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# The existence of nonminimal regular harmonic maps from $B^3$ to $S^2$

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# The Existence of Nonminimal Regular Harmonic Maps from $B^3$ to $S^2$

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## 1. - Introduction

There has been a great deal of interest in the harmonic map problem for maps from domains in $R^3$ to $S^2$, the two-dimensional sphere (see [BCL], [HL1,2], [ABL], [AL]). The large solutions for harmonic maps in two dimensions have been studied by H. Brézis and J. Coron [BC]. In addition to the geometric significance of these maps, they also arise as an important special case in the theory of nematic liquid crystals (see [deG], [E], [HKL1,2], [HL1,2]). A general existence and regularity theorem was proven by R. Schoen and K. Uhlenbeck (see [SU1,2,3]) for energy minimizing maps. In the case of three dimensional domains, their result asserts that minimizers are regular except for isolated points in the interior. There are several important earlier works which require some restriction on the target manifold. These include the works of J. Eells and J. H. Sampson [ES] for targets of *nonpositive* curvature, and the works of S. Hildebrandt, H. Kaul, and K. O. Widman [HKW1,2] for maps whose images lie in a convex coordinate neighbourhood.

If one considers a map $\varphi : \partial B^3 \longrightarrow S^2$ which has nonzero degree, then any minimizer which agrees with $\varphi$ on $\partial B^3$ necessarily possesses at least one interior singular point. A natural question arises as to whether the singular points are determined by the boundary data, or one can specify singular behavior as well as boundary data. This question is unresolved.

The first special case of the problem of prescribing singularities is the question of whether a boundary map $\varphi$ of zero degree has a regular harmonic extension to $B^3$. On the positive side, the theorem of S. Hildebrandt, H. Kaul and K. O. Widman [HKW2] asserts that a map $\varphi$, whose image lies in an open hemisphere of $S^2$, has a regular harmonic extension (also having an image in this hemisphere). On the other hand, R. Hardt and F. H. Lin [HL2] have constructed zero degree boundary maps $\varphi$ *for which* any minimizer must possess interior singular points. Thus, any regular harmonic extension, in such a case,

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would have larger energy than the infimum over all $W^{1,2}$ extensions of $\varphi$.

In this paper, we show that, for a general class of axially symmetric maps $\varphi$ of zero degree, there is a regular axially symmetric harmonic extension of $\varphi$ to $B^3$. In particular, in many cases these harmonic maps have larger energy than a minimizer. In the last section of this paper, we will present some examples of axially symmetric boundary maps $\varphi$ whose smooth harmonic extensions are not energy minimizers.

Our proof reduces to the analysis of a scalar partial differential equation in two variables. This equation is degenerate on the boundary corresponding to the axis of symmetry. Our main theorem can now be stated.

THEOREM. *Let $\varphi : \partial B^3 \longrightarrow S^2$ be a regular non-surjective axially symmetric map. There exists a regular axially symmetric harmonic extension of $\varphi$ to $B^3$.*

To make precise the symmetry condition we require: let $B^3 \subset R^3$ be the unit ball and $S^2 \subset R^3$ be the unit sphere; i.e.

$$B^3 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 < 1\},$$

and

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}.$$

We are going to use the following coordinates in $B^3$,

$$[0, 2\pi) \times D \ni (\theta, r, z) \longmapsto (r\cos\theta,\ r\ \sin\ \theta, z) \ni B^3,$$

where

$$D = \{(r, z) \in R^2 | r^2 + z^2 < 1,\ r > 0\}.$$

The Euclidean metric is then given by

$$r^2 \mathrm{d}\theta^2 + \mathrm{d}r^2 + \mathrm{d}z^2.$$

Choose coordinates $(\theta, \phi)$ in $S^2$ as follows,

$$[0, 2\pi) \times [0, \pi) \ni (\theta, \phi) \longmapsto (\ \sin\ \phi\cos\theta,\ \sin\ \phi\ \ \sin\ \theta, \cos\phi).$$

The metric on $S^2$ then takes the form

$$\sin\ ^2\phi \mathrm{d}\theta^2 + \mathrm{d}\phi^2.$$

The axially symmetric harmonic maps we consider have the form

$$\Phi(\theta, r, z) = (\theta, \phi(r, z)).$$

Using these coordinates, our problem is now to solve the following equation:

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right) - \frac{\sin\ 2\phi}{2r} = 0, \quad \text{in } D, \tag{1}$$

with the boundary condition $\phi = \varphi$ on $\partial D$, where $\varphi$ satisfies

(2) $$\varphi(0, z) = 0 \quad \text{and} \quad 0 \leq \varphi((1 - z^2)^{\frac{1}{2}}, z) \leq \pi - \varepsilon,$$

where $\varepsilon$ is a positive number.

We will solve the above equation by finding a smooth critical point of the energy functional:

(3) $$E(\phi) = \frac{1}{2} \int_D \left\{ r \left( \frac{\partial \phi}{\partial r} \right)^2 + r \left( \frac{\partial \phi}{\partial z} \right)^2 + \frac{\sin^2 \phi}{r} \right\} drdz,$$

which is defined in the Hilbert space

$$H = \{ u \in W^{1,2}(B^3, S^2) | u \text{ is axially symmetric } \},$$

where $u(\theta, r, z) = (\theta, \phi(r, z))$.

## 2. - Several Lemmas

LEMMA 1. *Any critical point $\phi$ of $E$ is a local minimum in the sense that*

(4) $$E(\phi) \leq E(\phi + \psi),$$

*provided $\psi \in C_0^\infty(D)$ and*

$$\text{dist}(\partial D, \text{supp}(\psi)) \geq C \text{diameter}(\text{supp}(\psi)),$$

*where $C$ is a fixed constant.*

PROOF. Suppose $\phi \in H$ is a critical point of $E$ and $\psi$ is a map in $C_0^\infty(D)$.

Using integration by parts, equation (1), and Taylor's theorem, we have

$$
\begin{aligned}
&2(E(\phi) - E(\phi+\psi)) \\
&= \int_D \left\{ r(|\nabla\phi|^2 - |\nabla(\phi+\psi)|^2) + \frac{\sin^2\phi - \sin^2(\phi+\psi)}{r} \right\} \mathrm{d}r\mathrm{d}z \\
&= \int_D \left\{ -r|\nabla(\psi)|^2 + \frac{\sin^2\phi - \sin^2(\phi+\psi) - \psi\sin 2\phi}{r} \right\} \mathrm{d}r\mathrm{d}z \\
&\le -\int_D r|\nabla(\psi)|^2 \mathrm{d}r\mathrm{d}z + \int_D \frac{\psi^2}{r}\mathrm{d}r\mathrm{d}z \\
&\le -\ \mathrm{dist}\ \{\partial D,\ \mathrm{supp}\ (\psi)\} \int_D |\nabla(\psi)|^2 \mathrm{d}r\mathrm{d}z \\
&\quad + \frac{\int_D (\psi)^2 \mathrm{d}r\mathrm{d}z}{\mathrm{dist}\ \{\partial D,\ \mathrm{supp}\ (\psi)\}}.
\end{aligned}
$$

Since the first eigenvalue of $-\Delta$ on a bounded domain $\Omega \subset R^n$ is bounded from below by $\{\text{diameter } (\Omega)\}^{-2}$, as long as the required condition is satisfied by supp $(\psi)$, inequality (4) holds. □

The next lemma is a consequence of Lemma 1 and standard elliptic regularity theory (see [M]).

LEMMA 2. *The critical points of the equation (1) in $H$ are regular in the interior of $D$.*

Using Lemma 1, we now derive the following existence result.

LEMMA 3. *Suppose $\phi_1$ and $\phi_2$ are critical points of $E$ and $0 \le \phi_1 \le \phi_2$ on $D$. Let $\varphi_i$ be the boundary values of $\phi_i, = 1, 2$. Suppose $\varphi$ is a function on $\partial D$ lying between $\varphi_1$ and $\varphi_2$, i.e. $\varphi_1 \le \varphi \le \varphi_2$. Then*

$$c \equiv \inf \{E(\phi) | \phi_1 \le \phi \le \phi_2, \phi|_{\partial D} = \varphi\}$$

*is a critical value of $E$ and is achieved by a critical point $\phi$, i.e. there exists $\phi \in H$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi + t\psi)|_{t=0} = 0, \textit{ for all } \psi \in C_0^\infty(D). \tag{5}$$

*Moreover, $\phi_1 \le \phi \le \phi_2$ on $D$ and $\phi|_{\partial D} = \varphi$.*

PROOF. Consider a minimizing sequence $\{\phi_n\}$ which has a weak limit $\phi$. It is easy to see that $E(\phi) = c$; in fact, $\phi$ is the strong limit of $\{\phi_n\}$. Therefore, we may assume

$$\phi_n \longrightarrow \phi \text{ a.e., } \phi_1 \le \phi \le \phi_2 \text{ and } \phi|_{\partial D} = \varphi.$$

In order to show that $\phi$ is a critical point of $E$, we need to prove that (5) holds for all $\psi \in C_0^\infty$. But it is enough to show that (5) holds for those $\psi \in C_0^\infty$ satisfying

$$\text{dist } (\partial D, \text{ supp } (\psi)) \geq C \text{ diameter } \{\text{supp } (\psi)\}.$$

Let $\psi$ be such a $C_0^\infty$ function and consider

$$\phi_t = \phi + t\psi, \text{ for all } t \in R^1.$$

We define

$$S_t = \{(r, z) \in D | \phi + t\psi > \phi_2\},$$
$$M_t = \{(r, z) \in D | \phi + t\psi < \phi_1\}.$$

Since $\phi_1 \leq \phi \leq \phi_2, S_t$ and $M_t$ satisfy the following conditions:

$$\text{dist } (\partial D, S_t) \geq C \text{ diameter } (S_t),$$

$$\text{dist } (\partial D, M_t) \geq C \text{ diameter } (M_t).$$

Define

$$\phi^t = \begin{cases} \phi_2 & \text{on } S_t \\ \phi + t\psi & \text{on } D - (S_t \bigcup M_t), \\ \phi_1 & \text{on } M_t \end{cases}$$

$$\phi_1^t = \begin{cases} \phi + t\psi & \text{on } M_t \\ \phi_1 & \text{on } D - M_t, \end{cases}$$

$$\phi_2^t = \begin{cases} \phi + t\psi & \text{on } S_t \\ \phi_2 & \text{on } D - S_t. \end{cases}$$

By the definition of $\phi$, we know that $E(\phi) \leq E(\phi^t)$. Using Lemma 1, we have

$$E(\phi^t) - E(\phi + t\psi) = \{E(\phi_1) - E(\phi_1^t)\} + \{E(\phi_2) - E(\phi_2^t)\} \leq 0.$$

Combining these two inequalities, we get

$$E(\phi) \leq E(\phi + t\psi).$$

Therefore (4) holds for every $\psi$. The lemma is proved. □

We are going to use Lemma 3 to prove our theorem. We will take the trivial solution, the zero function, as our $\phi_1$. The rest of the proof is to show that there exists a $\phi_2$, for any given boundary data $\varphi$, satisfying the condition of Lemma 3. Our $\phi_2$ will be a $z$-independent solution, i.e. a function of $r$ which is a solution of the following ordinary differential equation

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) - \frac{\sin 2\phi}{2r} = 0. \tag{6}$$

## 3. - The construction of $\phi_2$

Let $r = e^t, t \in (-\infty, 0]$. The equation (6) then becomes

$$\frac{d^2}{dt^2}\phi(t) - \frac{\sin 2\phi(t)}{2} = 0. \tag{7}$$

We are going to use a variational method to solve the above equation. The corresponding functional is

$$E_1(\phi) = \int_{-\infty}^{0} \left\{ \left| \frac{d\phi}{dt} \right|^2 + \sin^2\phi \right\} dt.$$

We are going to solve equation (7) with the following boundary condition:

$$\phi(-\infty) = 0, \quad \phi(0) = \tau < \pi. \tag{8}$$

This can be done by minimizing the functional $E_1$. It is easy to see that

$$\begin{aligned} &\inf \{E_1(\phi) | \phi \in W^{1,2}(-\infty, 0), \phi(0) = \tau\} \\ = &\inf \{E_1(\phi) | \phi \in W^{1,2}(-\infty, 0), \phi' \geq 0 \text{ and } \phi(0) = \tau\}. \end{aligned}$$

Since

$$\frac{\sin \tau}{\tau}\phi \leq \sin \phi \leq \phi, \text{ when } 0 \leq \phi \leq \tau,$$

the problem (7), (8) has a monotone increasing positive solution.

Let $\phi_\tau$ be such a solution. Multiply the equation (7) by $\phi'_\tau$ and integrate to get

$$\{\phi'_\tau\}^2 - \sin^2\phi_\tau = \text{constant}. \tag{9}$$

Since $\phi_\tau \in W^{1,2}(-\infty, 0)$, the constant is zero.

Because

$$\phi'_\tau \geq 0 \text{ and } 0 \leq \phi_\tau < \pi,$$

we have

$$\phi'_\tau(t) = \sin \phi_\tau(t). \tag{10}$$

Thus we have solved the following first order ordinary differential equation

$$\phi'(t) = \sin \phi(t), \quad -\infty \leq t \leq 0; \phi(0) = \tau < \pi. \tag{11}$$

Since $\phi \equiv \pi$ is the solution of (11) with the initial value $\tau = \pi$, we see that

$$\phi_\tau \longrightarrow \pi, \text{ as } \tau \longrightarrow \pi, \tag{12}$$

uniformly in any finite interval $[-T, 0]$. Furthermore,

(13) $$\phi_\iota(t) < \phi_\tau(t), \text{ when } \iota < \tau.$$

Take a $\nu \in (0, \pi)$ and let $t_\nu(\tau)$ be the root of $\phi_\tau(t) = \nu$, then

(14) $$t_\nu(\tau) \longrightarrow -\infty, \text{ as } \tau \to \pi.$$

Moreover, since $\sin \phi(t) \le \phi(t)$, we have

(15) $$(\nu e^{-t_\nu(\tau)})e^t \le \phi_\tau(t), \quad t \le t_\nu(\tau).$$

Now let us go back to the equation (6). Define $\phi^\tau(r)$ by

$$\phi^\tau(r) \equiv \phi_\tau(\log r).$$

The function $\phi^\tau$ is then a solution of (6) and will be our $\phi_2$ with proper choice of $\tau$.

The inequality (15), in terms of $r$, is

(16) $$(\nu e^{-t_\nu(\tau)})r \le \phi^\tau(r), \quad r \le e^{t_\nu(\tau)}.$$

Suppose the function $\varphi$ on $\partial D$ satisfies the following conditions:

(a) $\varphi = 0$, for $r = 0$;
(b) Lipschitz's condition at both $(0, 1)$ and $(0, -1)$;
(c) $\varphi \le \max_{\partial D}(\varphi) < \pi$.

From (b), we have two positive constants $k$ and $\delta$ such that

(17) $$\varphi(r, z) \le kr, \text{ when } r \le \delta \text{ and } (r, z) \in \partial D.$$

First, from (14), there is a $\tau^0$ such that

(18) $$\nu e^{-t_\nu(\tau^0)} > k.$$

For this $\tau^0$, we have

(19) $$\varphi \le \phi^{\tau^0}, \text{ on } \partial D,$$

when $r < \delta_0 \equiv \min\{\delta, e^{t_\nu(\tau^0)}\}$. By (13), we know that (19) holds for all $\tau \le \tau^0$, with the same $\delta_0$.

Secondly, from (12), there is a $\tau^1 \le \tau^0$ such that

(20). $$\max_{\partial D}(\varphi) \le \phi^{\tau^1}(r), \text{ for } r \in [\delta_0, \pi].$$

Therefore, $\phi_2 \equiv \phi^{\tau^1}$ will serve as our upper barrier. Using Lemma 3, we know that the problem (1), (2) has a regular solution $\phi$ whenever the boundary value $\varphi$ is regular and non-surjective.

## 4. - The proof of the Theorem

Let $\varphi$ be a regular boundary function satisfying the conditions of the theorem and $\phi$ be the corresponding regular solution of the problem (1), (2). Define the map $\Phi : B^3 \to S^2$ by

$$\Phi(\theta, r, z) = (\theta, \phi(r, z)).$$

$\Phi$ is an axially symmetric map with the boundary values

$$\varphi \ (\theta, r, z) \longrightarrow (\theta, \varphi(r, z)) \text{ on } \partial B^3.$$

We are going to show that $\Phi$ is harmonic and smooth on $B^3$.

We know that $\Phi$ is harmonic away from the axis, i.e. $\Phi$ satisfies the equation

$$\Delta\Phi = -|\nabla\Phi|^2\Phi, \quad x^2 + y^2 > 0 \tag{21}$$

or, equivalently,

$$\int_{B^3} \nabla\Phi\nabla\Psi dv = \int_{B^3} |\nabla\Phi|^2(\Phi, \Psi)dv, \tag{22}$$

for any $\Psi \in C_0^\infty(B^3, R^3)$ with $\Psi(0, 0, z) = 0$.

We will prove that $\Phi$ is harmonic in $B^3$ by showing that (22) holds for all $\Psi \in C_0^\infty(B^3, R^3)$.

Let $\Psi$ be a $C_0^\infty(B^3, R^3)$ map, and consider the following cut-off function

$$\eta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{when } 2 \leq t \leq \infty, \text{ and } |\ \eta'(t)| \leq 2. \end{cases}$$

Let $\eta_k(x, y, z) = \eta(k(x^2 + y^2)^{1/2})$. Then,

$$\begin{aligned}
&\int_{B^3} \{\nabla\Psi\nabla\Phi - |\nabla\Phi|^2(\Phi, \Psi)\}dxdydz \\
&= \int_{B^3} \{\nabla\Phi\nabla(\eta_k\Psi) - |\nabla\Phi|^2(\Phi, \eta_k\Psi)\}dxdydz \\
&+ \int_{B^3} \nabla\Phi\nabla((1 - \eta_k)\Psi)dxdydz - \int_{B^3} |\nabla\Phi\ |^2\ (\Phi, (1 - \eta_k)\Psi)dxdydz.
\end{aligned}$$

The first integral on the right side is zero. The second and the third integrals tend to zero as $k \longrightarrow \infty$, because the measure of supp $(\nabla_{\eta_k})$ goes to zero and $\{\int_{B^3} |\nabla_{\eta_k}|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z\}$ is bounded. Hence $\Phi$ is harmonic on $B^3$.

We know that $\Phi$ is smooth away from the axis. Since by construction

$$\phi(r,z) \leq \phi_2(r), \text{ and } \phi_2 \longrightarrow 0, \text{ as } r \to 0,$$

$\Phi$ is continuous on the axis. Therefore $\Phi$ is smooth everywhere in $B^3$ (see $[H]$). Our main theorem has been proved.

## 5. - An example

In this section we will show that the regular harmonic extension we get is not a energy minimizer among the $W^{1,2}$ extensions. As a matter of fact it is not even a energy minimizer among the axially symmetric $W^{1,2}$ extensions. We are going to show this by constructing a sequence of axially symmetric maps $\{\Phi_k\} \subset W^{1,2}(B^3, S^2)$ all of them have boundary values, say $\{\varphi_k\}$, satisfying the conditions of our theorem. Therefore, $\varphi_k$ have regular axially symmetric harmonic extensions on $B^3$ which will be denoted as $\overline{\Phi}_k$. The difference between $\{\Phi_k\}$ and $\{\overline{\Phi}_k\}$ is

$$E(\Phi_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty,$$

but $\{E(\overline{\Phi}_k)\}$ has a positive lower bound. Since $\{\Phi_k\}$ are axially symmetric, using the previous coordinates, we can write

$$\Phi_k(\theta, r, z) = (\theta, \phi_k(r,z)).$$

For this reason, we only need to give the definition of $\phi_k'$s:

$$\phi_k(r,z) = \begin{cases} \left(1 - \frac{r}{k}\right)\pi & \text{if } |z| \leq 1 - 2/k \\ 2\left(1 - \frac{r}{k}\right) \sin^{-1}\left(\frac{r}{\sqrt{r^2+z^2}}\right) & \text{otherwise} \end{cases}$$

where $k = 4, 5, 6, \ldots$. Its boundary value is

$$\varphi_k = \phi_k|_{S^1_{1/2}},$$

where $S^1_{1/2} = \{(r,z) \in \partial D \mid r^2 + z^2 = 1\}$. Since $\phi_k$ only takes $\pi$ and zero on the axis, $\Phi_k$ is well defined. Recognize that, in the coordinates we are using, the famous harmonic map from $B^3$ to $S^2$ namely $x \mapsto x/\ |x|$ can be written as

$$(\theta, r, z) \mapsto \left(\theta,\ \sin^{-1}\left(\frac{r}{\sqrt{r^2+z^2}}\right)\right).$$

Therefore, $\{\Phi_k\} \subset W^{1,2}(B^3, S^2)$. Furthermore, it is easy to see that

$$E(\Phi_k) \longrightarrow 0 \text{ as } k \longrightarrow 0.$$

It is obvious that $\varphi_k$ is $C^1$ on $S^1_{1/2}$ except two points $(*, \pm(1 - 2/k))$ and $\varphi_k < \pi$. Hence, by our theorem, $\varphi_k$ has a smooth axially symmetric harmonic extension $\overline{\Phi}_k = (\theta, \overline{\phi}_k)$.

Now, we are going to estimate $E(\overline{\Phi}_k)$.

$$E(\Phi_k) = \pi \int_{-1}^{1} \int_{0}^{\sqrt{1-z^2}} \left[ r \mid \nabla \overline{\phi}_k|^2 + \frac{\sin^2 \overline{\phi}_k}{r} \right] drdz$$
$$\geq \int_{-1/2}^{1/2} \left\{ \int_{0}^{\sqrt{1-z^2}} \left[ r \left( \frac{\partial \overline{\phi}_k}{\partial r} \right)^2 + \frac{\sin^2 \overline{\phi}_k}{r} \right] dr \right\} dz.$$

For any $z \in [-1/2, 1/2]$, $\overline{\phi}_k(r, z)$ is a smooth function of $r \in [0, \sqrt{1 - z^2}]$ with $\overline{\phi}_k(0, z) = 0$ and $1/2 < \overline{\phi}_k(\sqrt{1 - z^2}, z) = (1 - \sqrt{1 - z^2}/k)\pi < \pi$.

From Section 3, we know that

$$\inf_{\{k, |z| \leq 1/2\}} \left\{ \int_{0}^{\sqrt{1-z^2}} \left[ r \left( \frac{\partial \overline{\phi}_k}{\partial r} \right)^2 + \frac{\sin^2 \overline{\phi}_k}{r} \right] dr \right\} > c_{1/2}$$

is positive. Therefore, $\inf_k \{E(\overline{\Phi}_k)\}$ is positive.

REMARK. The same result holds for those axially symmetric bounded $R^3$ domains diffeomorphic to $B^3$ and also for axially symmetric boundary maps which have the form of

$$(\theta, r, z) \longrightarrow (n\theta, \varphi(r, z)), \qquad r^2 + z^2 = 1,$$

where $n$ is an integer; in fact the Euler-Lagrange equation then becomes

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \phi}{\partial z}\right) - n^2 \frac{\sin 2\phi}{2r} = 0.$$

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