Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Zbigniew Slodkowski

Polynomial hulls in $\mathbb{C}^2$ and quasicircles

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 16, nº 3 (1989), p. 367-391

<http://www.numdam.org/item?id=ASNSP_1989_4_16_3_367_0>

Polynomial Hulls in $\mathbb{C}^2$ and Quasicircles

ZBIGNIEW SLODKOWSKI

0. - Introduction

Ever since H. Rossi has proved the celebrated local maximum modulus principle, efforts were made to elucidate it further by finding analytic varieties, preferably analytic discs, in the set $\widehat{X}\backslash X$, where $\widehat{X}$ is the polynomial hull of a compact set $X \subset \mathbb{C}^n$. As it is well known, the general problem has negative answer, and several counterexamples were given, the most refined being that of J. Wermer [15]. Various positive results were obtained when X is a smooth manifold satisfying additional special assumptions; see Alexander [1], Bedford and Gaveau [3], Bishop [5], Forstnerič [8], and the references listed in these papers.

In this paper we will study another special situation when X is a subset of $\mathbb{C}^2$ which projects ontó a circle and has connected and simply connected fibers. We recall now the first general result in this direction, which was obtained independently by Alexander and Wermer in [2], and by the author in [14].

THEOREM 0.1. [2], [14]. *Let $X \subset \partial D \times \mathbb{C}$ be a compact set, where $D = \{z \in \mathbb{C} : |z| < 1\}$. Assume that for every $\varsigma \in \partial D$ the fiber $\{w \in \mathbb{C} : (\varsigma, w) \in X\}$ is geometrically convex. Then the set $\widehat{X}\backslash X$ (if nonempty) is equal to the union of graphs of bounded analytic functions with boundary values in $X(\varsigma)$, $\varsigma \in \partial D$.*

(The result is true also when $X \subset \partial D \times \mathbb{C}^n$ but the case $n > 1$ is not directly related to the context of this paper).

The convexity assumption of the last theorem was eliminated by F. Forstnerič [8] at the price, however, of adding the assumption that the hull in question already contains an analytic disc. We summarize next some of the Forstnerič's results.

THEOREM 0.2. [8]. *Let $M \subset \partial D \times \mathbb{C}$ be a $C^{(k)}$-regular hypersurface, $k \geq 2$, such that the fibers $M(\varsigma) = \{w \in \mathbb{C} : (\varsigma, w) \in M\}$ are $C^{(k)}$-regular Jordan curves for $\varsigma \in \partial D$. Assume that $0 \in \widehat{M(\varsigma)}\backslash M(\varsigma)$ for every $\varsigma \in \partial D$.*

Pervenuto alla Redazione il 14 Ottobre 1988.
This research was in part supported by National Science Foundation grant DMS - 870227.

Assume further that $\pi : M \to \partial D$ *is a submersion, where* $\pi(z,w) = z$. *Then the hull* $\widehat{M}$ *is covered by the graphs of continuous analytic functions in* $\overline{D}$. *Furthermore, there is a* $C^{(k-2)}$*-smooth embedding* $\Phi : \overline{D} \times \partial D \to \overline{D} \times \mathbb{C}$, *where* $\Phi(z,\xi) = (z, f_\xi(z))$, $z \in \overline{D}$, $\xi \in \partial D$, *such that*

(i) $f_\xi(\cdot)$ *are analytic functions;*

(ii) $\Phi|_{D\times\partial D}$ *is a* $C^{(k-1)}$*-regular imbedding;*

(iii) $\Phi(\overline{D} \times \partial D)$ *is equal to the relative boundary of* $\widehat{M}$ *in* $\overline{D} \times \mathbb{C}$, *and is a* $C^{(k-2)}$*-regular submanifold bordered by* $M = \Phi(\partial D \times \partial D)$.

The purpose of this paper is to provide a simultaneous generalization of the above two results. No a priori existence of analytic discs will be assumed. In fact, our main effort will be directed toward showing that $\widehat{X} \setminus X$ contains an analytic disc, if $X \subset \partial D \times \mathbb{C}$ has connected and simply connected fibers (and $\widehat{X} \neq X$). To achieve this, we use, among other tools, some techniques from quasiconformal geometry, which seems to be a novelty in this context.

The detailed formulation of our main results is given in Section 1.

The problem studied in this paper have some affinity with the subject of nonlinear H^∞-optimization developed by J.W. Helton and his coworkers. We refer the reader to Bence et al. [4], for a concise introduction and for further references.

1. - Results

It is convenient to formulate first the most general of the theorems proved here.

THEOREM 1.1. *Let* $X \subset \partial D \times \mathbb{C}$ *be a compact set such that for every* $\varsigma \in \partial D$ *the fiber* $X(\varsigma) = \{w \in \mathbb{C} : (\varsigma, w) \in X\}$ *is a simply connected continuum. Assume that* $\widehat{X} \setminus X$ *is nonempty. Then* $\widehat{X} \setminus X$ *is equal to the union of the graphs of all* $H^\infty(D)$ *functions whose cluster values at* $\varsigma \in \partial D$ *belong to* $X(\varsigma)$. *(Below, an analytic disc will always denote such a graph). Furthermore,*

(a) *the relative boundary of* $\widehat{X} \cap D \times \mathbb{C}$, *denoted by* S, *is covered by analytic discs, and every two distinct analytic discs contained in* S *are disjoint;*

(b) *if* V *denotes the relative interior of* $\widehat{X}$ *in* $\overline{D} \times \mathbb{C}$, *then* $V \setminus X$ *is covered by analytic discs, and every analytic disc contained in* $\widehat{X}$ *and intersecting* S *must be fully contained in* S*;*

(c) *for every* $z \in D$ *the fiber* $Y(z) = \{w \in \mathbb{C} : (z,w) \in \widehat{X}\}$, *of* $\widehat{X}$, *is a simply connected continuum and its topological boundary (in* $\mathbb{C}$*) is equal to the fiber of* S, *i.e.* $S(z) = \{w \in \mathbb{C} : (z,w) \in S\}$.

We introduce the following terminology.

DEFINITION 1.2. Let X be as in Theorem 1.1. We say that its polynomial hull $\widehat{X}$ is *nondegenerate* if $\widehat{X}\backslash X$ is nonempty and is not equal to a single analytic disc.

The meaning of this notion is further explained by the next corollary, which will follow directly from Theorem 1.1.

COROLLARY 1.3. *Under assumptions and in the notation of Theorem 1.1, the following conditions are equivalent:*

(i) *there is $z_0 \in D$ such that $Y(z_0)$ is not reduced to a single point;*

(ii) *for every $z \in D$ the fiber $Y(z)$ contains more than one point;*

(iii) *$\widehat{X}\backslash X$ is not equal to a single analytic disc.*

In this paper we will study almost exclusively nondegenerate polynomial hulls. Note that a degenerate hull contains an analytic disc by definition (if $\widehat{X}\backslash X \neq 0$), while establishing the same for nondegenerate hulls will require considerable work.

For the most part we will be working with sets X slightly less general than those of Theorem 1.1, namely such that fibers $X(\varsigma)$ are Jordan domains for $\varsigma \in \partial D$ and $\{(\varsigma, w) : \varsigma \in \partial D,\ w \in \partial X(\varsigma)\}$ is a topological torus. In connection with this situation, we specify now some assumptions and notations which will be upheld throughout the paper.

ASSUMPTIONS AND NOTATIONS 1.4. M will denote a compact subset of $\partial D \times \mathbb{C}$ such that for every $\varsigma \times \partial D$ the fiber $M(\varsigma) = \{w \in \mathbb{C} : (\varsigma, w) \in M\}$ is a Jordan curve with given orientation preserving parametrization $w = b(\varsigma, e^{is})$, $0 \leq s < 2\pi$. Assume that the map

$$(\varsigma, \xi) \to (\varsigma, b(\varsigma, \xi)) : \partial D \times \partial D \to M \tag{1.1}$$

is a homeomorphism onto. (Other varying regularity assumptions will be frequently added). Once M is given, X is defined as

$$\bigcup_{|\varsigma|=1} \{\varsigma\} \times \widehat{M(\varsigma)},$$

and then S, V, Y have the same meaning as in Theorem 1.1, namely $Y = \widehat{X} = \widehat{M}$, $S = \partial_{D\times\mathbb{C}}(Y\backslash X)$, $V = \operatorname*{Int}_{\overline{D}\times\mathbb{C}} Y$ (and $Y(z)$, $S(z)$, $V(z)$ are fibers of Y, S, V over $z \in \overline{D}$). Consistently with Definition 1.2, we say that $\widehat{M}$ is *nondegenerate* if the corresponding $\widehat{X}$ is nondegenerate, i.e. $\widehat{M} \cap (D \times \mathbb{C})$ is not equal to a single analytic disc.

The most complete description of the hull is obtained when M is at least $C^{(2)}$-smooth.

THEOREM 1.5. *Let M satisfy Assumptions and Notations 1.4. If, in addition, parametrization* (1.1) *is $C^{(k)}$-smooth, with $k \geq 2$, $\frac{\partial b}{\partial s}(\varsigma, e^{is}) \neq 0$ on M, and $\widehat{M}$ is nondegenerate, then*

$$\overline{S}\backslash S = M \text{ and } \partial(\widehat{M}) = (S \cup M) \cup X, \tag{1.2}$$

and there exists a continuous analytic function $g : \overline{D} \to \mathbb{C}$ *such that*

$$g(\varsigma) \in \widehat{M(\varsigma)} \backslash M(\varsigma) \text{ for } \varsigma \in \partial D. \tag{1.3}$$

Furthermore, there is a $C^{(k-2)}$*-regular homeomorphism* $\Phi : \overline{D} \times \partial D \to S \cup M$*, which is a* $C^{(k-1)}$*-regular immersion on* $D \times \partial D$*. (Properties* (a)-(c) *of Theorem* 1.1 *hold as well).*

Of course, once the existence of selection g satisfying (1.3) is established, the remaining properties hold by Forstnerič result. In fact, the existence of Φ is simply taken over from Theorem 0.2. As for the relations (1.2) however, we have to prove them first, before we will be able to deduce the existence of g.

Theorem 1.1. and Corollary 1.3 will be derived (in Section 5) from Theorem 1.5. The latter is proved in Sections 2 through 5. In the general outline, we consider a continuous family of smooth tori M^t, $1 \leq t \leq R$, with $M^1 = M$, such that the Jordan curve $M^t(\varsigma)$ encloses $M(\varsigma)$, for $\varsigma \in \partial D$ and $t > 1$ and M^t satisfies assumptions of Theorem 0.2 for t close to R. If we denote by I the largest interval consisting of t for which M^t admits a continuous analytic function with property (1.3), we have to show that $T \in I$ (see Notation 2.1), which implies that $T = 1$ as well. Elementary properties of $\widehat{M^T}$ are obtained in Section 2. A crucial step is to show that $\widehat{M^T}$ has nonempty interior. To this end we prove in Section 3 that the fibers $Y^t(z) = \{w : (z, w) \in \widehat{M^T}\}$, $t > T$, are quasicircles with a universal constant, which implies that their limit $Y^T(z)$ is a Jordan region. The next essential step (Section 4) consists in showing that the relative interior V^T of M^T in $\overline{D} \times \mathbb{C}$ is not too small near ∂D, specifically that $V^T(\varsigma) \supset \widehat{M^T}(\varsigma) \backslash M^T(\varsigma), \varsigma \in \partial D$. The proof of Theorem 1.5 is completed in Section 5.

In the remaining part of the paper (Sections 6-8) we study the hull $\widehat{M}$ under weaker than $C^{(2)}$ regularity assumptions of M. The picture we obtain is not as complete as in the $C^{(2)}$ case, but it is still considerably more precise than that of Theorem 1.1.

We show that *if the Jordan curves* $M(\varsigma)$, $\varsigma \in \partial D$, *are quasicircles (with a uniform constant), then so are the fibres* $Y(z)$, $z \in D$, *and the hull* $\widehat{M}$ *has nonempty interior* (Theorem 7.1).

To obtain stronger results we assume, essentially, that curves $M(\varsigma)$, $\varsigma \in \partial D$, are locally graphs of Lipschitz functions. We call the specific technical assumption (cf. Definition 6.1) the *continuous cone condition* and prove (cf. Theorem 6.3) that under it the Jordan regions $M(\varsigma)$, $\varsigma \in \partial D$, admit a continuous family of holomorphic inward pointing vector fields, i.e. have the property of *uniform transversal holomorphic contractibility*. This property (cf. Definition 4.1.) was introduced by Helton and Howe [9] and Theorem 6.3 generalizes an earlier result of Helton et al. [10, Theorem 2.2].

Assuming that M satisfies the continuous cone condition, we prove in Section 7 that *the boundary of* $\widehat{M}$ *in* $\mathbb{C}^2$ *is a topological manifold equal to* $X \cup (S \cup M)$; in particular $S \cup M$ is closed.

The closest approximation to Theorem 1.5. is obtained when the parametrization (1.1) of M is $C^{(1+\alpha)}$-regular, $\alpha > 0$. Then, in addition to properties described above, $S \cup M$ is covered by graphs of analytic functions that belong to $C^{1+\delta}(\overline{D})$, and the interior V of $\widehat{M}$ contains the graph of a continuous analytic function in $\overline{D}$ (Theorem 8.1).

2. - The approximation scheme

Let M be as in Theorem 1.5. As observed by Forstnerič [8], there is a smooth family of tori M^t, $t \in [1, R]$, satisfying the assumptions of Theorem 1.5 and such that

(2.1) $$M^1 = M; \quad 0 \in \operatorname{Int} M^R(\varsigma), \quad \varsigma \in \partial D;$$

(2.2) $$M^t(\varsigma) \subset \widehat{M^r}(\varsigma), \quad \varsigma \in \partial D, \quad t \leq r;$$

(2.3) $$M^t(\varsigma) = \bigcap_{r>t} M^r(\varsigma).$$

Specifically, if $\varphi_1 : (\partial D)^2 \to \partial D \times \mathbb{C}$ denotes the parametrization (1.1), then there is a family of $C^{(2)}$-smooth immersions $\varphi_t : (\partial D)^2 \to \partial D \times \mathbb{C}$, such that

(2.4) $(\varsigma, \xi, t) \to \varphi_t(\varsigma, \xi) : (\partial D)^2 \times [1, R] \to \partial D \times \mathbb{C}$ is a $C^{(2)}$-regular map; every φ_t is an embedding of the form $(\varsigma, b_t(\varsigma, \xi))$, with $\frac{\partial}{\partial s} b_t(\varsigma, e^{is}) \neq 0$, $\varsigma, e^{is} \in \partial D$, and the ranges $\varphi_t(\partial D \times \partial D)$, which we denote by M^t, $1 \leq t \leq R$, satisfy (2.1) and (2.2).

Then, (2.4) implies relation (2.3).

(To construct φ_t, one can proceed as follows. If $n(\varsigma, w)$ is the outward unit normal vector to $M(\varsigma)$ at w, and $a \in C^\infty(\partial D)$ is such that $a(\varsigma) \in \widehat{M(\varsigma)} \backslash M(\varsigma)$, $\varsigma \in \partial D$, then the vector field $w \to \frac{n(\varsigma, w)}{w - a(\varsigma)} : M(\varsigma) \to \mathbb{C} \backslash \{0\}$ has a winding number zero. Extending the $C^{(1)}$ function $\frac{n(\varsigma, w)}{w - a(\varsigma)}$ to a nonvanishing one on $\partial D \times \mathbb{C}$, multiplying the extension by $w - a(\varsigma)$, and then approximating the product by some $x \in C^\infty(\partial D \times \mathbb{C})$, we can achieve that

(i) $x(\varsigma, w)$ is a transversal, outward-pointing vector, at $w \in M(\varsigma)$,

(ii) $x(\varsigma, w) \neq 0$ for $\varepsilon \leq |w - a(\varsigma)| \leq \frac{1}{\varepsilon}$.

Let g_s, $s \in R$, be the one-parameter smooth diffeomorphism group on $\partial D \times \mathbb{C}$ uniquely determined by the vector field $x(\cdot, \cdot)$, and let $\varphi_t = g_{t-1} \circ \varphi_1$, $1 \leq t \leq R$. Clearly, if $\frac{1}{\varepsilon}$ and R are large enough, properties (2.1)-(2.4) hold).

NOTATION 2.1. Similarly as in Assumptions and Notations 1.4 and in Theorem 1.1, we let for $t \in [1, R]$:

$$X^t = \bigcup_{\varsigma \in \partial D} \{\varsigma\} \times M^t(\varsigma); \quad Y^t = \widehat{X}^t = \widehat{M}^t;$$

$V^t = \{$the relative interior of Y^t in $\overline{D} \times \mathbb{C}\}$;
$S^t = \{$the relative boundary of $Y^t \cap (D \times \mathbb{C})$ in $D \times \mathbb{C}\}$.
Whenever $Z \subset \mathbb{C}^2$, we denote

$$Z(z) : = \{w \in \mathbb{C} : (z, w) \in Z\} = \{\text{the fiber over } z \in \mathbb{C}\}.$$

Note that $\widehat{M^t(\varsigma)} = X^t(\varsigma) = Y^t(\varsigma)$ if $\varsigma \in \partial D$. Finally, let

T be the greatest lower bound of $t \in [1, R]$ for which there is $g \in A(\overline{D})$

(=the class of analytic functions on D with continuous extensions to $\overline{D}$) such that

$$g(\varsigma) \in X^t(\varsigma) \backslash M^t(\varsigma) = \operatorname{Int} \widehat{M^t(\varsigma)}, \quad \varsigma \in \partial D. \tag{2.5}$$

By (2.1) and (2.4), $T < R$. For $t \in (T, R]$ the hulls Y^t have the following properties established by Forstnerič [8].

PROPOSITION 2.2. *If $T < t \leq R$, then*

(a) $Y^t \backslash X^t$ *is covered by the graphs of functions in* $H^\infty(D)$;
(b) S^t *is covered by disjoint graphs of functions in* $H^\infty(D)$;
(c) $Y^t \subset V^r$, *provided* $t < r \leq R$.

Properties (a) and (b) correspond to parts (iv) and (v) of Theorem 3 in [8]; (c) was established in [8, Section 5].

The strategy of the proof of Theorem 1.5 is to show that also M^T satisfies condition (2.5) (with some $g \in A(\overline{D})$), provided $\widehat{M}$ is nondegenerate. We first collect some properties of Y^T which follow from Proposition 2.2 by simple limit arguments.

PROPOSITION 2.3. *Let M satisfy all the assumptions of Theorem 1.5 with possible exception for nondegeneracy. Let T be as in Notation 2.1. Then,*

(i) $Y^T \subset V^t$, $\quad t > T$;
(ii) S^T *is covered by disjoint graphs of bounded analytic functions in* D;
(iii) $S^T(z) = \partial Y^T(z)$ *for* $z \in D$;
(iv) $V^T \backslash X^T$ *is covered by graphs of functions in* $H^\infty(D)$;
(v) $V^T \backslash X^T$ *is nonempty if and only if* $V^T(z)$ *is nonempty for every* $z \in D$;
(vi) *as* $t \searrow T$, *the sets* $S^t(z)$ *converge to* $S^T(z)$, $z \in D$, *relative to the Hausdorff-distance topology;*
(vii) $Y^T(z)$, $z \in \overline{D}$, *are simply-connected continua.*

PROOF. (i) If $s \in (T, t)$, then $X^T \subset X^s$ by (2.2), and so $Y^T = \widehat{X}^T \subset \widehat{X}^s = Y^s$, by the definition of the polynomial hull. By Proposition 2.2(c), $Y^s \subset V^t$. Hence, $Y^T \subset V^T$, for $t > T$.

(ii) Since $X^T = \bigcap_{t>T} X^t$, by the definition of the polynomial hull, we get

$$Y^T = \bigcap_{t>T} Y^t. \tag{2.6}$$

Let $(a,b) \in Y^T \setminus X^T$. By Proposition 2.2(a), for every $t \in (T,R]$ there is $f^t \in H^\infty(D)$ such that $f^t(a) = b$, and $\mathrm{gr}(f^t)$ (=the graph of f^t) is contained in Y^t. By the normal family argument, there is a subsequence $f^{t(n)}$, $t(n) \searrow T$, convergent uniformly on compact subsets of D to some $f \in H^\infty(D)$. Clearly, $f(a) = b$. Thus

(2.7) $\quad Y^T \setminus X^T$ is covered by the graphs of functions $f \in H^\infty(D)$.

We claim now

(2.8) $\quad$ if $(a,b) \in S^T$, $\mathrm{gr}(g) \subset Y^T$, and $g(a) = b$, then $\mathrm{gr}(g) \subset S^T$,

and such g is unique.

By (2.6), there is a sequence $(a_n, b_n) \to (a,b)$ such that $(a_n, b_n) \in S^{r(n)}$ where $r(n) \searrow T$. By Proposition 2.2(b), there are $h_n \in H^\infty(D)$, $n = 1, 2 \cdots$, such that $h_n(a_n) = b_n$ and $\mathrm{gr}(h_n) \subset S^{r(n)}$. By the normal family argument, the sequence (h_n) contains a subsequence converging to some $h \in H^\infty$, uniformly on compact subsets of D. Without loss of generality, we assume that (h_n) is already convergent. Clearly, $h(a) = b$. By part (i), $\mathrm{gr}(h_n) \cap Y^T = \emptyset$, $n = 1, 2, \cdots$, and so $(h_n - g)(z) \neq 0$ in D. Since $h_n - g \to h - g$ uniformly on compacts, and since $(h-g)(a) = 0$, Hurwitz theorem implies that $h \equiv g$ in D. This proves (2.8).

Combining (2.7) and (2.8), we obtain (ii).

(iii) If a, b, h_n are the same as in the previous part, then $h_n(a) \to b$ and $h_n(a) \notin Y^T(a)$, which implies $S^T(a) \subset \partial Y^T(a)$. The reverse inclusion is obvious, because $V^T(a)$ is open in C.

(iv) Let $(a,b) \in V^T \setminus X^T$. By (2.7), there is $f \in H^\infty(D)$ such that $f(a) = b$ and $\mathrm{gr}(f) \subset Y^T$. Since $\mathrm{gr}(f) \not\subset S^T$, $\mathrm{gr}(f)$ cannot intersect S^T, by (2.8). Hence, $\mathrm{gr}(f) \subset V^T$, which proves (iv).

(v) is obvious by (iv).

(vi) Since $Y^t(z)$, $t \geq T$, form a monotone nonincreasing family of compact sets, and $\partial Y^t(z) = S^t(z)$, for $t \geq T$ and $z \in D$, by (iii), the conclusion follows by (2.6).

(vii) By the Forstnerič's result (cf. Theorem 0.2), $Y^t(z)$, $t > T$, are Jordan regions, hence $Y^T(z)$ is connected and simply connected by (2.6). Q.E.D.

PROPOSITION 2.4. *The conditions* (i)-(iii) *of Corollary 1.3 are equivalent when* $X := X^T$.

PROOF. The implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii). We consider two cases. Case (1). $V^T \neq \emptyset$. Then $V^T(z) \neq \emptyset$ for every $z \in D$ by Proposition 2.3(v), and then $Y^T(z)$ is uncountable for every $z \in D$. Case (2). If $V^T = \emptyset$, then $Y^T \backslash X^T = S^T$. By Proposition 2.3(ii), S^T is covered by mutually disjoint analytic discs, at least two, if (iii) holds, thus each $Y^T(z)$ has more than one point. Q.E.D.

Since $\widehat{M^T} \supset \widehat{M}$, the next observation is now obvious.

COROLLARY 2.5. *If $\widehat{M}$ is nondegenerate (cf. Definition 1.2, Assumptions and Notations 1.4), then* diam $Y^T(z) > 0$ *for every* $z \in D$.

3. - Application of quasicircles

In this section we show that surfaces S^t, $t > T$, do not collapse in the limit, as $t \to T$. More specifically, we prove that $S^t(z)$, $t \geq T$, $z \in D$, are quasicircles. For the convenience of the reader we recall the definition and necessary characterizations.

A Jordan curve in the extended plane is said to be a K- quasicircle, $K \in [1, +\infty)$, if it is the image of a circle under a K- quasiconformal mapping of the extended plane onto itself. See Lehto [11, I. § 6] for further background. We will not work with the definition; instead, we will rely on the following two characterizations.

CHARACTERIZATION 3.1. (BY ARC CONDITION). A Jordan curve C in the finite plane is a K- quasicircle with some $K \in [1, +\infty)$ if and only if there is a constant $h > 0$, such that for every two distinct points $w_1, w_2 \in C$,

$$\min(\operatorname{diam} C_1, \operatorname{diam} C_2) \leq h|w_1 - w_2|, \tag{3.1}$$

where C_1, C_2 are the two arcs into which points w_1, w_2 divide curve C. (See Lehto [11, I. 6.5]).

CHARACTERIZATION 3.2. (BY CROSS RATIO). A Jordan curve C is a K-quasicircle for some K, if there is a constant $\lambda > 0$, such that for every four distinct, cyclically ordered points $w_1, w_2, w_3, w_4 \in C$,

$$|(w_2, w_1, w_3, w_4)| \leq \lambda, \tag{3.2}$$

where $(w_2, w_1, w_3, w_4) = \dfrac{(w_3 - w_2)(w_4 - w_1)}{(w_3 - w_1)(w_4 - w_2)}$. See Pommerenke [12, Lemma 9.4].

REMARK 3.3. Each of the constants K, h, λ has a uniform upper bound in terms of any of the remaining constants.

We do not have a reference for the following fact, although it is undoubtedly well known.

PROPOSITION 3.4. *Let $(C_n)_{n=1}^{\infty}$ be a sequence of K- quasicircles. Assume that they converge, relative to the Hausdorff distance, to a compact set Z. Then Z is either a point or a K^*- quasicircle, where K^* is a constant depending only on K.*

PROOF (SKETCH). Assume that Z is not a point. Then inf diam $C_n > 0$. We show first that Z is a Jordan curve. As it is well known, this holds if for every two points $a, b \in Z$, with $a \neq b$, there are two connected compact sets Z' and Z'', such that $Z' \cup Z'' = Z$, $Z' \cap Z'' = \{a, b\}$. By the convergence assumption, there are points $a_n, b_n \in C_n$ such that $a_n \to a$ and $b_n \to b$. Let C'_n, C''_n denote the two closed arcs of C_n with endpoints a_n, b_n. By the compactness of Hausdorff topology on the space of compact subset of C contained in a common closed circle, one can select convergent subsequences of C'_n, C''_n. To simplify the notation, we assume (without loss of generality) that $C'_n \to Z'$, $C''_n \to Z''$, where Z', Z'' are some compact sets. Clearly, $Z' \cup Z'' = Z$ and $a, b \in Z' \cap Z''$.

It remains to show that $Z' \cap Z'' = \{a, b\}$. Suppose there is $p \in Z' \cap Z'' \backslash \{a, b\}$. Then, there are sequences (p'_n), (p''_n), such that $p'_n \in C'_n$, $p''_n \in C''_n$, $p'_n \to p$, $p''_n \to p$ and $p'_n \neq a_n, b_n$, $p''_n \neq a_n, b_n$, for large n. The points p'_n, p''_n divide C_n into two arcs A_n, B_n, where $a_n \in A_n$, $b_n \in B_n$. By the arc condition (Characterization 3.1), min(diam A_n, diam $B_n) \leq h|p'_n - p''_n|$, where h is a common arc condition constant for all C_n, cf. Remark 3.3. This leads to a contradiction, because $|p'_n - p''_n| \to 0$, while lim diam $A_n \geq$ lim $|a_n - p'_n| \geq |a-p|$ and, likewise, lim diam $B_n \geq |b - p|$. Hence, $p = a$ or b and Z is a Jordan curve.

Finally, Z satisfies the arc condition with the same constants h as C_n do. Indeed, if a, b, Z', Z'' are as above, then, clearly, Z', Z'' are arcs and $\min(\operatorname{diam} Z', \operatorname{diam} Z'') = \lim_n \min(\operatorname{diam} C'_n, \operatorname{diam} C''_n) \leq \lim_n h|a_n - b_n| = h|a - b|$. By Remark 3.3, the proof is complete. Q.E.D.

We can now show that V^T is nonempty.

LEMMA 3.5. *Let M, M^t, M^T be as in Notation 2.1. Assume that $\widehat{M}$ is nondegenerate. Then, there is a constant K depending only on M, such that the sets $S^t(z)$, $z \in D$, $t \in [T, R]$, are K- quasicircles. In particular, $S^T(z)$, $z \in D$, are Jordan curves.*

PROOF. *Assertion 1.* There is a constant K' such that all the curves $M^t(\varsigma)$, $\varsigma \in \partial D$, $1 \leq t \leq R$, are K'- quasicircles.

This is a well-know fact (and a simple exercise) which follows from the assumptions about the existence of C^2-smooth parametrization satisfying condition (2.4). It is also a special case of more general Lemma 6.4 below.

Assertion 2. Let M' be a $C^{(2)}$-manifold satisfying the assumptions of Theorem 0.2. Assume further that the curves $M'(\varsigma)$, $\varsigma \in \partial D$, are quasicircles with a common constant λ of Characterization 3.2. Then, all the curves $S'(z)$ (=the boundary of $\{w \in \mathbb{C} : (z, w) \in \widehat{M'}\}$) are quasicircles with the same constant λ in (3.2).

(Of course, in this proof $M' = M^t$, $t > T$).

Fix $z_0 \in D$ and let w_1, w_2, w_3, w_4 be any four cyclically ordered points on the Jordan curve $S'(z_0)$. If $\Phi(z, \xi) = (z, f_\xi(z))$ is the parametrization with properties described in Theorem 0.2, let $\xi(i)$, $1 \leq i \leq 4$, be the unique points of ∂D, such that $f_{\xi(i)}(z_0) = w_i$. Denote $f_{\xi(i)} = f_i$. Clearly, $\xi(1), \xi(2), \xi(3), \xi(4)$ are cyclically ordered on ∂D and, since Φ is a homeomorphism, for every $\varsigma \in \partial D$, the points $f_1(\varsigma), f_2(\varsigma), f_3(\varsigma), f_4(\varsigma)$ are cyclically ordered on $M'(\varsigma)$. Let $h(\varsigma) = (f_2(\varsigma), f_1(\varsigma), f_3(\varsigma), f_4(\varsigma))$ (=the cross ratio (3.2)). By (3.2), $|h(\varsigma)| \leq \lambda$ for $\varsigma \in \partial D$. Since $h \in A(\overline{D})$, and $h(z_0) = (w_2, w_1, w_3, w_4)$, by the maximum principle the condition (3.2) holds for $S'(z_0)$. This proves Assertion 2.

By Assertion 1 and Assertion 2, applied to $M' = M^t$, $t \in (T, R]$, we obtain now that all the curves $S^t(z)$, $z \in D$, $t \in (T, R]$, are K- quasicircles with some uniform constant K (cf. Remark 3.3). By Proposition 3.4, also $S^T(z)$, $z \in D$, are quasicircles, with uniform, but possibly larger, constant K^*. Q.E.D.

4. - Application of uniform holomorphic contractibility

As we already know, S^T is covered by graphs of bounded analytic functions in D. We have to show that they have continuous (or better) boundary values. It suffices to prove (by a result of Čirka [6]), that their cluster values are contained in M^T. To this end we will establish in this section (Lemma 4.4) that $\overline{S^T} \backslash S^T = M^T$, which we do with the help of the notion of uniform transversal holomorphic contractibility introduced by Helton and Howe [9].

DEFINITION 4.1. [9], [10]. A family of compact (plane) sets $W(\varsigma)$, $\varsigma \in \partial D$, is said to be *uniformly transversally holomorphically contractible* (shortly: UTHC) if there is a domain $W \supset \bigcup_\varsigma W(\varsigma)$, a continuous function $v : \partial D \times W \to \mathbb{C}$, and constants $\gamma, \alpha > 0$, such that for every $\varsigma \in \partial D$ the function $w \to v(\varsigma, w)$ is holomorphic and

$$\min\{|w + tv(\varsigma, w) - y| : w \in W(\varsigma), y \notin \operatorname{Int} W(\varsigma)\} \geq t\gamma \tag{4.1}$$

for every $\varsigma \in \partial D$ and for every $0 < t \leq \alpha$.

REMARK 4.2. If UTHC holds, then we can assume that there exists a bounded *holomorphic* function $v : \{r_1 < |z| < r_2\} \times W \to \mathbb{C}$, where $r_1 < 1 < r_2$, such that (4.1) still holds (for $\varsigma \in \partial D$), with possibly smaller α, γ, W.

It is this seemingly stronger version of UTHC that we well apply. (To prove Remark 4.2, note that any function $V'(\varsigma, w)$, such that $|v - V'| \leq \varepsilon$ on $\partial D \times W$, also satisfies condition (4.1) with constants $\gamma - \varepsilon, \alpha$. By applying Runge theorem in w and smooth partition of unity on $|\varsigma| = 1$, we find first V' equal to a finite sum of products of rational functions in w multiplied by smooth functions of ς. Approximating those smooth coefficients by rational functions of z, we get Remark 4.2).

REMARK 4.3. It is a special case of Theorem 2.2 of Helton et al. [10], that if M is like in Theorem 1.5, then the family of Jordan regions $\{\widehat{M(\varsigma)}\}_{\varsigma\in\partial D}$ (and in the same way $\{\widehat{M^t(\varsigma)}\}_{\varsigma\in\partial D}$ for $1 \leq t \leq R$) has the UTHC property. (The result of Helton et al. [10] is actually more general in that the parametrization $b(\varsigma, \xi)$ is only assumed to be continuous in ς, and $\frac{\partial^2 b}{\partial s^2}(\varsigma, e^{is}) \in L^\infty \cap (L^\infty)^{-1}$. A still more general theorem giving a sufficient and necessary condition for UTHC is obtained below, cf. Theorem 6.3).

With later application in mind, we prove the next lemma in greater generality than needed in Secion 5.

LEMMA 4.4. *In the Notation 2.1, if $\widehat{M}$ is nondegenerate, then*

$$V^T \supset X^T \backslash M^T \text{ and } \overline{S^T} \backslash S^T \subset M^T. \tag{4.2}$$

More generally, let $X^*(\varsigma)$, $\varsigma \in \partial D$, be a family of compact Jordan domains satisfying UTHC. Denote $X^* = \{(\varsigma, w) : |\varsigma| = 1,\ w \in X^*(\varsigma)\}$, $Y^* = \widehat{X}^*$, and $V^* =$ the relative interior of Y^* in $\overline{D} \times \mathbb{C}^*$. Assume that X^* is compact and $\{(\varsigma, w) : |\varsigma| = 1,\ w \in \operatorname{Int} X^*(\varsigma)\}$ is an open subset of $\partial D \times \mathbb{C}$. Assume further that $\overline{V^*(z)} = Y^*(z)$ for $z \in D$, and there is $h \in H^\infty(D)$, such that $\operatorname{gr}(h) \subset V^*$. Then

$$V^*(\varsigma) \supset X^*(\varsigma) \backslash \partial X^*(\varsigma), \ \varsigma \in \partial D. \tag{4.3}$$

For the proof, we need an auxiliary fact.

NOTATION 4.5. For a bounded open domain G in C, let $A(G)$ denote the uniform closure on $\overline{G}$ of rational functions with poles outside $\overline{G}$. Denote by $A(G \times \mathbb{C})$ the closure, with respect to the topology of uniform convergence on compact subsets of $\overline{G} \times \mathbb{C}$, of the sums

$$\sum_{i=1}^{N} f_i(z) w^i, \text{ where } f_i \in A(G),\ N = 1, 2, \cdots.$$

LEMMA 4.6. *Let G be as in Notation 4.5. Let H be a bounded, relatively open subset of $\partial G \times \mathbb{C}$, such that all its fibers $H(\varsigma)$, $\varsigma \in \partial G$, are connected and simply connected. Assume that each point $\varsigma \in \partial G$ is a peak point for the algebra $A(G)$, and that there is a function $f_0 \in A(G)$, such that*

$$f_0(\varsigma) \in H(\varsigma), \ \varsigma \in \partial G. \tag{4.4}$$

Let F be the hull of $\overline{H}$ relative to the algebra $A(G \times \mathbb{C})$. Then, $H \subset U := \operatorname{Int} F$.

The proof is delayed to the Appendix A.

PROOF OF LEMMA 4.4. Observe first that the special case (4.2) follows from the general case.

Indeed, if we let $X^* := X^T$, then $Y^* = Y^T$, $V^* = V^T$, and then UTHC holds by Remark 4.3. By Lemma 3.5, $S^T(z)$, $z \in D$, are Jordan curves and so, by Proposition 2.3(iii), $Y^T(z) = \overline{V^T(z)}$, $z \in D$. This implies also that $V^T \backslash X^T$ is nonempty, and so, by Proposition 2.3(iv), there is $h \in H^\infty(D)$ with $\mathrm{gr}(h) \subset V^T$. The remaining assumptions follow from the set-up of Notation 2.1. Then, (4.3) implies that $V^T \supset X^T \backslash M^T$. Since $S^T \cap V^T = \emptyset$, also $\overline{S^T} \cap V^T = \emptyset$, and so $\overline{S^T} \backslash S^T \subset M^T$, which confirms (4.2).

We will consider now the general case. Let $v : W \times \{r_1 < |z| < r_2\} \to \mathbb{C}$ be a holomorphic function satisfying all conditions of Definition 4.1 and Remark 4.2, in particular (4.1), with $W(\varsigma) = X^*(\varsigma)$.

Fix $r \in (r_1, 1)$. Choose a positive number d, such that

$$0 < d < \min\{|h(z) - y| : |z| = r, \ (z, y) \notin V^*\}. \tag{4.5}$$

Let $h_t(z) = h(z) + tv(z, h(z))$, for $r_1 < |z| < 1$, $t > 0$. Clearly, $h_t(\cdot)$ are bounded holomorphic functions. Denote by $Cl(h, \varsigma)$, $|\varsigma| = 1$, the set of all cluster values of h at ς, i.e. $w \in Cl(h, \varsigma)$, if there is a sequence (z_n), $r_1 < |z_n| < 1$, $z_n \to \varsigma$, such that $h(z_n) \to w$. An elementary compactness argument yields for every t:

$$Cl(h_t, \varsigma) = \{w = w^* + tv(\varsigma, w^*) : w^* \in Cl(h, \varsigma)\}, \ \text{for } |\varsigma| = 1. \tag{4.6}$$

Let $C = \sup|v|$. Then, by (4.5),

$$(z, h_t(z)) \in V^*, \ \text{for } |z| = r \text{ and } 0 < t < \frac{d}{C}. \tag{4.7}$$

Combining cluster formula (4.6) with property (4.1), we get, for every $0 < t \le \alpha$, the inclusion:

$$\{(\varsigma, w) : \varsigma \in \partial D, \ w \in Cl(h_t, \varsigma)\} \subset \{(\varsigma, w) : \varsigma \in \partial D, \ w \in \mathrm{Int}\ X^*(\varsigma)\}.$$

Since the latter set is open in $\partial D \times \mathbb{C}$ (by assumption), and the former is compact, there is $\rho_0 < 1$ (depending on t), such that

$$h_t(\rho\varsigma) \in \mathrm{Int}\ X^*(\varsigma), \ \varsigma \in \partial D, \ \text{if } \rho_0 \le \rho < 1. \tag{4.8}$$

If we fix $t \in \{0, \min(\frac{d}{C}, \alpha)\}$, then we can choose, in view of (4.7), a number $\rho \in (\rho_0, 1)$, such that $(z, h_t(\rho z)) \in V^*$, whenever $|z| = r$. Letting $f_0(z) = h_t(\rho z)$, we can summarize the above argument as follows.

Assertion 1. There is $r \in (0, 1)$ and a function $f_0 \in A(\overline{G})$, where $G = \{z \in \mathbb{C} : r < |z| < 1\}$, such that

$$f_0(\varsigma) \in \mathrm{Int}\ X^*(\varsigma), \ \varsigma \in \partial D, \ \text{and } (z, f_0(z)) \in V^*, |z| = r.$$

We will apply now Lemma 4.6 with G and f_0 as in Assertion 1, and with

$$H := \{(z,w) : |z| = r,\ (z,w) \in V^*\} \cup \{(\varsigma, w) : |\varsigma| = 1, w \in \text{Int } X^*(\varsigma)\}. \tag{4.9}$$

The assumptions of Lemma 4.6 are fulfilled by Assertion 1 and so, if $F = \{\text{the hull of } \overline{H} \text{ relative to } A(G \times \mathbb{C})\}$ and $U = \text{Int } F$, relative to $\overline{G} \times \mathbb{C}$, then $U \supset H$, in particular,

$$U \supset \text{Int } X^* (\text{relative to } \partial D \times \mathbb{C}). \tag{4.10}$$

Observe now some further properties of the hull F. By the Rossi's local maximum modulus principle, the set $F \backslash \overline{H}$ has the local maximum property. (In general, a locally closed set $Y \subset \mathbb{C}^n$ is said to have the local maximum property, if for every compact set K in $\mathbb{C}^n$, such that $K \cap Y$ is compact, and for every polynomial $p(z)$, $\max_{K \cap Y} |p| \le \max_{(\partial K) \cap Y} |p|$). Recall that, by assumptions, $\overline{V^*(z)} = Y^*(z)$ for $|z| = r$; in particular, $\overline{V^*(z)} = \overline{H(z)}$ is polynomially convex. This and the definition of the $A(G \times \mathbb{C})$-hull imply, in the standard fashion, that

$$F \cap (\partial G \times \mathbb{C}) = Y^* \cap (\partial G \times \mathbb{C}). \tag{4.11}$$

(See [14, Proof of Lemma 5(i)] for a practically identical argument).

Let now

$$Z = (F \backslash \overline{H}) \cup (Y^* \backslash X^*). \tag{4.12}$$

By (4.11), Z is closed in $D \times \mathbb{C}$ and

$$\overline{Z} \backslash Z \subset X^*. \tag{4.13}$$

Assertion 2. The set Z has a local maximum property with respect to polynomials.

By the already mentioned Rossi's principle, the set $Y^* \backslash X^* = \widehat{X}^* \backslash X^*$ has a local maximum property. Now, by (4.12), Z is the union of two sets with a local maximum property and is locally closed (i.e. relatively open in its closure). According to [13, Proposition 3.5(b)], Z must have a local maximum property as well.

By Assertion 2 and (4.13), $Z \subset Y^*$. Consequently, $Y^* \supset F \supset U$, and so $V^* \supset U$. Then, by (4.10), $V^* \supset \text{Int } X^*$ (relative to $\partial D \times \mathbb{C}$), i.e. (4.3) holds.

Q.E.D.

REMARK 4.7. There is a smooth function $a : \overline{D} \to \mathbb{C}$, such that $a(z) \in V^T(z)$ for every $z \in \overline{D}$. This is a direct consequence of the last proof. Clearly, if $\delta > 0$ is small enough, then $r - \delta > r_1$, $r + \delta < 1$, and the segment joining $h_t(\rho z)$, $h(z)$ is contained in $V^T(z)$ for $r - \delta \le |z| \le r + \delta$. Let φ_1, φ_2 be smooth functions on $\overline{D}$, such that $\varphi_j \ge 0$, $\varphi_1 + \varphi_2 \equiv 1$, and $\varphi_1(z) = 0$ for $|z| \le r - \delta$, $\varphi_2(z) = 0$ for $|z| \ge r + \delta$. Then, $a(z) = \varphi_1(z) h_t(\rho z) + \varphi_2(z) h(z)$ has the required properties.

5. - Proofs of main results

PROOF OF THEOREM 1.5. Pick, by Proposition 2.3(ii), a function $f \in H^\infty(D)$, such that $\mathrm{gr}(f) \subset S^T$. Since $\overline{S^T}\backslash S^T \subset M^T$ by Lemma 4.4, we obtain that $\overline{\mathrm{gr}(f)}\backslash \mathrm{gr}(f) \subset M^T$. Since M^T is a totally real $C^{(2)}$- smooth manifold in $\mathbb{C}^2$, we are in the situation of Theorem 33 in Čirka [6], which implies that $f \in C^{1+\alpha}(\overline{D})$, for every $\alpha < 1$. Let $n(\varsigma, w)$ denote the inward unit normal vector at w for the $C^{(2)}$- smooth Jordan curve $M^T(\varsigma)$. Then, $n : M^T \to \mathbb{C}$ is $C^{(1)}$- smooth. Let $h(\varsigma) = n(\varsigma, f(\varsigma))$. Then, $h : \partial D \to \mathbb{C}\backslash\{0\}$ is $C^{(1)}$- smooth.

Assertion 1. The winding number of h is zero.

We sketch an argument for this intuitively obvious statement. Choose a constant $\beta > 0$ such that, if we let $f_0(\varsigma) := f(\varsigma) + \beta h(\varsigma)$, $\varsigma \in \partial D$, then $f_0(\varsigma) \in V^T(\varsigma) = X^T(\varsigma)\backslash M^T(\varsigma)$ for $\varsigma \in \partial D$. Let $a \in C^\infty(\overline{D})$ be the function from Remark 4.7. We claim that there is a continuous homotopy $\{F_t\}_{0\le t\le 1}$, such that $F_t \in C(\partial D)$, $F_t(\varsigma) \in V^T(\varsigma)$ for $0 \le t \le 1$, $\varsigma \in \partial D$, and $F_0 = f_0$, $F_1 = a|_{\partial D}$. (One way to see this is by using the well-known fact that the open set $\{(\varsigma, w) : |\varsigma| = 1,\ w \in V^T(\varsigma)\} \subset \partial D \times \mathbb{C}$ is homeomorphic with $\partial D \times \{|w| < 1\}$ by a homeomorphism which preserves fibers over $\varsigma \in \partial D$. This follows, for example, from the Riemann mapping theorem, as in Proposition 7.4). Since $F_t - f$ do not vanish on ∂D, $0 \le t \le 1$, functions $\beta h = F_0 - f|_{\partial D}$ and $F_1 - f|_{\partial D}$ have the same winding number. The latter function has continuous non-vanishing extensions to $\overline{D}$, namely $a - f$, which proves Assertion 1.

The next, well-known, observation was used by Forstnerič in a similar context [7, Proposition 3.1].

Assertion 2. If $h : \partial B \to \mathbb{C}\backslash\{0\}$ belongs to the Hölder class C^ρ for some $\rho > 0$ and has a winding number zero, then there is a positive function $r : \partial D \to (0, +\infty)$, such that the function $r(\varsigma)h(\varsigma)$ has a continuous analytic and non-vanishing extension $h_1 : \overline{D} \to \mathbb{C}$.

Since $h_1(\varsigma)$ is an inward normal vector to $M^T(\varsigma)$ at $f(\varsigma)$, there is a positive constant $\varepsilon > 0$, such that $f(\varsigma) + \varepsilon h_1(\varsigma) \in V^T(\varsigma)$, for $\varsigma \in \partial D$. Thus, the function $g = f + \varepsilon h_1$, which belongs to $A(\overline{D})$, satisfies condition (2.5) with $t = T$.

This contradicts the definition of T (cf. Notation 2.1) unless $T = 1$. Hence, $g(\varsigma) \in \widehat{M(\varsigma)}\backslash M(\varsigma)$, $\varsigma \in \partial D$, which proves (1.3). The remaining statements of Theorem 1.5 follow from Lemma 4.4, Proposition 2.3, and the Forstnerič's result (Theorem 0.2). Q.E.D.

PROOF OF THEOREM 1.1 (SKETCH). We leave without a proof the following claim, intuitively obvious: there exist manifolds $M_n, n = 1, 2, \cdots$, as in Assumptions and Notations 1.4 and Theorem 1.5, such that

$$\widehat{M_n(\varsigma)}\backslash M_n(\varsigma) \supset \widehat{M_{n+1}(\varsigma)},\ n = 1, 2, \cdots,\ \varsigma \in \partial D, \tag{5.1}$$

$$\bigcap_{n=1} \widehat{M_n(\varsigma)} = X(\varsigma),\ \varsigma \in \partial D. \tag{5.2}$$

With $X_n = \{(\varsigma, w) : \varsigma \in \partial D, \ w \in \widehat{M_n(\varsigma)}\}$, the meaning of the symbols Y_n, V_n, S_n is in accordance with Assumptions and Notations 1.4. The following relation follows from the definition of the polynomial hull:

$$Y = \bigcap_{n=1}^{\infty} Y_n. \tag{5.3}$$

It is now fairly clear that the properties of Y, V, S required in Theorem 1.1 follow from those of Y_n, V_n, S_n (which hold by Theorem 1.5), in practically the same way as those of Y^T, V^T, S^T were obtained from properties of Y^t, V^t, S^t, $t > T$, in the proof of Proposition 2.3. The difference is that now the family M_n is not continuous (in n), but (5.3) replaces it sufficiently. Furthermore, ∂X (relative to $\partial D \times \mathbb{C}$) is not a manifold as in Proposition 2.3, but this property was not used in the proof of Proposition 2.3 at all. The only relation which might require explanation is

$$Y_{n+1} \subset V_n \tag{5.4}$$

which replaces (c) of Proposition 2.2. To see this, apply observation (2.8) with $S^T = S_n$. (Y_{n+1} is covered by analytic discs with boundaries in $X_{n+1} \subset V_n$, and so, by (2.8), these discs are disjoint with S_n). Q.E.D.

PROOF OF COROLLARY 1.3. The Corollary 1.3 follows from Theorem 1.1 in the same way as did Proposition 2.4 from Proposition 2.3. Q.E.D.

6. - Holomorphic contractibility for families of Lipschitz domains

We will return to properties of polynomial hulls in Sections 7 and 8. As a preparation, we generalize now, to families of Jordan domains whose boundaries are locally graphs of Lipschitz functions (in some uniform fashion), the theorem of Helton et al. [10], referred to in Remark 4.3.

DEFINITION 6.1. We say that the compact Jordan regions $W(\varsigma), \varsigma \in \partial D$, satisfy the continuous cone condition (shortly: CCC), if there are numbers $\alpha \in (0, \frac{\pi}{2})$ and $\beta > 0$, and a continuous function $x(\varsigma, w)$, $\varsigma \in \partial D$, $w \in \partial W(\varsigma)$, such that $|x(\varsigma, w)| = \beta$ and

$$C(\varsigma, w) \subset \operatorname{Int} W(\varsigma) \cup \{w\}, \tag{6.1}$$

where

$$C(\varsigma, w) = \left\{ u \in \mathbb{C} : |u - w| \leq \beta, \left| \frac{\operatorname{Arg}\ (u - w)}{x(\varsigma, w)} \right| \leq \alpha \right\}. \tag{6.2}$$

To avoid cumbersome arguments, we assume also that

$$\{(\varsigma, w) : |\varsigma| = 1, \ w \in W(\varsigma)\} \text{ is compact;} \tag{6.3}$$

$$\{(\varsigma, w) : |\varsigma| = 1,\ w \in \operatorname{Int} W(\varsigma)\} \text{ is open in } \partial D \times \mathbb{C}. \tag{6.4}$$

REMARK 6.2. One can show (by the arguments akin to those of the proof of Lemma 6.4) that CCC is equivalent to a suitable uniform Lipschitz property of the curves $\partial W(\varsigma)$, $\varsigma \in \partial D$. As we do not need the result, we omit further details.

An example of CCC is provided by Proposition 8.2 below.

THEOREM 6.3. *If a family of Jordan domains $W(\varsigma)$, $|\varsigma| = 1$, satisfies the continuous cone condition (cf. Definition 6.1), then it has the property of uniform transversal holomorphic contractibility (cf. Definition 4.1).*

Actually, the CCC is not only sufficient for UTHC but it is also necessary, as it is clear from looking at (4.1). The proof of the theorem is based on the next lemma.

LEMMA 6.4. *If Jordan domains $W(\varsigma)$, $|\varsigma| = 1$ satisfy CCC, then there is $K < +\infty$, such that all the Jordan curves $\partial\, W(\varsigma)$, $|\varsigma| = 1$, are K- quasicircles.*

PROOF. If we replace $x(\varsigma, w)$ in (6.2) by its (close enough) uniform approximation $X'(\varsigma, w)$ and replace α, β by sufficiently smaller positive numbers, the new $C'(\varsigma, w)$ defined in terms of x', α', β' will satisfy (6.1). Thus, the following holds.

Assertion 1. Without loss of generality, one can assume that function $x(\varsigma, w)$ in Definition 6.1 has smooth extension to $\partial D \times \mathbb{C}$, in particular, that there is $L < +\infty$, such that

$$|x(\varsigma, a) - x(\varsigma, b)| \le L|a - b|,\quad a, b \in W(\varsigma),\ |\varsigma| = 1. \tag{6.5}$$

We will work for a while with a single curve $\partial W(\varsigma)$, but our estimates will be uniform. Denote $C(w) := C(\varsigma, w)$, $x(w) := x(\varsigma, w)$. Fix points $a, b \in \partial\, W(\varsigma)$ and note the following simple geometric relations.

(6.6) If φ denotes the absolute angle between vectors $x(a)$ and $x(b)$, then $\varphi \le \beta^{-1}\pi|x(a) - x(b)|$.

(6.7) The straight-line sides of $C(a), C(b)$ intersect, provided $\dfrac{|a - b|}{\tan(\alpha - \varphi/2)} < \beta$.

In case (6.7) holds, denote the bounded component of the complement of the union $C(a) \cup C(b) \cup [a, b]$ by Δ. Then

$$\operatorname{diam} \Delta < \frac{|a - b|}{\tan\,(\alpha - \varphi/2)}. \tag{6.8}$$

Choose now $\delta_0 > 0$, so that $\pi\beta^{-1}\delta_0 < \frac{1}{2}\alpha, \frac{1}{2}\pi - \alpha$ and $\dfrac{\delta_0}{\tan\,\alpha/2} < \beta$. Then, if $|a - b| < \delta_0$, these relations force $\varphi < \frac{1}{2}\alpha$, $\varphi + \alpha < \frac{1}{2}\pi$, and the sides of $C(a)$, $C(b)$ to intersect.

The points a, b split $\partial W(\varsigma)$ into two closed arcs. If one of them intersects the open segment (a, b), call this arc γ. If both of them are disjoint from this straight-line segment, call γ the one with the property that Δ is contained in the unbounded open component of $\mathbb{C}\backslash(\gamma \cup [a, b])$.

Consider now a point c traversing the open arc $\gamma\backslash\{a, b\}$. Then either $c \in \Delta$, or $C(c) \cap [a, b] \neq \emptyset$, but still $a, b \notin C(c)$, by (6.1). In the first case, $|c - a| \leq \frac{|a-b|}{\tan \alpha/2}$, by (6.8). In the second case, a similar geometric argument shows that

$$|c - a| \leq \frac{|a-b|}{\tan\left(\alpha - \frac{\varphi}{2}\right)} \leq \frac{|a-b|}{\tan \alpha/2}.$$

We conclude that:

(6.9) there is $\delta_0 > 0$ and $h_0 \left(= \frac{1}{\tan\frac{\alpha}{2}}\right)$, such that every arc $\gamma \subset \partial W(\varsigma), |\varsigma| = 1$, with diam $\gamma \leq \delta_0$, satisfies arc condition $\frac{|a-c|}{|a-b|} \leq h_0$, whenever $a, b, c \in \gamma$, with b between a and c.

As the curves $\partial W(\varsigma)$, $|\varsigma| = 1$, have jointly bounded diameter, condition (6.9) implies the usual arc condition (Characterization 3.1). Q.E.D.

PROOF OF THEOREM 6.3. We need the following fact, presumably well-known.

Assertion 1. Let W be a compact Jordan domain and $w \to x(w) : \partial W \to \mathbb{C}\backslash\{0\}$ be a continuous function, such that $w + x(w) \in \operatorname{Int} W$ for every $w \in \partial W$. Then, $x(\cdot)$ has a winding number one.

Let $\Psi_t : W \to \mathbb{C}$, $0 \leq t \leq 1$, be a continuous family of embedings such that $\Psi_0 = \mathrm{Id}_W$, $\Psi_1(W) = D$. (We take for granted the existence of such a family). For every $t \in [0, 1]$, $w \to \Psi_t(w + x(w)) - \Psi_t(w)$ maps ∂W into $\mathbb{C}\backslash\{0\}$, and so all these maps have the same winding number. Since $\overline{D}$ is convex, it is clear that $w \to \Psi_1(w + x(w)) - \Psi_1(w)$ is homotopic, via maps $\partial W \to \mathbb{C}\backslash\{0\}$ with the map $w \to N(\Psi_1(w))$, where $N(\Psi_1(w))$ is the (inward) unit normal vector to ∂D at $\Psi_1(w)$, that is $-\frac{\Psi_1(w)}{|\Psi_1(w)|}$. This map has a winding under one, and so the same holds for $x(w) = \Psi_0(x(w) + w) - \Psi_0(w)$.

Assertion 2. For a given $\varsigma \in \partial D$, there is a polynomial $p(w)$, such that

(6.10) $$w + p(w) \in \operatorname{Int} C(\varsigma, w), \text{ for every } w \in \partial W(\varsigma).$$

Let $g : \overline{D} \to W(\varsigma)$ be a homeomorphism which is conformal in D. Since $\partial W(\varsigma)$ is a quasicircle (by Lemma 6.4), g has a quasiconformal extension to $\mathbb{C}$, cf. [12, Theorem 9.14(ii)], and so is Hölder-continuous, cf. [12, Eq. 9.12(10), p. 288]. By Assertion 1, Proof of Lemma 6.4, assume that $x(w) = x(\varsigma, w)$ is smooth in w (and is defined in $\mathbb{C}$). Then, the composition $h_0(\xi) = x(g(\xi))$ is Hölder-continuous and $h_0 : \partial D \to \mathbb{C}\backslash\{0\}$. Seeing that g is orientation-preserving, h_0 has a winding number one by Assertion 1. Then, the function

$h(\xi) = \xi^{-1}h_0(\xi)$, $h : \partial D \to \mathbb{C}\backslash\{0\}$ has a winding number zero and is Hölder-continuous. By the already applied Assertion 2 in Section 5, Proof of Theorem 1.5, there is a positive continuous function r or ∂D, such that the product $r(\xi)h(\xi)$ has a non-vanishing extension $h_1 \in A(\overline{D})$. By all this,

$$r(g^{-1}(w))x(w) = g^{-1}(w)h_1(g^{-1}(w)), \quad w \in \partial W(\varsigma). \tag{6.11}$$

Since $r(g^{-1}(w))$ is positive and continuous, we can choose $\varepsilon > 0$, so that, by (6.2), $w + \varepsilon g^{-1}(w)h_1(g^{-1}(w)) \in \text{Int } C(\varsigma, w)$, for $w \in \partial W(\varsigma)$. The function $\varepsilon g^{-1}(w)h_1(g^{-1}(w))$ is continuous on $W(\varsigma)$ and is analytic in Int $W(\varsigma)$. By Mergelyan theorem, it can be approximated uniformly on $W(\varsigma)$ by polynomials; if $p(\varsigma)$ is an approximation close enough, then (6.10) holds.

We will construct now the function $v(\varsigma, w)$ required by Definition 4.1. For every ς, let $p_\varsigma(w)$ denote a polynomial satisfying Assertion 2. Since $x(\varsigma, w)$ is a continuous function (on $\partial D \times \mathbb{C}$), by the compactness of $\{(\varsigma, w) : \varsigma \in \partial D,\ w \in \partial W(\varsigma)\}$, we can cover ∂D by finite number of open arcs $\gamma_1, \cdots, \gamma_n$ with midpoints $\varsigma(1), \cdots, \varsigma(n)$, so that

$$w + p_{\varsigma(i)}(w) \in \text{Int } C(\varsigma, w), \quad w \in \partial W(\varsigma),\ \varsigma \in \gamma_i,\ i = 1, \cdots, n. \tag{6.12}$$

Let $\{\varphi_1, \cdots, \varphi_n\}$ be a partition of unity, with $\varphi_1, \cdots, \varphi_n \geq 0$, subordinated to the covering $\{\gamma_1, \cdots, \gamma_n\}$. Let $v(\varsigma, w) = \sum_{i=1}^{n} \varphi_i(\varsigma)p_{\varsigma(i)}(w)$. By the convexity of $C(\varsigma, w)$ and (6.12),

$$w + tv(\varsigma, w) \in \text{Int } C(\varsigma, w), \quad \text{for } t \in (0, 1] \text{ and } w \in \partial W(\varsigma). \tag{6.13}$$

Clearly, $w \to v(\varsigma, w)$ is a polynomial in w for each ς. By (6.13), $w+v(\varsigma, w)t$ maps $\partial W(\varsigma)$ into Int $W(\varsigma)$, cf. (6.1), and so maps $W(\varsigma)$ into Int $W(\varsigma)$ by the open range theorem. Finally, (4.1) follows easily from (6.13). Q.E.D.

7. - Topological boundary of the polynomial hull

The next two theorems provide, among other observations, sufficient conditions for the interior of the hull to be non-empty.

THEOREM 7.1. *Let M (and X, Y, S, V) be as in Assumptions and Notations 1.4. Assume, further, that $\widehat{M}$ is nondegenerate and that all curves $M(\varsigma)$, $|\varsigma| = 1$, are K- quasicircles with a common $K < +\infty$. Then, V is nonempty, $S(z)$, $z \in D$, are K^*- quasicircles with a common $K^* < +\infty$, and S is a topological surface in $D \times \mathbb{C}$, homeomorphic with $D \times \{|\xi| = 1\}$ by a mapping preserving fibers over $z \in D$.*

THEOREM 7.2. *Let M be as in Assumptions and Notations 1.4. Assume that $\widehat{M}$ is nondegenerate. If, in addition, the family of Jordan regions $\widehat{M(\varsigma)}$, $|\varsigma| = 1$,* *satisfies the continuous cone condition (Definition 6.1), then*

$$\overline{S}\backslash S = M,$$

and the topological boundary of $Y = \widehat{M}$ *is equal to* $X \cup (M \cup S)$, *is a topological surface, and there exists a fiber-preserving homeomorphism of* $\overline{D} \times \{|\xi| = 1\}$ *onto* $S \cup M$.

We prove first a generalization of Lemma 3.5.

LEMMA 7.3. *Under assumptions of Theorem 7.1, sets* $S(z)$, $z \in D$, *are* K- *quasicircles with a uniform constant.*

PROOF. *Assertion.* There is a sequence of $C^{(2)}$-regular manifolds M_n, $n = 1, 2, \cdots$, satisfying assumptions of Theorem 1.5, such that

$$M(\varsigma) = \bigcap_{\infty} M_n(\varsigma), \tag{7.1}$$

$$\operatorname{Int} M_n(\varsigma) \supset M_{n+1}(\varsigma), \quad \varsigma \in \partial D, \quad n = 1, 2, \cdots, \tag{7.2}$$

and the Jordan curves $M_n(\varsigma)$, $\varsigma \in \partial D$, $n = 1, 2, \cdots$, are K- quasicircles with a uniform constant K.

We take the assertion for granted in the rest of this proof. (An elementary argument can be given along the following lines. Consider a fixed division of $\mathbb{C}$ into the net of squares of size ε. A covering of $\widehat{M(\varsigma)}$ by such squares produces a quasicircle $X'_\varepsilon(\varsigma)$ with a good constant. By selecting points $\varsigma_1, \cdots, \varsigma_n$ in $\partial D \cong [0, 2\pi)$ closely enough, we can now construct a polyhedron X_ε in $\partial D \times \mathbb{C}$, without vertical faces, with $X_\varepsilon(\varsigma_i) = X'_\varepsilon(\varsigma_i)$, and intermediate sections $X_\varepsilon(\varsigma)$, $\varsigma \in (\varsigma_i, \ \varsigma_{i+1})$ obtained by suitable interpolation. Smoothing the polyhedron X_ε slightly at the edges and vertices, we obtain the required approximation M_ε. The details are tedious and are omitted. For a reader who is not satisfied by this argument, we sketch another proof of Lemma 7.3 in Appendix B).

We apply now Assertion 2 of the Proof of Lemma 3.5 to $M' = M_n$, where M_n are as in the assertion above. We conclude that all the curves $S_n(z)$, $z \in D$, $n = 1, 2, \cdots$, are quasicircles with a uniform constant independent of z, n. Recall that, as usual, $S_n(z) = \partial Y_n(z)$, where $Y_n = \widehat{M}_n$. As noted earlier, cf. (5.3), $Y = \widehat{M} = \bigcap_n Y_n$ and, in the same way as in Proportion 2.3(vi), $S_n(z)$ converge to $S(z) = \partial Y(z)$, as $n \to \infty$, for $z \in D$. Since $\widehat{M}$ is nondegenerate (by assumption), diam $S(z) > 0$ by Corollary 1.3, and so, by Proposition 3.4, all $S(z)$, $z \in D$, are quasicircles with a common constant K. Q.E.D.

In the next proposition we summarize classical results, of Caratheodory and others, on the Riemann mapping, in a form convenient for our applications.

PROPOSITION 7.4. *Let* H *be a locally compact metric space. Let* $E(h)$, $h \in H$, *be a family of compact Jordan regions, such that for every compact subset*

$H_0 \subset H$

(7.3) *the set* $\{(h,w) : h \in H_0,\ w \in \partial E(h)\}$ is compact.

Assume that the sets $E(h)$, $h \in H_0$, *are uniformly locally connected, whenever* H_0 *is compact. Denote* $\Delta = \{u \in \overline{\mathbb{C}} | u| > 1\}$. *Let* $g_h : \Delta \to \mathbb{C}\backslash E(h)$ *be the unique Riemann mapping normalized by the conditions:* $g_h(\infty) = \infty$, $g_h(u) = bu + \sum_k b_k u^{-k}$ *with* $b > 0$, *Then, the map*

$$(h,u) \to (h, g_h(u)) : H \times \Delta \to \{(h,w) : h \in H,\ w \notin E(h)\} \tag{7.4}$$

can be extended to a homeomorphism

$$H \times \overline{\Delta} \to \{(h,w) : h \in H,\ w \notin \operatorname{Int} E(h)\}, \tag{7.5}$$

which maps $H \times \partial\Delta$ *onto* $\{(h,w) : h \in H,\ w \in \partial E(h)\}$.

EXPLANATION. First observe that the map (7.4) is continuous, i.e. if $h(n) \to h$ in H, then $g_{h(n)} \to g_h$ locally uniformly in Δ. According to the *Caratheodory kernel theorem*, cf. [12, Theorem 1.8], $g_{h(n)} \to g_h$, if the sequence of domains $g_{h(n)}(\Delta) = \overline{\mathbb{C}}\backslash E(h(n)) =: F_n$ converges to its kernel, and this kernel is $g_h(\Delta) = \overline{\mathbb{C}}\backslash E(h) =: F$. By Exercise 3 in [12, Section 1.4, p. 31], (F_n) converges to F, if (i) every compact subset of F is contained in F_n for large n, and (ii) for every $w \in \partial F$, there exist points $w_n \in \partial F_n$, such that $w_n \to w$. Now, (i) follows from (7.3), while (ii) holds because of (7.3) and the fact that ∂F_n are (closed) Jordan curves.

Of course, g_h has an extension to a homeomorphism $g_h^* : \overline{\Delta} \to \overline{\mathbb{C}}\backslash \operatorname{Int} E(h)$. Finally, the content of [12, Theorem 9.11] is that $g_{h(n)}^* \to g_h^*$ *uniformly* on $\overline{\Delta}$, provided the sequence $(E(h(n)))$ is uniformly locally connected, which we have assumed.

Recall that the sets $(E(h))_{h \in H_0}$ are uniformly locally connected, cf. [12, Section 9.3.1], if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for every $h \in H_0$ and for every a, $b \in E(h)$, such that $|a-b| < \delta$, there are connected compact sets with $a, b \in A_h \subset E(h)$ and diam $A_h < \varepsilon$.

REMARK 7.5. By the characterization of quasicircles in terms of *linear local connectivity*, cf. [11, Section 6.4], if $\partial E(h), h \in H$, are K- quasicircles, then the sets $E(h)$, $h \in H$, are uniformly locally connected.

PROOF OF THEOREM 7.1. By Lemma 7.3, $S(z)$, $z \in D$, are quasicircles, and so V is nonempty. By Remark 7.5, sets $V(z) \cup S(z) = Y(z)$ form a uniformly locally connected family of sets, if $z \in \overline{D}(0,r)$, $r < 1$. Condition (7.3) holds for sets $S(z)$, as S is closed in $D \times \mathbb{C}$ and bounded. Thus, the last statement of Proposition 7.4 implies that S and $D \times \{|\xi| = 1\}$ are homeomorphic.

Q.E.D.

PROOF OF THEOREM 7.2. By the continuous cone condition and Lemma 6.4, curves $M(\varsigma)$, $\varsigma \in \partial D$, are K- quasicircles, with a uniform K. By Theorem 7.1, $S(z)$, $z \in D$, are K^*- quasicircles, i.e. if we let in Proposition 7.4: $H = \overline{D}, E(z) = \widehat{M(z)}$ for $z = 1$ and $E(z) = V(z) \cup S(z)$ for $|z| < 1$, by Remark 7.5, $\{E(z)\}_{z \in \overline{D}}$ form a uniformly locally connected family.

Since domains $\widehat{M}(\varsigma)$, $|\varsigma| = 1$, have UTHC (by Theorem 6.3), and since we know that $\overline{V(z)} = Y(z)$, Lemma 4.4 yields in turn that $\widehat{M(\varsigma)} = V(\varsigma) \cup M(\varsigma)$ and $\overline{S} \backslash S \subset M$. The latter condition implies that (7.3) is fulfilled by our family $E(z)$. By the last statement of Proposition 7.4, $S \cup M$ is a topological manifold with boundary M, homeomorphic with $\overline{D} \times \{|\xi| = 1\}$ by a fiber-preserving homeomorphism. Q.E.D.

Very likely, the surface $X \cup (M \cup S)$ has much stronger properties related to quasiconformal geometry. Is it a quasi-sphere?

8. - The hull in the $C^{1+\alpha}$-regular case

As an example of application of the technical results of the last two sections, we prove now, under weaker assumptions, an analogue of Theorem 1.5.

THEOREM 8.1. *Let M (as well as Y, X, V, S) be as in Assumptions and Notations 1.4. In addition, let M be $C^{1+\alpha}$-regular, $\alpha \in (0,1)$, that is parametrization $b(\cdot,\cdot)$ in (1.1) is $C^{1+\alpha}$-regular and $\frac{\partial b}{\partial \theta}(\varsigma, e^{i\theta}) \neq 0$ on M. Assume, further, that $\widehat{M}$ is nondegenerate. Then, $V \neq 0$, $\overline{S} \backslash S = M$, $\widehat{M} = Y = \overline{V} = V \cup (X \cup S \cup M)$, the boundary of $\widehat{M}$, $X \cup (S \cup M)$ is a topological surface. All fibers $S(z)$, $z \in D$, are quasicircles. The interior V of $\widehat{M}$ in $\overline{D} \times \mathbb{C}$ contains the graph of a continuous analytic function on $\overline{D}$. The surface $S \cup M$ is the union of graphs of functions of the class $\bigcap_{\beta < \alpha} C^{1+\beta}(\overline{D}) \cap A(\overline{D})$.*

We conjecture that the surface S is actually $C^{1+\beta}$ regular for $0 < \beta < \alpha$, but the argument below does not yield this. It can be shown however, which we will do elsewhere, that just assuming C^1-regularity of M will not suffice for C^1-regularity of $S \cup M$.

We need the following elementary observation.

PROPOSITION 8.2. *Assume that the regions $W(\varsigma)$ are bounded by Jordan curves $\partial W(\varsigma)$, $|\varsigma| = 1$, parametrized by $w = b(\varsigma, e^{i\theta})$, so that*

(i) *$b(\varsigma, e^{i\theta})$ is continuous on $\partial D \times \partial D$ and, for each ς, one-to-one;*

(ii) *$\frac{\partial b}{\partial \theta}(\varsigma, e^{i\theta})$ is continuous in $(\varsigma, e^{i\theta})$ and vanishes nowhere.*

Then, the regions $W(\varsigma)$, $|\varsigma| = 1$, satisfy the continuous cone condition.

PROOF (SKETCH). Let $n(\varsigma, w)$ be the unit inward normal vector to $\partial W(\varsigma)$

at w. By condition (ii), we can choose $\delta > 0$ and $\alpha \in (0, \frac{1}{2}\pi)$ with the following property: if, for $\varsigma \in \partial D$ and $w \in \partial W(\varsigma)$, $\gamma_{\varsigma.w} = \{b(\varsigma, e^{i\varphi}) : \theta - \delta < \varphi < \theta + \delta\}$, where $w = b(\varsigma, e^{i\theta})$, then $\gamma_{\varsigma.w} \cap \left\{u \in \mathbb{C} : \left|\frac{\text{Arg }(u-w)}{n(\varsigma, w)}\right| \leq \alpha\right\} = \{w\}$.

The set $\gamma_{\varsigma.w}$ is an open arc of $\partial W(\varsigma)$ containing w; let $\gamma^*_{\varsigma.w}$ be the complementing closed arc. The fact that $(\varsigma, e^{i\theta}) \to (\varsigma, b(\varsigma, e^{i\theta}))$ is a homeomorphism implies, by the simple compactness argument, that

$$\min\{\text{dist}(w, \gamma^*_{\varsigma.w}) : \varsigma \in \partial D,\ \ w \in \partial W(\varsigma)\} = \beta > 0.$$

If let now $x(\varsigma, w) = \frac{1}{2}\beta n(\varsigma, w)$, and define $C(\varsigma, w)$ by (6.2), then the two relations established above immediately yield (6.1). Q.E.D.

Proof of Theorem 8.1. By the last proposition, the Jordan domains $\widehat{M(\varsigma)}$, $\varsigma \in \partial D$, satisfy CCC and so the quasicircle property of $S(z)$, $z \in D$, together with all the topological properties of the surface $S \cup M$, follows from Theorem 7.2.

By Theorem 1.1, for every $(a, b) \in S$, there is $f \in H^\infty(D)$ such that $f(a) = b$ and $\text{gr}(f) \subset S$. Since $\overline{S}\backslash S \subset M$, the cluster set of $\text{gr}(f)$ is contained in M, which is a totally real manifold. By a result of Čirka [6, Theorem 33], $f \in C^{1+\beta}(D)$ for every $\beta < \alpha$. From this moment we continue like in the Proof of Theorem 1.5 in Section 5, and obtain eventually a function $g \in A(\overline{D})$ with the property

$$g(\varsigma) \in V(\varsigma) = \text{Int } \widehat{M(\varsigma)} \text{ for every } \varsigma \in \partial D. \tag{8.1}$$

(The only difference is that now the gradient field $n(\varsigma, w)$ - if we continue to use the notation of the Proof of Lemma 1.5 - is only Hölder-continuous, but this still suffices for the Hölder continuity of h, which is enough for the proof to work). Q.E.D.

Appendix A

Proof of Lemma 4.6. *Assertion 1.* Let $a \in \partial G$ and V be a neighbourhood of a. For every $\varepsilon > 0$ and $\delta > 0$ there is a function $p \in A(G)$ (cf. Notation 4.5) such that

$$p(a) = 1,\ \ \|p\| = 1; \tag{A.1}$$

$$|p(z)| \leq \varepsilon \text{ if } z \in \overline{G}\backslash V; \tag{A.2}$$

$$p(z) \neq 0 \text{ for } z \in \overline{G}, \tag{A.3}$$

$$|\text{Im} p(z)| \leq \delta,\ \ z \in \overline{G}. \tag{A.4}$$

To construct p, choose first $h \in A(\overline{G})$ such that $h(a) = 1$, $\|h\| \leq 1$,

$|h(z)| \leq \frac{1}{2}$ on $\overline{G}\backslash V$. As in [7, Proof of Lemma 2.1, Case 2], one can find a continuous conformal map χ of $\{|u| \leq 1\}$ onto a subset of $\{\xi : 0 < \operatorname{Re}\xi,\ |\xi| \leq 1,\ |\operatorname{Im}\xi| \leq \delta\}$, such that $\chi(1) = 1$ and $|\chi(u)| \leq \varepsilon$ for $|u| \leq \frac{1}{2}$. Then, the function $p = \chi \circ h$ belongs to $A(\overline{G})$ and satisfies (A.1)-(A.4).

Assertion 2. Let $f \in A(\overline{G})$ and $a \in \partial G$, $b \in H(a)$. Assume that $\mathrm{gr}(f|_{\partial D}) \subset H$, $\mathrm{gr}(f) \subset U$, and that the line segment $[f(a), b]$ joining $f(a)$ and b is contained in $H(a)$. Then, there is $g \in A(\overline{G})$ such that $g(a) = b$ and $\mathrm{gr}(g) \subset U$.

To see this, choose r with $0 < r <$ minimum distance of $[f(a), b]$ and $\partial H(a)$. Choose $\delta > 0$ and V, neighbourhood of a in $\overline{G}$ small enough, so that whenever $|w - b| \leq r$, $z \in V$, and $|\xi| \leq 1$, $|\operatorname{Im}\xi| \leq \delta$, $\operatorname{Re}\xi \geq 0$, then $f(z) + \xi(W - f(a)) \in H(z)$. Choose $\varepsilon > 0$ with $\varepsilon < \min\{\mathrm{dist}(f(\varsigma), \partial H(\varsigma)) : \varsigma \in \partial G\}$. Let p be a function satisfying Assertion 1 with δ, ε, V. Let now $g_u(z) = f(z) + p(z)(u - f(a))$, for $|u - b| < r$. It is clear, by the preceding construction, that $g_u(\varsigma) \in H(\varsigma)$, $\varsigma \in \partial G$, and so $\mathrm{gr}(g_u) \subset F$ (see Notation 4.5), for $|u - b| < r$. Furthermore, since $p(z) \neq 0$ in $\overline{G}$, the set $\{(z, g_u(z)) : z \in \overline{G}, |u - b| < r\}$ is relatively open in $\overline{G} \times \mathbb{C}$, and so, $\mathrm{gr}(g_u) \subset U$. Hence, the function $g := g_b$ satisfies all the requirements of Assertion 2.

We conclude now the proof of the lemma. Denote $f_0(a) = b_0$, and consider an arbitrary $b \in H(a)$. Since $H(a)$ is connected, there is polygonal line contained in $H(a)$ and joining b_0 and b, with vertices $b_0, b_1, \cdots, b_{n-1}, b_n = b$. Applying inductively Assertion 2 to consecutive segments $[b_i, b_{i+1}]$, $i = 0, 1, \cdots, n-1$, we obtain functions f_i, $i = 1, \cdots, n$, such that $\mathrm{gr}(f_i) \subset U$, $\mathrm{gr}(f_i|_{\partial G}) \subset H$, and $f_i(a) = b_1$. Thus, $b \in U(a)$, for every $b \in H(a)$. Q.E.D.

Appendix B

We will sketch another proof of Lemma 7.3, based on the following steps.

Assertion 1. Under assumptions of Lemma 7.3 (i.e. Theorem 7.1), there is a sequence of topological manifolds $M_n \subset \partial D \times \mathbb{C}$, $n = 1, 2, \cdots$, such that (7.1), (7.2) hold and all fibers $M_n(\varsigma)$, $\varsigma \in \partial D$, $n = 1, 2, \cdots$, are K- quasicircles with a uniform K.

(Note that we do not require any smoothness properties from M_n).

We consider a new quantity whose finiteness characterizes quasicircles. In the notation of Characterization 3.1, denote, for a Jordan curve C, $\mu(C) = \sup\left(\mathrm{diam}\ C_1 \cdot \mathrm{diam}\ \dfrac{C_2}{|w_1 - w_2|}\right)$, where w_1, w_2 are all pairs of distinct point of C.

Assertion 2. Let $C(z)$, $z \in \overline{D}$, be Jordan curves, such that $\mathrm{gr}(C) = \{(z, w) : z \in \overline{D},\ w \in C(z)\}$ is a compact set. Assume that $S' = \mathrm{gr}(C) \cap (D \times \mathbb{C})$ is covered by disjoint graphs of analytic functions and that all $C(\varsigma)$, $\varsigma \in \partial D$, are quasicircles with $\mu(C(\varsigma)) \leq \mu_0 < +\infty$. Then, $\mu(C(z))$ is a log-subharmonic function in D and is bounded by μ_0. In particular, $C(z)$ are quasicircles.

The proof of Assertion 2 is similar in spirit to that of Assertion 2 in the Proof of Lemma 3.5, but longer.

As for the Assertion 1, we can obtain it from Proposition 7.4, applied to $H = \partial D$ and $E(h) = \widehat{M(h)}$. Then, the maps g_h have K- quasiconformal extensions to the whole plane and so, for every $r > 1$, the curve $\{g_h(u) : |u| = r\}$ is a K- quasicircles. We define $M_n(\varsigma)$ as such curve with $h = \varsigma$, $r = \frac{n+1}{n}$.

Now, if $S_n := \partial_{D\times\mathbb{C}} M_n$, we can prove that $\overline{S}_n \backslash S_n \subset M_n$. (Namely, as $\widehat{M(\varsigma)}$ are encircled by $M_n(\varsigma)$, and $\widehat{M}$ contains $\mathrm{gr}(h)$, $h \in H^\infty$, we get that, for some $\delta < 1$, the function $f_0(z) = h(\delta z)$, $z \in \overline{D}$, has the property $f_0(\varsigma) \in \mathrm{Int}\ \widehat{M_n(\varsigma)}$. By Lemma 4.6, we conclude that $\overline{S}_n \backslash S_n \subset M_n$). Applying Assertion 2 with $C(\varsigma) = M_n(\varsigma)$, $\varsigma \in \partial D$ and $C(z) = S_n(z)$, for $z \in D$, one obtains that $S_n(z)$ are quasicircles with a uniform constant. The proof is completed in the similar way as was the Proof of Lemma 7.3 in Section 7. We omit further details.

REFERENCES

[1] H. ALEXANDER, *Hulls of deformations in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **266** (1981), 243-257.

[2] H. ALEXANDER - J. WERMER, *Polynomial hulls with convex fibers*, Math. Ann. **27** (1985), 99-109.

[3] E. BEDFORD - B. GAVEAU, *Envelopes of holomorphy of certain 2-spheres in* $\mathbb{C}^2$, Amer. J. Math. **105** (1983), 975-1009.

[4] J. BENCE - J.W. HELTON - D.E. MARSHALL, *Optimization over* H^∞.

[5] E. BISHOP, *Differentiable manifolds in complex Euclidean spaces*, Duke Math. J. **32** (1965), 1-21.

[6] E.M. ČIRKA, *Regularity of boundaries of analytic sets*, Math. USSR-Sb. **117** (1982), 291-334, (Russian) Math. USSR-Sb. **45** (1983), 291-336.

[7] F. FORSTNERIČ, *Polynomially convex hulls with piecewise smooth boundaries*, Math. Ann. **276** (1986), 97-104.

[8] F. FORSTNERIČ, *Polynomial hulls of sets fibered over the circle*, Indiana Univ. Math. J. **37** (1988), 869-889.

[9] J.W. HELTON - R.E. HOWE, *A bang-bang theorem for optimization over spaces of analytic functions*, J. Approx. Theory **47** (1986), 101-121.

[10] J.W. HELTON - D.F. SCHWARTZ - S.E. WARCHAWSKI, *Local optima in* H^∞ *produce a constant objective function*, Complex Variables, **8** (1987), 65-81.

[11] O. LEHTO, *University functions and Teichmüller spaces*, Springer-Verlag, New York 1986.

[12] CHR. POMMERENKE, *Univalent functions*, Vandenhoeck and Ruprecht in Göttingen, 1975.

[13] Z. SLODKOWSKI, *Local maximum property and q-plurisubharmonic functions in uniform algebras*, J. Math Anal. Appl. **115** (1986). 105-130.

[14] Z. SLODKOWSKI, *Polynomially convex hulls with convex sections and interpolating spaces*, Proc. Amer. Math. Soc. **96** (1986), 255-260.
[15] J. WERMER, *Polynomially convex hulls and analyticity*, Ark. Mat., **20** (1982), 129-135.

Department of Mathematics,
University of Illinois at Chicago
Chicago, Illinois 60680