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### Asymptotic expansions of quasiperiodic solutions

# Asymptotic Expansions of Quasiperiodic Solutions

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## 1. - Introduction

We first describe the existence problem of quasiperiodic solutions in a general setting and consider a Lagrangian function $F = F(t, x, p)$,

(1.1) $$F(t, x, p) \text{ defined on } T^{n+1} \times \mathbb{R}^n,$$

i.e. periodic in $(t, x) \in \mathbb{R}^{n+1}$ with periodic 1, $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$. The aim is to find special solutions of the associated Euler-equations

(1.2) $$\frac{\mathrm{d}}{\mathrm{d}t} F_p(t, x(t), \dot{x}(t)) = F_x(t, x(t), \dot{x}(t)).$$

We shall call, in the following, a solution $x(t)$ quasiperiodic with frequencies $\omega$, if it is of the form

(1.3) $$x(t) = U(t, \omega t),$$

where $\omega \in \mathbb{R}^n$ is a given vector with rationally independent components, and where

(1.4) $$U(t, \vartheta) - \vartheta =: u(t, \vartheta) \quad \text{is defined on } T^{n+1}$$

i.e. is periodic in $(t, \vartheta)$. Inserting (1.3) into (1.2), one obtains the nonlinear partial differential equation for $U$:

(1.5) $$DF_p(t, U, DU) = F_x(t, U, DU),$$

where

(1.6) $$D = D(\omega) = \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j} + \frac{\partial}{\partial t}.$$

The differential operator D depends on the frequencies $\omega = (\omega_1, \cdots, \omega_n)$. It is the differentiation in the direction $(\omega, 1)$. Restricting our attention to functions of the special form

$$(1.7) \qquad F(t, x, p) = \frac{1}{2}|p|^2 + f(t, x),$$

the equation to be solved becomes

$$(1.8) \qquad D^2 u = f_x(t, \vartheta + u)$$

for $u(t, \vartheta) = U(t, \vartheta) - \vartheta$ being a function on $T^{n+1}$. In order to solve (1.8) we shall assume $f$ to be analytic and the frequencies $\omega$ to satisfy the diophantine conditions:

$$(1.9) \qquad |\langle \omega, j \rangle + m| \geq \gamma(|j|)^{-\tau}$$

for two constants $\gamma > 0$ and $\tau \geq n$ and for all $(j, m) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}$.

It is well know that under these conditions on $f$ and $\omega$ the equation (1.8) has a solution, provided $f$ is sufficiently small (in an appropriate sense). This is a consequence of the KAM theory, and we refer to [CC], [SZ] and [M1]. However, if $f$ is not small, then (1.8) may not admit any solutions for frequencies contained in a compact region of $\mathbb{R}^n$, see [Ma]. We shall not impose any smallness conditions on $f$ in the following. Instead we shall construct quasiperiodic solutions having sufficiently large frequencies. We point out, that the system under consideration, decribed by a Lagrangian function in the special form of (1.7), can be viewed as beeing "close to an integrable system" in the region in which $|p|$ is large. Introducing

$$(1.10) \qquad \omega(\alpha) = \frac{1}{\alpha}\omega,$$

we look for quasiperiodic solutions having frequencies $\omega(\alpha)$ for sufficiently small $\alpha \in R$ with $\alpha \neq 0$. We shall abbreviate

$$(1.11) \qquad E(u) =: D^2 u - f_x(t, \vartheta + u),$$

with $D = D(\omega(\alpha))$.

In the second section we shall prove that there is unique formal powerseries expansion in $\alpha$:

$$(1.12) \qquad \tilde{u} \sim \sum_{j=2}^{\infty} \alpha^j u_j(\vartheta, t),$$

with analytic functions $u_j$ on $T^{n+1}$, which solves the equation $E(\tilde{u}) = 0$ formally, and satisfies

$$\int_{T^{n+1}} u_j \, dt \, d\vartheta = 0, \text{ for all } j.$$

However, in general, the series diverges as it is well known, and our aim is to show that the formal series can be interpreted as an asymptotic expansion for the true quasiperiodic solutions $u_\alpha$, as $\alpha$ tends to zero. For this purpose $\alpha$ is required to belong to the subset

$$(1.13) \quad A(\omega) = \left\{ \alpha \in \mathbb{R} : \left| \frac{1}{\alpha} \langle \omega, j \rangle + m \right| \geq \gamma |j|^{-\tau}, \text{ for all } (j, m) \in \mathbb{Z}^n \times \mathbb{Z} \backslash 0 \right\}.$$

If $\gamma$ is sufficiently small and $\tau > n + 1$, we will see that the set $\{\alpha \in A(\omega) : |\alpha| \leq \epsilon\}$ has positive Lebesgue measure for every $1 > \epsilon > 0$. Setting now for every $N \geq 2$

$$(1.14) \qquad \tilde{u}_N := \sum_{j=2}^{N} \alpha^j u_j(t, \vartheta),$$

one concludes that, in proper norms,

$$(1.15) \qquad |E(\tilde{u}_N)| \leq C_N |\alpha|^{N-1}$$

for all $|\alpha| \leq 1$, with a constant $C_N$ independent of $\alpha$. Consequently, $\tilde{u}$ can be interpreted as an approximate solution of $E(u) = 0$, if only $\alpha$ is small. Moreover, $\tilde{u}_N$ is stable in the sense that the matrixfunction on $T^{n+1}$,

$$(1.16) \qquad V_\vartheta^T F_{pp}(t, V, DV) V_\vartheta,$$

with $V =: \vartheta + \tilde{u}_N(\vartheta, t)$, is close to the identity matrix. Thus the assumptions of the KAM theory are met and one concludes that there is an $\alpha^* = \alpha^*(N)$, such that for $\alpha \in A(\omega)$ satisfying $|\alpha| < \alpha^*$ there is a unique analytic solution $u_\alpha$ of (1.8) having frequencies $\omega(\alpha)$, hence solving

$$(1.17) \qquad E(u_\alpha) = 0,$$

moreover

$$\int_{T^{n+1}} u_\alpha = 0.$$

In addition, one has an estimate of the form

$$(1.18) \qquad |u_\alpha - \tilde{u}_N| \leq C_N |E(u_N)|.$$

This establishes the existence of uncountably many quasiperiodic solutions for every analytic $f$. We point out again, that $f$ is not assumed to be small. Moreover, on account of (1.18) and (1.15) one concludes that for every $N \geq 2$ there are constants $C_N > 0$ and $\alpha^* = \alpha^*(N)$ such that

$$(1.19) \qquad \left| u_\alpha - \sum_{j=2}^{N} \alpha^k u_j \right|_\infty \leq C_N |\alpha|^{N+1}$$

for all $\alpha \in A(\omega)$ satisfying $|\alpha| \leq \alpha^*$. This shows that indeed the formal series (1.12) serves as an asymptotic expansion for the solutions having large frequencies $\omega(\alpha)$. The precise statement and the details of this argument are given in section 3. For simplicity we shall only treat the case in which $f$ is analytic. We point out that the asymptotic expansion holds true also for $f \in C^\infty(T^{n+1})$, in which case also the solutions $u_\alpha$ belong to $C^\infty(T^{n+1})$.

It should be mentioned that in the special case $n = 1$ the existence of quasiperiodic solutions having large frequencies can be used in order to prove that all solutions of

$$(1.20) \qquad \ddot{x} - f_x(t, x) = 0, \quad (t, x) \in T^2$$

are bounded, i.e.

$$\sup_t |\dot{x}(t)| < \infty.$$

This has already been pointed out in [M1] and we shall recall the argument. We shall write (1.20) as a system

$$(1.21) \qquad \dot{x} = y, \quad \dot{y} = f_x(t, x), \quad \dot{t} = 1,$$

which is considered as a vectorfield on the phase space $T^2 \times \mathbb{R}$. Assume now that $U$ is a solutions of

$$\mathrm{D}^2 U = f_x(t, U)$$

$$(1.22) \qquad U(t, \vartheta) - \vartheta = u(t, \vartheta) \quad \text{on } T^2,$$

$$\mathrm{D} = \frac{\omega}{\alpha} \frac{\partial}{\partial \vartheta} + \frac{\partial}{\partial t}.$$

Then the map $\psi : T^2 \to T^2 \times \mathbb{R}$, defined by $(t, \vartheta) \to (t, x = U(t, \vartheta), y = \mathrm{D}U(t, \vartheta))$, describes an embedding of the torus $T^2$ into the phase space. In view of (1.22), the vectorfield (1.21) is tangential to $\psi(T^2) \subset T^2 \times \mathbb{R}$ so that its flow leaves this embedded torus invariant. If now $a_1 = \min \mathrm{D}U \leq \mathrm{D}U \leq a_2 = \max \mathrm{D}U$, then $\psi(T^2) \subset T^2 \times [a_1, a_2]$, and since $\psi(T^2)$ is invariant under the flow we conclude, for every solution $(t, x(t), y(t))$ satisfying $y(t^*) < a_1$ for some $t^* \in \mathbb{R}$, that $y(t) < a_2$ for all $t \in \mathbb{R}$. Since $\mathrm{D}U = \frac{\omega}{\alpha} + O(\alpha)$, we can construct for every $C > 0$ a quasiperiodic solution $U$ satisfying $\mathrm{D}U > C$ by choosing $\alpha$ sufficiently small. This proves the claim, that all solutions are bounded. One can show that the analyticity of $f$ is not necessary for the argument. It is sufficient to assume $f$ to be sufficiently smooth, e.g. $f \in C^6(T^2)$, for the smooth case we refer to [M2]. Similar arguments allow to prove the boundedness of solutions of other equations, for example for the Euler equation associated to

$$F(t, x, p) = \frac{1}{2}p^2 + \sqrt{1 + p^2} f(t, x)$$

on $T^2 \times \mathbb{R}$. The above argument was used also in the more subtle proof in [DZ] of the boundedness of solutions for a nonlinear Duffing equation on $\mathbb{R}^2 \times \mathbb{R}$.

Observe that this note deals only with systems of very restricted nature and it is desirable to have asymptotic expansion for a more general class of Euler equations associated to

$$F(t, x, p) = g(p) + f(t, x, p)$$

on $T^{n+1} \times \mathbb{R}^n$, with

$$\frac{|f(t, x, p)|}{|g(p)|} \to 0, \quad \text{as } |p| \to \infty.$$

## 2. - The formal expansion

In order to solve $E(u) = 0$ we set formally

$$(2.1) \qquad u =: \sum_{j=0}^{\infty} \alpha^j u_j(\vartheta, t),$$

and recall that

$$(2.2) \qquad E(u) := D^2 u - f_x(t, \vartheta + u)$$

contains the parameter $\alpha$ also in the differential operator D. Introducing the operator

$$\partial =: \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j}$$

we can write

$$(2.3) \qquad D^2 = \frac{1}{\alpha^2} \partial^2 + \frac{2}{\alpha} \partial D_t + D_t^2,$$

where $D_t$ denotes partial derivative with respect to $t$. Expanding $\alpha^2 E(u) = 0$ into powers of $\alpha$ we find the following equations to be solved for the functions $u_j$:

$$\partial^2 u_0 = 0$$

$$(2.4) \qquad \partial^2 u_1 + 2\partial D_t u_0 = 0$$

$$\partial^2 u_j + 2\partial D_t u_{j-1} + D_t^2 u_{j-2} = \varphi_{j-2},$$

for $j \geq 2$, where

$$(2.5) \qquad \varphi_j = \varphi_j(u_0, \cdots, u_j) = \frac{1}{j!} \left( \frac{\mathrm{d}}{\mathrm{d}\alpha} \right)^j f_x \left( t, \vartheta + \sum_{s=0}^{j} \alpha^s u_s \right) \Bigg|_{\alpha=0}$$

is a polynomial in $u_1, \cdots, u_j$.

We shall show that there are unique analytic solutions $u_j$ defined on $T^{n+1}$, if we normalize

$$\int_{T^{n+1}} u_j \mathrm{d}\vartheta \mathrm{d}t = 0.$$

We first observe that the linear equation $\partial u = g$ on $T^{n+1}$ admits a unique analytic solution $u$ with meanvalue zero, provided $g$ is analytic and has vanishing meanvalue. Since we will need it we formulate this well know result in quantitative terms. Denote by $H_\sigma$ the space of holomorphic functions $g(t, x)$ defined in the complex strip $\sum_\sigma = \{(x, t) \in C^{n+1} : |\text{Im } x_i| < \sigma, |\text{Im } t| < \sigma\}$ and periodic in all its variables, and abbreviate

$$|g|_\sigma =: \sup_{\Sigma_\sigma} |g|.$$

LEMMA 1. *Let $\omega$ satisfy the diophantine conditions (1.9). Assume $g \in H_\sigma$ satisfies $|g|_\sigma < \infty$ and $\int g \mathrm{d}x = 0$. Then there is a unique analytic and periodic solution $u$ satisfying*

$$(2.6) \qquad \partial u = g \text{ on } \sum_\sigma, \text{ and } \int_{T^n} u \mathrm{d}x = 0.$$

*Moreover, there is a constant $C = C(n, \tau)$ such that*

$$(2.7) \qquad |u|_{\sigma-\delta} \leq \frac{1}{\gamma}\delta^{-\tau}C|f|_\sigma, \text{ for all } 0 < \delta \leq \sigma.$$

For a proof we refer e.g. to [R]. We notice that here the variable $t$ is only a parameter. To construct the solutions one proceeds inductively.

a) First we show that $u_0 = u_1 = 0$. Indeed from the first two equations in (2.4) we conclude, in view of Lemma 1, that $u_0 = u_0(t)$ and $u_1 = u_1(t)$ are independent of the $\vartheta$-variable. Integration of

$$(2.8) \qquad \partial^2 u_2 + 2\partial \mathrm{D}_t u_1 + \mathrm{D}_t u_1 + \mathrm{D}_t^2 u_0 = f_x(t, \vartheta + u_0)$$

in the $\vartheta$-variable gives $\mathrm{D}_t^2 u_0(t) = 0$ and hence $u_0 = 0$, if the meanvalue should vanish. Integrating now

$$\partial^2 u_3 + 2\partial \mathrm{D}_t u_2 + \mathrm{D}_t^2 u_1 = f_{xx}(t, \vartheta)u_1(t)$$

in the $\vartheta$-variable over $T^n$ we find $\mathrm{D}_t^2 u_1(t) = 0$ and hence $u_1(t) = 0$.

b) Next we proceed by induction and assume that

(2.9) $$\partial^2 u_j + 2\partial \mathrm{D}_t u_{j-1} + \mathrm{D}_t^2 u_{j-1} = \varphi_{j-2},$$

$$\int_{T^n} (\mathrm{D}_t^2 u_j - \varphi_j)\mathrm{d}\vartheta = 0$$

hold true for $0 \leq j \leq n$, where quantities with negative subscripts are defined to be zero. In order to prove the statement for $j = n+1$ we first solve

(2.10) $$\partial^2 u_{n+1} = \varphi_{n-1} - 2\partial \mathrm{D}_t u_n - \mathrm{D}_t^2 u_{n-1}.$$

On account of the induction assumption the meanvalue over $T^n$ of the right hand side vanishes, and by Lemma 1 there is a solution

(2.11) $$u_{n+1} = a + b,$$

where $a = a(\vartheta, t)$ is uniquely determined, if we set

(2.12) $$\int_{T^n} a(\vartheta, t)\mathrm{d}\vartheta = 0,$$

$b = b(t)$ is arbitrary. It will be determined by the condition

$$\int_{T^n} (\mathrm{D}_t^2 u_{n+1} - \varphi_{n+1})\mathrm{d}\vartheta = 0$$

or

(2.13) $$\mathrm{D}_t^2 b = \int_{T^n} \left(\varphi_{n+1} - \mathrm{D}_t^2 a\right) \frac{\mathrm{d}\vartheta}{(2\pi)^n} = \int_{T^n} \varphi_{n+1} \frac{\mathrm{d}\vartheta}{(2\pi)^n}.$$

Observe that the average over $T^n$ of $\varphi_{n+1}$ does *not* depend on b. Indeed $\varphi_{n+1}$ is, in view of (2.5), of the form

$$\varphi_{n+1} = f_{xx}(t, \vartheta)u_{n+1} + \tilde{\varphi},$$

where $\tilde{\varphi}$ depends on $u_n, u_{n-1}, \cdots, u_1$ only. Therefore, since the meanvalue of $f_{xx}(t, \vartheta)b(t)$ is zero, the meanvalue of $\varphi_{n+1}$ is independent of b.

Now the necessary and sufficient condition for a solution of (2.13) is the vanishing of the meanvalue in the $t$ variable:

(2.14) $$\int_{T^{n+1}} \varphi_{n+1}\mathrm{d}\vartheta \mathrm{d}t = 0.$$

Assuming (2.14) to hold true there is a unique solution b of (2.13) having meanvalue zero and the induction is completed. It remains to prove (2.14).

c) For the proof of (2.14) we need

LEMMA 2. *For every* $u \in C^2(T^{n+1})$.

$$
(2.15) \qquad \int_{T^{n+1}} (1 + u_\vartheta)^T E(u) \mathrm{d}\vartheta \mathrm{d}t = 0,
$$

*where $u_\vartheta$ is the Jacobian matrix in the $\vartheta$-variable.*

PROOF. Set $1 + u_\vartheta = V$, by integration:

$$
\int_{T^{n+1}} V^T E(u) \mathrm{d}\vartheta \mathrm{d}t = \int_{T^{n+1}} \left( V^T \mathrm{D} F_p - V^T F_x \right) \mathrm{d}\vartheta \mathrm{d}t
$$

$$
= - \int_{T^{n+1}} \left( \mathrm{D} V^T F_p + V^T F_x \right) \mathrm{d}\vartheta \mathrm{d}t
$$

$$
- \int_{T^{n+1}} \frac{\partial}{\partial \vartheta} F(t, \vartheta + u, \mathrm{D}(\vartheta + u)) \mathrm{d}\vartheta \mathrm{d}t = 0.
$$

Inserting the expansion for $\alpha^2 E(u)$ into (2.15) one finds the identities

$$
\int_{T^{n+1}} \varphi_j = \int_{T^{n+1}} \sum_{s+\ell=j+2} u_{s,\vartheta}^T \Phi_\ell \mathrm{d}\vartheta \mathrm{d}t,
$$

where
$$
\Phi_\ell =: \partial^2 u_\ell + 2\partial \mathrm{D}_t u_{\ell-1} + \mathrm{D}_T^2 u_{\ell-2} - \varphi_{\ell-2},
$$

for all $j \geq 0$, and for every formal series $u$. The claim (2.14) follows immediately if we set $j = n + 1$, since the integrand on the right hand side vanishes: indeed if $s = 0$ and $s = 1$, then $u_0 = u_1 = 0$. If $s \geq 2$, then by the induction assumption and by (2.10), $\Phi_\ell = 0$ for all $\ell \leq n + 1$. This finishes the proof of the unique formal power series.

## 3. - Existence and asymptotic character

In this section we give the necessary details in order to prove (1.17)-(1.19). First we observe that the set

$$
A := \left\{ \alpha > 0 : \left| \frac{1}{\alpha} \langle \omega, j \rangle + m \right| \geq \gamma |j|^{-\tau} \text{ for all } j, m \in \mathbb{Z}^{n+1}, j \neq 0 \right\}
$$

has positive Lebesgue measure $\mu$ provided the constant $\gamma$ is sufficiently small. Here $\omega$ is a fixed vector with rationally independent components and $\tau$ is a constant satysfying $\tau > n + 1$. More precisely:

LEMMA 3. *Fix* $0 < \lambda < 1$. *Then there is a constant* $\gamma^* = \gamma^*(\lambda)$ *such that for* $0 < \gamma \leq \gamma^*$

(3.1)                     $\mu\left\{\alpha \in A \,|\, 0 < \alpha \leq \epsilon\right\} \geq \epsilon(1 - \lambda),$

*for every* $0 < \epsilon \leq 1$.

PROOF. Assume $\gamma \leq \frac{1}{2}$, we prove that $\mu(B_\epsilon) \leq \epsilon\lambda$ if $\gamma$ is sufficiently small, where $B_\epsilon = (0, \epsilon) \backslash A$ is the complement. We have

$$\mu(B_\epsilon) \leq \mu\left(\bigcup_{(j.m) \neq 0} A_{jm}\right)$$

where

$$A_{jm} = \left\{0 < \alpha \leq \epsilon : \left|\frac{1}{\alpha} - \frac{m}{\langle\omega, j\rangle}\right| < \frac{\gamma}{|\langle\omega, j\rangle||j|^\tau}\right\}.$$

In view of $\gamma \leq \frac{1}{2}$ one verifies readily that

$$\sum_{(j,m)} \mu(A_{jm}) \leq 4\gamma \sum_{j \neq 0} \frac{|\langle\omega, j\rangle|}{|j|^\tau} \sum_{|m| \geq \frac{1}{\epsilon}|\langle\omega,j\rangle|} \frac{1}{m^2}.$$

Since the sum over $m$ is dominated by $\frac{2\epsilon}{|\langle\omega,j\rangle|}$ we conclude that

$$\mu(B_\epsilon) \leq 8\gamma\epsilon \sum_{j \neq 0} \frac{1}{|j|^\tau}.$$

In view of $\tau > n + 1$, the right hand side is equal to $8\gamma\epsilon C$. Therefore, defining $\gamma^*(\lambda) = \min\left\{\frac{1}{2}, \frac{\lambda}{8C}\right\}$, one concludes that $\mu(B_\epsilon) \leq \lambda\epsilon$ as claimed.
     Now, we can state our main result.

THEOREM. *Assume* $\gamma < \gamma^*$. *Assume* $f$ *is real analytic in the (closure of the) complex strip* $\sum_\sigma$ *for some* $1 \geq \sigma > 0$. *For every* $N \geq 2$, *there exist positive constants* $\alpha^* = \alpha^*(N)$ *and* $C_N$ *with the following properties:*

     *For* $\alpha \in A(\omega)$ *satisfying* $|\alpha| < \alpha^*$ *there is a unique* $u_\alpha$ *real-analytic in, say,* $\sum_{\sigma/8}$ *and of mean value 0 such that*

(3.2)                     $E(u_\alpha) = 0 \quad in \ \sum_{\sigma/8}$

*and*

(3.3)
$$\left| u_\alpha - \sum_{j \geq 2}^{N} \alpha^j u_j \right|_{\sigma/8} \leq C_N |\alpha|^{N+1}.$$

The proof rests on the discussion in section 2 and on the following KAM result, for which we refer to [SZ] (Theorem 1) and [CC] (Lemma 6).

LEMMA 4. *Let $f$ be as in the above Theorem. Let $\omega$ satisfy* (1.9) *and let $v \in H_\sigma$ with $|v|_\sigma \leq \sigma, |v_\vartheta|_\sigma \leq \frac{1}{2}$. There exists a costant $C = C(n, f, \sigma, \gamma, \tau)$ such that if*

(3.4)
$$C|E(v)|_\sigma \leq 1,$$

*then there is a unique real analytic $u \in U_{\sigma/2}$ satisfying*

(3.5)
$$E(u) = 0, \quad \int (u - v) = 0, \quad |u - v|_{\sigma/2} < C|E(v)|_\sigma.$$

PROOF OF THE THEOREM. Applying iteratively Lemma 1 and the Cauchy estimates (to control derivatives in terms of functions) to the $u_i's$ constructed in section 2, one finds estimates of the form

$$|u_i|_{\sigma/2} \leq K_i, 2 \leq i \leq N,$$

with constants $K_i$ depending on $n, f$ and $\gamma, \tau$. Thus one can find an $\alpha_0^*$ so small, that for $|\alpha| < \alpha_0^*$ one has

(3.6)
$$|\tilde{u}_N|_{\sigma/4} < \frac{\sigma}{4}, |\tilde{u}_{N,\vartheta}|_{\sigma/4} \leq \frac{1}{2},$$

where, as above, $\tilde{u}_N =: \sum_{j=2}^{N} \alpha^j u_j$. Moreover, Taylor's formula leads to the bound

(3.7)
$$|E(\tilde{u}_N)|_{\sigma/4} \leq K_N^* |\alpha|^{N-1}.$$

Now, if we set

$$\alpha^* = \min \left\{ \alpha_0^*, (CK_N^*)^{\frac{1}{1-N}} \right\},$$

the Theorem follows from Lemma 4 simply replacing $\omega$ by $\frac{\omega}{\alpha}$ $(\alpha \epsilon A(\omega))$, $\sigma$ by $\frac{\sigma}{4}$ and $v$ by $\tilde{u}_N$. In this case (3.3) holds with $C_N =: CK_N^*$.

This theorem gives a precise meaning to the asymptotic character of the series $\sum \alpha^i u_i$ which, as mentioned in the introduction, is in general divergent. It would, therefore, also be desirable to have good estimates for the functions

$u_j$. In the special case in which $n = 1$ the operator $\partial$ is simply the differential operator $\omega \frac{\partial}{\partial \vartheta}$. We may therefore assume $\omega = 1$ and find the following estimates:

PROPOSITION. *Assume* $n = 1 = \omega$, *and assume that* $f$ *is analytic and bounded on the strip* $\sum_\sigma$ *with* $0 < \sigma \le 1$. *Then the unique formal power series in section 2 satisfies*

$$|u_{j+2}|_{\sigma/2} \le B^{j+2} j^{2j} \text{ for all } j \ge 0.$$

*Here* $B = \left(\frac{30M}{\sigma}\right)^2$ *with* $M = \max\{|f|_\sigma, |f_x|_\sigma, 1\}$.

We shall use the following

LEMMA 4. *For all* $j \ge 1$:

$$\sum_{k_1 + 2k_2 + \cdots + jk_j = j} \prod_{s=1}^{j} \frac{1}{k_s!} < e^4.$$

PROOF. Using the generating functions, the left hand side of the inequality is equal to

$$\frac{1}{j!}\left(\frac{d}{d\alpha}\right)^j \exp\left(\sum_1^\infty \alpha^s\right)\Bigg|_{\alpha=0}$$

$$= \frac{1}{j!}\left(\frac{d}{d\alpha}\right)^j \exp\left(\frac{\alpha}{1-\alpha}\right)\Bigg|_{\alpha=0} = \frac{e^{-1}}{j!}\sum_{n=1}^\infty \frac{(n+j-1)(n+j-2)\cdots n}{n!},$$

so that the claim follows from

$$\frac{(n+j-1)(n+j-2)\cdots n}{j!} < 4^n \text{ for all } n, j \ge 1.$$

PROOF OF THE PROPOSITION. Recall that $u_0 = u_1 = 0$, and

(3.8)                        $u_j = a_j + b_j, \quad j \ge 2,$

is determined by

(3.9)                        $\int a_j(\vartheta, t) d\vartheta = 0, \int b_j(t) dt = 0,$

(3.10)                       $\partial_\vartheta^2 a_{j+2} = -2\partial_\vartheta \partial_t a_{j+1} - \partial^2 u_j - \varphi_j$

(3.11)                       $\partial_t^2 b_{j+2} = \int \varphi_{j+2} d\vartheta,$

where $\varphi_0 = f_x(\vartheta, t), \varphi_1 = 0$ and where, for $j \geq 2$

(3.12)
$$\varphi_j = \sum_{k \in P_j} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^{j} \frac{u_s^{k_s}}{k_s!}$$

$$= \frac{1}{j!} \left( \frac{d}{d\alpha} \right)^j f_x \left( t, \vartheta + \sum_{n \geq 2} \alpha^n u_n \right) \Bigg|_{\alpha = 0}.$$

here

$$P_j = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j\}.$$

and $|k| = k_2 + k_3 + \cdots + k_j$. Setting

$$P_{j+2}^* = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j + 2\}$$

we can rewrite equation (3.11) as

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2} d\vartheta = \int \left\{ f_{xx} a_{j+2} + \sum_{P_{j+2}^*} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^{j+2} \frac{u_s^{k_s}}{k_s!} \right\}.$$

Integrating the first term by parts and inserting the equation (3.10) for $a_{j+2}$ gives

(3.13)
$$\partial_t^2 b_{j+2} = - \int f \left( 2\partial_\vartheta \partial_t a_{j+1} + \partial_t^2 u_j - \varphi_j \right) + \int \Psi_j d\vartheta,$$

where

(3.14)
$$\Psi_j = \Psi_j(u_j, u_{j-1}, \cdots, u_2) = \sum_{P_{j+2}^*} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^{j} \frac{u_s^{k_s}}{k_s!}.$$

We proof first the Lemma for $j = 0$. From

$$\partial_t^2 b_2 = \int f f_x d\vartheta = 0$$

we conclude that $b_2 = 0$ so that $u_2 = a_2$. Since the meanvalue of $a_2$ vanishes we conclude that

$$|u_2|_\sigma \leq |\partial_\vartheta^2 u_2|_\sigma = |f_x|_\sigma \leq M,$$

which proves the Lemma for $j = 0$.

Assume now $j \geq 1$. We shall show that

(3.15)
$$|u_{i+2}|_{\sigma_i} \leq B^{i+1} j^{2i} \quad \text{for all } 0 \leq i \leq j,$$

where

$$\text{(3.16)} \qquad \sigma_i = \sigma\left(1 - \frac{i}{2j}\right).$$

The Lemma then follows by setting $i = j$. The estimate (3.15) will be proved by induction in $i$. In the case $i = 0$, (3.15) is already proved above for $\sigma_0 = \sigma$ and we shall assume now that

$$\text{(3.17)} \qquad |u_{s+2}|_{\sigma_s} \leq B^{s+1} j^{2s}, 0 \leq s \leq i - 1,$$

where, of course, $1 \leq i \leq j$. From (3.9), (3.10) and (3.13) we conclude

$$\text{(3.18)} \qquad \begin{aligned} |u_{i+2}|_{\sigma_i} &\leq |a_{i+2}|_{\sigma_i} + |b_{i+2}|_{\sigma_i} \\ &\leq |\partial_\vartheta^2 a_{i+2}|_{\sigma_i} + |\partial_t^2 b_{i+2}|_{\sigma_i} \\ &\leq 4M|\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} + 2M|\partial_t^2 u_i|_{\sigma_i} + 2M|\varphi_i|_{\sigma_i} + |\Psi_i|_{\sigma_i}. \end{aligned}$$

We estimate each term separately. Using the Cauchy estimates and the induction hypothesis (3.17) one finds

$$\text{(3.19)} \qquad \begin{aligned} |\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} &\leq \frac{1}{(\sigma_{i-1} - \sigma_i)^2} |u_{i+1}|_{\sigma_{i-1}} = \left(\frac{2j}{\sigma}\right)^2 |u_{i+1}|_{\sigma_i} \\ &\leq \frac{4}{\sigma^2} B^i j^{2i}, \end{aligned}$$

similarly

$$\text{(3.20)} \qquad |\partial_t^2 u_i|_{\sigma_i} \leq \left(\frac{j}{\sigma}\right)^2 |u_i|_{\sigma_{i-2}} \leq \frac{1}{\sigma^2} B^{i-1} j^{2(i-1)}.$$

Observe now that $\sigma_{s-2} \geq \sigma_s \geq \sigma_i$, and $B \geq \frac{2}{\sigma}, i \geq 1$ and that $|k| \geq 1$ for $k \in P_i$, then

$$\begin{aligned} |\varphi_i|_{\sigma_i} &\leq M \sum_{P_i} \frac{1}{(\sigma - \sigma_i)^{|k|}} \prod_{s=2}^{i} \frac{|u_s|_{\sigma_{s-2}}^{k_s}}{k_s!} \\ &\leq M \sum_{P_i} \left(\frac{2j}{i\sigma}\right)^{|k|} \frac{B^i j^{2i}}{B^{|k|} j^{4|k|}} \prod_{s=2}^{i} \frac{1}{k_s!} \\ &\leq \frac{2M}{\sigma} B^{i-1} j^{2i-3} \sum_{P_i} \prod_{s=1}^{i} \frac{1}{k_s!}, \end{aligned}$$

so that, by Lemma 4,

$$\text{(3.21)} \qquad |\varphi_i|_{\sigma_i} \leq \frac{1}{\sigma} 2e^4 M B^{i-1} j^{2i-3}.$$

.

Observing that, if $k \in P^*_{i+2}$, then $2k_2 + \cdots + jk_j = j + 2$ and $|k| \geq 2$, one concludes similarly

(3.22)                        $$|\Psi_i|_{\sigma_i} \leq 4e^4 M B^i j^{2(i-1)}.$$

Adding up we find from (3.18)-(3.22) that

$$|u_{i+2}|_{\sigma_i} \leq \frac{1}{\sigma^2}(16 + 2 + 8e^4)M^2 B^i j^{2i},$$

$$< \frac{900 M^2}{\sigma^2} B^i j^{2i} = B^{i+1} j^{2i},$$

where we have used the definition of the constant $B$. This finishes the proof of the proposition.

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