension is given by

$$\mathcal{F}(T) = \frac{1}{2}\int_\Omega |Du_T|^2 dx + M(T - T_{u_T})$$

or equivalently

$$\mathcal{F}(T) = \frac{1}{2}\int_\Omega |Du_T|^2 dx + M(T \llcorner \mathcal{M}\setminus\mathcal{M}_+).$$

EXAMPLE 4. Finally we consider the Dirichlet integral for mapping from $\Omega \subset \mathbb{R}^2$ into $S^2 \subset \mathbb{R}^3$ which was our starting point. For fixed $u(x)$, its integrand is given by

$$f : \left\{ G \in M_{3\times 2} \mid G^T n = 0, \ n = u(x)/|u(x)| \right\} \to \mathbb{R}, \quad f(G) = \frac{1}{2}|G|^2.$$

The associated parametric integrand is given by

$$F(n,\xi) = \begin{cases} \frac{1}{2}|G_{\xi_1/\xi_{\bar{0}0}}|^2 + \|\xi_2 - \xi_{\bar{0}0} M_2(G_{\xi_1/\xi_{\bar{0}0}})\|_f & \text{if } \xi_{\bar{0}0} > 0, \ G^T_{\xi_1/\xi_{\bar{0}0}} n = 0 \\ \|\xi_2\|_f & \text{if } \xi_{\bar{0}0} = 0, \ \xi_1 = 0 \\ +\infty & \text{otherwise;} \end{cases}$$

we omit the details of the computations which go along the lines of the previous examples. In particular we see that the parametric extension is finite if and only if T belongs to $\text{cart}^{2,1}(\Omega, S^2)$ and in this case $\mathcal{F}(T)$ coincides with the value of the parametric extension of the Dirichlet integral to $\text{cart}(\Omega, \mathbb{R}^3)$. Thus, in view of theorem 3, $\mathcal{F}(T)$ is lower semicontinuous in $\text{cart}^{2,1}(\Omega, S^2)$ with respect to the weak convergence and, if $T \in \text{cart}^{2,1}(\Omega, S^2)$, i.e. $T = T_{u_T} + \sum_{i=1}^{k} d_i [\![\{x_i\} \times S^2]\!]$, we have

$$\mathcal{F}(T) = \frac{1}{2}\int_\Omega |Du_T|^2 dx + \sum_{i=1}^{k} |d_i| \mathcal{H}^2(S^2).$$

EXAMPLE 5. Here we collect a few more examples leaving to the reader the simple details. Consider the variational integral

$$\int |Du|^p dx,$$

where $1 < p < 2$ or $2 < p < +\infty$, defined for smooth mappings from a domain Ω of $\mathbb{R}^2$, and denote by f the integrand $f(G) = |G|^p$. It is easily seen that for $p, 1 < p < 2$, the parametric integrand F evaluated on simple 2-vectors is given by

$$F(\xi) = \begin{cases} \xi_{\bar{0}0}^{1-p} |\xi_1|^p & \forall \xi \in \Lambda_+ \ , \ \xi \text{ simple} \\ 0 & \forall \xi \in \Lambda_0 \ , \ \xi \text{ simple with } \xi_1 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

i.e. the parametric extension is finite for currents T with $u_T \in H^{1,p}(\Omega, \mathbb{R}^2)$ and $T - T_{u_T}$ completely vertical, i.e. with tangent vector of the form $(0, 0, \xi_2)$. Observe that this extension is *degenerate* as it gives no control on ξ_2, thus in this sense the original variational integral is *not regular*. For $p > 2$ we have

$$F(\xi) = \begin{cases} \xi_{\bar{0}0}^{1-p} |\xi_1|^p & \forall \xi \in \Lambda_+ \ , \ \xi \text{ simple} \\ +\infty & \text{otherwise,} \end{cases}$$

corresponding to the fact that the parametric extension of $\int |Du|^p dx$, $p > 2$, is finite if and only if T has no vertical part and, more precisely, T is the current T_u for some $u \in H^{1,p}(\Omega, \mathbb{R}^2)$. In this case we can say that the original variational integral is regular.

We shall now give the parametric extension of the variational integral

$$\int_{\mathbb{R}^3} (|Du|^2 + |M_2(Du)|^\sigma)\, dx, \qquad u : \mathbb{R}^3 \to \mathbb{R}^3,$$

with integrand

$$f(G) = |G|^2 + |M_2(G)|^\sigma, \qquad G \in M_{3\times 3},$$

where $\sigma \in \mathbb{R}$ and $\sigma > 1$. In order to do that, we observe that if $v_1, \ldots, v_n$ are n vectors in $\mathbb{R}^n$ and G denotes the $n \times n$-matrix with columns $v_1 \ldots v_n$, we have, because of the isoperimetric inequality for parallelograms,

$$|\det G| = |v_1 \wedge \ldots \wedge v_n| \leq n^{\frac{-n}{n-1}} \left[\sum_i |v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_n| \right]^{\frac{n}{n-1}}$$

$$\leq n^{\frac{-n}{2(n-1)}} \left[\sum_{i=1}^n |v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_n|^2 \right]^{\frac{n}{2(n-1)}}$$

$$= n^{\frac{-n}{2(n-1)}} |M_{n-1}(G)|^{\frac{n}{n-1}}$$

and actually

$$\sup \frac{n^{\frac{n}{2(n-1)}} |\det G|}{|M_{n-1}(G)|^{\frac{n}{n-1}}} = 1,$$

the supremum being taken on the multiples of the identity matrix. Taking into account this observation, one easily sees that the parametric extension F of f on simple vectors $\xi \in \Lambda_+$ is given by

$$F(\xi) = \frac{|\xi_1|^2}{\xi_{\bar{0}0}} + |\xi_2|^\sigma, \qquad \forall \xi \in \Lambda_+,$$

while for $\xi \in \Lambda_0$ we have

$$F(\xi) = \begin{cases} 0 & \xi_1 = \xi_2 = 0 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for } \sigma < \frac{3}{2}$$

$$F(\xi) = \begin{cases} 3^{3/4}|\xi_3| & \xi_1 = \xi_2 = 0 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for } \sigma = \frac{3}{2}$$

$$F(\xi) = \quad +\infty \qquad \text{always} \qquad \text{for } \sigma > \frac{3}{2},$$

that is the parametric extension (or the initial integral) is regular, i.e.controls all components of the current T if and only if $\sigma \geq 3/2$.

Similar extensions could be computed when considering the same functional for mappings from $\mathbb{R}^3$ or S^3 into S^3, but we shall not deal with those cases; we observe that these kinds of functionals appear in Skyrme's model for meson fields (see e.g. [29] and its references).

EXISTENCE RESULTS. Let us come back to problem (4.2). As we have seen in example 3, we can extend the Dirichlet integral to $\text{cart}^{2,1}(\Omega, S^2)$ as a parametric integral, and this extension is lower semicontinuous and it is given by

$$\mathcal{D}(T) := \frac{1}{2}\int_\Omega |Du_T|^2 dx + M(T - T_{u_T})$$

$$= \frac{1}{2}\int_\Omega |Du_T|^2 dx + 4\pi \sum_{i=1}^{k} |d_i|$$

for $T = T_{u_T} + \sum_{i=1}^{k} d_i [\![\{x_i\} \times S^2]\!]$. Thus, we can now minimize $\mathcal{D}(T)$ among the currents T in $\text{cart}^{2,1}(\Omega, S^2)$ with prescribed degree.

In order to do that, we shall still confront ourselves with one more difficulty. Given a sequence $\{T_k\} \subset \text{cart}^{2,1}(\Omega, S^2)$, the singular parts of the T_k's may disappear on the boundary $\partial\Omega \times S^2$, for example if $x_i \to x_0 \in \partial\Omega$, we have

$$[\![\{x_i\} \times S^2]\!] \rightharpoonup 0 \quad \text{in } \Omega.$$

The currents $[\![\{x_i\} \times S^2]\!]$ have Dirichlet's integral constantly equal to 4π and degree one, and we have loss of energy and degree in the limit. We have two ways of overcoming this difficulty. Let us describe the first possibility.

Let Ω be any smooth bounded domain in $\mathbb{R}^2$ and let $\tilde{\Omega}$ be a bounded domain with $\tilde{\Omega} \supset\supset \Omega$. Suppose γ be the restriction on $\partial\Omega$ of a smooth function, that we call again γ, from $\tilde{\Omega}$ into S^2, not necessarily non-constant; for convenience we shall assume that $\gamma = \underline{u}$ in Ω. For $m \in Z$ we consider the class

$$\bar{E}_m := \{T \in \text{cart}^{2,1}(\tilde{\Omega}, S^2) \mid T \llcorner \pi^{-1}(\tilde{\Omega}\backslash\bar{\Omega}) = [\![G_\gamma]\!] \llcorner \pi^{-1}(\tilde{\Omega}\backslash\bar{\Omega}) \text{ and}$$

$$\deg(T, \tilde{\Omega}, y) - \deg([\![G_\gamma]\!], \tilde{\Omega}, y) = m \qquad \forall y \in S^2\backslash\gamma(\partial\Omega)\}.$$

In case Ω is the unit ball, we can take as γ the function $\mathcal{U}$ defined in the beginning of this section, and we have $\bar{E}_m \supset E_m$, but of course this is not necessary. By proposition 2 sec. 3, $\bar{E}_m$ is closed with respect to the weak convergence of currents, the parametric extension $\mathcal{D}(T)$ of the Dirichlet integral to $\text{cart}^{2,1}(\tilde{\Omega}, S^2)$ is lower semicontinuous and coercive on $\bar{E}_m$, therefore we ge

THEOREM 4. *In each class $\bar{E}_m, m \in Z$, there exists a minimizer of the 'Dirichlet integral'*

$$\mathcal{D}(T) := \frac{1}{2}\int_\Omega |Du_T|^2 dx + M(T - T_{u_T}).$$

Because of theorem 2, each minimizer T in $\bar{E}_m, m \in Z$, is the weak limit of a minimizing sequence of currents $[\![G_{u_k}]\!]$ associated to smooth functions u_k in $\tilde{\Omega}$ with $u_k = \gamma$ on $\tilde{\Omega}\backslash\bar{\Omega}$, in particular

$$\inf\left\{\frac{1}{2}\int_{\tilde{\Omega}} |Du|^2\, dx \mid u \in C^1(\tilde{\Omega}, S^2), u = \gamma \text{ on } \tilde{\Omega}\backslash\Omega,\right.$$
$$\left.\deg(u,\Omega,y) - \deg(\underline{u},\Omega,y) = m, \qquad \forall y \in S^2\backslash\gamma(\partial\Omega)\right\}$$
$$= \inf\{\mathcal{D}(T) | T \in \bar{E}_m\},$$

i.e. no Lavrentiev phenomenon occurs; moreover $\mathcal{D}(T)$ is the *relaxed* functional of the Dirichlet integral or, in other words, it is the *Lebesgue extension* of the Dirichlet integral

$$\mathcal{D}(T) = \inf\left\{\liminf_{k\to+\infty} \frac{1}{2}\int_{\tilde{\Omega}} |Du_k|^2 dx \ :\ u_k \text{ smooth},\right.$$
$$\left. u_k = \gamma \text{ on } \tilde{\Omega}\backslash\bar{\Omega},\ [\![G_{u_k}]\!] \rightharpoonup T\right\}.$$

In fact, more generally, we have

$$\mathcal{D}(T,\Omega) = \inf\left\{\liminf_{k\to+\infty} \frac{1}{2}\int_{\Omega} |Du_k|^2 \ :\ u_k \text{ smooth},\ [\![G_{u_k}]\!] \rightharpoonup T \text{ in } \Omega\right\}$$

for all $T \in \text{cart}^{2,1}(\Omega, S^2)$ and the class $\text{cart}^{2,1}(\Omega, S^2)$ can be seen as the class of the *limit points* of smooth sequences $\{u_k\}$ with $\sup_k \frac{1}{2}\int_\Omega |Du_k|^2 dx < +\infty$ with the obvious identifications.

Let T be a minimizer in $\bar{E}_m$ for some $m \in$ Z. In general, T is not the current associated to the function $u_T \in H^{1,2}(\Omega, S^2)$, and it may have a non-empty vertical part. In this case, the vertical part may also project on the boundary of Ω, and its location depends on the minimizing sequence considered and can be changed without changing energy. In fact, if $T = T_{u_T} + \sum_{i=1}^k d_i [\![\{x_i\} \times S^2]\!]$ is a minimizer in $\bar{E}_m$, then any current S of the form

$$S = T_{u_T} + \sum_{j=1}^h \tilde{d}_j [\![\{y_i\} \times S^2]\!], \quad \tilde{d}_j \in Z,$$

with

$$\sum_{j=1}^h \tilde{d}_j = \sum_{i=1}^k d_i \qquad \text{and } \sum_{j=1}^h |\tilde{d}_j| = \sum_{i=1}^k |d_i|,$$

is a minimizer, too. Moreover, it is easily seen that u_T is a harmonic map which minimizes the Dirichlet integral and its parametric extension in the class $E_{m'}$, where m' is the degree of u_T, i.e. the degree of T minus $\sum_{i=1}^k d_i$.

In our context, the question of the existence of a minimizer in E_m becomes then the question of whether there is a minimizer in $\bar{E}_m$ with no vertical part. Therefore it is related to the regularity question of whether a minimizer in $\bar{E}_m$ has a singular part or not. We shall now give some partial answer to this question.

Set

$$\lambda_k := \inf\{\mathcal{D}(T) \mid T \in \bar{E}_k\}$$

and observe that

$$\lambda_0 = \frac{1}{2}\int_\Omega |D\underline{u}|^2 dx + \frac{1}{2}\int_{\tilde{\Omega}\setminus\Omega} |D\gamma|^2 dx.$$

Trivially we have

$$\lambda_{\pm 1} \leq \lambda_0 + 4\pi,$$

therefore we obtain, compare [15], [39],

PROPOSITION 2. *Let $m = \pm 1$ and let T be a minimizer of $\mathcal{D}$ in $\bar{E}_m$. Suppose there exists a function $v \in H^{1,2}(\tilde{\Omega}, S^2)$ with $T_v \in \bar{E}_m$ and such that*

$$\frac{1}{2}\int_{\tilde{\Omega}} |Dv|^2 dx < \lambda_0 + 4\pi,$$

then T has no vertical part, i.e. $T = T_{u_T}$.

PROOF. We have

$$\begin{aligned}\lambda_0 + M(T - T_{u_T}) &\leq \frac{1}{2}\int_{\tilde{\Omega}} |Du_T|^2 dx + M(T - T_{u_T}) \\ &= \lambda_m \leq \frac{1}{2}\int_{\tilde{\Omega}} |Dv|^2 dx < \lambda_0 + 4\pi,\end{aligned}$$

thus $M(T - T_{u_T}) < 4\pi$, i.e. $T = T_{u_T}$.

q.e.d.

For all non-constant γ on $\partial\Omega$, Brézis-Coron [15] have proved the existence of such a function v for either $m = 1$ or $m = -1$. This yields the existence of a harmonic map which is different from the absolute energy minimizing map $\underline{u}$.

More generally, we obviously have

$$\lambda_\ell \leq \lambda_k + 4\pi|k - \ell|$$

for all $k, \ell \in \mathbb{Z}$, and moreover (compare [43])

PROPOSITION 3. *Let T be a minimizer in $\overline{E}_m$. If*

$$\lambda_m < \lambda_\ell + 4\pi|\ell - m|, \qquad \text{for } |\ell - m| \leq |m|, \tag{4.19}$$

then T has no vertical part.

PROOF. Let T be a minimizer, $T = T_{u_T} + T - T_{u_T}$, and let ℓ be the degree of u_T and, consequently $m - \ell$ the degree of $T - T_{u_T}$. The current $T_\gamma + m[\![\{x_0\} \times S^2]\!], x_0 \in \Omega$, belongs to $\bar{E}_m$, hence

$$\frac{1}{2}\int_\Omega |D\underline{u}|^2 dx + \frac{1}{2}\int_{\tilde{\Omega}\setminus\bar{\Omega}} |D\gamma|^2 dx + 4\pi|m-\ell|$$

$$\leq \frac{1}{2}\int_{\tilde{\Omega}} |Du_T|^2 dx + 4\pi|m-\ell| = \mathcal{D}(T) \leq \mathcal{D}(T_\gamma + m[\![\{x_0\} \times S^2]\!])$$

$$= \frac{1}{2}\int_\Omega |D\underline{u}|^2 dx + \frac{1}{2}\int_{\tilde{\Omega}\setminus\Omega} |D\gamma|^2 dx + 4\pi|m|,$$

thus we get $|\ell - m| \leq m$.
Now we claim that $\ell = m$, i.e. $T - T_{u_T} = \phi$. Suppose in fact that $\ell \neq m$. Since u_T minimizes with its degree, we have

$$\lambda_\ell + 4\pi|m-\ell| = \mathcal{D}(T) = \lambda_m < \lambda_\ell + 4\pi|m-\ell|$$

a contradiction.

q.e.d.

Notice that (4.19) is an obvious necessary condition for the regularity of *all* minimizers in $\bar{E}_m$. Much more delicate is the question of the regularity in dependence of the boundary datum γ; for a result in this direction we refer to [60].

Let us discuss now a second approach to problem (4.2). Let Ω be the unit ball in $\mathbb{R}^2$ and let $\mathcal{U}$ be the function defined in the beginning of this section. Clearly problem (4.2) is equivalent to minimizing

$$\int_{S^2} |Du|^2 dx$$

among mappings $u : S^2 \to S^2$ with $\deg(u, S^2) = m$ and $u = \mathcal{U} \circ \pi_S$ on S^2_-. We shall now proceed introducing the class $\mathrm{cart}^{2,1}(S^2, S^2)$ and showing that we can actually work essentially as before in $\mathrm{cart}^{2,1}(S^2, S^2)$.

Let X^n be an n-dimensional oriented Riemannian manifold. Denote by $\mathcal{D}_n(X^n \times \mathbb{R}^N)$ the space of n-dimensional currents in $X^n \times \mathbb{R}^N$, i.e. the space of linear functionals on the space $\mathcal{D}^n(X^n \times \mathbb{R}^N)$ of C^∞ n-forms with compact support in $X^n \times \mathbb{R}^N$, which are continuous (in the sense of distributions). Because of the product structure in $X^n \times \mathbb{R}^N$, we can define the components of $T \in \mathcal{D}_n(X^n \times \mathbb{R}^N)$. In fact, since for every $x \in X^N$ and $y \in \mathbb{R}^N$, the space of n-covectors $\Lambda^n(T_x X^n \times T_y \mathbb{R}^N)$ can be decomposed, as we have already seen for the dual space of n-vectors above, into a direct sum of orthogonal factors as

$$\Lambda^n(T_x X^n \times T_y \mathbb{R}^N) = \bigoplus_{k=0}^{\min(n.N)} \Lambda^{n-k} T_x X^n \Lambda^k T_y \mathbb{R}^N,$$

every n-form ω in $\mathcal{D}^n(X^n \times \mathbb{R}^N)$ can be uniquely decomposed as

$$\omega = \sum_{k=0}^{\min(n.N)} \omega_k,$$

with

$$\omega_k(x,y) \in \Lambda^{n-k} T_x X^n \wedge \Lambda^k T_y \mathbb{R}^N;$$

the *components of* $T \in \mathcal{D}_n(X^n \times \mathbb{R}^N)$ are defined as the currents T_k, $k = 0, 1, \ldots, \min(n, N)$, given by

$$T_k(\omega) := T(\omega_k),$$

so that

$$T(\omega) = \sum_{k=0}^{\min(n,N)} T_k(\omega_k).$$

Having defined the components of T, it is easily seen that, exactly as we have previously done, we can define the class of cartesian currents $\operatorname{cart}(X^n, \mathbb{R}^N)$ and, for every multiindex $p = (p_0, p_1, \ldots, p_{\min(n.N)})$, $p_i \geq 1$, $i = 0, \ldots, \min(n, N)$, the class $\operatorname{cart}^p(X^n, \mathbb{R}^N)$, compare sec. 2.

Suppose now that Y^r be a properly immersed submanifold of $\mathbb{R}^N$ of dimension r. Then we define

$$\operatorname{cart}^p(X^n, Y^r) := \{T \in \operatorname{cart}^p(X^n, \mathbb{R}^N) \mid \operatorname{spt} T \subset X^n \times Y^r\};$$

for $p = (n, n, \frac{n}{2}, \frac{n}{3}, \ldots, 1)$ we set $\operatorname{cart}^{n.1}(X^n, Y^r)$ instead of $\operatorname{cart}^p(X^n, Y^r)$.

According to the remark at the end of sec. 3, if $r = n$ and for instance X^n and Y^n are compact manifold without boundary, the degree of $T \in \operatorname{cart}(X^n, Y^n)$ is well defined. Moreover if $Y^n = S^n$, or more generally if the n-cohomology with compact support of Y^n is $\mathbb{R}$, then, with the same proof of theorem 1, we get

THEOREM 5. *Let $T \in \operatorname{cart}^{n.1}(X^n, S^n)$. Then there exists a finite number of points x_i, $i = 1, \ldots, k$ in X^n and k integers d_i, such that*

$$T = T_{u_T} + \sum_{i=1}^{k} d_i [\![\{x_i\} \times S^n]\!]$$

with $u_T \in H^{1.n}(X^n, S^n)$.

Let us now come back to the second approach to problem (4.2). Consider the class

$$\bar{E}_m(S^2, S^2) := \{T \in \operatorname{cart}^{2,1}(S^2, S^2) \mid \deg(T, S^2, S^2) = m\}, \quad m \in \mathbb{Z},$$

or, for any 2-dimensional and boundaryless Riemannian manifold X^2,

$$\bar{E}_m(X^2, S^2) := \{T \in \operatorname{cart}^{2,1}(X^2, S^2) \mid \deg(T, X^2, S^2) = m\}, \quad m \in \mathbb{Z},$$

and also the class

$$\bar{E}_{m,\Gamma}(X^2, S^2)$$

of the currents $T \in \bar{E}_m(X^2, S^2)$ which are prescribed on some open set Γ in X^2, $\Gamma = \emptyset$ being admissible, for instance, if $X^2 = S^2$ and $\Gamma = S^2_-$, for the currents which are given on S^2_-. These classes are closed with respect to the weak convergence of currents, the parametric extension of the Dirichlet integral is lower semicontinuous and coercive on them and it is given by

$$\mathcal{D}(T) := \frac{1}{2} \int_{X^2} |Du_T|^2 d\mathcal{H}^2 + M(T - T_{u_T}).$$

Therefore we get at once the following theorem which in a sense extends theorem 4.

THEOREM 6. *In each class $\bar{E}_{m,\Gamma}(X^2, S^2)$, in particular for $X^2 = S^2$ or $X =$ the two dimensional torus T^2, there exists a minimizer of the parametric extension of the Dirichlet integral.*

Of course analogous results to the ones of proposition 1 and 2 can be stated, but we shall omit them. Let us remark that again no Lavrentiev phenomenon occurs, i.e. for instance

$$\inf\{\mathcal{D}(T) \mid T \in \bar{E}_m(X^2, S^2)\}$$
$$= \inf\left\{\frac{1}{2}\int_{X^2} |Du|^2 d\mathcal{H}^2 \mid u : X^2 \to S^2, \qquad u \text{ smooth, degree of } u = m\right\}.$$

Since by the isoperimetric inequality we have

$$\frac{1}{2}\int_{X^2} |Du|^2 d\mathcal{H}^2 \geq \mathcal{H}^2(u(X^2))$$

for smooth mappings $u : X^2 \to S^2$, we may conclude for $X^2 = S^2$ or T^2 that all currents $[\![\{x_0\} \times S^2]\!]$, $x_0 \in S^2$ or T^2 , are minimizers of the 'Dirichlet integral' in the class of mappings from S^2 or T^2 into S^2 with degree 1, compare [42], [13], [26].

Finally, let us consider mappings from S^n into S^n, $n \geq 3$. In this case, the Dirichlet integral is *degenerate*, in fact it does not control all components of the current associated to such maps, compare example 3, and in particular we cannot expect to have any control of the degree under weak convergence. Instead, the situation becomes exactly the same as for the Dirichlet integral if we consider the functional

$$\frac{1}{n^{n/2}} \int_{S^n} |Du|^n d\mathcal{H}^n. \tag{4.20}$$

In fact, given n-vectors $v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$, denote by G the matrix which has $v_1, \ldots, v_n$ as columns. From the isoperimetric inequality for parallelograms we get

$$\sup_G \frac{n^{\frac{n}{2(n-1)}} |\det G|}{|M_{n-1}(G)|^{\frac{n}{n-1}}} = 1.$$

Then, as in the case of the Dirichlet integral, it is not difficult to see that the parametric extension of the functional (4.20) is finite and lower semicontinuous on $\text{cart}^{n,1}(S^n, S^n)$ (or in general in $\text{cart}^{n,1}(X^n, S^n)$), coincides with the Lebesgue extension of (4.20), and is given by

$$\mathcal{F}(T) := \frac{1}{n^{n/2}} \int_{S^2} |Du_T|^n d\mathcal{H}^n + M(T - T_{u_T}). \tag{4.21}$$

Therefore we can state

THEOREM 7. *In each class $\bar{E}_{m,\Gamma}(S^n, S^n), m \in Z$, there exists a minimizer of* (4.21).

5. - Energy minimizing maps from a domain of $\mathbb{R}^3$ into S^2

In this section we shall discuss the problem of minimizing the Dirichlet integral among maps from a domain $\Omega \subset \mathbb{R}^3$ into S^2 under various "boundary conditions". There is a large literature about such variational problems as they appear both in a geometrical context (in the study of harmonic maps, see e.g. [23], [24], [25]) and in the physical context (in the so called nonlinear sigma model and in the theory of liquid crystals, see e.g. [16], [14], [1], [2], [28], [34], [35], [36] and the references there). Actually, in the study of liquid crystals one considers the more general functional

$$\tilde{\mathcal{E}}(u) := \int_\Omega k_1(\text{div}u)^2 + k_2(u \cdot \text{curl}u)^2 + k_3|u \wedge \text{curl}u|^2 + \alpha[\text{tr}(Du)^2 - (\text{div}u)^2]\, dx,$$

where k_1, k_2, k_3, α are positive constants, which reduces to the Dirichlet integral (apart from the factor $\frac{1}{2}$) in the special case $k_1 = k_2 = k_3 = \alpha$.

The usual approach is to seek a minimizer in $H^{1,2}(\Omega, S^2)$ under boundary and/or defect conditions. The minimizers are in general singular and, according to the work of Schoen-Uhlenbeck [57], see also [32], the singularities are isolated and called defects; moreover, the minimizers map small spheres around any singular point to S^2 with topological degree plus or minus one [16]. The singularities appear not only for topological reasons (in fact there is no continuous extension on B^3, the unit ball of $\mathbb{R}^3$, if $g : \partial B^3 \to S^2$ has topological degree different from zero), but because they enable to reduce the energy. Hardt-Lin [37] have in fact shown that in general, even for zero degree boundary maps g, we have

$$\inf\left\{\int_{B^3} |Du|^2 dx \ : \ u \in H^{1,2}(B^3, S^2), \ u = g \text{ on } \partial B^3\right\}$$
$$< \inf\left\{\int_{B^3} |Du|^2 dx \ : \ u \in H^{1,2}(B^3, S^2) \cap C^0(B^3, S^2), \ u = g \text{ on } \partial B^3\right\},$$

that is a Lavrentiev phenomenon occurs.

As in section 4 and with analogous motivations, we shall see in the sequel that we are naturally led in some respects to consider the parametric extension of the Dirichlet integral and to look for minimizers in suitable subsets of $\text{cart}^{2,1}(\Omega, S^2)$. In this way the feature of the problem changes much more than in the analogous case in sec. 4. In general *line singularities will appear instead of point singularities* and small balls around isolated singularities are mapped into S^2 with zero topological degree. Moreover, in some specific situations *no Lavrentiev phenomenon occurs*, and we conjecture that it does not occur in general.

POINT SINGULARITIES IN $\mathbb{R}^3$ AND THE DIPOLE. Consider first the problem, studied in [16], [1], of minimizing the Dirichlet integral in the class of functions with prescribed point singularities together with their local degree. More precisely, given N-points in $\mathbb{R}^3$, $a_1 \dots, a_N$, try to minimize the Dirichlet integral

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 dx$$

in the class

$$E := \left\{u \in C^1\left(\mathbb{R}^3 \setminus \bigcup_{i=1}^{N}\{a_i\}, S^2\right) \ \middle|\ \mathcal{E}(u) < +\infty \ , \ \deg(u, a_i) = d_i, \ \forall i\right\},$$

where the d_i's are given integers, $d_i \in Z$, $d_i \neq 0$, which satisfy the compatibility condition

$$\sum_{i=1}^{N} d_i = 0,$$

and $\deg(u, a_i)$ denotes the degree of u restricted to any small sphere around a_i. Equivalently, try to minimize $\mathcal{E}$ among smooth mappings from $\mathbb{R}^3 - \{a_1, \dots a_N\}$ to S^2 which map $\mathbb{R}^3$ minus some bounded region of $\mathbb{R}^3$ into the south pole of S^2, and which, for each i, map small spheres around a_i to S^2 with degree d_i.

The answer to this problem is the following. In the terminology of [16] $\inf \mathcal{E}$ is 4π the lenght L of the *minimal connection* associated to the points $a_1, \ldots, a_N$, or equivalently, see [1], $\inf_E \mathcal{E}$ equals 4π the least mass of 1-dimensional currents T in $\mathbb{R}^3$ with

$$\partial T = \sum_{i=1}^{N} d_i [\![a_i]\!],$$

where $[\![a_i]\!]$ denotes the zero dimensional current which maps smooth functions φ to $\varphi(a_i)$. Moreover, any minimizing sequence $\{u_k\}$ in E converges weakly in $H^{1,2}$ to zero, hence

$$\inf_E \mathcal{E}(u) > \inf_{\tilde{E}} \mathcal{E}(u),$$

$$\tilde{E} := \{u \in H^{1,2}(\mathbb{R}^3 \backslash \{a_1, \ldots, a_N\}, S^2) \cap$$
$$C^0(\mathbb{R}^3 \backslash \{a_1, \ldots, a_N\}, S^2) | \deg(u, a_i) = d_i, \ \forall i\}$$

and the minimum in E is not attained.

Since it is relevant for us, we shall now repeat the arguments of [16], [1] in the special case $N = 2$ and $d_1 = -1, d_2 = 1$, which in a sense is the most relevant one and referred as the *dipole* case in [16].

By construction one first proves that

$$\inf_E \mathcal{E}(u) \leq 4\pi L.$$

The construction is the following. Choose a smooth curve C connecting a_1 to a_2 and orient C by a smoothly varying unit tangent vector ς which points away from a_1 to a_2. The length of C is the mass of the one dimensional current $S = \tau(C, 1, \varsigma)$. Choose now two smoothly varying normal vectorfields η_1, η_2 along C which are perpendicular to each other and for which the 3-vector $\varsigma(x) \wedge \eta_1(x) \wedge \eta_2(x)$ equals the orienting 3- vector $e_1 \wedge e_2 \wedge e_3$ of $\mathbb{R}^3$. Also, consider the map $\gamma : \mathbb{R}^2 \to S^2$ which is the following slight modification of the inverse of the stereographic projection π_S : $\gamma(y) = \pi_S^{-1}(y)$ for $|y| < R, R$ fixed and large, $\gamma(y) =$ the south pole q for $|y| \geq 2R$; γ is a suitable interpolation in $R < |y| < 2R$. Finally consider the map $f : \mathbb{R}^3 \to S^2$ depending on a smoothly varying radius function δ on C with $\delta(a_1) = \delta(a_2) = 0, \delta > 0$ otherwise, defined as follows: for each point P in $\mathbb{R}^3$ of the type $x + s\eta_1(x) + t\eta_2(x)$ for $x \in C$ and $s^2 + t^2 < \delta^2(x)$, we set

$$f(P) = \gamma\left(\frac{2Rs}{\delta(x)}, \frac{2Rt}{\delta(x)}\right),$$

otherwise $f(P) = q$. Then one shows, see e.g. [1], that $\mathcal{E}(f)$ is nearly equals to 4π times the lenght of C.

The inequality

$$\inf_E \mathcal{E}(u) \geq 4\pi L$$

can be proved by means of the coarea formula, see [1]. The basic observation is that

$$\frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 dx \geq \int_{\mathbb{R}^3} |M_2(Du)|\, dx,$$

thus, by the coarea formula

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 dx &\geq \int_{\mathbb{R}^3} |M_2(Du)|\, dx = \int_{S^2} \mathcal{H}^1(u^{-1}(y))\, d\mathcal{H}^2(y)\\ &= \int_{S^2} M(<\mathbb{R}^3, u, y>)\, d\mathcal{H}^2\\ &\geq \mathcal{H}^2(S^2)\inf\{M(S) \mid \partial S = [\![a_2]\!] - [\![a_1]\!]\} = 4\pi L,\end{aligned}$$

where $<\mathbb{R}^3, u, w>$ is the *slice* of the current $\mathbb{R}^3$ by the map u, compare [30], [59], or equivalently the current $\tau(u^{-1}(y), 1, \varsigma), \varsigma$ being the natural induced orientation, and one sees, compare also the sequel of this section, that

$$\partial\tau(u^{-1}(y), 1, \varsigma) = [\![a_2]\!] - [\![a_1]\!].$$

The construction above gives also, for C tending to the minimal connection and $\delta(x)$ tending to zero, a minimizing sequence for our problem. Of course such a sequence converges weakly to zero in $H^{1,2}$, but it is not difficult to see that instead the currents $[\![G_f]\!]$ converge to the current

$$L \times [\![S^2]\!],$$

where now L is the least mass 1-dimensional current with $\partial L = [\![a_2]\!] - [\![a_1]\!]$. As we shall see soon, in fact $L \times [\![S^2]\!]$ realizes the infimum of the parametric extension of the Dirichlet integral among the $T \in \text{cart}^{2,1}_{\text{loc}}(\mathbb{R}^3 - \{a_1, a_2\}, S^2)$ with finite Dirichlet's norm and $\partial T = [\![\{a_2\} \times S^2]\!] - [\![\{a_1\} \times S^2]\!]$. Thus we are again naturally lead to study variational problems for the Dirichlet integral in $\text{car}^{2,1}(\Omega, S^2)$.

THE CLASSES $H^{1,2}(\Omega, S^2), \text{cart}^{2,1}(\Omega, S^2)$ AND THE DIRICHLET INTEGRAL. We shall first write explicitly the components of a current in $\text{cart}(\Omega, S^2)$, where Ω is a domain in $\mathbb{R}^3$. This, besides being useful in the sequel, will make more evident the relationships between our computations and some of the computations in [16].

We observe that, since $\text{spt}\, T \subset \Omega \times S^2$ and T is rectifiable, $T = \tau(\mathcal{M}, \theta, \xi)$, $\xi(x, y) \in T_x\Omega \times T_y S^2$, hence $T(\eta)$ depends only on the projection ω of the 3-form η on $\mathcal{D}^3(\Omega \times S^2)$ given, for each x, y, by the orthogonal projection of $\eta(x, y)$ into

$\Lambda^3(T_x\Omega \times T_y S^2)$. Denote by $n(y)$ the outer normal to $S^2 := \{y \in \mathbb{R}^3 : |y| = 1\}$ at y, then we can write the 1-forms on S^2 as

$$\omega = \omega_i(y)dy_i \quad \text{with} \quad \omega_i(y)n_i(y) = 0$$

and the 2-forms on S^2 as a multiple of the volume form on S^2

$$\omega_{S^2} = (-1)^{i-1} n_i(y)\widehat{dy}_i.$$

Thus a generic 3-form on $\Omega \times S^2$ can be written as

$$\begin{aligned} \omega(x,y) =& \omega_0(x,y)dx_1 \wedge dx_2 \wedge dx_3 + \omega_{ij}(x,y)\widehat{dx}_i \wedge dy_j \\ &+ \omega_i(x,y)dx_i \wedge \hat{\pi}^{\#}\omega_{S^2}, \end{aligned} \tag{5.1}$$

where

$$\omega_{ij}(x,y)n_j(y) = 0.$$

Given now a 3-form

$$\eta = \sum_{|\alpha|+|\beta|=3} \eta_{\alpha\beta}\, dx^\alpha \wedge dy^\beta$$

in $\Omega \times \mathbb{R}^3$, it is easy to see that its tangential part on $\mathcal{D}^3(\Omega \times S^2)$ is given by (5.1) with

$$\begin{aligned} \omega_0(x,y) &:= \eta_{\bar{0}0}(x,y) \\ \omega_{ij}(x,y) &:= \eta_{\bar{i}j} - \eta_{\bar{i}h} n_h(y) n_j(y) \\ \omega_i(x,y) &:= \sum_{|\beta|=2} \eta_{i\beta}(-1)^{\bar{\beta}-1} n_{\bar{\beta}}(y), \end{aligned}$$

and $T(\eta) = T(\omega)$.

Let $u : \Omega \to S^2$ be a smooth function or a function in $H^{1,2}(\Omega, S^2)$, and let T be the current T_u associated to u. We have

$$T(\eta) = T_0(\eta_0) + T_1(\eta_1) + T_2(\eta_2) + T_3(\eta_3),$$

where T_i and η_i are respectively the components of T and η, and

$$\begin{aligned} T_0(\eta_0) &= \int_\Omega \omega_0(x, u(x))\, dx \\ T_1(\eta_1) &= \int_\Omega \omega_{ij}(x, u(x))(-1)^{i-1} D_i u^j\, dx \\ T_2(\eta_2) &= \int \omega_i(x, u(x))\, dx_i \wedge u^{\#}\omega_{S^2} = \int_\Omega \omega_i(x, u(x))(-1)^{i+j} n_j M_{\bar{j}\bar{i}}(Du)\, dx \\ &= \int_\Omega \omega_i(x, u) D_i(x)\, dx \\ T_3(\eta_3) &= 0, \end{aligned}$$

where D_i is the D-field in [16]

$$D := u \cdot u_{x_2} \wedge u_{x_3} e_1 + u \cdot u_{x_3} \wedge u_{x_1} e_2 + u \cdot u_{x_1} \wedge u_{x_2} e_3. \tag{5.2}$$

Observe that for any 1-form γ in Ω

$$\pi_{\#}(T \llcorner \hat{\pi}^{\#} \omega_{S^2})(\gamma) = \int_\Omega < \gamma, D > dx.$$

The following theorem gives a characterization of the elements of $\mathrm{cart}^{2,1}(\Omega, S^2)$.

THEOREM 1. *Let* $T = \tau(\mathcal{M}, \theta, \xi) \in \mathrm{cart}^{2,1}(\Omega, S^2)$. *T decomposes as*

$$T = T_{u_T} + S, \tag{5.3}$$

where $u_T \in H^{1,2}(\Omega, S^2)$ *and S is its singular part,* $S = T \llcorner \mathcal{M} \backslash \mathcal{M}_+$. *Moreover there exists a rectifiable 1-dimensional current* $L = \tau(\mathcal{L}, \gamma, \varsigma)$ *in* Ω *such that*

$$S = L \times [\![S^2]\!], \tag{5.4}$$

i.e. $S = \tau(\mathcal{L} \times S^2, \gamma \circ \pi, \varsigma \wedge \xi_{S^2})$ *or, in other words,* $\mathcal{M} \backslash \mathcal{M}_+ = \mathcal{L} \times S^2$, $\xi = \varsigma \wedge \xi_{S^2}$ *and* $\theta = \gamma \circ \pi$ *on* $\mathcal{M} \backslash \mathcal{M}_+$, *where* ξ_{S^2} *is the orienting 2-vector of* S^2.

PROOF. Although the situation is substantially different, the idea of the proof and actually the proof of this theorem is the same of the proof of theorem 1 sec. 4. For future references we split it in three steps.

Step 1. We already know the decomposition $T = T_{u_T} + S$, so we ought to prove only the second part of the theorem. Contrary to the case in which Ω is a domain of $\mathbb{R}^2$, in this case we cannot say that T_{u_T} has no boundary in $\Omega \times \mathbb{R}^3$, and actually, as we know and shall also see later, T_{u_T} does have boundary in general. But we claim that

$$\partial T_{u_T}(\omega) = 0 \tag{5.5}$$

for all 2-forms in $\Omega \times S^2$ of the type

$$\omega = \omega_{ij} dx_i \wedge dy_j + \omega_i \widehat{dx}_i, \quad (\omega_{ij} n_j = 0), \tag{5.6}$$

or in other words ∂T_{u_T} has only a completely vertical part. This is easily seen by considering a smooth approximation u_ϵ of u in $H^{1,2}(\Omega, \mathbb{R}^3)$. Then we have $u_\epsilon \to u$ in $L^2, D_{u_\epsilon} \to Du$ in L^2 and $M_2(Du_\epsilon) \to M_2(Du)$ in L^1 and, as $d\omega$ has the form

$$d\omega = \omega_0 dx_1 \wedge dx_2 \wedge dx_3 + \bar{\omega}_{ij} dx_i \wedge dy_j + \omega_{ij} dx_i \wedge \widehat{dy}_j,$$

we conclude that

$$T_{u_\epsilon}(d\omega) =$$
$$\int_\Omega \left[\bar{\omega}_0(x,u_\epsilon) + \bar{\omega}_{ij}(x,u_\epsilon)(-1)^{i-1}D_i u_\epsilon^j + \omega_{ij}(x,u_\epsilon)(-1)^{i-1}M_{\bar{j}\bar{i}}(Du_\epsilon)\right]dx$$

converges to $T_u(d\omega)$. As $\partial T_{u_\epsilon}(\omega) = 0$, since T_{u_ϵ} has no boundary in $\Omega \times \mathbb{R}^3$, we get the conclusion.

Step 2. From step 1 we also see that S_0 and S_1 are zero, i.e. S is 'vertical' and may be non-zero only on forms of the type $\omega_i dx_i \wedge \pi^{\#}\omega_{S^2}$. Consider one such form, then, compare theorem 1 sec. 4, we have

$$\begin{aligned}
&\omega_i(x,y)dx_i \wedge \hat{\pi}^{\#}\omega_{S^2} \\
&= \bar{\omega}_i(x)\,dx^i \wedge \hat{\pi}^{\#}\omega_{S^2} + dx_i \wedge d_y\eta, \\
&\bar{\omega}_i(x) := \int_{S^2} \omega_i(x,y)d\mathcal{H}^2(y),
\end{aligned}$$

where η is a 1-form on $\Omega \times S^2$. Therefore, taking into account step 1, we deduce

$$S(\omega_i(x,y)dx_i \wedge \hat{\pi}^{\#}\omega_{S^2}) = S(\bar{\omega}_i(x)\,dx_i \wedge \hat{\pi}^{\#}\omega_{S^2}).$$

Step 3. We are now ready to prove that $S = L \times [\![S^2]\!]$, compare theorem 1 sec 4. Recalling that for $\|S\| = \theta(x,y)\mathcal{H}^3 \llcorner \mathcal{M}$ we have

$$S(\omega_i dx_i \wedge \hat{\pi}^{\#}\omega_{S^2}) = \int \xi_i(x,y)\omega_i(x,y)d\|S\|(x,y),$$

we consider the vector valued measure with components $S_i = \xi_i\|S\|$, so that $S = S_i e_i \wedge \xi_{S^2}$ and we define the vector valued measure on Ω, (L_1, L_2, L_3), by

$$L_i(A) := \frac{1}{4\pi}S_i(A \times S^2) = \frac{1}{4\pi}\pi_{\#}S_i(A), \quad \forall A \subset \Omega,$$

or equivalently by

$$\int \varphi(x)dL_i(x) := \frac{1}{4\pi}\int \varphi(x)dS_i(x,y).$$

Observe that

$$L_i e_i = L,$$

where

$$L = \frac{1}{4\pi}\pi_{\#}(S \llcorner \hat{\pi}^{\#}\omega_{S^2}), \tag{5.7}$$

and that L is a 1-dimensional rectifiable current with density one. By step 2 we have for all regular $\varphi(x,y)$

$$\int \varphi(x,y)dS_i(x,y) = \int \bar{\varphi}(x)dS_i(x,y) = 4\pi \int \varphi(x,y)d(\mathcal{H}^2 \llcorner S^2)(y)dL_i(x),$$

$$\bar{\varphi}(x) = \frac{1}{4\pi}\int_{S^2} \varphi(x,y)d\mathcal{H}^2(y),$$

thus

$$S_i = L_i \times (\mathcal{H}^2 \llcorner S^2).$$

In conclusion

$$S = L \times [\![S^2]\!].$$

q.e.d.

Let T be a current in $\text{cart}^{2,1}(\Omega, S^2)$, $T = T_{u_T} + L \times [\![S^2]\!]$. In general T_{u_T} does not belong to $\text{cart}^{2,1}(\Omega, S^2)$, since it may have a non-empty boundary in $\Omega \times S^2$, and in fact

$$\partial T_{u_T} = -\partial L \times [\![S^2]\!].$$

In particular the currents T_u associated to functions $u \in H^{1,2}(\Omega, S^2)$ do not belong in general to $\text{cart}^{2,1}(\Omega, S^2)$. For example if we consider the map $x/|x|$ from B^3 into S^2 and the associated current $T_{x/|x|}$ we have

$$\partial T_{x/|x|} \llcorner \pi^{-1}(B^3) = -[\![\{0\} \times S^2]\!].$$

Of course $T_{x/|x|}$ can be "completed" to a current $T \in \text{cart}^{2,1}(B^3, S^2)$, with the property that $u_T = x/|x|$, by simply defining

$$T := T_{x/|x|} + L \times [\![S^2]\!],$$

where L is a dipole with end points $\{0\}$ and any $x_0 \in \partial B^3$, and L is oriented so that

$$\partial L = [\![0]\!] - [\![x_0]\!] \ ;$$

for example L can be the segment or any smooth curve coming out from x_0 and going to 0. This is actually a general fact. From the weak approximation result in [9], in fact we know that each $u \in H^{1,2}(\Omega, S^2)$ is the weak limit in $H^{1,2}(\Omega, S^2)$ of a sequence of smooth maps $u_k : \Omega \to S^2$. The currents $[\![G_{u_k}]\!]$ then converge, passing possibly to a subsequence, to some T in $\text{cart}^{2,1}(\Omega, S^2)$, and $u_T = u$.

In general, it seems difficult to get control on the boundary of T_u in $\pi^{-1}(\Omega)$ for $u \in H^{1,2}(\Omega, S^2)$. However the situation drastically simplifies if we assume that $M(\partial T_u)$ is finite, which is equivalent to $M(\partial L) < +\infty$, if $T = T_u + L \times [\![S^2]\!]$ is a current in $\text{cart}^{2,1}(\Omega, S^2)$, which 'completes' T_u. In this case, by the boundary rectifiability theorem in sec. 2, ∂L is a rectifiable 0–dimensional current, i.e. ∂L

is a finite combination of integrations on points $\{a_i\} \subset \Omega$, i.e. ∂T_u has masses only on $\bigcup_i \{a_i\} \times S^2$. And we see that $\partial T_u \llcorner \pi^{-1}(a_i)$ is described exactly by the "degree of $\partial T_u \llcorner \pi^{-1}(a_i)$." In fact trivially $\partial T_u \llcorner \pi^{-1}(a_i)$ is a boundaryless current in $\{a_i\} \times S^2$, thus the constancy theorem yields the existence of integers m_i (∂T_u is rectifiable) such that

$$\partial T_u \llcorner \pi^{-1}(a_i) = m_i [\![\{a_i\} \times S^2]\!].$$

We call $-m_i$ the *degree of* ∂T_u *at* a_i. Of course the degree of ∂T_u is defined at every point $x_0 \in \Omega$ and, for $x_0 \notin \{a_i\}$, degree of ∂T_u at x_0 is zero. One also sees that the degree of ∂T_u at x_0 is the "topological degree of $u_{|\partial B_r} : \partial B_r \to S^2$" for small spheres ∂B_r around x_0. In order to prove that, we observe that the "topological degree of $u_{|\partial B_r} : \partial B_r(x_0) \to S^2$", denoted by $\deg(\partial T_u, \partial B_r, S^2)$, is given by

$$\deg(\partial T_u, \partial B_r, S^2) = \frac{1}{4\pi} \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(\partial B_r)(\hat{\pi}^{\#}\omega_{S^2}), \tag{5.8}$$

where we compute $\partial(T_u \llcorner \pi^{-1}(B_r))$ thinking of $T_u \llcorner (B_r \times S^2)$ as a current in $\mathbb{R}^3 \times S^2$; the degree of ∂T_u at x_0 is given by

$$\deg(\partial T_u, x_0) = -\frac{1}{4\pi} \partial T_u \llcorner \pi^{-1}(x_0)(\hat{\pi}^{\#}\omega_{S^2}) \tag{5.9}$$

and finally we observe that, since ∂T_u has no boundary in $B_r(x_0) \backslash \{x_0\}$ for small r, we have

$$\begin{aligned}
&\partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(\partial B_r) + \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(x_0) \\
&= \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(\bar{B}_r) - \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner (\pi^{-1}(B_r) \backslash \pi^{-1}(x_0)) \\
&= \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(\bar{B}_r) - \partial T_u \llcorner \pi^{-1}(B_r \backslash \{x_0\}) \\
&= \partial(T_u \llcorner \pi^{-1}(B_r)) \llcorner \pi^{-1}(\bar{B}_r).
\end{aligned}$$

Therefore, denoting by χ_A the characteristic function of A,

$$\begin{aligned}
\deg(\partial T_u, \partial B_r, S^2) - \deg(\partial T_u, x_0) &= \frac{1}{4\pi} \partial(T_u \llcorner \pi^{-1}(B_r))(\chi_{\pi^{-1}(\bar{B}_r)} \wedge \hat{\pi}^{\#}\omega_{S^2}) \\
&= T_u \llcorner \pi^{-1}(B_r)(d\chi_{\pi^{-1}(\bar{B}_r)} \wedge \hat{\pi}^{\#}\omega_{S^2}) = 0.
\end{aligned}$$

For the reader's convenience, let us give now a more explicit expression of

$\deg(\partial T_u, \partial B_r(x_0), S^2)$ and deg $(\partial T_u, x_0)$. We have

$$\deg(\partial T_u, x_0) = -\frac{1}{4\pi}\partial T_u \llcorner \pi^{-1}(x_0)(\hat{\pi}^{\#}\omega_{S^2})$$
$$= -\frac{1}{4\pi}\partial T_u(\chi_{\pi^{-1}(x_0)}\hat{\pi}^{\#}\omega_{S^2})$$
$$= -\frac{1}{4\pi}T_u(d\chi_{\pi^{-1}(x_0)} \wedge \hat{\pi}^{\#}(\omega_{S^2})) = -\frac{1}{4\pi}(T_u \llcorner \hat{\pi}^{\#}\omega_{S^2})(d\chi_{\pi^{-1}(x_0)}).$$

For a sequence of smooth functions $\varphi_k(x)$ converging to the characteristic functions of $\{x_0\}$, we then obtain, compare the computations for the components of T,

$$\deg(\partial T_u, x_0) = \lim_{k\to\infty} \frac{1}{4\pi} T_u \llcorner \hat{\pi}^{\#}(\omega_{S^2}))(\varphi_{k.x_i} dx_i)$$
$$= \lim_{k\to\infty} \frac{1}{4\pi}\int_\Omega D_i(x)\varphi_{k.x_i}\, dx,$$

where D is the field defined in (5.2). Therefore we conclude, compare [16], that in the sense of distributions

$$\operatorname{div} D = 4\pi \sum_{i=1}^{N} \deg(\partial T_u, a_i)\,\delta_{a_i}. \tag{5.10}$$

In case u is regular in a neighbourhood of x_0, we also have

$$\deg(\partial T_u, \partial B_r, S^2) = \frac{1}{4\pi}[\![G_\psi]\!](\hat{\pi}^{\#}\omega_{S^2}),$$

where ψ denotes the restriction of u to ∂B_r. If J_ψ denotes the 2×2 Jacobian determinant of the map ψ, we get the classical expression for the degree

$$\deg(\partial T_u, x_0) = \deg(\partial T_u, \partial B_r, S^2) = \frac{1}{4\pi}\int_{\partial B_r} J_\psi\, d\mathcal{H}^2.$$

We observe that if (ξ_1, ξ_2) are normal coordinates on ∂B_r, from the fact that $\psi \cdot \psi_{\xi_1} = \psi \cdot \psi_{\xi_2} = 0$ and thus $\psi_{\xi_1} \wedge \psi_{\xi_2} = J_\psi \psi$, one deduces the following classical expression for J_ψ

$$J_\psi = \psi \cdot \psi_{\xi_1} \wedge \psi_{\xi_2}.$$

The degree of ∂T at $x_0 \in \Omega$ is defined in the same way for $T \in \text{cart}^{2,1}(\Omega, S^2)$ and, in this case, it is trivially zero. The previous arguments also show that, as in the smooth case, the boundary of T (considered of course as a current in $\mathbb{R}^3 \times S^2$) has degree zero (in the sense of section 3). In particular, given a boundaryless 2-dimensional current S in $\partial\Omega \times S^2$ with non-zero degree, for example the graph of a smooth function $\psi : \partial\Omega \to S^2$ with non-zero degree, there is no $T \in \text{cart}^{2,1}(\Omega, S^2)$ such that $\partial T = S$. But, because of the weak approximation result in [9], it does exist $T \in \text{cart}^{2,1}(\Omega, S^2)$ with $u_T = \psi$ in the sense of traces in $H^{1,2}$. For instance, as we have seen,

$$T = T_{x/|x|} + L \times S^2, \quad \partial L = [\![0]\!] - [\![x_0]\!], \; x_0 \in \partial B^3,$$

belongs to $\text{cart}^{2,1}(B^3, S^2)$, $u_T = x$ on ∂B^3 (in the sense of $H^{1,2}$), $\deg(x, S^2, S^2) = 1$, but

$$\partial T = \{\text{integration on the graph of } id : S^2 \to S^2\} + [\![\{x_0\} \times S^2]\!].$$

Assume still that $M(\partial T_{u_T}) < +\infty$ or equivalently $M(\partial L) < +\infty$ for $T = T_{u_T} + L \times [\![S^2]\!] \in \text{cart}^{2,1}(\Omega, S^2)$. We observe that defects, i.e. points in which the degree of ∂T_{u_T} is non-zero (which coincide with $\text{spt}\partial T_{u_T}$), are always connected by *line singularities*, the current L. Of course u_T may also have singular points, that is points in which u_T is not continuous, in which there is no boundary of T_{u_T} (compare example 1 sec. 2); on such kind of points ∂T_{u_T} *must have zero degree.*

Let us consider now the Dirichlet integral

$$\frac{1}{2}\int_\Omega |Du|^2 dx$$

for mappings $u : \Omega \subset \mathbb{R}^3 \to S^2 \subset \mathbb{R}^3$, which has integrand f given for each fixed $n \in S^2$ by $\frac{1}{2}|G|^2$, where $G \in \{G \in M_{3\times 3} | G^T \cdot n = 0\}$. As in example 3 sec. 4, one easily sees that the associated parametric integrand is given, in the notations of sec. 4, by

$$F(n,\xi) = \begin{cases} \frac{1}{2}|G_{(\xi_1/\xi_{\bar{0}0})}|^2 + \|\xi_2 - \xi_{\bar{0}0} M_2(G_{(\xi_1/\xi_{\bar{0}0})})\|_f & \text{if } G^T_{\xi_1\backslash\xi_{\bar{0}0}} \cdot n = 0 \\ & \text{and } \xi_{\bar{0}0} > 0 \\ |\xi_2| & \text{if } \xi_{\bar{0}0} = 0 \text{ and } \xi_1 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore we conclude that the parametric extension of the Dirichlet integral is finite exactly on $\text{cart}^{2,1}(\Omega, S^2)$; there it coincides with the parametric extension of the Dirichlet integral for mappings from $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$, thus it is lower semicontinuous (and trivially coercive) on $\text{cart}^{2,1}(\Omega, S^2)$ and, finally, it is given by

$$\mathcal{D}(T;\Omega) := \frac{1}{2}\int_\Omega |Du_T|^2 dx + 4\pi M(L), \quad \text{for } T \in \text{cart}^{2,1}(\Omega, S^2), \qquad (5.11)$$
$$T = T_{u_T} + L \times [\![S^2]\!].$$

The dipole problem, or the more general point singularities problem in $\mathbb{R}^3$, can be now formulated as follows. Given N points $\{a_1, \dots a_N\}$ in $\mathbb{R}^3$ and N

non-zero integers d_i with $\sum_{i=1}^{N} d_i = 0$, or equivalently given the two dimensional current in $\mathbb{R}^3 \times S^2$

$$S = \sum_{i=1}^{N} d_i [\![\{a_i\} \times S^2]\!],$$

minimize the Dirichlet $\mathcal{D}(T)$ in (5.11) in the class

$$\{T \in \mathrm{cart}^{2,1}_{\mathrm{loc}}(\mathbb{R}^3, S^2) | \mathcal{D}(T, \mathbb{R}^3) < +\infty \ , \ \partial T = S\}.$$

Then, taking into account the coarea formula (compare above), it is easily seen that a minimizer is given by

$$L \times [\![S^2]\!],$$

where L is a least mass 1-dimensional current in $\mathbb{R}^3$ with

$$\partial L = \sum_{i=1}^{N} d_i [\![a_i]\!].$$

Finally, we observe that in case of mappings into a flat manifold the situation simplifies drastically. The proof of theorem 1 yields in fact that $\mathrm{cart}^{2,1}(\Omega, \mathbb{R}^2) = H^{1,2}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^3$.

Variational problems for the parametric extension of the Dirichlet integral in $\mathrm{cart}^{2,1}(\Omega, S^2)$

Let Ω be a domain in $\mathbb{R}^3$ with a finite number of holes and, omitting a finite number of points,

$$\Omega = A \backslash \left(\bigcup_{i=1}^{N} a_i \cup \bigcup_{i=1}^{n} H_i \right),$$

where A is a smooth simply connected domain of $\mathbb{R}^3$, the H_i are disjoint sets, each H_i equals the closure of a ball type smooth domain strictly contained in A, and the a_i are points in $A \backslash \bigcup_{i=1}^{n} H_i$. We split the family $\{1, \ldots, n\}$ as a disjoint union of three subsets I_1, I_2, I_3; we allow one or two of them to be empty. In case I_1 is not empty, we choose, for each $i \in I_1$, $\tilde{H}_i \subset\subset H_i$, $\tilde{H}_i$ equals the closure of a ball type smooth domain, and for the other indices we set $\tilde{H}_i = H_i$. We also split the exterior boundary of Ω, i.e. ∂A, as a disjoint union of two "smooth" subsets $\partial A = \Gamma_1 \cup \Gamma_2$ (one of which could be empty), and we choose, when $\Gamma_1 \neq \emptyset$, an open set $\tilde{A} \supset A$ such that $\tilde{A} \supset A \cup \mathring{\Gamma}_1$, $\tilde{A} \cap \mathring{\Gamma}_2 = \emptyset$. Finally we set

$$\tilde{\Omega} = \tilde{A} \backslash \left(\bigcup_{i=1}^{N} a_i \cup \bigcup_{i=1}^{n} \tilde{H}_i \right)$$

and we fix a current $S \in \text{cart}^{2,1}(\tilde{\Omega}, S^2)$.

We can now formulate the following variational problem $\mathcal{P}$: minimize the Dirichlet integral in (5.11) in the class E of currents $\text{cart}^{2,1}(\tilde{\Omega}, S^2)$ such that

i) $\partial T \llcorner \pi^{-1}(a_i)$ is prescribed, or in other words the degree of ∂T is prescribed on a_i (we also assume the degrees prescribed are non-zero).

ii) On $\partial H_i, i \in I_1$, we prescribe the boundary of T in the following way: we require that all currents T in E coincide with S on $\pi^{-1}(\overset{\circ}{H_i} \setminus \tilde{H}_i)$; on $\partial H_i, i \in I_2$, we prescribe the trace of the $H^{1,2}$ function u_T associated to T, requiring that $u_T = u_S$, and finally for $i \in I_3$ we prescribe the degree of $\partial(T \llcorner \Omega) \llcorner \pi^{-1}(\partial H_i)$.

iii) On the exterior boundary of Ω, i.e. on ∂A, we prescribe either the degree of $\partial(T \llcorner \pi^{-1}(\Omega)) \llcorner \pi^{-1}(\partial A)$ or ∂T on $\pi^{-1}(\Gamma_1)$, requiring that $T = S$ on $\pi^{-1}(\tilde{\Omega} \setminus \tilde{A})$, and the trace of u_T on $\Gamma_2, u_T = u_S$ on Γ_2.

It is easily seen that the class E is non-empty except when I_2 and Γ_2 are empty. In this last case we must (and do) require that the compatibility condition "the total degree of $\partial(T \llcorner \pi^{-1}(\Omega))$ is zero" holds in order E to be non-empty.

THEOREM 2. *Problem $\mathcal{P}$ has a minimizer in E.*

PROOF. Since $\mathcal{D}$ is lower semicontinuous and coercive on $\text{cart}^{2,1}$ and since the traces and 'boundaries' are preserved by weak convergence, the existence of a minimizer follows at once by the standard methods if we prove that, if T_k is a minimizing sequence converging weakly to T, then the degree of $\partial(T_k \llcorner \pi^{-1}(\Omega))$ on ∂H_i (or/and $\partial\Omega$) equals the prescribed degree of $\partial(T_k \llcorner \pi^{-1}(\Omega))$ on ∂H_i (or/and $\partial\Omega$). In order to prove this, we choose a smooth decreasing family of ball type domains $H_i(\sigma)$, $\sigma \in [0, \epsilon)$, such that $H_i(0) = H_i$ and the $\partial H_i(\sigma)$ give a foliation of $H_i(\epsilon) \setminus H_i$. Then we observe first that, for σ sufficiently small, the degree of $\partial(T_k \llcorner \pi^{-1}(H_i(\sigma) \setminus H_i))$ on $\partial H_i(\sigma)$ equals the degree of $\partial(T_k \llcorner \pi^{-1}(\Omega))$ on ∂H_i, and that, compare e.g. [59] p.182, we can choose σ small such that

$$\partial(T_k \llcorner \pi^{-1}(H_i(\sigma) \setminus H_i)) \llcorner \pi^{-1} \partial H_i(\sigma) \rightharpoonup \partial(T \llcorner \pi^{-1}(H_i(\sigma) \setminus H_i)) \llcorner \pi^{-1}(\partial H_i(\sigma)).$$

Since the degree is preserved by weak convergence (see sec. 3), the conclusion follows at once.

q.e.d.

EXAMPLE 1. Suppose Ω omits N points $\{a_i\}$ and has no holes, and suppose that $\bar{E}$ is the class of currents in $\text{cart}^{2,1}(\Omega, S^2)$ with $\partial T = -\sum_{i=1}^{N} d_i [\![a_i]\!], d_i \neq 0$, $\sum_{i=1}^{N} d_i = 0$, and with prescribed boundary $\partial T \llcorner \pi^{-1}(A)$ equals $[\![G_\psi]\!]$, where ψ is a constant on the exterior boundary of Ω. Then, compare [16], [1], [14], taking into account the coarea formula , one sees that a minimizer of problem $\mathcal{P}$ is given by $T_\psi + L \times [\![S^2]\!]$ where L is a least mass 1-dimensional current in Ω with $\partial L = -\sum_i d_i [\![a_i]\!]$. If, instead of ∂T on $\pi^{-1}(\partial A)$, we only prescribe $u_T = \psi =$ constant on ∂A, the situation changes completely and one easily convinces oneself that a minimizer is given by $T_\psi + L \times [\![S^2]\!]$ where this time L solves

$$M(L) \to \min,$$
$$\partial L = -\sum_i d_i [\![a_i]\!], \qquad \text{in } \Omega.$$

Observe that in this case the compatibility condition $\sum_{i=1}^{N} d_i = 0$ is not necessary. A similar situation occurs if points are replaced by holes (or we have both holes and points), provided we compute minimal connections between holes with respect to the metric $\delta(a,b)$ in [16], compare [16], [1], [14].

Of course, if the trace and/or the boundary data are *not constant*, the function u_T cannot be taken as constant, and we cannot expect a simple expression not even for the infimum of the Dirichlet integral.

EXAMPLE 2. Consider $\Omega = A \setminus \bigcup_{i=1}^{N} \{a_i\}$ and let $T_1 = T_{u_{T_1}} + L_1 \times [\![S^2]\!]$, $T_2 = T_{u_{T_2}} + L_2 \times [\![S^2]\!]$ be respectively solutions of problem

$$\mathcal{P}_1 : \begin{cases} \mathcal{D}(T) \to \min, \ \ \partial T \llcorner \pi^{-1}(\Omega) = -\sum_{i=1}^{N} d_i [\![a_i]\!], \ \ \sum_{i=1}^{N} d_i = 0 \ , \\ \partial T \llcorner \pi^{-1}(\partial A) = \partial S \llcorner \pi^{-1}(\partial A) \end{cases}$$

and of problem

$$\mathcal{P}_2 : \mathcal{D}(T) \to \min, \qquad \partial T \llcorner \pi^{-1}(\Omega) = -\sum_{i=1}^{N} d_i [\![a_i]\!], \ u_T = u_S, \ \text{on } \partial\Omega.$$

Suppose we also know that u_{T_1} and u_{T_2} have a discrete set of singular points (which would be the case for $H^{1,2}$ energy minimizing maps [57], [58], [40]), and denote by $x_1^{(1)}, \dots, x_k^{(2)}$ and $x_1^{(2)}, \dots, x_h^{(2)}$ respectively the singular points with non-zero degrees $d_i^{(1)}$ and $d_i^{(2)}$. Obviously both u_{T_1} and u_{T_2} are harmonic mappings in Ω (in general with no harmonic extension in A, see [46]). Then we see that L_1 must be the least mass 1-dimensional current in Ω with

$$\partial L_1 = -\sum_{i=1}^{N} d_i [\![a_i]\!] + \sum_{i=1}^{k} d_i^{(1)} [\![x_i^{(1)}]\!],$$

while L_2 must be the least mass (in Ω) 1-dimensional current with

$$\partial L_2 = -\sum_{i=1}^{n} d_i [\![a_i]\!] + \sum_{i=1}^{h} d_i^{(2)} [\![x_i^{(2)}]\!] + \sum_{i=1}^{\ell} d_i^{(3)} [\![x_i^{(3)}]\!],$$

where $x_i^{(3)}$ are free points on ∂A, with 'degree' $d_i^{(3)}$, so that

$$\sum_i \left(-d_i + d_i^{(2)} + d_i^{(3)}\right) = 0.$$

We conjecture that if S is integration on the graph of a smooth function g on ∂A, then the least energy in problem $\mathcal{P}_1$ coincides with

$$\inf\left\{\frac{1}{2}\int_\Omega |Du|^2 dx : u \in C^1\left(\Omega \setminus \bigcup_{i=1}^N \{a_i\}\right),\ \deg(u, a_i) = d_i, u = g \text{ on } \partial A\right\}.$$

But for that we only have a heuristic argument. One can approximate the current L in mass by a poliedral 1-dimensional chain C, and $C \times [\![S^2]\!]$ by a smooth function in $H^{1,2}$ (compare [1]), outside a small neighbourhood of C one should be able to approximate u_T in $H^{1,2}$ (compare [9]) and then interpolate with a small energy map. This would give the desired approximation.

While, for the point singularities problem, our description has many points of contact with the ones in [16], [1], [14], the situation changes strongly when considering the problem of free singularities.

It is well known that the problem

$$\frac{1}{2}\int_\Omega |Du|^2 dx \to \min, \qquad u \in H^{1,2}(\Omega, S^2),\ u = g \text{ on } \partial\Omega, \tag{5.12}$$

has a solution which is a harmonic map. But $C^1(\Omega, S^2) \cap H^{1,2}(\Omega, S^2)$ is not dense in $H^{1,2}(\Omega, S^2)$, a necessary condition for the approximability of u in $H^{1,2}$ by smooth functions being that the current T_u has no boundary in $\pi^{-1}(\Omega)$. So we claim that the Dirichlet integral is defined in a sense 'arbitrarily' in $H^{1,2}$. More precisely, the minimizers of (5.12) can create holes on its graphs without paying in energy, or in other words they can create new boundaries lowering the energy. Thus, in spite of its appearance, problem (5.12) is a problem with a free boundary more than a Dirichlet problem.

If the degree of g is zero, we can formulate the following Dirichlet problem

$$\mathcal{D}(T) \to \min,\ T \in E := \{T \in \text{cart}^{2,1}(\Omega, S^2),\ \partial T = [\![G_g]\!] \text{ on } \pi^{-1}(\partial\Omega)\}; \tag{5.13}$$

and it has a solution. But if the degree of g is different from zero, E is empty and no Dirichlet problem can be formulated. However we can always formulate the following problem (regardless of any degree condition)

$$\mathcal{D}(T) \to \min, \qquad T \in E := \{T \in \text{cart}^{2,1}(\Omega, S^2), u_T = g \text{ on } \partial\Omega\} \tag{5.14}$$

which is partly a Dirichlet problem and partly a free boundary problem.

We would like to point out that the solution of (5.14) or of (5.15) look completely different from the solutions of (5.12). In fact if $T = T_{u_T} + L \times [\![S^2]\!]$

is a solution of (5.14) (or of (5.13)), in general u_T *is not harmonic* although, if u_T has a discrete singular set, it is harmonic outside this set. For example if $g = x$ on $\partial B^3, \Omega = B^3$, the unique solution of (5.12) is $x/|x|$, see [16]; we claim that u_T is different from $x/|x|$. Suppose in fact that $u_T = x/|x|$, then necessarily L has to be the current integration on a segment joining the origin to a boundary point (say for simplicity (1,0,0)). Consider now a smooth family of diffeomorphisms $\phi_t : B^3 \to B^3$ such that $\phi_t(0) = (t, 0, 0)$ and such that ϕ_0 equals the identity on B^3, and set

$$T_t := T_{\phi_t(x)/|\phi_t(x)|} + L_t \times [\![S^2]\!],$$

L_t being the integration over the segment $[(t,0,0),(1,0,0)]$. Then $T_t \in \text{cart}^{2,1}(\Omega, S^2)$, $u_{T_t} = x$ on ∂B^3 and, since $x/|x|$ is an energy minimizing harmonic map, we have

$$\frac{1}{2}\int_{B^3} \left| D\frac{\phi_t(x)}{|\phi_t(x)|}\right|^2 dx = \frac{1}{2}\int_{B^3}\left|D\frac{x}{|x|}\right|^2 + o(t),$$

hence, we deduce

$$\mathcal{D}(T_t) = \frac{1}{2}\int_{B^3}\left|D\frac{x}{|x|}\right|^2 dx + 4\pi(1-t) + o(t)$$

and consequently

$$\frac{d}{dt}\mathcal{D}(T_t) = -4\pi < 0,$$

i.e. T is not a minimizer.

We conjecture that when g is smooth and the degree of g is zero

$$\inf\{\mathcal{D}(T) \ : \ T \in \text{cart}^{2,1}(\Omega, S^2), \ \partial T = [\![G_g]\!] \text{ on } \pi^{-1}(\partial\Omega)\}$$
$$= \inf\left\{\frac{1}{2}\int_\Omega |Du|^2 dx \ : \ u \in C^1(\bar\Omega, S^2), u = g\right\},$$

but we have no proof of that. Also, of course the regularity theory of Schoen-Uhlenbeck [57], [58], as well as the estimates on the defects of minimizers in [2], [36], do not apply to solutions of problems (5.13), (5.14), and the regularity theory remains completely open for those problems. Finally, it is worthwhile remarking that the discussion in this section extends with just formal changes to the analogous n-dimensional problem of minimizing

$$\frac{1}{(n-1)^{(n-1)/2}}\int_{B^n} |Du|^{n-1} d\mathcal{H}^n$$

among maps $u : B^n \to S^{n-1}$ with prescribed data. Also one could consider integrals with homogeneity of a surface instead of a line, compare [16], [14], [1], but we shall not do it.

THE LIQUID CRYSTAL FUNCTIONAL. We conclude this section by giving an explicit formula for the parametric extension of the liquid crystal functional. We recall that the liquid crystal functional can be written as

$$\tilde{\mathcal{E}}(u) := \int_\Omega \Big[\alpha|Du|^2 + (k_1-\alpha)(\text{div } u)^2 + (k_2-\alpha)(u\cdot \text{curl } u)^2 + (k_3-\alpha)|u\wedge \text{curl } u|^2\Big]\,dx$$

and we assume that $\alpha > 0$ and $k_i > \alpha$, for $i = 1,2,3$. Its integrand $W(n,G)$ is defined for all $n \in S^2$ and all 3×3 matrices G satisfying $G^T n = 0$ and, compare [28], it is invariant under orthogonal transformations Q :

$$W(Qn, QGQ^T) = W(n,G).$$

As we have seen in section 4, the parametric integrand associated to W is given, on the simple 3-vectors ξ, by

$$F(n,\xi) = \begin{cases} \xi_{\bar{0}0} W(n, G_{(\xi_1/\xi_{\bar{0}0})}) & \text{if } \xi_{\bar{0}0} > 0,\ G^T_{(\xi_1/\xi_{\bar{0}0})}\cdot n = 0 \\ \|\xi_2\|_W & \text{if } \xi_{\bar{0}0} = 0,\ \xi_1 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where we recall

$$\|\xi_2\|_W = \sup\{\phi(\xi_2) \mid \phi : V_2 \to \mathbb{R} \text{ linear, } \phi(M_2(G)) \le W(n,G)\ \forall G \text{ with } G^T\cdot n = 0\};$$

hence we ought to compute only $\|\xi_2\|_W$. Let $x\in\Omega$, $n\in S^2$ and let $\xi_2 \in \Lambda_3(T_x\Omega \times T_n\mathbb{R}^3)$.

One sees that $\|\xi_2\|_W$ is finite if and only if ξ_2 is a simple 3-vector in $\Lambda_3(T_x\Omega\times T_nS^2)$ and, in this case,

$$\xi_2 = t\wedge\tilde{\epsilon}_1\wedge\tilde{\epsilon}_2$$

where $\tilde{\epsilon}_1\wedge\tilde{\epsilon}_2$, is the canonical 2-vector on S^2 at n and $t\in T_x\Omega$. Moreover, we also see that $\|\xi_2\|_W$ is the largest convex function below

$$\Gamma(\xi_2) := \inf\{W(n,G) \mid G^T\cdot n = 0,\ M_2(G) = \xi_2\}.$$

Consider now the orthogonal transformation Q from $\mathbb{R}^3$ into $\mathbb{R}^3$ which maps

n	into	$(0,0,1)$
$\frac{t-(t.n)n}{\lvert t-(t.n)n\rvert}$	into	$(1,0,0)$
$\frac{t\wedge n}{\lvert t\wedge n\rvert}$	into	$(0,1,0)$

and consequently

$$Qt = (A, 0, B), \text{ with } A := \sqrt{|t|^2 - (t,n)^2}\ ,\ B = (t,n).$$

Denote the matrix QGQ^T by H; the condition $G^T \cdot n = 0$ then reads as

$$H = \begin{pmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

and the condition $M_2(G) = \xi_2$ amounts to

$$\begin{aligned} ad - bc &= B \\ af - eb &= 0 \\ cf - ed &= A. \end{aligned} \tag{5.16}$$

Therefore because of the invariance property of W under the transformation Q, we get

$$\Gamma(\xi_2) = \inf\{W((0,0,1), H) \mid H \text{ of the type (5.15) satisfying (5.16)}\}.$$

By a standard calculus computation that we omit, one then finds

$$\Gamma(\xi_2) = 2\left\{\sqrt{k^2B^2 + kk_3A^2} + (k-\alpha)B\right\},$$

where

$$k := \min(k_1, k_2).$$

(One first minimizes on the set $ad = \xi$, $bc = \eta$ for ξ, η fixed and then on ξ, η).

Since $\Gamma(\xi_2)$ is convex we then get

$$\|\xi_2\|_W = \Gamma(\xi_2)$$

and therefore the parametric extension $\mathcal{F}$ of $\tilde{\mathcal{E}}$ is finite exactly on $\text{cart}^{2,1}(\Omega, S^2)$ and, for $T = T_{u_T} + L \times [\![S^2]\!]$, is given by

$$\begin{aligned} \mathcal{F}(T) = \int_\Omega W(u_T, Du_T)\, dx + 2\int_{\mathcal{L}\times S^2} \gamma(x)\Big\{&\sqrt{k^2(t,n)^2 + kk_3(1-(t,n)^2)} \\ &+ (k-\alpha)(t,n)\Big\} d\mathcal{H}^1(x) d\mathcal{H}^2(n), \end{aligned}$$

where $L = \tau(\mathcal{L}, \gamma, t)$. An easy computation then gives

$$\mathcal{F}(T) = \int_\Omega W(u_T, Du_T)dx + 4\pi\sqrt{kk_3}\ \varphi(\gamma)M(L),$$

where

$$\gamma = \frac{k}{k_3} - 1, \qquad \varphi(\gamma) = \begin{cases} \sqrt{1+\gamma} + \frac{1}{\sqrt{\gamma}} \log(\sqrt{\gamma} + \sqrt{1+\gamma}) & \text{if } \gamma > 0 \\ 2 & \text{if } \gamma = 0 \\ \sqrt{1+\gamma} + \frac{\arcsin\sqrt{|\gamma|}}{\sqrt{|\gamma|}} & \text{if } \gamma < 0. \end{cases}$$

Of course we can formulate for $\mathcal{F}(T)$ the same problem $\mathcal{P}$ above, and in the same way we obtain existence of a minimizer.

6. - Some extensions

Consider a variational integral

$$\mathcal{F}(u) = \int_\Omega f(x, u, Du) dx \tag{6.1}$$

with integrand f defined on a class of smooth mappings from a domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^N$. Let F be the parametric integrand associated to f and let $\mathcal{F}(T)$ be the corresponding parametric integral defined for all currents T which are *graphs*, i.e. satisfy $\pi_\# T = [\![\Omega]\!]$ and $T_{\bar{0}0} \geq 0$. We say that $\mathcal{F}(u)$ is *regular* on a subclass C of the currents which are graphs, if $\mathcal{F}$ is coercive and semicontinuous on C. In sections 4 and 5 we have seen examples of regular variational integrals and we have illustrated the way one can formulate and solve several "boundary value problems" for regular integrals.

Our guiding principle was to work in the "smallest class obtained as the set of limit points of sequences of smooth functions with equibounded $\mathcal{F}$-energy", and this naturally led us to work with currents which are graphs without boundary in $\pi^{-1}(\Omega)$. But in many respects it is interesting and useful to study also variational problems among currents which are graphs *with* possible boundaries in $\pi^{-1}(\Omega)$. We have already encountered examples of this type in section 5; another example appears in the study of cavitation [6], [33].

In order to study variational problems among graphs with boundaries or holes in $\pi^{-1}(\Omega)$, we can proceed as follows. Set

$$N(T) = M(T) + M(\partial T),$$

and consider the class

$$\operatorname{cart}_b(\Omega, \mathbb{R}^N) := \{T \in \mathcal{D}_n(\Omega \times \mathbb{R}^N) : \pi_\# T = [\![\Omega]\!],\ T_{\bar{0}0} \geq 0,\ N(T) < +\infty\}.$$

Suppose $\mathcal{F}$ be a regular functional on $\operatorname{cart}_b(\Omega, \mathbb{R}^N)$. Then, in a standard way, we can formulate the classical boundary value problems for $\mathcal{F}(T)$ in $\operatorname{cart}_b(\Omega, \mathbb{R}^N)$; but, since $\mathcal{F}$ a priori does not control the mass in $\pi^{-1}(\Omega)$ of the boundary of a minimizing sequence, so that these boundaries can become wilder and wilder, we cannot expect existence of a minimizer. The situation can be handled instead, and we can easily prove *existence theorems*, if we assume that $\mathcal{F}(T)$ is *also* coercive with respect to the mass of ∂T on $\pi^{-1}(\Omega)$, for instance, if we replace $\mathcal{F}$ by $\mathcal{F}+\mathcal{G}$ where $\mathcal{G}$ is a parametric (lower semicontinuous) functional on ∂T which satisfies

$$\mathcal{G}(\partial T) \geq M(\partial T \llcorner \pi^{-1}(\Omega)).$$

EXAMPLE 1. Consider the problem of minimizing the Dirichlet integral (5.11) among currents which are prescribed on the boundary of $\partial\Omega$ and may have free boundaries in $\pi^{-1}(\Omega)$ only in at most a discrete set of points $\{a_1 \dots, a_N\}$, with free degrees d_i satisfying the constraint $\sum_{i=1}^{N} |d_i| \leq k$, k being an a priori given constant. The previous discussion shows that this problem has a solution.

The following two subclasses of $\mathrm{cart}_b(\Omega, \mathbb{R}^N)$ seem to be of special interest. We denote by $C_b^1(\Omega, \mathbb{R}^N)$ the class of functions u for which there exists a closed set K in Ω with $\mathcal{H}^n(K)=0$ and such that $u \in C^1(\Omega \setminus K)$ and for which T_u is well defined in Ω (i.e. u and the minors of Du are summable in Ω) and satisfies $N(\partial T_u) < +\infty$. The smaller subclass of $C_b^1(\Omega, \mathbb{R}^N)$ of functions u, for which K is a finite union of submanifolds of Ω of dimensions less or equal than $n-1$, will be denoted by $C_r^1(\Omega, \mathbb{R}^N)$. Of course

$$C_r^1(\Omega, \mathbb{R}^N) \subset C_b^1(\Omega, \mathbb{R}^N) \subset \mathrm{cart}_b(\Omega, \mathbb{R}^N).$$

The smallest sequentially closed sets, with respect to the weak convergence of currents with N-equibounded norms, containing repectively $C_r^1(\Omega, \mathbb{R}^N)$ and $C_b^1(\Omega, \mathbb{R}^N)$, will be denoted by $\mathrm{Cart}_{br}(\Omega, \mathbb{R}^N)$ and $\mathrm{Cart}_b(\Omega, \mathbb{R}^N)$. Exactly in the same way we did in the previous sections, we can now define the classes $\mathrm{Cart}_b^p(\Omega, \mathbb{R}^N)$ and $\mathrm{Cart}_r^p(\Omega, \mathbb{R}^N)$ for a multiindex $p=(p_0, p_1, \dots, p_{\min(n,N)})$, and also analogous classes in which Ω and $\mathbb{R}^N$ are replaced respectively by an n-dimensional oriented Riemannian manifold X^n and by an imbedded submanifold Y^r of $\mathbb{R}^N$.

EXAMPLE 2. Let $\Omega, \tilde{\Omega}$ be bounded domains in $\mathbb{R}^3$ with $\Omega \subset\subset \tilde{\Omega}$ and let $\mathcal{D}(T)$ be the parametric extension of the Dirichlet integral (or the parametric extension of the liquid crystal functional). Suppose S be a given current in $\mathrm{Cart}^{2,1}(\Omega, S^2)$. The problem

$$\begin{cases} \mathcal{D}(T) + M(\partial T \llcorner \pi^{-1}(\tilde{\Omega})) \\ T \in E := \{T \in \mathrm{Cart}_{br}(\tilde{\Omega}, S^2) | T = S \text{ on } \pi^{-1}(\tilde{\Omega} \setminus \bar{\Omega})\}, \end{cases}$$

according to our previous discussion, has a solution T. The minimizer T shows now very interesting features. In fact for T it is not convenient to create a dipole $L \times [\![S^2]\!]$, L being for example 'the segment $[x_0, x_1]$' in Ω, with a connection of large lenght, as such a dipole would contribute to the energy as $4\pi M(L)$, while, creating boundaries at $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$ would contribute to the energy as $2 \cdot 4\pi$. On the other hand, it is not convenient to create boundaries, for example, at points x_0, x_1 which are too near, as the dipole $[\![[x_0, x_1]]\!] \times [\![S^2]\!]$ would have smaller energy $4\pi|x_1 - x_0|$ than $2 \cdot 4\pi$. Also the minimizer T might find convenient to 'fracture', for example, along a 2- dimensional surface. Therefore, in principle, many different kinds of singularities may coexist for T. Of course a real analysis remains to be done.

A problem, which has many points of contact with the previous one, has been formulated by De Giorgi-Ambrosio [21]. In our approach, it could be stated as the problem of minimizing

$$\mathcal{D}(T) + \mathcal{H}^{n-1}(\pi(\mathcal{L})), \qquad \partial T \llcorner \pi^{-1}(\Omega) = \tau(\mathcal{L}, \theta, \varsigma), \tag{6.3}$$

in E (or in E where Cart_{rb} is replaced by Cart_b). The delicate question here is of course the question of the coercivity and semicontinuity of the functional (6.3), question which does not seem to have a simple answer. We mention that problem (6.3) can be seen as a generalization of a codimension 1 problem in the study of segmentation of images, cfr. [51], [47], [11], [22].

We think that the abstract setting described above is very convenient to give a mathematical static model of *fractures* in the nonlinear theory of hyperelastic materials. In the sequel of this section we shall discuss this idea more precisely.

NONLINEAR HYPERELASTICITY AND FRACTURES. Let us first recall some ideas and facts concerning the mathematical formulation of the static equilibrium problem for *perfectly* elastic bodies (compare [33] and its references). A *deformation* of an elastic body is described by a function u from a domain Ω in $\mathbb{R}^3$, taken as a *reference configuration*, into $\mathbb{R}^3$ which is *orientation preserving* and *globally invertible*. A material is called *hyperelastic* if its mechanical properties are characterized by a *stored energy* function $W(x, G)$ in terms of which its *total stored energy* is given by

$$\mathcal{E}(u, \Omega) = \int_\Omega W(x, Du)\, dx, \tag{6.4}$$

Du being the deformation gradient. One is then interested in finding a deformation which satisfies suitable boundary conditions and which minimizes $\mathcal{E}$. Of course one has to specify both mathematical and physical reasonable *constitutive* conditions on the stored energy W. In doing that for *perfectly* elastic bodies, we have a few general principles that we shall briefly describe.

1. A deformation is described modulus changes of the reference parameters. This already suggests that our problem has a parametric character more than an apparent non-parametric one, or in other words what is relevant is the graph of u more than the map u, or still in other words, our problem lives in the product $\Omega \times \mathbb{R}^3$.

2. For a perfectly elastic material it is natural to require that the stored energy

depends on the deformations of line surface and volume elements, i.e.

$$W(x,G) = F(x, M(G)). \tag{6.5}$$

Moreover, since the more an elastic body is pulled (or compressed) the longer it grows (or shortens), it is natural to require that W be 'elliptic', which might be expressed as $F(x,M)$ be convex with respect to M.

3. It is reasonable to require that, in order to stretch a fiber to infinite lenght or compress it to zero, we need an infinite amount of energy. This turns into requiring *coercivity of* W *at infinity and at zero*. At infinity we require

$$F(x, M(Du)) \geq |M(Du)|^p, \quad p \geq 1,$$

and observing that compression in the reference configuration is seen as stretching in the deformed configuration, that is coercivity at zero is equivalent to coercivity at infinity in the deformed configuration, we are led, compare [33], to require that

$$F(x, M(Du)) \geq |M(Du)|^p + \frac{|M(Du)|^q}{(\det Du)^{q-1}}, \qquad p, q \geq 1. \tag{6.6}$$

In conclusion the equilibrium problem for hyperelastic materials can be formulated as the problem of minimizing a functional of the type (6.4) (with integrand W given by a convex function $F(x,\xi)$ on the simple 3-vectors with $\xi_{\bar{0}0} = 1$ and satisfying (6.6)), among orientation preserving and globally invertible mappings. Actually it would be more correct to refer to (6.6) as the energy associated to a perfectly elastic body if $p, q > 1$, and as the energy associated to a perfectly "elastoplastic body" if $p, q \geq 1$, compare [33]. Extending $\mathcal{E}$ to the class $\mathrm{Dif}^{p,q}(\Omega, \hat{\Omega})$ (compare sec. 2 and [33]) with prescribed "Dirichlet" conditions, one then proves existence of a minimizer, see [33]. More generally, we need not fix the image $\hat{\Omega}$ of Ω, and we may work on the class $\mathrm{Dif}^{p,q}(\Omega)$, see [33], defined as the weak sequential closure (in the sense of currents) of smooth diffeomorphisms u of Ω into $u(\Omega)$ with equibounded $\|\cdot\|_{\mathrm{Dif}^{p,q}(\Omega)}$ norms; and again, under natural conditions, one proves the existence of a minimum energy deformation.

By definition, the elements T of $\mathrm{Dif}^{p,q}(\Omega)$ have no boundary in $\Omega \times \mathbb{R}^n$. But as we have seen in the beginning of this section, we may also work with currents with boundaries. Consider in fact the subclass of $C^1_r(\Omega, \mathbb{R}^3)$, $\mathrm{Dif}_r(\Omega)$, of u for which there exists a finite union K of submanifolds of Ω of dimension less or equal to $n-1$ and such that u is a smooth diffeomorphism from $\Omega \backslash K$ into $u(\Omega \backslash K)$ with $M(\partial T_u) + \|T_u\|_{\mathrm{Dif}^{p,q}} < +\infty$, and its sequential closure (in the sense of currents) with equibounded $\|T_u\|_{\mathrm{Dif}^{p,q}}$ and $M(\partial T_u)$ norms, denoted $\mathrm{Dif}^{p,q}_{br}(\Omega)$. We may regard the elements of $\mathrm{Dif}^{p,q}_{br}(\Omega)$ as *deformations with fractures*.

If we now introduce a lower semicontinuous functional defined on the "boundaries"

$$\mathcal{E}_b(\partial T \llcorner \pi^{-1}(\Omega))$$

which is also coercive with respect to the mass of $\partial T \llcorner \pi^{-1}(\Omega)$, we can minimize

$$\mathcal{E}(T) + \mathcal{E}_b(\partial T \llcorner \pi^{-1}(\Omega))$$

in suitable subclasses (determined by the boundary data) of $\mathrm{Dif}^{p,q}_{br}(\Omega)$. $\mathcal{E}_b(\partial T \llcorner \pi^{-1}(\Omega))$ can be interpreted as the energy that should be spent in order to produce the *fracture* $\partial T \llcorner \pi^{-1}(\Omega)$.

A special case of this problem is the problem of cavitation [6], where one looks for radial deformations which may produce a fracture or cavitation only at the origin. For such a problem it is reasonable to consider homogeneous bodies with stored energies of the type

$$W(G) = \Phi(v_1, v_2, \ldots, v_n),$$

where Φ is a symmetric function of the eigenvalues $v_1, \ldots, v_n$ of $(G^T G)^{1/2}$ and assume that W satisfies Legendre-Hadamard condition, compare [6] (see also [33]). Assuming the body *perfectly* elastic, the energy corresponding to a radial deformation

$$u(x) = \mathcal{U}(|x|)\frac{x}{|x|}$$

which is regular in $B_1(0)\backslash\{0\}$ and with vertical part $[\![\{0\} \times B(0, \mathcal{U}(0))]\!]$, when regarded as an element of $\mathrm{dif}(B_1)$, is given, compare [33], by

$$\tilde{\mathcal{E}}(u) = \int_{B_1(0)} W(Du)\, dx + |B(0, \mathcal{U}(0))| \lim_{\epsilon \to 0} \Phi\left(\frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}\right).$$

In this case there will be no cavitation or fracture, see [33], [45], in the sense that for any regular stationary point in $B_1(0)\backslash\{0\}$ of

$$\int_{B_1(0)} W(Du)\, dx$$

with $u = u_0$ on $\partial B_1(0)$, $u_0(x) = r_0 x$, we have

$$\tilde{\mathcal{E}}(u) \geq \int_{B_1(0)} W(Du_0)\, dx.$$

Suppose instead that the body is *not perfectly elastic* and, in our model, we decide to evaluate the energy needed to produce a fracture at zero, i.e. the term $\mathcal{E}_b(\partial T_u \llcorner \pi^{-1}(0))$, as

$$\mathcal{E}_b(\partial T_u \llcorner \pi^{-1}(0)) = \gamma M(\partial T_u \llcorner \pi^{-1}(0))^{n/(n-1)},$$

then, compare [33], we conclude that the body actually fractures at zero, or cavitates provided

$$\frac{\gamma}{\omega} < 1,$$

where ω is the isoperimetric constant

$$\omega = |B(0,1)|/|\partial B(0,1)|^{n/(n-1)}$$

and the displacement at ∂B_1, i.e. r_0, is sufficiently large. In particular the body will surely cavitate for r_0 large, if $\gamma = 0$, i.e. if it does not have to pay in energy in order to fracture at zero, see [6].

REFERENCES

[1] F. ALMGREM - W. BROWDER - E.H. LIEB, *Co-area, Liquid crystals, and minimal surfaces,* In DDT - a selection of Papers, Springer-Verlag, 1987.

[2] F. ALMGREM - E.H. LIEB, *Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds*, Preprint.

[3] L. AMBROSIO - S. MORTOLA - V.M. TORTORELLI, *Functionals with linear growth defined on vector valued BV functions*, preprint.

[4] J.M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity,* Arch. Rat. Mech. Anal. **63** (1977).

[5] J.M. BALL, *Global invertibility of Sobolev functions and the interpenetration of matter,* Proc. Roy. Soc. Edinburgh A **88** (1981) 315-328.

[6] J.M. BALL, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity.* Philos. Trans. Roy. Soc. A **306** (1982) 577-611.

[7] J.M. BALL - F. MURAT, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*. J. Funct. Anal. **58** (1984) 225-253.

[8] F. BETHUEL, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, preprint.

[9] F. BETHUEL, *Approximation dans des espaces de Sobolev entre deux variété et groupes d'homotopie,* preprint.

[10] F. BETHUEL - X. ZHENG, *Density of smooth functions between two manifolds in Sobolev spaces*, preprint.

[11] J. BLAT - JH. MOREL, *Elliptic problem image segmentation and their relation to fracture theory*, preprint.

[12] R. BOTT - L.W. TU, *Differential forms in Algebraic Topology,* Springer-Verlag, New York, 1982.

[13] S. BREZIS, *Metastable harmonic maps,* In "Metastability and incompletely posed problems" Eds. S. Antman, J.L. Ericksen, D. Kinderlehrer, J. Müller, Springer-Verlag 1987.

[14] S. BREZIS, *S^k-valued maps with singularities,* In "Topics in Calculus of Variations" Ed. M. Giaquinta, Lecture Notes in Math. n.1365, Springer-Verlag 1989.

[15] S. BREZIS - J.M. CORON, *Large solutions for harmonic maps in two dimensions,* Commun. Math. Phys. **92** (1983) 203-215.

[16] S. BREZIS - J.M. CORON - E.H. LIEB, *Harmonic maps with defects,* Comm. Math. Phys. **107** (1986), 649-705.

[17] H. BUSEMAN G. EWALD - G. SHEPARD, *Convex bodies and convexity on Grassmann cones,* Part I to IV Math. Ann. **151** (1963) 1-41.

[18] P.J. CIARLET - J. NEČAS, *Unilateral problems in nonlinear, three-dimensional elasticity,* Arch. Rat. Mech. Anal. **97** (1987) 171-188.

[19] B. DACOROGNA, *Remarques sur les notions de polyconvexité, quasi-convexité et convexité de rang 1.* J. Math. pures et Appl. **64** (1985) 403-438.

[20] G. DAL MASO, *Integral representation on* $BV(\Omega)$ *of* Γ*-limits of variational integrals.* Manuscripta Math. **30** (1980) 387-416.

[21] E. DE GIORGI - L. AMBROSIO, *Un nuovo tipo di funzionale del calcolo delle variazioni,* preprint.

[22] E. DE GIORGI - M. CARRIERO - A. LEACI, *Existence theorem for a minimum problem with free discontinuity set,* preprint.

[23] J. EELLS - L. LEMAIRE, *A report on harmonic maps,* Bull. London Math. Soc. **10** (1978) 1-69.

[24] J. EELLS - L. LEMAIRE, *Selected topics in harmonic maps,* Amer. Mat. Soc. Regional Conf. Series in Math. **50**, 1983.

[25] J. EELLS - L. LEMAIRE, *Another report on harmonic maps,* Bull. London Math. Soc. **20** (1988) 385-524.

[26] J. EELLS - J.B. WOOD, *Restrictions on harmonic maps of surfaces,* Topology **15** (1976) 263-266.

[27] I. EKELAND - R. TEMAM, *Convex analysis and variational problems,* North Holland, Amsterdam 1976.

[28] J. ERICKSEN - D. KINDERLEHRER, *Theory and applications of liquid crystals,* IMA Series vol 5, Springer-Verlag 1987.

[29] M.J. ESTEBAN, *A direct variational approach to Skyrme's model for meson fields,* Comm. Math. Phys. **105** (1986) 571-591.

[30] H. FEDERER, *Geometric measure theory,* Springer-Verlag, New York, 1969.

[31] H. FEDERER - W. FLEMING, *Normal and integral currents,* Ann. of Math. **72** (1960) 458-520.

[32] M. GIAQUINTA - E. GIUSTI, *The singular set of the minima of certain quadratic functionals,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 45-55.

[33] M. GIAQUINTA - G. MODICA - J. SOUČEK, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Archive for Rat. Mech. Anal. **106** (1989) 97-159. Erratum and addendum, to appear in Archive for Rat. Mech. Anal.

[34] R. HARDT - D. KINDERLEHRER - F.H. LIN, *Existence and partial regularity of static liquid crystal configurations,* Comm. Math. Phys. **105** (1986), 547-570.

[35] R. HARDT - D. KINDERLEHRER - F.H. LIN, *A remark about the stability of smooth equilibrium configurations of static liquid crystals*, Mol. Cryst. Liq, Cryst. **139** (1986) 189-194.

[36] R. HARDT - D. KINDERLEHRER - F.H. LIN, *Stable defects of minimizers of constrained variational principles,* Ann. Inst. Henri Poincaré **5** (1988) 207-322.

[37] R. HARDT - F.H. LIN, *A remark on* H^1 *mappings*, Manuscripta Math. **56** (1986) 1010.

[38] R. HARDT - L. SIMON, *Seminar on Geometric Measure Theory,* Birkhäuser, Boston, 1986.

[39] J. JOST, *The Dirichlet problem for harmonic maps from a surface with boundary into a 2-sphere with non-constant boundary values,* J. Diff. Geom. **19** (1984) 393-401.

[40] J. JOST, *Harmonic maps between surfaces,* Lecture notes in Math. **1062**, Springer-Verlag 1984.

[41] R.V. KOHN - G. STRANG, *Optimal design and relaxation of variational problems,* Comm. Pure Appl. Math. to appear.

[42] L. LEMAIRE, *Applications harmoniques de surfaces riemanniennes,* J.Diff. Geom. **13** (1978) 51-78.

[43] P.L. LIONS, *The concentration compactness principle in the calculus of variations. The limit case, part* 2, Revista Matematica Iberoamericana **1** (1985) 45-121.

[44] F.C. LIU, *A Lusin type property of Sobolev functions,* Indiana Univ. Math. J. **26** (1977) 645-671.

[45] P. MARCELLINI, *The stored energy of some discontinuous deformations in nonlinear elasticity,* In Partial Differential Equations and the Calculus of Variations: Essays in Honor of Ennio De Giorgi, Birkäuser, Boston, 1989, to appear.

[46] M. MEIER, *Removable singularities of harmonic maps and an application to minimized submanifolds,* Indiana Univ. Math. J. **35** (1986) 705-726.

[47] J.M. MOREL, S. SOLIMINI, *Segmentation of images by variational methods,* preprint.

[48] F. MORGAN, *Geometric measure theory,* Academic Press, 1988.

[49] C.B. MORREY, *Multiple integrals in the calculus of variations,* Springer-Verlag, Berlin, 1966.

[50] S. MÜLLER, *Weak continuity of determinants and nonlinear elasticity,* C.R. Acad. Sci. Paris **307** (1988) 501-5?6.

[51] D. MUMFORD, J. SHAH, *Boundary detection by minimizing functionals,* Proceedings of the IEEE. Conference on computer vision and pattern recognition, San Francisco, 1985.

[52] L. NIRENBERG, *Topics in Nonlinear Functionals Analysis,* New York University, Lectures Notes, New York (1974).

[53] T. QI, *Almost-everywhere injectivity in nonlinear elasticity,* Proc. Roy. Soc. Edinburgh **109** A (1988) 79-95.

[54] Y.G. RESHETNYAK, *On the stability of conformal mappings in multidimensional spaces,* Siberian Math. J. **8** (1967) 69-85.

[55] Y.G. RESHETNYAK, *Stability theorems for mappings with bounded distorsion,* Siberian Math. J. **9** (1968) 499-512.

[56] Y.G. RESHETNYAK, *Weak convergence of completely additive vector functions on a set,* Sibirskii Mat. Zhurnal **9** (1968) 1386-1394.

[57] R. SCHOEN - K. UHLENBECK, *A regularity theory for harmonic maps,* J. Diff. Geom. **17** (1982) 307-335.

[58] R. SCHOEN - K. UHLENBECK, *Boundary regularity and miscellaneous results on harmonic maps,* J. Diff. Geom. **18** (1983) 253-268.

[59] L. SIMON, *Lectures on Geometric Measure Theory,* Proc. of the Centre for Math. Analysis vol. **3** Australian National University, Canberra, 1983.

[60] A. SOYEUR, *The Dirichlet problem for harmonic maps from the disc into 2-sphere,* preprint.

[61] V. ŠVERÁK, *Regularity properties of deformations with finite energy,* Arch. Rat. Mech. Anal. **98** (1987).

[62] B. WHITE, *A new proof of the compactness theorem for integral currents,* preprint.

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