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Corner Mellin Operators and Reduction of Orders with Parameters

B.W. SCHULZE

1. - Introduction

This paper is part of the program to establish a calculus of pseudo-differential operators (ψDO's) on manifolds with singularities. The singularities, in our context, may be generated successively by 'conifications' and 'edgifications' of given geometric objects, starting with $\mathbb{R}_+$ and a closed compact C^∞ manifold X. The conification of X is then $X^\wedge := \mathbb{R}_+ \times X$ (geometrically it is thought as a cone $\setminus\{\text{vertex}\}$ with base X). A 'manifold' M with conical singularities is intuitively defined as a compact topological space with exceptional points $v_1, \ldots, v_N$, so that $M\setminus\{v_1, \ldots, v_N\}$ is a C^∞ manifold, and M may be identified locally, near any vertex v_j, with $\overline{\mathbb{R}}_+ \times X_j \setminus \{o\} \times X_j$, for some closed compact C^∞ manifold X_j, where we also keep in mind the local $\mathbb{R}_+$ actions $\lambda(t, x) = (\lambda t, x)$, $(t, x) \in \mathbb{R}_+ \times X_j$, $\lambda \in \mathbb{R}_+$. A precise definition may be found, for instance, in [S2]. Now we can pass to a further conification $M^\wedge = \mathbb{R}_+ \times M$ which is the local model of a corner, so that we can define manifolds with corners globally and so on. The edgification is defined by $\mathbb{R}^q \times M$, where M is a manifold with conical singularities. This gives rise to an evident global definition of manifolds with edges. In particular, a manifold with corners always has outgoing edges of dimension $q = 1$. We also can talk about manifolds with boundaries and boundary value problems. From the point of view of edges, they are included anyway, since $\mathbb{R}^q \times \mathbb{R}_+$, as the local model, is the edgification of the cone with zero-dimensional base. The program is now, parallel to this geometric picture, to realize function spaces and operator algebras with symbolic structures for studying the solvability for natural classes of differential operators on the underlying spaces with singularities.

It is custom (and well motivated, cf. e.g. [S2], [S4]) to say that the operators of Fuchs type (= the 'totally characteristic' ones) are natural for the

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cone. In the coordinates $(t,x) \in \quad_+ \times X$, close to a vertex, they are of the form

$$\mathrm{A}(t,x,\partial_t,\mathrm{D}_x)=\sum_{j=0}^{\mu} A_j(t)\ \left(-t\ \frac{\partial}{\partial t}\right)^j, \tag{1}$$

with $A_j(t) := A_j(t,x,\mathrm{D}_x) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$, where $\mathrm{Diff}^\nu(X)$ denotes the space of all smooth differential operators on X of order ν (smooth means with C^∞ coefficients in local coordinates).

Let us denote by $\mathrm{Diff}^\mu(M)$ the class of all (smooth) differential operators on $M \backslash \{v_1,\ldots,v_N\}$ of order μ, M being a manifold with conical singularities $v_1,\ldots,v_N$, where any $\mathrm{A} \in \mathrm{Diff}^\mu(M)$ admits, close to v_p, a representation of the form (1), with respect to the corresponding base X_p, $p=1,\ldots,N$. For the corner $M^\wedge = \mathbb{R}_+ \times M$, we define $\mathrm{Diff}^\mu(M^\wedge)$ as the space of all differential operators on $M^\wedge \backslash (\mathbb{R}_+ \times \{v_1,\ldots,v_N\})$ of order μ which are, close to $r=0$, of the form

$$\mathrm{A}=\sum_{k=0}^{\mu} A_k(r)\left(-r\ \frac{\partial}{\partial r}\right)^k, \tag{2}$$

with certain $A_k(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-k}(M))$, and similarly close to $r=\infty$.

Let us also mention the form of the natural operators over $\mathbb{R}^q \times M$ (cf. [S4]). We say that $\mathrm{A} \in \mathrm{Diff}^\mu(\mathbb{R}^q \times M)$ if we have locally, close to $\mathbb{R}^q \times \{v_p\}$,

$$\mathrm{A}=\sum_{|\alpha|\le\mu} A_\alpha(y)(t\mathrm{D}_y)^\alpha, \tag{3}$$

with certain $A_\alpha(y) \in C^\infty(\mathbb{R}^q, \mathrm{Diff}^{\mu-|\alpha|}(X_p^\wedge))$, X_p being the base of the cone belonging to v_p, $p=1,\ldots,N$. The variable y runs along the edge $\mathbb{R}^q$, $\alpha=(\alpha_1,\ldots,\alpha_q)$ is a multi-index.

An analogous definition applies for any open subset $\Omega \subset \mathbb{R}^q$, which yields the class $\mathrm{Diff}^\mu(\Omega \times M)$. It is then easy to see that the operators (2) also belong to $\mathrm{Diff}^\mu(\mathbb{R}_+ \times M)$, in the sense of the interpretation of $\mathbb{R}_+$ as open subset of $\mathbb{R}^1$, considered as edge of dimension $q=1$.

The explicit expressions (1), (2) and (3) show that the operators degenerate close to the singularities in some typical way. In other words, apart from the geometric picture, we could talk as well about the solvability for classes of degenerate operators. Another point of view is to emphasize the non-compactness of space \{singularities} and to perform a calculus on a non-compact manifold with a special Riemannian metric, for instance, $\mathbb{R}^n$ or a space with 'cylindrical' exits.

Many authors have studied such problems under the different aspects and motivations (such as applications in physics and technical disciplines, but also in index theory, geometry, topology) with very different degree of generality.

Let us mention, for instance, Kondrat'ev [K1], Plamenevskij [P1], Grisvard [G1], M. Dauge [D3], Teleman [T1], [T2], Brüning, Seeley [B1], Melrose, Mendoza [M1], Rempel, Schulze [R1], [R2], [S2], Unterberger [U1]. Further references were given in [K2].

The goal of the present paper is to develop the tools for a systematic calculus of ψDO's on manifolds with corners including the asymptotic properties of the solutions. This is a very complex program and of independent interest. It yields a deeper insight into the structure of cone algebras and further useful properties, such as reductions of orders and the parameter-depending theory. The calculus for the corner itself will be subject of a forthcoming paper (cf. [S5]).

The present paper is organized as follows. In Chapter 2 we investigate pseudo-differential operators with operator-valued symbols, based on the Mellin transform. It may be considered as a pseudo-differential calculus, where the symbols are totally characteristic with respect to several variables. The behaviour near the corners both of the distributions and of the symbols is controlled. Chapter 3 contains the material on parameter-depending cone ψDO's. In particular we establish a result on reduction of orders for the cone. This is, in Chapter 4, the starting point for the corner Sobolev spaces and the corner Mellin operators. The structure of the cone algebra shows the scheme how to generate a corner algebra with the analogous Mellin operators.

2. - Mellin Pseudo-Differential Calculus on $(\mathbb{R}_+)^m$ with ψDO-Valued Symbols

2.1. *Symbol Spaces and Continuity in* $\mathcal{H}^s((\mathbb{R}_+)^m, E)$

This section presents the material on Mellin pseudo-differential operators on $Q^m := (\mathbb{R}_+)^m$ that we need later on in the case $m = 2$.

The calculus for $m \geq 2$ has analogous applications for corners of higher orders, whereas the case $m = 1$ corresponds to conical singularities.

Let us first introduce some notations on the Mellin transform on $\mathbb{R}_+$. Let $u \in C_0^\infty(\mathbb{R}_+)$ and

$$(1) \qquad (Mu)(z) = \int_0^\infty t^{z-1}\, u(t)\mathrm{d}t, \;\; z \in \mathbb{C},$$

be its Mellin transform. Then the inverse is given by

$$(2) \qquad (M^{-1}g)(t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}}} t^{-z} g(z)\mathrm{d}z,$$

where $\Gamma_\rho = \{z : \mathrm{Re}\; z = \mathrm{Re}\; \rho\}$. It is well-known that $u \to Mu|_{\Gamma_{\frac{1}{2}}}$ extends, by continuity, to an isomorphism

$$M : L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}}),$$

where the inverse is an extension of (2). In virtue of $M^{-1}zM = -t\,\frac{\partial}{\partial t}$ defined on the subspace of all $u \in L^2(\mathbb{R}_+)$ with $t\,\frac{\partial}{\partial t}\,u \in L^2(\mathbb{R}_+)$, we are motivated to introduce Mellin ψDO's by

$$op_M(a)u = M^{-1}a(t,z)Mu,$$

where $a(t,z)$ runs over some space of symbols. It will be defined below in more detail.

We are interested in a generalization of this idea in two directions. First, we consider t as a variable on $Q^m = (\mathbb{R}_+)^m$ and deal with the m-dimensional analogue of the Mellin transform. Moreover we admit the symbols to have values in a space of operators which also has a symbolic structure, in our case $L^\mu(\mathbb{R}^n)$, the space of ψDO's over $\mathbb{R}^n$ of order μ, defined by means of the Fourier transform. In Chapter 4 we shall also deal with the cone algebra instead.

In addition we study the behaviour up to $t = 0$, which is a specific novelty compared with the standard calculus of ψDO's.

In order to unify the different versions for the calculus we want to give an axiomatic description of the generalities.

Let E be a Banach space and $\{x_\lambda : \lambda \in \mathbb{R}_+\}$ be a group of linear continuous operators on E, $x_\lambda x_\rho = x_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}_+$. Denote by $\tau = (\tau_1, \dots, \tau_m)$ the covariable in $\mathbb{R}^m$ (dual to $t = (t_1, \dots, t_m) \in Q^m$) and let $[\tau]$ be a strictly positive function with $[\tau] \geq \frac{3}{2}$, $[\tau] = |\tau|$, for $|\tau| \geq 2$. Then, we set

$$x(\tau) = x_{[\tau]}, \quad \tau \in \mathbb{R}^m.$$

1. DEFINITION. Let E, $\tilde{E}$ be Banach spaces and x_λ and $\tilde{x}_\lambda$ be fixed groups of linear continuous operators on E and $\tilde{E}$, respectively. Then $S^\mu(Q^p \times \mathbb{R}^m; E, \tilde{E})$, $\mu \in \mathbb{R}$, denotes the space of all

$$a(r,\tau) \in C^\infty\left(Q^p \times \mathbb{R}^m,\ \mathcal{L}(E,\tilde{E})\right),$$

for which

$$\left\|\tilde{x}(\tau)^{-1}(\mathrm{D}_r^\alpha \mathrm{D}_\tau^\beta a(r,\tau))\ x(\tau)\right\|_{\mathcal{L}(E,\tilde{E})} \leq c[\tau]^{\mu-|\beta|}, \tag{3}$$

for all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^m$ and $r \in K \subset\subset Q^p$, $\tau \in \mathbb{R}^m$, with a constant $c = c(\alpha, \beta, K) > 0$.

We consider $S^\mu(\dots)$ in the Fréchet space topology, given by the best constants in the estimates (3).

Next we want to introduce Sobolev spaces of vector valued distributions on Q^m. We identify the variable τ with a point on $\Gamma^m_{\frac{1}{2}} = \Gamma_{\frac{1}{2}} \times \ldots \times \Gamma_{\frac{1}{2}}$. Then the m-dimensional Mellin transform

$$(Mu)(z) = \int_0^\infty \ldots \int_0^\infty t_1^{z_1 - 1} \ldots t_m^{z_m - 1} \; u(t_1, \ldots, t_m) \mathrm{d}t_1 \ldots \mathrm{d}t_m,$$

$z = (z_1, \ldots, z_m)$, $u \in C_0^\infty(Q^m)$, extends to an isomorphism

$$M : L^2(Q^m) \to L^2(\Gamma^m_{\frac{1}{2}}).$$

It is a simple exercise to check that the one-dimensional standard formulas for the Mellin transform have the corresponding analogue in higher dimensions. In particular (2) extends to the m-dimensional case. Write, for $a(t, t', \tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$,

$$op_M(a)u(t) = M^{-1}_{\tau \to t} \; M_{t' \to \tau} \; a(t, t', \tau) u(t').$$

Then

$$op_M(a) : C_0^\infty(Q^m, E) \to C^\infty(Q^m, \tilde{E}) \tag{4}$$

is continuous.

2. Definition. Let E, x_λ be as above, and $s \in \mathbb{R}$. Then $\mathcal{H}^s(Q^m, E)$ denotes the closure of $C_0^\infty(Q^m, E)$ with respect to the norm

$$\|u\|_{\mathcal{H}^s(Q^m, E)} = \left\{ \int_{\mathbb{R}^m} [\tau]^{2s} \; \|x^{-1}(\tau) M_{t \to \tau} u(\tau)\|^2_E \; \mathrm{d}\tau \right\}^{\frac{1}{2}}.$$

Point out that the interpretation of $x^{-1}(\tau) Mu(\tau)$ is that $x^{-1}(\tau)$ acts on the values of Mu in E, for every fixed τ.

Let us consider a typical example which is a motivation of our definition.

Let F denote the Fourier transform in $\mathbb{R}^n$, and let $\mathcal{H}^s(Q^m \times \mathbb{R}^n)$ be the closure of $C_0^\infty(Q^m \times \mathbb{R}^n)$ with respect to the norm

$$\left\{ \int ([\tau]^2 + |\xi|^2)^s \; |(M_{t \to \tau} F_{x \to \xi} u)(\tau, \xi)|^2 \, \mathrm{d}\tau \; \mathrm{d}\xi \right\}^{\frac{1}{2}}. \tag{5}$$

Then a straightforward calculation shows that (5) equals

$$\left\{ \int [\tau]^{2s} \; \left\{ \int (1 + |\xi|^2)^s \; |(MFu)(\tau, [\tau]\xi)|^2 \, [\tau]^n \; \mathrm{d}\xi \right\} \mathrm{d}\tau \right\}^{\frac{1}{2}}.$$

Write $(x_\lambda v)(x) = \lambda^{\frac{n}{2}} v(\lambda x)$, v defined on $\mathbb{R}^n$. Then $x_\lambda F v = F x_\lambda^{-1} v$, for all $\lambda \in \mathbb{R}_+$; in other words, our norm expression is equivalent to

$$\left\{ \int [\tau]^{2s} \ \| \ \dot{x}^{-1}(\tau) \ Mu(\tau, \cdot)\|^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}\tau \right\}^{\frac{1}{2}} .$$

Thus, in this case, $E = H^s(\mathbb{R}^n)$, x_λ is a homothety up to a power of λ, and $\mathcal{H}^s(Q^m \times \mathbb{R}^n) = \mathcal{H}^s(Q^m, H^s(\mathbb{R}^n))$. This also is an example for the following property. There are constants σ and c so that

$$\|x(\tau)\|_{\mathcal{L}(E)}, \ \|x^{-1}(\tau)\|_{\mathcal{L}(E)} \le c[\tau]^\sigma, \tag{6}$$

for all $\tau \in \mathbb{R}^m$.

It can be proved that (6) is equivalent with $\|x_\lambda\|_{\mathcal{L}(E)} \le L$, for all $\lambda \in [e^{-a}, e^a]$, with certain constants L, $a > 0$. This is satisfied, in particular, when $x_\lambda : \mathbb{R}_+ \to \mathcal{L}(E)$ is continuous in the strong topology.

In our applications, we also have the following property.

$\mathcal{H}^s(Q^m, E)$ is a $C_0^\infty(\overline{Q^m})$-module, and $\|\mathcal{M}_\varphi\|_{\mathcal{L}(\mathcal{H}^s(Q^m,E))} \to 0$ as $\varphi \to 0$ in $C_0^\infty(\overline{Q^m})$. Here $\mathcal{M}_\varphi$ is the operator of multiplication by φ.

In the abstract setting, we shall suppose once and for all that this module property holds. In addition, we assume (6) with $\sigma = \sigma(E, x_\lambda)$.

Let us mention the following general result (cf. [S6]).

3. THEOREM. *Let E, E^0 be Hilbert spaces and $E \hookrightarrow E^0$ dense, $x_\lambda \in \mathcal{L}(E) \cap \mathcal{L}(E^0)$, and x_λ a group of unitary operators on E^0. Then, $\mathcal{H}^s(Q^m, E)$ is a $C_0^\infty(\overline{Q^m})$-module and $\mathcal{M}_\varphi \to 0$ in $\mathcal{L}(\mathcal{H}^s(Q^m, E))$, for $\varphi \to 0$ in $C_0^\infty(\overline{Q^m})$, $s \in \mathbb{R}$ arbitrary.*

The proof is based on similar considerations that are used for proving the continuity of pseudo-differential operators in Sobolev spaces (cf. e.g. [R3], Section 1.2.3.5.). In the action of $\mathcal{M}_\varphi$ the covariable disappears.

Denote by $\mathcal{H}^s_{\mathrm{loc}}(Q^m, E)$ ($\mathcal{H}^s_{\mathrm{comp}}(Q^m, E)$) the space of all $u \in \mathcal{D}'(Q^m, E)$, with $\varphi u \in \mathcal{H}^s(Q^m, E)$, for any $\varphi \in C_0^\infty(Q^m)$ ($u \in \mathcal{H}^s(Q^m, E)$ with compact support with respect to $t \in Q^m$).

For notational convenience, it is useful also to consider the symmetry of the spaces $\mathcal{H}^s(Q^m, E)$ induced by the transformations

$$(I_j u)(t_1, \ldots, t_m) = t_j^{-1} u(t_1, \ldots, t_j^{-1}, \ldots, t_m),$$

$j = 1, \ldots, m$. In view of

$$(MI_j u)(z_1, \ldots, z_m) = (Mu)(z_1, \ldots, 1 - z_j, \ldots, z_m),$$

then follows (up to equivalence of norms)

$$\|u\|_{\mathcal{H}^s(Q^m,E)} = \|I_j u\|_{\mathcal{H}^s(Q^m,E)}.$$

In other words, I_j induces isomorphisms

$$I_j : \mathcal{H}^s(Q^m, E) \to \mathcal{H}^s(Q^m, E),$$

for all $s \in \mathbb{R}$.

4. DEFINITION. $S^\mu(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E})$, $\mu \in \mathbb{R}$, denotes the space of all $a(r,\tau) \in C^\infty(\overline{Q^p} \times \mathbb{R}^m, \ \mathcal{L}(E, \tilde{E}))$ for which

(i) $\|\tilde{x}(\tau)^{-1}(D_r^\alpha D_\tau^\beta a(r,\tau))x(\tau)\|_{\mathcal{L}(E,\tilde{E})} \leq c[\tau]^{\mu-|\beta|}$ for all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^m$ and $r \in K \subset\subset \overline{Q^p}$, for every compact set K, and all $\tau \in \mathbb{R}^m$, with a constant $c = c(\alpha, \beta, K) > 0$,

(ii) $a_j(r,\tau) := a(r_1, \ldots, r_j^{-1}, \ldots, r_p, \tau) \in C^\infty(\overline{Q^p} \times \mathbb{R}^m; \mathcal{L}(E, \tilde{E}))$, and a_j satisfies the estimates of (i) for $j = 1, \ldots, p$.

$S^\mu(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E})$ is a Fréchet space with natural semi-norms that follows immediately from the definition.

5. REMARK. Let $E, \tilde{E}$ be Banach spaces, $L = \mathcal{L}(E, \tilde{E})$ the space of linear continuous operators in the norm topology, and x_λ and $\tilde{x}_\lambda$ groups of operators on E and $\tilde{E}$, respectively. Then,

$$\tilde{x}_\lambda^{-1} \ \mathbf{A} \ x_\lambda := \delta_\lambda \mathbf{A}, \ \ \lambda \in \mathbb{R}_+$$

is a group of linear continuous operators on L, and (6) implies an analogous estimate for $\delta(\tau) = \delta_{[\tau]}$. In particular, we also can define the spaces $\mathcal{H}^s(Q^m, L)$, $s \in \mathbb{R}$.

It is clear that, for $\delta(\tau) := \delta_{[\tau]}$,

$$\|\delta(\tau)\mathbf{A}\|_{\mathcal{L}(E,\tilde{E})}, \ \|\delta(\tau)^{-1}\mathbf{A}\|_{\mathcal{L}(E,\tilde{E})} \leq c_A[\tau]^{\sigma+\tilde{\sigma}}, \tag{7}$$

with a constant $c_A > 0$ and $\sigma, \tilde{\sigma}$ given by (6), for $x(\tau)$ and $\tilde{x}(\tau)$, respectively.

The spaces $S^\mu(Q^p \times \mathbb{R}^m; E, \tilde{E})$, in Definition 1, depend on the groups $x_\lambda, \tilde{x}_\lambda$ which are usually kept fixed. Sometimes, it will be useful also to consider the trivial actions, i.e. when the groups only consist of the identity. Denote the corresponding space by $S^\mu_{(1)}(Q^p \times \mathbb{R}^m; E, \tilde{E})$. Then we have the following simple

6. LEMMA. *For every $\mu \in \mathbb{R}$, we have continuous embeddings*

$$S^\mu(Q^p \times \mathbb{R}^m; E, \tilde{E}) \hookrightarrow S^{\mu+\sigma+\tilde{\sigma}}_{(1)}(Q^p \times \mathbb{R}^m; E, \tilde{E}),$$

$$S^\mu_{(1)}(Q^p \times \mathbb{R}^m; E, \tilde{E}) \hookrightarrow S^{\mu+\sigma+\tilde{\sigma}}(Q^p \times \mathbb{R}^m; E, \tilde{E}).$$

In particular,

$$S^{-\infty}(Q^p \times \mathbb{R}^m; E, \tilde{E}) = S^{-\infty}_{(1)}(Q^p \times \mathbb{R}^m; E, \tilde{E}).$$

An analogous statement holds for the symbol spaces over $\overline{Q^p}$. The proof is obvious.

7. PROPOSITION. *Let* $a_j \in S^{\mu_j}(Q^p \times \mathbb{R}^m; E, \tilde{E})$, $j \in \mathbb{N}$, *be a sequence* $\mu_j \to -\infty$, *as* $j \to \infty$. *Then there exists an* $a \in S^{\mu}(Q^p \times \mathbb{R}^m; E, \tilde{E})$, $\mu = \max\{\mu_j\}$, *such that*

$$\operatorname{ord}\left(a - \sum_{j=0}^{N} a_j\right) \to -\infty, \text{ as } N \to \infty.$$

If $\tilde{a}$ is another such symbol in S^μ, then $a - \tilde{a} \in S^{-\infty}(Q^p \times \mathbb{R}^m; E, \tilde{E})$. An analogous statement holds for $a_j \in S^{\mu_j}(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E})$, then $a \in S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E})$, $a - \tilde{a} \in S^{-\infty}(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E})$.

As usual, we write $a \sim \sum_{j=0}^{\infty} a_j$. If χ is an excision function, i.e. $\chi \in C^\infty(\mathbb{R}^m)$, $\chi(\tau) = 0$ close to $\tau = 0$, $\chi(\tau) = 1$ for $|\tau| \geq$ const., then

$$a(t,\tau) = \sum_{j=0}^{\infty} \chi(c_j^{-1}\tau) a_j(t,\tau) \tag{8}$$

converges in $S^\mu(Q^p \times \mathbb{R}^m; E, \tilde{E})$ $(S^\mu(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E}))$, for $c_j \to \infty$, sufficiently fast, and thus $a \sim \sum a_j$. The proof of the convergence is completely analogous as for scalar symbols.

Remark that the excision of a symbol, by means of χ, is a rather brutal procedure in connection with operator-valued symbols. It may be compared with the excision of a standard ψDO symbol $a(x,\xi) \in S^\mu(\mathbb{R}^{m+n} \times \mathbb{R}^{m+n})$, $\xi = (\xi', \xi'') \in \mathbb{R}^{m+n}$, by an excision function with respect to ξ' only. In other words, the negligible symbols in $S^{-\infty}(\ldots; E, \tilde{E})$ have no particular regularity with respect to the operation $E \to \tilde{E}$. In our applications, such a regularity is very essential. So we shall replace, later on, (8) by another more precise method of obtaining asymptotic sums.

An operator function

$$a(t,\tau) : Q^p \times \mathbb{R}^m \to \mathcal{L}(E, \tilde{E})$$

is called homogeneous for $|\tau| \geq c$ of order μ, if

$$a(t, \lambda\tau) = \lambda^\mu \, \tilde{x}_\lambda a(t,\tau) \, x_\lambda^{-1},$$

for all $t, \tau, |\tau| \geq c$, $\lambda \geq 1$. Denote by $S^{(\mu)}(Q^p \times \mathbb{R}^m; E, \tilde{E})$ the space of all $a \in C^\infty(Q^p \times \mathbb{R}^m, \mathcal{L}(E, \tilde{E}))$ which are homogeneous of order μ for $|\tau| \geq c$, $c = c(a)$. Analogous notations will be used with respect to $\overline{Q^p}$. By $S^\mu_{cl}(Q^p \times \mathbb{R}^m; E, \tilde{E})$, we denote the subspace of all $a(r,\tau) \in S^\mu(\ldots)$ for which $a \sim \sum a_{\mu-j}$, with $a_{\mu-j} \in S^{(\mu-j)}(\ldots)$. Analogous notations will be used with respect to $\overline{Q^p}$.

In our applications we often have the following situation. E is a Hilbert space and there is another Hilbert space E^0 (considered as a 'reference space'), such that $E \hookrightarrow E^0$ is dense and the E^0-scalar product $(\cdot,\cdot)_0$ extends to a non-degenerate pairing

$$(\cdot,\cdot)_0 : E \times E' \to \mathbb{C},$$

E' being the dual of E. Moreover x_λ is a group of unitary operators in E^0, which induces the corresponding actions in E, E' (the actions in E, E' are not unitary in general). It can easily be proved that then $x(\tau)$ is in E' of the same growth with respect to τ as in E.

Let us use the abbreviation $\{E, E^0, E'; x_\lambda\}$, for the couple of data with the mentioned relations, where we also admit the converse orders of the spaces, and call the couple, like that, a Hilbert space triplet with unitary action.

Now let $\{E, E^0, E'; x_\lambda\}$, $\{\tilde{E}, \tilde{E}^0, \tilde{E}'; \tilde{x}_\lambda\}$ be Hilbert space triplet with unitary actions. Then we have the following

8. THEOREM. *Let $a(t,t',\tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$, then $op_M(a)$ extends to a continuous operator*

$$op_M(a) : \mathcal{H}^s(Q^m, E) \to \mathcal{H}^{s-\mu}(Q^m, \tilde{E}),$$

for all $s \in \mathbb{R}$. If $a \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$, then

$$op_M(a) : \mathcal{H}^s_{\mathrm{comp}}(Q^m, E) \to \mathcal{H}^{s-\mu}_{\mathrm{loc}}(Q^m, \tilde{E})$$

is continuous, $s \in \mathbb{R}$.

PROOF. There is a canonical isomorphism

$$S^\mu(\overline{Q^p} \times \mathbb{R}^m; E, \tilde{E}) \doteq F_1 \otimes_\pi F_2,$$

where F_1 is the subspace of all τ independent elements, F_2 the subspace of all r-independent elements, both in the induced topologies. For $p = 2m$, $r = (t,t')$, we have in addition $F_1 = F_0 \otimes_\pi F_0'$, where $F_0(F_0')$ is the space of all t' independent (t independent) vectors. By a well-known theorem of projective tensor products of Fréchet spaces, every $a(t,t',\tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$ can be written as a converging sum

$$a(t,t',\tau) = \sum_{j=0}^{\infty} \lambda_j f_j(t) g_j(t') h_j(\tau),$$

with $\sum |\lambda_j| < \infty$, $f_j \to 0$ in F_0, $g_j \to 0$ in F_0', $h_j \to 0$ in F_2. Applying Theorem 3, we get

$$\mathcal{M}_{g_j} \to 0 \text{ in } \mathcal{L}(\mathcal{H}^s(Q^m, E)),\ \mathcal{M}_{f_j} \to 0 \text{ in } \mathcal{L}(\mathcal{H}^{s-\mu}(Q^m, \tilde{E})).$$

Moreover it is trivial that $op_M(h_j) \in \mathcal{L}(\mathcal{H}^s(Q^m, E),\ \mathcal{H}^{s-\mu}(Q^m, \tilde{E}))$ also tends to zero, for $j \to \infty$. Thus

$$op_M(a) = \sum_{j=0}^{\infty} \lambda_j \mathcal{M}_{f_j} op_M(h_j) \mathcal{M}_{g_j}$$

converges in $\mathcal{L}(\mathcal{H}^s(Q^m, E),\ \mathcal{H}^{s-\mu}(Q^m, \tilde{E}))$. From this, we obtain immediately also the second statement of the theorem. □

From now on, we assume that all occurring spaces $E, \tilde{E}, \dots$ belong to Hilbert space triplet, with unitary actions. To every continuous operator $a : E \to \tilde{E}$, we then have the 'formal adjoint' $a^{(*)}$, defined via the reference scalar products, i.e. $(au, v)_{\tilde{E}^0} = (u, a^{(*)}v)_{E^0}$, for all $u \in E^0$, $v \in \tilde{E}^0$, for which the scalar products are finite. Then $a^{(*)}$ induces a continuous operator

$$a^{(*)} : \tilde{E}' \to E', \text{ with } \|a\|_{\mathcal{L}(E,\tilde{E})} = \|a^{(*)}\|_{\mathcal{L}(\tilde{E}',E')}.$$

The point-wise formal adjoint $a(r, \tau) \to a^{(*)}(r, \tau)$ leads to isomorphisms

$$(*) : S^\mu(\overline{Q^p} \times R^m; E, \tilde{E}) \to S^\mu(\overline{Q^p} \times R^m; \tilde{E}', E')$$

with $(*)^2 = id$, and the same over Q^p. Moreover, we have a non-degenerate sesqui-linear pairing

$$(\cdot, \cdot)_{\mathcal{H}^0(Q^m, E^0)} : \mathcal{H}^s(Q^m, E) \times \mathcal{H}^{-s}(Q^m, E') \to \mathbb{C},$$

so that $\mathcal{H}^{-s}(Q^m, E') = \mathcal{H}^s(Q^m, E)'$.

For every operator

$$\mathbf{A} : \mathcal{H}^s(Q^m, E) \to \mathcal{H}^{s-\mu}(Q^m, \tilde{E})$$

which is continuous for all $s \in \mathbb{R}$, we obtain a formal adjoint $\mathbf{A}^*$ which induces continuous operators

$$\mathbf{A}^* : \mathcal{H}^s(Q^m, \tilde{E}') \to \mathcal{H}^{s-\mu}(Q^m, E'), \tag{9}$$

for all $s \in \mathbb{R}$.

Our next objective is to study the distributional kernels of the operators $op_M(a)$. Let us deal with amplitude functions over $\overline{Q^{2m}}$. The case Q^{2m} is completely analogous.

First, we perform the calculations formally. If $a(t, r, \tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ is an amplitude function, $t, r \in \overline{Q^m}$, then

$$op_M(a)u(t) = \int_0^\infty K\left(t, r, \frac{t}{r}\right) u(r) \, \frac{dr}{r}, \tag{10}$$

$u \in C_0^\infty(Q^m)$, where $\frac{t}{r} := \left(\frac{t_1}{r_1}, \ldots, \frac{t_m}{r_m}\right)$, $\frac{dr}{r} := \frac{dr_1}{r_1} \ldots \frac{dr_m}{r_m}$, and

$$K(t,r,\rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau}\, a(t,r,\tau)d\tau, \tag{11}$$

$$\rho^{-\frac{1}{2}-i\tau} := \rho_1^{-\frac{1}{2}-i\tau_1} \ldots \rho_m^{-\frac{1}{2}-i\tau_m},$$

$$a(t,r,\tau) = \int_0^\infty \rho^{\frac{1}{2}+i\tau}\, K(t,r,\rho)\, \frac{d\rho}{\rho}. \tag{12}$$

The integrals have a classical interpretation for μ sufficiently negative. In general, they are to be taken in the distributional sense. In particular (11), (12) are inverse to each other, in the sense of the Mellin transform which extends to operator-valued distributions on Q^m, in the same way as in the scalar case. We shall tacitely treat the integrals as converging ones. The precise justification follows by oscillatory integral arguments for the Mellin ψDO's. Instead of (11), we also can write

$$K(t,r,\rho) = \left(-\rho\, \frac{\partial}{\partial\rho}\right)^\alpha (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} \left(\frac{1}{2}+i\tau\right)^{-\alpha} a(t,r,\tau)d\tau, \tag{13}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\left(-\rho\, \frac{\partial}{\partial\rho}\right)^\alpha = \left(-\rho_1\, \frac{\partial}{\partial\rho_1}\right)^{\alpha_1} \ldots \left(-\rho_m\, \frac{\partial}{\partial\rho_m}\right)^{\alpha_m}$. This enables us to reduce (11) to a converging integral which is then to be differentiated in the distributional sense.

For every $M \in \mathbb{N}$, there exists a ν such that

$$K(t,r,\rho) \in C^M(\overline{Q^{2m}} \times Q^m, \mathcal{L}(E, \tilde{E})), \tag{14}$$

whenever $a \in S^\mu$ and $M + \mu < \nu$. This follows immediately from

$$\left(-\rho\, \frac{\partial}{\partial\rho}\right)^\alpha K(t,r,\rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} \left(\frac{1}{2}+i\tau\right)^{\alpha} a(t,r,\tau)d\tau$$

and the convergence of the integral on the right for $|\alpha| + \mu$ sufficiently negative.

Let us also express the distributional kernel of the formal adjoint operator of $op_M(a)$. From

$$\begin{aligned} &\int \left(\int K\left(t,r,\frac{t}{r}\right) u(r)\, \frac{dr}{r},\, v(t)\right)_{\tilde{E}^0} dt \\ &= \int \left(u(r),\, \int \frac{t}{r}\, K^{(*)}\left(t,r,\frac{t}{r}\right) v(t)\, \frac{dt}{t}\right)_{E^0} dr, \end{aligned}$$

we obtain

$$K^*(t,r,\rho) = \rho^{-1}\ K^{(*)}(r,t,\rho^{-1}). \tag{15}$$

A Mellin ψDO $op_M(a)$ is called properly supported if $K\left(t,r,\frac{t}{r}\right)$ has a proper support in $Q^m \times Q^m$.

9. DEFINITION. $ML^\mu(\overline{Q^m}; E, \tilde{E})$ is the space of all operators $\mathbf{A}+\mathbf{G}$, where $\mathbf{A} = op_M(a)$ is a properly supported Mellin ψDO with an amplitude function $a(t,r,\tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ and $\mathbf{G}$ an operator which induces continuous mappings

$$\mathbf{G} : \mathcal{H}^s(Q^m, E) \to \mathcal{H}^\infty(Q^m, \tilde{E}),$$
$$\mathbf{G}^* : \mathcal{H}^s(Q^m, \tilde{E}') \to \mathcal{H}^\infty(Q^m, E'),$$

for every $s \in \mathbb{R}$ (cf. formula (9)). Moreover $ML^\mu(Q^m; E, \tilde{E})$ is the space of all $\mathbf{A}+\mathbf{G}$, where $\mathbf{A} = op_M(a)$ is properly supported, $a(t,r,\tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$, and $\mathbf{G}$ an operator which induces continuous operators

$$\mathbf{G} : \mathcal{H}^s_{\text{comp}}(Q^m, E) \to \mathcal{H}^\infty_{\text{loc}}(Q^m, \tilde{E}),$$
$$\mathbf{G}^* : \mathcal{H}^s_{\text{comp}}(Q^m, \tilde{E}') \to \mathcal{H}^\infty_{\text{loc}}(Q^m, E'),$$

for every $s \in \mathbb{R}$. In an analogous manner, we define the spaces $ML^\mu_{cl}(\ldots)$ over $\overline{Q^m}$ and Q^m, respectively, where the amplitude functions are assumed to be classical.

Set $\Delta_\tau = -\sum_{k=1}^{m} \frac{\partial^2}{\partial \tau_k^2}$, $v(\rho) = \sum_{k=1}^{m} \log^2 \rho_k$. Then

$$v(\rho)^{-N}\ \Delta_\tau^N\ \rho^{-\frac{1}{2}-i\tau} = \rho^{-\frac{1}{2}-i\tau}.$$

Integration by parts in (11), yields

$$K(t,r,\rho) = v(\rho)^{-N}(2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} \Delta_\tau^N a(t,r,\tau) d\tau, \tag{16}$$

for every $N \in \mathbb{N}$. Since $\Delta_\tau^N a \in S^{\mu-2N}$, the integral on the right side becomes smoother the larger N is. Thus

$$K(t,r,\rho) \in C^\infty\left(\overline{Q^{2m}} \times (Q^m \setminus \{1,\ldots,1\}), \mathcal{L}(E,\tilde{E})\right). \tag{17}$$

Set $L = \mathcal{L}(E,\tilde{E})$ and fix $t = t_0,\ r = r_0$. For

$$f(\rho) := K(t_0, r_0, \rho), \tag{18}$$

$$a(z) := a(t_0, t_0, \tau),\ \text{Re}\ z_j = \tau_j,\ j = 1,\ldots,m, \tag{19}$$

we have $a = Mf$, with the Mellin transform M applied to L-valued distributions. Then (16) gives

$$f(\rho) = v(\rho)^{-N}(2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} \Delta_\tau^N a(\tau) d\tau = v^{-N} \ M^{-1}(\Delta^N a).$$

Choosing N sufficiently large, we can deal with converging integrals, since

$$\begin{aligned} \|\Delta^N a(\tau)\|_L &= \|\tilde{x}(\tau)\tilde{x}^{-1}(\tau)\Delta^N a(\tau) x(\tau) x^{-1}(\tau)\|_L \\ &\le c\|\tilde{x}(\tau)\|_{\mathcal{L}(\tilde{E})} \|x^{-1}(\tau)\|_{\mathcal{L}(E)} \ [\tau]^{\mu-2N} \le c[\tau]^{\mu-2N+2\sigma} \end{aligned}$$

and σ is a fixed constant, only depending on $E, \tilde{E}$. Now assume for a moment $a(\tau) \in S^{-\infty}$. Then the symbol estimates and (6) show that

$$\|\tau^\alpha D_\tau^\beta a(\tau)\|_L \le c_{\alpha\beta},$$

with constants $c_{\alpha\beta}$, for all multi-indices α, β, i.e. $a(\tau) \in \mathcal{S}(\mathbb{R}^m, L)$ (the Schwartz space of L-valued functions). Conversely $a \in \mathcal{S}(\mathbb{R}^m, L)$ implies $a \in S^{-\infty}$. Set

$$\mathcal{H}^\infty(Q^m, L)^\infty = M^{-1}\mathcal{S}(\mathbb{R}^m, L).$$

Then

$$a(\tau) \in S^{-\infty} \ (\mathbb{R}^m; L) \text{ implies } f(\rho) \in \mathcal{H}^\infty(Q^m, L)^\infty.$$

The space $\mathcal{S}(\mathbb{R}^m, L)$ can also be characterized by the condition

$$\left\{ \int \|\tau^\alpha D_\tau^\beta a(\tau)\|_L^2 \, d\tau \right\}^{\frac{1}{2}} < \infty, \qquad \text{for all } \alpha, \beta.$$

Then $f(\rho) \in \mathcal{H}^\infty(Q^m, L)^\infty$ is equivalent to

(20) $$\left\{ \int \left\| \left(-\rho \frac{\partial}{\partial \rho}\right)^\alpha \log^\beta \rho \ f(\rho) \right\|_L^2 d\rho \right\}^{\frac{1}{2}} < \infty, \text{ for all } \alpha, \beta.$$

10. PROPOSITION. *$a(t, r, \tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ implies*

(21) $$(1 - \psi(\rho))K(t, r, \rho) \in C^\infty(\overline{Q^{2m}}, \ \mathcal{H}^\infty(Q^m, \mathcal{L}(E, \tilde{E}))^\infty),$$

for every $\psi \in C_0^\infty(Q^m)$, with $\psi(\rho) \equiv 1$ close to $\rho = \{1, \dots, 1\}$. Moreover, $a(t, r, \tau) \in S^{-\infty}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ is equivalent to

(22) $$K(t, r, \rho) \in C^\infty(\overline{Q^{2m}}, \mathcal{H}^\infty(Q^m, \mathcal{L}(E, \tilde{E}))^\infty).$$

An analogous statement holds for Q^{2m} instead of $\overline{Q^{2m}}$.

PROOF. The equivalence of (22) with $a \in S^{-\infty}$ is just the result of the preceding discussion applied to C^∞ functions over $\overline{Q^{2m}}$, with values in the considered spaces. It remains to verify (21). For simplicity, we also will assume independence of the variables $t, r \in \overline{Q^{2m}}$. By definition of $\mathcal{H}^s(Q^m, L)$, it is clear that, for every $s \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that

$$v(\rho)^N f(\rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} \Delta_\tau^N(\tau) d\tau \in \mathcal{H}^s(Q^m, L),$$

The same is true of

$$\left(-\rho \frac{\partial}{\partial \rho}\right)^\alpha v(\rho)^N f(\rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} (1 + i\tau)^\alpha \Delta_\tau^N a(\tau) d\tau,$$

for every given α, provided N is large enough. This remains in force, if we replace $f(\rho)$ by $(1 - \psi(\rho))f(\rho) =: f^0(\rho)$, which belongs to $C^\infty(Q^m, L)$. In particular for $s = 0$, it follows

$$\int \left\| \left(-\rho \frac{\partial}{\partial \rho}\right)^\alpha v(\rho)^N \; f^0(\rho) \right\|_L^2 d\rho < \infty. \tag{23}$$

It remains to show that (20) holds for f^0, for arbitrary β. To this end, we choose another function $\psi_1(\rho)$, with analogous properties as $\psi(\rho)$ and $\psi\psi_1 = \psi_1$. Then

$$f^0(\rho) = (1 - \psi_1(\rho))(1 - \psi(\rho))f(\rho)$$

and

$$v(\rho)^N \; \log^{-\gamma} \rho = \log^\beta \rho,$$

for every fixed β, with appropriate large N and another multi-index γ. Now

$$\left(-\rho \frac{\partial}{\partial \rho}\right)^\alpha \log^\beta \rho \; f^0(\rho) = \left(-\rho \frac{\partial}{\partial \rho}\right)^\alpha \left\{(1 - \psi_1(\rho)) \; \log^{-\gamma} \rho\right\} \; v^N(\rho) f^0(\rho)$$

and

$$\left| \left(-\rho \frac{\partial}{\partial \rho}\right)^\eta \left\{(1 - \psi_1(\rho)) \log^{-\gamma} \rho\right\} \right| < \text{const.},$$

for all $\rho \in Q^m$ and all $\eta \in \mathbb{N}^m$, shows that (23) indeed implies (20), for all β. □

11. REMARK. Let $K(t, r, \rho)$ be associated with some $a(t, r, \tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$, via (11), and $\varphi(\rho) \in C_0^\infty(Q^{2m})$. Then $\tilde{K}(t, r, \rho) := \varphi(\rho) K(t, r, \rho)$ is associated with some $\tilde{a}(t, r, \tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$.

For notational convenience, we now replace for a moment τ by y and write the symbols in the form $a\left(t, r, \frac{1}{2} + iy\right)$.

12. PROPOSITION. *For every* $a\left(t,r,\frac{1}{2}+iy\right) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$, *there exists a function* $a_1(t,r,z) \in C^{\infty}(\overline{Q^{2m}} \times \mathbb{C}^m; \mathcal{L}(E,\tilde{E}))$ *which is holomorphic in* $z \in \mathbb{C}^m$, *for every fixed* t, r, *so that*

$$a\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right) \in S^{\mu-1}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E}), \tag{24}$$

for every $\gamma \in \mathbb{R}$. *An analogous statement holds, if we replace* $\overline{Q^{2m}}$ *by* $\overline{Q^p}$ *or* Q^p, *for any* p, *or for classical symbols.*

PROOF. Fix a function $\psi(\rho)$ as in Proposition 10. From

$$K(t,r,\rho) = \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-iy} a\left(t,r,\frac{1}{2}+iy\right) dy,$$

we can pass to another symbol

$$a_1(t,r,z) := \int_0^{\infty} \rho^z \ \psi(\rho) K(t,r,\rho) \ \frac{d\rho}{\rho},$$

which is an entire function in $z \in \mathbb{C}^m$ and C^{∞} of $(t,r) \in \overline{Q^{2m}}$. For $\gamma = 0$, we obtain (24) from Proposition 10. The difference is even in $S^{-\infty}$. Now, let $\gamma = (\gamma_1, \ldots, \gamma_m)$ be arbitrary. First, assume $\gamma_2 = \ldots = \gamma_m = 0$. Then

$$\begin{aligned}
&a\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right)\\
&= a\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}+iy\right)\\
&+ a_1\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right)
\end{aligned}$$

equals, modulo $S^{-\infty}$,

$$\begin{aligned}
&\int_0^{\infty} \rho^{\frac{1}{2}+iy}(1-\rho_1^{-\gamma_1})\psi(\rho)K(t,r,\rho) \ \frac{d\rho}{\rho}\\
&= \int_0^{\infty} \rho^{\frac{1}{2}+iy}\tilde{\psi}(\rho) \ \log \ \rho_1 K(t,r,\rho) \ \frac{d\rho}{\rho},
\end{aligned} \tag{25}$$

with $\tilde{\psi}(\rho) = (1-\rho_1^{-\gamma_1}) \ \log^{-1} \ \rho_1\psi(\rho)$. The function $\tilde{\psi}(\rho)$ belongs to $C_0^{\infty}(Q^m)$. From $i \log \ \rho_1 K(t,r,\rho) = \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-iy}\partial_{y_1} a\left(t,r,\frac{1}{2}+iy\right) dy$, we know that log

$\rho_1 K(t,r,\rho)$ is the kernel of a symbol in $S^{\mu-1}$. Applying Remark 11, we obtain that (25) belongs to $S^{\mu-1}$. For $\gamma = (\gamma_1, \ldots, \gamma_m)$, we can proceed inductively, by writing the difference (24) as a sum of differences, where in every item only one γ_j is changed. Modulo $S^{-\infty}$, the difference equals

$$\begin{aligned}
&a_1\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right)\\
&= a_1\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma_{(1)}+iy\right)\\
&+ a_1\left(t,r,\frac{1}{2}-\gamma_{(1)}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma_{(2)}+iy\right)\\
&+ \ldots + a_1\left(t,r,\frac{1}{2}-\gamma_{(m-1)}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right),
\end{aligned}$$

where $\gamma_{(1)} = (\gamma_1, 0, \ldots, 0)$, $\gamma_{(2)} = (\gamma_1, \gamma_2, 0, \ldots, 0), \ldots, \gamma_{(m-1)} = (\gamma_1, \ldots, \gamma_{m-1}, 0)$. For the single terms, we can proceed similarly as before. This yields our assertion. □

13. REMARK. As in the standard calculus of ψDO's, every $\mathbf{A} = op_M(a)$, with $a \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$, can be written as $\mathbf{A} = \mathbf{A}_0 + \mathbf{G}$, where $\mathbf{A}_0$ is properly supported and $\mathbf{G} \in ML^{-\infty}(\overline{Q^m}; E, \tilde{E})$. An analogous assertion holds over Q^m.

The formula (11) defines a space of distributional kernels belonging to amplitude functions $a \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ that will be denoted by $T^\mu(\overline{Q^{2m}} \times Q^m; E, \tilde{E})$. In other words, we have, by definition, an isomorphism

$$M^{-1} : S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E}) \to T^\mu(\overline{Q^{2m}} \times Q^m; E, \tilde{E}). \tag{26}$$

In an analogous sense, we also use the spaces

$$T^\mu(Q^p \times Q^m; E, \tilde{E}),\ T^\mu_{cl}(Q^p \times Q^m; E, \tilde{E}), \ldots,$$

equipped with the corresponding locally convex topologies induced by the spaces in the preimages under M^{-1}. Denote by $T^\mu_{(1)}(\ldots)$ the spaces which are image of $S^\mu_{(1)}(\ldots)$ under M^{-1}, equipped with the corresponding topologies.

Choose a function $\psi \in C_0^\infty(Q^m)$, $\psi(\rho) \equiv 1$ close to $\rho = \{1, \ldots, 1\}$. Set $\psi_1(c\rho) = \psi\left(1 - c(1-\rho_1), \ldots, 1 - c(1-\rho_m)\right)$, $c > 0$ a constant, and

$$K(a)(t,r,\rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2}-i\tau} a(t,r,\tau)\mathrm{d}\tau, \tag{27}$$

$a \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$. Note that a consequence of Proposition 10 is

$$a - M\psi_1(c\rho)K(a) \in S^{-\infty}(Q^{2m} \times \mathbb{R}^m; E, \tilde{E}),$$

for every $c > 1$.

14. THEOREM. *Let $a_j \in S^{\mu_j}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$, $j \in \mathbb{N}$, be a sequence, $\mu_j \to -\infty$, as $j \to \infty$. Then, there are a sequence of amplitude functions $a_{j.0} \in S^{-\infty}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$ and a sequence of constants c_j such that $K := \sum_{j=0}^{\infty} \psi_1(c_j\rho)\ K(a_j - a_{j.0})(t, r, \rho)$ converges in the space $T^{\mu}(\overline{Q^{2m}} \times Q^m; E, \tilde{E})$, $\mu := \max\ \{\mu_j\}$. If a denotes the amplitude function belonging to K, via (26), then $a \sim \sum_{j=0}^{\infty} a_j$. An analogous statement holds if we replace $\overline{Q^{2m}}$ by $\overline{Q^p}$ or Q^p, for any p.*

PROOF. By Lemma 6, we have the continuous embeddings

$$T^{\mu} \hookrightarrow T^{\mu+\sigma+\tilde{\sigma}}_{(1)},\ T^{\mu}_{(1)} \hookrightarrow T^{\mu+\sigma+\tilde{\sigma}}.$$

To prove the asserted convergence, it suffices to consider the semi-norm systems in the spaces with subscript (1). Indeed, we may write

$$K = \sum_{j=0}^{N} \psi_1(c_j\rho)\ K(\tilde{a}_j) + \sum_{j=N+1}^{\infty} \psi_1(c_j\rho)\ K(\tilde{a}_j) = K_N + K'_N,$$

$\tilde{a}_j := a_j - a_{j.0}$. If K'_N converges in $T^{\mu_N}_{(1)}$, $\mu_N = \max_{j \geq N+1} \{\mu_j\}$, then it also converges in $T^{\mu_N+\sigma+\tilde{\sigma}}$ and hence also in T^{μ}, for N so large that $\mu_N + \sigma + \tilde{\sigma} \leq \mu$. Thus we may ignore, in the definition of the semi-norm systems, the group actions. Without loss of generality, we assume that $\mu_{N+1} < \mu_N$. Next, observe that, for any given semi-norm π on the space $T^{\mu_N}_{(1)}$, it suffices to prove that

$$\pi(\psi_1(c\rho)K(\tilde{a}_j)) \to 0, \text{ as } c \to \infty, \text{ for } j \geq j_1(\pi), \tag{28}$$

after the appropriate choice of $a_{j.0}$. Then, we can ensure the convergence of our sum with respect to π, for $c \to \infty$ sufficiently fast. Since π runs over a countable system, a diagonal argument then yields a sequence c_j which fits for all semi-norms. From (14), we obtain a sequence $M_j \in \mathbb{N}$, with $M_j \to \infty$, as $j \to \infty$, such that $K(a_j) \in C^{M_j}(\overline{Q^{2m}} \times Q^m, \mathcal{L}(E, \tilde{E}))$, for $j \geq j_0$, with some j_0 large enough. Let $\tilde{\psi}(\rho)$ be in $C_0^{\infty}(Q^m)$, $\tilde{\psi}\psi = \psi$. For abbreviation, consider for a moment the case of t, r independent amplitude functions. By Taylor expansion of $K(a_j)$ at $\rho = \{1, \ldots, 1\}$, we can write

$$K(a_j)(\rho) = \sum_{|\alpha| \leq N_j - 1} (1-\rho)^{\alpha} \int_{-\infty}^{\infty} P_{\alpha}(\tau) a_j(\tau) \mathrm{d}\tau + K_j^0(\rho).$$

Here $P_{\alpha}(\tau)$ are polynomials in τ of order $|\alpha|$, and N_j is chosen in such a way that $b_{\alpha j} = \int P_{\alpha}(\tau) a_j(\tau) \mathrm{d}\tau$ converges in $\mathcal{L}(E, \tilde{E})$, for $0 \leq |\alpha| \leq N_j - 1$, but

$N_j \to \infty$, as $j \to \infty$, and $K_j^0(\rho)$ at least continuous in ρ. These conditions can obviously be satisfied. From Proposition 10, we get

$$a_{j,0}(\tau) := M_{\rho\to\tau}\left\{\tilde{\psi}_1(\rho)\sum_{|\alpha|\le N_j-1}(1-\rho)^\alpha b_{\alpha j}\right\}\in S^{-\infty}(\ldots;E,\tilde{E}),$$

and $K(\tilde{a}_j)(\rho)$ is flat at $\rho=\{1,\ldots,1\}$ of order N_j. Now we have to show that, for an appropriate choice of constants c_j, the sum $\sum\psi_1(c_j\rho)K(\tilde{a}_j)(\rho)$ converges in T^μ. As mentioned, we always may remove a finite partial sum, such that it suffices to deal with $T^\nu_{(1)}$, with ν so negative as we want.

Let us fix a semi-norm π in $T^\nu_{(1)}$ and prove the convergence of the remaining sum with respect to π under the corresponding choice of $c_j = c_j(\pi)$. For abbreviation, we want to discuss amplitude functions with constant coefficients. The general case is completely analogous. The symbol estimates for $S^\nu_{(1)}$, in the case of constant coefficients, are

$$\|\mathrm{D}_\tau^\alpha a(\tau)\|_L \le c[\tau]^{\nu-|\alpha|},$$

for all $\alpha\in\mathbb{N}^m$, $c=c(\alpha)$ constants. This can be replaced by

$$\|\tau^\beta\mathrm{D}_\tau^\alpha a(\tau)\|_L \le C[\tau]^{\nu+\nu_0},$$

for all $\alpha,\beta\in\mathbb{N}^m$, with $|\alpha|+\nu_0\ge|\beta|$ and any fixed $\nu_0\in\mathbb{N}$. In particular,

$$a\to\sup_\tau\ \|\tau^\beta\mathrm{D}_\tau^\alpha a(\tau)\|_L, \tag{29}$$

form a semi-norm system for $S^\nu_{(1)}(\ldots)_{\text{const.}}$ and $\nu=-\nu_0$, where α,β runs over the indicated set of multi-indices. We want to pass to semi-norms of the form

$$a\to\left\{\int\|\tau^\beta\mathrm{D}_\tau^\alpha a(\tau)\|_L^2\,\mathrm{d}\tau\right\}^{\frac{1}{2}}, \tag{30}$$

and show that they are equivalent to (29) up to a fixed loss of order, only depending on m. Observe that in (30), (29), we can interchange the order of application of $\tau^\beta,\mathrm{D}_\tau^\alpha$, up to equivalence. Let $f(\tau)$ be an operator function of the form

$$f(\tau)=\tau^{\gamma_1}\mathrm{D}_\tau^{\gamma_2}\ \tau^{\gamma_3}\mathrm{D}_\tau^{\gamma_4}\ldots\tau^{\gamma_M}\mathrm{D}_\tau^{\gamma_{M+1}}\ a(\tau),$$

with certain multi-indices $\gamma_j\in\mathbb{N}^m$, and f of sufficient decrease, for $|\tau|\to\infty$. Then

$$f(\tau)=i^m\int_{\tilde\tau<\tau}(\mathrm{D}_1\ldots\mathrm{D}_m f)(\tilde\tau)\mathrm{d}\tilde\tau,$$

$\mathrm{D}_j = -i\partial/\partial\tau_j$, and hence

$$\|f(\tau)\|_L \leq \int\limits_{\tilde{\tau}<\tau} \|\mathrm{D}_1 \dots \mathrm{D}_m f\|_L \; \mathrm{d}\tilde{\tau} \leq \int\limits_{R^m} \|\mathrm{D}_1 \dots \mathrm{D}_m f\|_L \; \mathrm{d}\tilde{\tau}.$$

Thus

$$\begin{aligned}\sup_\tau \; \|f(\tau)\|_L &\leq \int (1+|\tilde{\tau}|^2)^{-N}(1+|\tilde{\tau}|^2)^N \|\mathrm{D}_1 \dots \mathrm{D}_m f(\tilde{\tau})\|_L \; \mathrm{d}\tilde{\tau} \\ &\leq c \int \|(1+|\tilde{\tau}|^2)^N \mathrm{D}_1 \cdot \ldots \cdot \mathrm{D}_m f(\tilde{\tau})\|_L^2 \; \mathrm{d}\tilde{\tau},\end{aligned}$$

for N so large that $\int (1+|\tilde{\tau}|^2)^{-2N} \mathrm{d}\tilde{\tau} < \infty$, i.e. $-4N+m<-1$. It follows

$$\sup_\tau \|f(\tau)\|_L \leq c \int \; \|(1+|\tilde{\tau}|^2)^N \mathrm{D}_1 \cdot \ldots \cdot \mathrm{D}_m f(\tilde{\tau})\|_L^2 \; \mathrm{d}\tilde{\tau}. \tag{31}$$

By inserting $f(\tau) = \tau^{\tilde{\beta}} \mathrm{D}_\tau^{\tilde{\alpha}} a(\tau)$, we see that the system of semi-norms (30), for $|\alpha|+\nu_0+2N \geq |\beta|$, is stronger or equal than the system (29), for $|\alpha|+\nu_0 \geq |\beta|$. Conversely, we have

$$\begin{aligned}\int \|\tau^\beta \mathrm{D}_\tau^\alpha a(\tau)\|_L^2 \mathrm{d}\tau &= \int \; \left\|(1+|\tau|^2)^{-N}(1+|\tau|^2)^N \tau^\beta \mathrm{D}_\tau^\alpha a(\tau)\right\|_L^2 \; \mathrm{d}\tau \\ &\leq c \; \sup_\tau \; \left\|(1+|\tau|^2)^N \tau^\beta \mathrm{D}_\tau^\alpha a(\tau)\right\|_L^2,\end{aligned}$$

with the same N as above. Thus the expressions (30), for $|\alpha|+\nu_0+2N \geq |\beta|$, constitute a semi-norm system on $S_{(1)}^{-\nu_0-4N}(\dots)_{\text{const}}$, stronger or equal than that induced by $S_{(1)}^{-\nu_0}(\dots)_{\text{const}}$.

From $a(\tau)$, we pass to $K(a)(\rho)$ and use

$$K(\tau^\beta \mathrm{D}_\tau^\alpha a)(\rho) = \left(\rho \; \frac{\partial}{\partial\rho}\right)^\beta \; \log^\alpha \rho \; K(a)(\rho).$$

In virtue of an analogue of Parseval's equation for the Mellin transform, we can replace (30) by

$$a \rightarrow \left\{\int \left\| \left(\rho \; \frac{\partial}{\partial\rho}\right)^\beta \; \log^\alpha \rho \; K(a)(\rho) \right\|_L^2 \; \mathrm{d}\rho \right\}^{\frac{1}{2}}. \tag{32}$$

Note that Parseval's equation can be applied, since we talk about operators between Hilbert spaces. After composing with isomorphisms $R : E \underset{\triangleq}{\longrightarrow} E^0$, $\tilde{R} : \tilde{E} \underset{\triangleq}{\longrightarrow} E^0$ to a standard Hilbert space E^0, the norm of $\mathbf{A}$ is equivalent to

$\|\mathbf{A}^0(\mathbf{A}^0)^*\|_{\mathcal{L}(E^0)}$, $\mathbf{A}^0 = \tilde{R}\mathbf{A}R^{-1}$. As mentioned, we fix α, β and get a semi-norm π for which we want to show the convergence

$$\sum_{j=j_1}^{\infty} \psi_1(c_j\rho)K(\tilde{a}_j)(\rho),$$

$j_1 = j_1(\pi)$ chosen sufficiently large. According to (28), we show that, for large j,

$$\int \left\| \tilde{\psi}_1(\rho) \left(\rho \frac{\partial}{\partial \rho}\right)^{\beta} \log^{\alpha} \rho \ \psi_1(c\rho)K(\tilde{a}_j)(\rho) \right\|_L^2 \mathrm{d}\rho \to 0, \tag{33}$$

as $c \to \infty$, $\tilde{\psi}_1$ was defined above. Let us consider, for a moment, the case $m = 1$. For every $F \in \mathbb{N}$, there is a j such that

$$\tilde{\psi}_1(\rho)(1-\rho)^{-F} \ K(\tilde{a}_j)(\rho)$$

is continuously differentiable β-times in ρ. Let first $\beta = 0$. Then

$$\begin{aligned}
\tilde{\psi}_1(\rho) \ &\log^{\alpha} \rho \ \psi_1(c\rho)K(\tilde{a}_j)(\rho) \\
&= \tilde{\psi}_1(\rho) \ \log^{\alpha} \rho \ \psi_1(c\rho)(1-\rho)^F \left\{ \tilde{\psi}_1(\rho)(1-\rho)^{-F} K(\tilde{a}_j)(\rho) \right\} \\
&= \psi_1(c\rho)(1-\rho)^F \tilde{K}(\tilde{a}_j)(\rho), \\
\tilde{K}(\tilde{a}_j)(\rho) &= \tilde{\psi}_1(\rho) \ \log^{\alpha}(\rho) \ \tilde{\psi}_1(\rho)(1-\rho)^{-F} \ K(\tilde{a}_j)(\rho).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int \left\| \tilde{\psi}_1(\rho) \ \log^{\alpha} \rho \ \psi_1(c\rho)K(\tilde{a}_j)(\rho) \right\|_L^2 \mathrm{d}\rho \\
&= \int |\psi_1(c\rho)(1-\rho)^F|^2 \ \left\| \tilde{K}(\tilde{a}_j)(\rho) \right\|_L^2 \mathrm{d}\rho \\
&\le \sup_{\rho} \ \left\| \tilde{K}(\tilde{a}_j)(\rho) \right\|_L^2 \ \int \ |\psi_1(c\rho)(1-\rho)^F|^2 \mathrm{d}\rho.
\end{aligned}$$

For $\rho' = (1-\rho)$, we obtain

$$\begin{aligned}
\int |\psi_1(c\rho)(1-\rho)^F|^2 \ \mathrm{d}\rho &= \int |\psi(1-c\rho') \ \rho'^F|^2 \mathrm{d}\rho' \\
&= \int \left| \frac{\psi(1-\rho)\rho^F}{c^F} \right|^2 \ c^{-1} \ \mathrm{d}\rho \to 0, \ \text{as } c \to \infty.
\end{aligned}$$

In an analogous manner, we can deal with the ρ derivatives, where the arising powers of c, in the derivatives of $\psi_1(c\rho)$, are compensated by c^{-F}, when $F > \beta$. The case $m > 1$ is analogous as well.

Thus we obtain (33) which was the remaining point in the proof. □

2.2. *Further Elements of the Calculus*

The standard calculus of ψDO's contains further relations between symbolic and operator level, in particular composition rules and so on. The present section gives analogous results for Mellin ψDO's.

First, we want to compare the Mellin calculus in the interior of the t space with the calculus of ψDO's with respect to the Fourier transform.

If $\Omega \subseteq \mathbb{R}^p$ is an open set, we can define the symbol spaces $S^\mu(\Omega \times \mathbb{R}^m; E, \tilde{E})$ in an analogous manner, as in Section 2.1., with respect to the group actions $\varkappa(\xi)$, $\tilde{\varkappa}(\xi)$, $\xi \in \mathbb{R}^m$. Set

$$op_F(a) = F^{-1}\ aF,\ \ a \in S^\mu(\mathbb{R}^{2m} \times \mathbb{R}^m; E, \tilde{E}).$$

Then

$$op_F(a)u(x) = (2\pi)^{-m} \int e^{i(x-y)\xi}\ a(x, y, \xi)u(y)\mathrm{d}y\ \mathrm{d}\xi.$$

If $\Omega \subseteq \mathbb{R}^m$ is open, $u \in C_0^\infty(\Omega, E)$, we may admit, as usual, $a \in S^\mu(\Omega \times \Omega \times \mathbb{R}^m; E, \tilde{E})$.

The considerations of the preceding section have a classical analogue in the setting of the Fourier transform. In particular, we can define the classes of ψDO's $L^\mu(\Omega; E, \tilde{E})$, $L^\mu_{cl}(\Omega; E, \tilde{E})$, for every open set $\Omega \subseteq \mathbb{R}^m$. For $L^\mu(\Omega; E, \tilde{E})$, we shall use tacitly the common rules; for instance, the behaviour under diffeomorphisms.

We also have the notion of a complete symbol of an operator $\mathbf{A}$ in L^μ which is defined as an amplitude function $a(x, \xi) \in S^\mu(\Omega \times \mathbb{R}^m; E, \tilde{E})$, with $\mathbf{A} - op_F(a) \in L^{-\infty}(\Omega; E, \tilde{E})$.

1. Proposition. *Let* $b\left(t, r, \frac{1}{2} + i\tau\right) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$, $\mu \in \mathbb{R}$. *Then,* $op_M(b) \in L^\mu(Q^m; E, \tilde{E})$, *and every complete symbol* $a(x, \xi)$ *of* $op_M(b)$, *in the sense of the class* L^μ, *satisfies*

$$a(x, \xi) = b(x, x, -ix\xi) \bmod S^{\mu-1}(Q^m \times \mathbb{R}^m; E, \tilde{E})$$

(*here* $-ix\xi := (-ix_1\xi_1, -ix_2\xi_2, \ldots, -ix_m\xi_m)$).

Proof. Let $u \in C_0^\infty(Q^m)$, then

$$op_M(b)u(t) = (2\pi)^{-m} \int\limits_{-\infty}^{\infty} e^{-\left(\frac{1}{2}+i\tau\right)\log\ t} b(t, r, \frac{1}{2} + i\tau) \left\{ \int\limits_0^\infty e^{\left(\frac{1}{2}+i\tau\right)\log\ r} u(r) \frac{\mathrm{d}r}{r} \right\} \mathrm{d}\tau$$

$$= (2\pi)^{-m} \int\limits_{-\infty}^{\infty} \int\limits_0^\infty e^{i(\log\ t - \log\ r)\xi}\ b(t, r, -i\xi)u(r)\ \frac{\mathrm{d}r}{r}\ \mathrm{d}\xi.$$

Here we have substituted $-i\xi = \frac{1}{2} + i\tau$. The transformation $x = \log\ t$ defines a diffeomorphism $\boldsymbol{x} : \mathbb{R}^m \to Q^m$, $\boldsymbol{x}(x) = t = e^x$. We obtain

$$op_M(b)u(t) = (\boldsymbol{x}^*)^{-1}(2\pi)^{-m} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x-y)\xi}\ b(e^x, e^y, -i\xi)(\boldsymbol{x}^* u)(y)\mathrm{d}y\ \mathrm{d}\xi.$$

Thus $op_M(b) = (\boldsymbol{x}^*)^{-1}\mathbf{A}\boldsymbol{x}^*$, where $\mathbf{A}$ is a ψDO on $\mathbb{R}^m$, with the amplitude function $b(e^x, e^y, -i\xi)$. Since the push forward of a ψDO, under a diffeomorphism, is a ψDO, again, we obtain the first part of the assertion. It remains to express the complete symbol of $op_M(b)$.

A standard formula says that $\mathbf{A}$ has a complete symbol $b(e^x, e^x, -i\xi)$ mod $S^{\mu-1}(\mathbb{R}^m \times \mathbb{R}^m; E, \tilde{E})$. Thus

$$a(t,\tau) - b(t,t,-i\boldsymbol{x}'(x)\tau) \text{ mod } S^{\mu-1}(Q^m \times \mathbb{R}^m; E, \tilde{E}),$$

where $\boldsymbol{x}'(x)$ is the diagonal matrix with the diagonal elements $e^{x_i} = t_i$, $i = 1, \dots, m$. The latter relation is a consequence of the behaviour of complete symbols of ψDO's, under coordinate transformations. □

This calculation can also be performed in the converse direction, in other words,

$$L^\mu(Q^m; E, \tilde{E}) = ML^\mu(Q^m; E, \tilde{E}). \tag{1}$$

The difference, between the theories for L^μ and ML^μ over Q^m, consists in the operator convention. The ways to associate, with an amplitude function in $S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$, operators in L^μ or ML^μ are completely different. So we have to establish the symbolic rules for ML^μ regardless of (1). Moreover, as already emphasized in the beginning, we also want to control the behaviour for the subclass $ML^\mu(\overline{Q^m}; E, \tilde{E})$ up to ∂Q^m.

By a simple modification of 2.1. Theorem 8, we get the following.

Let $a(t,r,\tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$ and $op_M(a)$ be properly supported.

Then $op_M(a)$ induces continuous operators

$$C_0^\infty(Q^m, E) \to C_0^\infty(Q^m, \tilde{E}),\ C^\infty(Q^m, E) \to C^\infty(Q^m, \tilde{E}),$$
$$\mathcal{H}^s_{\mathrm{comp}}(Q^m, E) \to \mathcal{H}^{s-\mu}_{\mathrm{comp}}(Q^m, \tilde{E}),\ \mathcal{H}^s_{\mathrm{loc}}(Q^m, E) \to \mathcal{H}^{s-\mu}_{\mathrm{loc}}(Q^m, \tilde{E}).$$

Set

$$f_\tau(t) = t^{-\left(\frac{1}{2}+i\tau\right)}, \quad \tau \in \mathbb{R}^m \text{ fixed.}$$

Then $f_\tau(t)e \in C^\infty(Q^m, E)$, for every $e \in E$ (expressions like $\varphi(t)e$, $\varphi \in C^\infty(Q^m)$, $e \in E$, are understood as tensor products). Let $\mathbf{A} \in ML^\mu(Q^m; E, \tilde{E})$ be properly supported. Then, it can be applied to $f_\tau(t)e$. Define

$$\sigma_A(t,\tau)e = f_\tau^{-1}(t)\mathbf{A}\ f_\tau(\cdot)e. \tag{2}$$

Then,

$$\text{(3)} \qquad \mathbf{A}u(t) = op_M(\sigma_A)u,$$

for every $u \in C_0^\infty(Q^m, E)$.

2. DEFINITION. An amplitude function $a(t,\tau) \in S^\mu(Q^m \times \mathbb{R}^m; E, \tilde{E})$, with $\mathbf{A} - op_M(a) \in ML^{-\infty}(Q^m; E, \tilde{E})$, is called a complete Mellin symbol of $\mathbf{A} \in ML^\mu(Q^m; E, \tilde{E})$. Similarly an $a(t,\tau) \in S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$ is called a complete Mellin symbol of $\mathbf{A} \in ML^\mu(\overline{Q^m}; E, \tilde{E})$ if $\mathbf{A} - op_M(a) \in ML^{-\infty}(\overline{Q^m}; E, \tilde{E})$.

A complete symbol is never unique. The constructions in 2.1. Proposition 12 show that every choice of some $\psi(\rho) \in C_0^\infty(Q^m)$, with $\psi(\rho) = 1$ close to $\rho = (1, \dots, 1)$, gives rise to a complete symbol. But we shall see that they are equal modulo $S^{-\infty}$.

3. PROPOSITION. *Let* $\mathbf{A} \in ML^\mu(Q^m; E, \tilde{E})$ *be properly supported. Then,* (2) *is a complete symbol of* $\mathbf{A}$.

PROOF. We may assume that $\mathbf{A}$ is given in the form $\mathbf{A} = op_M(a)$, $a(t,r,\tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$. Then

$$op_M(a)u = M^{-1}a\ Mu = \Phi^{-1}\ F^{-1}\ a_F\ F\Phi u,$$

with $a_F(x,y,\xi) = a(e^x, e^y, -\xi)$ (cf. the notations in the proof of 2.1. Lemma 15). Since $(\Phi f_\xi)(x) = e^{-ix\xi}$, $f_\xi^{-1}(t)\ \Phi^{-1} = \Phi^{-1}\ e^{ix\xi}$, we get

$$f_\xi^{-1} op_M(a) f_\xi\ h = \Phi^{-1}\ \left\{e^{ix\xi}\ F^{-1}\ a_F F\ e^{-ix\xi} h\right\},$$

$h \in E$. On the right side, we have (except of the substitution Φ^{-1}) the standard formula of a complete symbol of a standard ψDO. So it is an amplitude function in our class. This was the point to be proved, since we already have the formula (3). □

Clearly every $\mathbf{A} \in ML^\mu(Q^m; E, \tilde{E})$ has a complete symbol $\sigma_A(t,\tau)$, since, by definition, $\mathbf{A}$ equals a properly supported operator modulo $ML^{-\infty}(Q^m; E, \tilde{E})$.

As a consequence of (1), we obtain

4. REMARK. If $a_i(t,\tau)$ are complete symbols of an $\mathbf{A} \in ML^\mu(Q^m; E, \tilde{E})$, $i = 1, 2$, then $a_1(t,\tau) - a_2(t,\tau) \in S^{-\infty}(Q^m \times \mathbb{R}^m; E, \tilde{E})$. Further $\mathbf{A} \to \sigma_{\mathbf{A}}$ induces an isomorphism

$$\begin{aligned} &ML^\mu(Q^m; E, \tilde{E})/ML^{-\infty}\ (Q^m; E, \tilde{E}) \\ &\doteq S^\mu(Q^m \times \mathbb{R}^m; E, \tilde{E})/S^{-\infty}\ (Q^m \times \mathbb{R}^m; E, \tilde{E}). \end{aligned}$$

The analogous statement holds with the subscripts *cl*.

5. THEOREM. *Let* $\sigma_A(t,\tau) \in S^\mu(Q^m \times \mathbb{R}^m; E, \tilde{E})$ *be a complete symbol of*

$\mathbf{A} = op_M(a) \in ML^\mu(Q^m; E, \tilde{E})$, $a(t, r, \tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$. Then,

$$\sigma_A(t, \tau) \sim \sum_\alpha \frac{1}{\alpha!} \mathrm{D}_\tau^\alpha \ (-r\partial_r)^\alpha \ a(t, r, \tau)|_{r=t}. \tag{4}$$

Here $(-r\partial_r)^\alpha = \left(-r_1 \dfrac{\partial}{\partial r_1}\right)^{\alpha_1} \cdots \left(-r_m \dfrac{\partial}{\partial r_m}\right)^{\alpha_m}$. If $\mathbf{A} \in ML^\mu(\overline{Q^m}; E, \tilde{E})$, then there is a complete symbol $\sigma_A(t, \tau) \in S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$. An analogous statement holds for the subspaces of classical objects.

PROOF. First, we shall prove the following statement. For every $a(t, r, \tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$ and given $N \in \mathbb{N}$, there exist symbols $b_N(t, \tau) \in S^\mu(Q^m \times \mathbb{R}^m; E, \tilde{E})$, $a_N(t, r, \tau) \in S^{\mu-N}(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$ such that

$$op_M(a) = op_M(b_N) + op_M(a_N),$$

where

$$b_N(t, \tau) = \sum_{|\alpha| \le N-1} \frac{1}{\alpha!} \mathrm{D}_\tau^\alpha \ (r\partial_r)^\alpha \ a(t, r, \tau) \Big|_{r=t}.$$

Applying the Taylor expansion near the diagonal $r = t$, we obtain

$$a(t, r, \tau) = \sum_{|\alpha| \le N-1} \frac{1}{\alpha!} \ (r - t)^\alpha \ \ (\partial_r^\alpha a(t, r, \tau)|_{r=t}) + f(r, t) a_{N.0}(r, t, \tau)$$

with some $f(r, t) \in C^\infty(Q^{2m})$, which is flat of order N at $r = t$, and $a_{N.0}(r, t, \tau) \in S^\mu(Q^{2m} \times \mathbb{R}^m; E, \tilde{E})$. Let $u \in C_0^\infty(Q^m)$, $\mathbf{A} = op_M(a)$. Then,

$$\mathbf{A}u(t) = (2\pi)^{-m} \int \left(\frac{r}{t}\right)^{\frac{1}{2}+i\tau} a(t, r, \tau) u(r) \ \frac{\mathrm{d}r}{r} \ \mathrm{d}\tau = \sum_{|\alpha| \le N-1} \mathbf{A}_\alpha u(t) + \mathbf{R}_N u(t),$$

where

$$\mathbf{A}_\alpha u(t) = (2\pi)^{-m} \int \left(\frac{r}{t}\right)^{\frac{1}{2}+i\tau} \frac{1}{\alpha!} \ \log^{-\alpha}\left(\frac{r}{t}\right) (r - t)^\alpha (-\mathrm{D}_\tau)^\alpha$$
$$(\partial_r^\alpha \ a(t, r, \tau)|_{r=t}) \, u(r) \ \frac{\mathrm{d}r}{r} \ \mathrm{d}\tau,$$

and $\mathbf{R}_N$ is the operator associated with $a_{N.0}$. The formula for $\mathbf{A}_\alpha$ follows from

$$\mathrm{D}_\tau^\alpha \left(\frac{r}{t}\right)^{\frac{1}{2}+i\tau} = \log^\alpha\left(\frac{r}{t}\right) \ \left(\frac{r}{t}\right)^{\frac{1}{2}+i\tau}$$

and integration by parts. By analogous considerations, as for standard ψDO's, it can be proved that $\mathbf{R}_N$ corresponds to an operator with an amplitude function in $S^{\mu-N}$. Now we set

$$F^\alpha(r, t) = \log^{-\alpha}\left(\frac{r}{t}\right) \ \left(\frac{r}{t} - 1\right)^\alpha$$

and

$$(T^\alpha b)(t) = \frac{1}{\alpha!}\,(-r)^\alpha\,\left(\frac{\partial}{\partial r}\right)^\alpha\, b(t,r)\Big|_{r=t},$$

for any function $b(t,r)$. Then $\mathbf{A}_\alpha$ has the amplitude function

$$a_\alpha(t,r,\tau) = F^\alpha(r,t)(T^\alpha \mathrm{D}_\tau^\alpha a)(t,\tau).$$

We can write

$$\mathbf{A}_\alpha = op_M(a_\alpha) = \sum_{|\beta|\le N-1} op_M(a_{\alpha\beta}) + \mathbf{R}_N,$$

with another remainder $\mathbf{R}_N$ of analogous structure as the above one and

$$a_{\alpha\beta}(t,r,\tau) = F^\beta(t,r)T^\beta F^\alpha\ \mathrm{D}_\tau^{\alpha+\beta}(T^\alpha a)(t,\tau).$$

The procedure can be iterated and we then obtain

$$\begin{aligned}\mathbf{A} = op_M(a_0) + \sum_{1\le|\alpha|\le N-1} op_M(a_{\alpha 0}) + \sum_{1\le|\alpha|.|\beta|\le N-1} op_M(a_{\alpha\beta 0})\\ + \ldots + \sum_{\substack{1\le|\alpha^j|\le N-1\\ j=1,\ldots,N}} op_M(a_{\alpha^1\ldots\alpha^N 0}) + \mathbf{R}_N.\end{aligned}$$

Here the 0 in the last subscript means $\alpha^{N+1}=0$. The remainder $\mathbf{R}_N$ has again an amplitude function in $S^{\mu-N}$ and the amplitude functions $a_0, a_{\alpha 0},\ldots$ are independent of r. We have

$$\begin{aligned}a_0(t,\tau) &:= a_0(t,t,\tau),\\ a_{\alpha 0}(t,\tau) &= \mathrm{D}_\tau^\alpha(T^\alpha a)(t,\tau),\\ a_{\alpha\beta 0}(t,\tau) &= T^\beta F^\alpha\ \mathrm{D}_\tau^{\alpha+\beta}(T^\alpha a)(t,y)\\ &\ \vdots\\ a_{\alpha^1\ldots\alpha^N 0}(t,\tau) &= T^{\alpha^N}\ F^{\alpha^{N-1}}\ T^{\alpha^{N-1}} F^{\alpha^{N-2}}\ldots\\ &\qquad T^{\alpha^2} F^{\alpha^1}\mathrm{D}_\tau^{\alpha^1+\ldots+\alpha^N}(T^{\alpha^1}a)(t,\tau),\\ \alpha^j &= (\alpha_1^j,\ldots,\alpha_m^j)\in\mathbb{N}^m.\end{aligned}$$

Since F^α depends of $\frac{r}{t}$, the expressions $T^\beta F^\alpha$ are constants. Thus

$$a_{\alpha^1\ldots\alpha^N 0}(t,\tau) = c_{\alpha^1\ldots\alpha^N}\mathrm{D}_\tau^{\alpha^1+\ldots+\alpha^N}(T^{\alpha^1}a)(t,\tau),$$

with certain constants $c_{\alpha^1\ldots\alpha^N}$. In view of

$$r^\alpha\,\left(\frac{\partial}{\partial r}\right)^\alpha = \sum_{1\le|\beta|\le|\alpha|} d_\beta\,\left(-r\,\frac{\partial}{\partial r}\right)^\beta$$

with certain constants d_β, our result can be summarized as follows. There is a system of differential operators $p_\nu\left(\mathrm{D}_\tau, -r\,\frac{\partial}{\partial r}\right)$, where $p_\nu(\eta,\rho)$ denotes a polynomial which is homogeneous of order ν in $(\eta,\rho) \in \mathbb{R}^{2m}$, such that, for every N, there is an N' with

$$\begin{aligned} op_M\left(\sum_{\nu=0}^{N'} p_\nu\left(\mathrm{D}_\tau, -r\,\frac{\partial}{\partial r}\right) a(t,r,\tau)\Big|_{r=t}\right) \\ = op_M(a) \bmod ML^{\mu-N}(Q^m; E, \tilde{E}). \end{aligned} \tag{5}$$

These p_ν are independent of a and μ. They are uniquely determined by the relations (5) if we insert all polynomials

$$a(t,r,\tau) = \sum a_\gamma(t,r)\tau^\gamma, \ \ a_\gamma \in C^\infty(Q^{2m}).$$

Then an elementary calculation shows that $p_\nu(\eta,\rho) = \sum\limits_{|\alpha|=\nu} \frac{1}{\alpha!}\,\eta^\alpha\rho^\alpha, \ \ \nu \in \mathbb{N}$. Together with Remark 4, we immediately get (4). In order to prove the second statement, we remember that $a(t,r,\tau) \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E})$. Then

$$p_\nu\left(\mathrm{D}_\tau, \ -r\,\frac{\partial}{\partial r}\right) a(t,r,\tau)\Big|_{r=t} \in S^{\mu-|\nu|}(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E}).$$

By 2.1. Theorem 14, the asymptotic sum in (4) can be carried out within the class $S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$. This proves the existence of a complete symbol in $S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$. □

6. REMARK. There is a canonical isomorphism

$$\begin{aligned} &ML^\mu(\overline{Q^m}; E, \tilde{E})/ML^{-\infty}(\overline{Q^m}; E, \tilde{E}) \\ &\hat{=}\ S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})/S^{-\infty}(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E}) \end{aligned}$$

and the same with the subscripts cl.

Indeed, from Remark 4, we know that two different complete symbols of $\mathrm{A} \in ML^\mu(\overline{Q^m}; E, \tilde{E})$ are equal modulo $S^{-\infty}(Q^m \times \mathbb{R}^m; E, \tilde{E})$. They are then also equal modulo $S^{-\infty}(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$. Thus

$$ML^\mu(\overline{Q^m}; E, \tilde{E}) \to S^\mu(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})/S^{-\infty}(\overline{Q^m} \times \mathbb{R}^m; E, \tilde{E})$$

is correctly defined. Moreover, it is obviously surjective.

By Remark 4., the kernel consists of $ML^{-\infty}(\overline{Q^m}; E, E)$ which is the intersection of $ML^{-\infty}(Q^m; E, \tilde{E})$ with $ML^\mu(\overline{Q^m}; E, \tilde{E})$.

Remember now that we have formal adjoints of operators in the scales (indicated by (*)) and in the associated spaces over Q^m (indicated by *).

7. THEOREM. *$\mathbf{A} \in ML^{\mu}(\overline{Q^m}; E, \tilde{E})$ implies $\mathbf{A}^* \in ML^{\mu}(\overline{Q^m}; \tilde{E}', E')$. For the associated complete symbols we have*

$$\sigma_{\mathbf{A}^*}(t,\tau) \sim \sum \frac{1}{\alpha!}\, \mathrm{D}_\tau^\alpha (-t\partial_t)^\alpha\, \sigma_{\mathbf{A}}(t,\tau)^{(*)}.$$

The proof is a straightforward generalization of the corresponding method for standard ψDO's and of the asymptotic formula in Theorem 5. In a similar way we obtain

8. THEOREM. *Let $\mathbf{A} \in ML^{\mu}(Q^m; E_2, E_3)$, $\mathbf{B} \in ML^{\nu}(Q^m; E_1, E_2)$ $\big(ML^{\mu}(\overline{Q^m}; E_2, E_3),\ ML^{\nu}(\overline{Q^m}; E_1, E_2)\big)$ and $\mathbf{A}$ or $\mathbf{B}$ be properly supported. Then,*

$$\mathbf{AB} \in ML^{\mu+\nu}(Q^m; E_1, E_3)(ML^{\mu+\nu}(\overline{Q^m}; E_1, E_3))$$

and

$$\sigma_{\mathbf{AB}}(t,\tau) \sim \sum_{\alpha} \frac{1}{\alpha!}\, \mathrm{D}_\tau^\alpha\, \sigma_{\mathbf{A}}(t,\tau)(-t\partial_t)^\alpha\, \sigma_{\mathbf{B}}(t,\tau).$$

2.3. *The Scale Axiom*

In the applications of our calculus, we do not have spaces E which are fixed, but scales $\{E^s\}$, where s runs over a parameter space, for instance, $s \in \mathbb{R}$. It may also happen that every E^s contains subspaces $\{E^s_{P,\Delta}\}$, where P runs over a system of 'asymptotic types' and Δ over a system of 'weight intervals'. The operator-valued symbols are then assumed to have extra properties with respect to such scales of spaces. In the abstract version, we restrict ourselves to the case of scales $\{E^s\}_{s\in\mathbb{R}}$. If the spaces, in a concrete situation, are enriched with more 'regularity data' P, Δ, the constructions admit adequate modifications.

We assume that

(i) E^s is a Hilbert space, $s \in \mathbb{R}$, and $E^{s'} \hookrightarrow E^s$ continuous, when $s' \geq s$

(ii) $E^\infty = \bigcap_s E^s$ is dense in every E^s

(iii) the E^0-scalar product $(\cdot,\cdot)_{E^0}$ extends to a non-degenerate pairing $E^s \times E^{-s} \to \mathbb{C}$, for every $s \in \mathbb{R}$

(iv) $\{E^s, E^0, E^{-s}; x_\lambda\}$ is a Hilbert space triplet with unitary actions, for all $s \in \mathbb{R}$,

(v) if $\{E^s\}$, $\{\tilde{E}^t\}$ are two scales and $a \in \bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu})$, then $\|a\|_{\mathcal{L}(E^s;\tilde{E}^{s-\mu})} \leq c\ \max\Big\{\|a\|_{\mathcal{L}(E^{s'},\tilde{E}^{s'-\mu})},\ \|a\|_{\mathcal{L}(E^{s''},\tilde{E}^{s''-\mu})}\Big\}$, when $s' \leq s \leq s''$, $c = c(s', s'')$ being a constant.

The interpretation of (v) is that, whenever we consider two scales, we suppose (v).

From now on, we assume that all occurring scales $\mathcal{E} = \{E^s\}$, $\tilde{\mathcal{E}} = \{\tilde{E}^s\}, \ldots$ satisfy the conditions (i)-(v). The space $E^\infty = \bigcap_s E^s$ is a Fréchet space in the system of norms $\{\|\cdot\|_{E^s} : s \in \mathbb{R}\}$.

We consider subspaces of operators

$$\mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}) \subset \bigcap_{s\in\mathbb{R}} \mathcal{L}(E^{s}, \tilde{E}^{s-\mu}),$$

given for every pair of scales $\mathcal{E}, \tilde{\mathcal{E}}$ and any $\mu \in \mathbb{R}$, with the following properties

(a) $\mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ is a Fréchet space,

(b) $\mathcal{A}^{\mu-1}(\mathcal{E}, \tilde{\mathcal{E}}) \subseteq \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ with continuous embedding

(c) $\mathbf{A} \in \mathcal{A}^{\mu}(\mathcal{E}, \mathcal{E}')$, $\mathbf{B} \in \mathcal{A}^{\nu}(\mathcal{E}', \tilde{\mathcal{E}})$ implies $\mathbf{BA} \in \mathcal{A}^{\mu+\nu}(\mathcal{E}, \tilde{\mathcal{E}})$, $\mathbf{I} \in \mathcal{A}^{0}(\mathcal{E}, \mathcal{E})$ for all $\mathcal{E}$,

(d) $\mathbf{A} \in \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ implies $\mathbf{A}^{*} \in \mathcal{A}^{\mu}(\tilde{\mathcal{E}}, \mathcal{E})$, $\mathbf{A}^{*}$ being the formal adjoint with respect to the o-scalar products

(e) $\mathcal{A}^{-\infty}(\mathcal{E}, \tilde{\mathcal{E}}) = \bigcap_{\mu\in\mathbb{R}} \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}) = \bigcap_{s\in\mathbb{R}} \mathcal{L}(E^{s}, \tilde{E}^{\infty})$

(f) if $\mathbf{A}_{j} \in \mathcal{A}^{\mu-j}(\mathcal{E}, \tilde{\mathcal{E}})$, $j \in \mathbb{N}$, is an arbitrary sequence then there exists an $\mathbf{A} \in \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ such that, for every $N \in \mathbb{N}\backslash\{0\}$,

$$\mathbf{A} - \sum_{j=0}^{N-1} \mathbf{A}_{j} \in \mathcal{A}^{\mu-N}(\mathcal{E}, \tilde{\mathcal{E}})$$

(clearly $\mathbf{A}$ is then unique modulo $\mathcal{A}^{-\infty}$),

(g) $\mathbf{A} \in \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ implies $\tilde{x}_{\lambda}^{-1}\mathbf{A}\, x_{\lambda} \in \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$, for all $\lambda \in \mathbb{R}_{+}$.

Note that we also might modify the conditions to the scales and the operator spaces by making a difference between the scales and the dual ones. For simplicity, we omit this more general case, but it occurs in certain applications.

We suppose that, for every $\mathcal{E}, \tilde{\mathcal{E}}$, the choice of $\mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ is fixed once and for all.

In the applications that we have in mind, the properties (a)-(g) are fulfilled. In addition, the operators have a symbolic structure which is involved in the Fréchet space structure of $\mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$. It will also be employed in abstract terms.

(a_1) For every $\mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$, there is a space of 'principal symbols' $\mathrm{Symb}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})$ with a Fréchet space structure and a surjective linear mapping

$$\sigma^{\mu} : \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}) \to \mathrm{Symb}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}),$$

with $\mathcal{A}^{\mu-1}(\mathcal{E}, \tilde{\mathcal{E}}) = \ker \sigma^{\mu}$,

(b_1) there is a linear mapping

$$op^{\mu} : \mathrm{Symb}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}) \to \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}}),$$

with $\sigma^{\mu} op^{\mu} =$ identity, and op^{μ} is continuous, with respect to the topology induced by $\bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu})$ (op^μ is called an operator convention)

(c_1) there is a bilinear mapping $\mathrm{Symb}^\mu(\mathcal{E}, \mathcal{E}') \times \mathrm{Symb}^\nu(\mathcal{E}', \tilde{\mathcal{E}}) \to \mathrm{Symb}^{\mu+\nu}(\mathcal{E}, \tilde{\mathcal{E}})$, and $\mathbf{A} \in \mathcal{A}^\mu(\mathcal{E}, \mathcal{E}')$, $\mathbf{B} \in \mathcal{A}^\nu(\mathcal{E}', \tilde{\mathcal{E}})$ implies $\sigma^{\mu+\nu}(\mathbf{BA}) = \sigma^\nu(\mathbf{B})\sigma^\mu(\mathbf{A})$,

(d_1) there is an involution $* : \mathrm{Symb}^\mu(\mathcal{E}, \tilde{\mathcal{E}}) \to \mathrm{Symb}^\mu(\tilde{\mathcal{E}}, \mathcal{E})$, with $\sigma^\mu(\mathbf{A}^*) = \sigma^\mu(\mathbf{A})^*$,

(e_1) the Fréchet space structure of $\mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}})$ is compatible with that of $\mathrm{Symb}^\mu(\mathcal{E}, \tilde{\mathcal{E}})$, with respect to the choice of the operator conventions.

The last requirement needs a definition. Let $\mathbf{A} = \mathbf{A}_0 \in \mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}}$ will be omitted for abbreviation), then in view of (a_1), (b_1), we have unique decompositions

$$\mathbf{A}_0 = \mathbf{A}_1 + op^\mu(\sigma^\mu(\mathbf{A}_0)), \ \ \mathbf{A}_1 \in \mathcal{A}^{\mu-1},$$
$$\mathbf{A}_j = \mathbf{A}_{j+1} + op^{\mu-j}(\sigma^{\mu-j}(\mathbf{A}_j)), \ \ \mathbf{A}_{j+1} \in \mathcal{A}^{\mu-j-1},$$

for all $j \in \mathbb{N}$. Thus the fixed choice of the $op^{\mu-j}$, for all j, gives us a well-defined sequence of linear mappings

$$\lambda_j : \mathcal{A}^\mu \to \bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu+j}), \ \ \sigma_j : \mathcal{A}^\mu \to \mathrm{Symb}^{\mu-j},$$

$\lambda_j(\mathbf{A}) := A_j$, $\sigma_j(\mathbf{A}) := \sigma^{\mu-j}(A_j)$. They induce a countable system of semi-norms in $\mathcal{A}^\mu$. The compatibility in (e_1) means, by definition, that the Fréchet space structure in $\mathcal{A}^\mu$ coincides with that induced by the sequence of λ_j, σ_j.

In the applications, we have many explicit possibilities of choosing operator conventions and the Fréchet space structures are then independent.

Let us finally strengthen the property of establishing asymptotic sums by the following axiom.

(f_1) Let $A_j(\rho) \in C^\infty(\mathbb{R}^m, \mathcal{A}^{\mu-j}(\mathcal{E}, \tilde{\mathcal{E}}))$ be a sequence, $j \in \mathbb{N}$, supp $\mathbf{A}_j$ contained in a fixed compact subset of $\mathbb{R}^m$, which is independent of j. Moreover, let $c_{j.\beta}$ be an arbitrary sequence of constants, $j \in \mathbb{N}$, $\beta \in \mathbb{N}^m$. Then, there is a sequence of $\tilde{A}_j(\rho) \in C^\infty(\mathbb{R}^m, \mathcal{A}^{\mu-j}(\mathcal{E}, \tilde{\mathcal{E}}))$, satisfying the same support condition, and $A_j(\rho) - \tilde{A}_j(\rho) \in C^\infty(\mathbb{R}^m, \mathcal{A}^{-\infty}(\mathcal{E}, \tilde{\mathcal{E}}))$, for all j, such that $\sum_{j=0}^{\infty} c_{j.\beta} \left(\frac{\partial}{\partial\rho}\right)^\beta \tilde{A}_j(\rho)$ converges in $C^\infty(\mathbb{R}^m, \mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}}))$, for all β.

Note that the convergence is a condition only to those semi-norms of $\mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}})$ which are induced by the above mappings λ_k. In fact, $\sigma_k : \mathcal{A}^\mu \to \mathrm{Symb}^{\mu-k}$ vanishes over $\mathcal{A}^{\mu-j}$, for $j > k$. The condition (f_1) looks a bit strong, but it is natural in all examples that have to be considered in connection with the corner operators.

From now on, in this section, we assume that all the mentioned objects are given and have the corresponding properties.

1. DEFINITION. $S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ is the subspace of all

$$a(t, r, \tau) \in C^{\infty}(Q^{2m} \times \mathbb{R}^m, \ \mathcal{A}^{\mu}(\mathcal{E}, \tilde{\mathcal{E}})),$$

with

(i) $\mathrm{D}^{\alpha}_{t,r}\mathrm{D}^{\beta}_{\tau} a(r, \tau) \in C^{\infty}(\overline{Q^{2m}} \times \mathbb{R}^m, \mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \tilde{\mathcal{E}}))$

$$\bigcap S^{\mu-|\beta|}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E}^{s-\mu+|\beta|}),$$

for all $s \in \mathbb{R}$ and all $\alpha \in \mathbb{N}^{2m}, \beta \in \mathbb{N}^m$, and the same for the functions a_j, as in (ii) of 2.1. Definition 4,

(ii) for every $\psi \in C_0^{\infty}(Q^m)$, $\psi \equiv 1$ close to $\rho = \{1, \ldots, 1\}$, we have $(1 - \psi(\rho))K(a)(t, r, \rho) \in \bigcap_{s \in \mathbb{R}} C^{\infty}(\overline{Q^{2m}}, \mathcal{H}^{\infty}(Q^m, \mathcal{L}(E^s, \tilde{E}^{\infty}))^{\infty})$ (1), where $K(a)(t, r, \rho)$ is defined by 2.1. (27).

In an analogous way, we also define the spaces over $\overline{Q^p}$, Q^p, for any p.

The condition (ii) will also be called the scale axiom. For simplicity, we now mainly consider the spaces over $\overline{Q^{2m}}$, the other cases are analogous.

The definition gives rise to a system of natural mappings

$$\iota_{s.\alpha,\beta} : S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \to$$
$$C^{\infty}\left(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \tilde{\mathcal{E}})\right) \cap S^{\mu-|\beta|}(\overline{Q^{2m}} \times \mathbb{R}^m; E, \tilde{E}^{s-\mu+|\beta|})$$

and similarly for the a_j, called $\iota_{s,\alpha,\beta,j}$, further

$$\omega_{s,k} : S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \to C^{\infty}(\overline{Q^{2m}}, \mathcal{H}^{\infty}(Q^m, \mathcal{L}(E^s, \tilde{E}^{\infty}))^{\infty}),$$

where ω_s is defined by $a \to (1 - \psi_1(2^k \rho))K(a)$, ψ_1 as in the proof of 2.1. Theorem 14, $c > 1$ a constant, $k \in \mathbb{N}$.

$S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ is a Fréchet space, in the projective limit topology, with respect to the system of these mappings, since a countable set of real s suffices. Moreover, it is independent of the choice of the function ψ_1.

Note that the conditions in Definition 1 are preserved under natural operations such as point-wise compositions.

2. DEFINITION. $ML^{\mu}(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})$ is the space of all operators $\mathbf{A} + \mathbf{G}$, $\mathbf{A} = op_M(a)$, with $a(t, r, \tau) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, and $\mathbf{G}$ induces continuous operators

$$\mathbf{G} : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^{\infty}(Q^m, \tilde{E}^{\infty}),$$
$$\mathbf{G}^* : \mathcal{H}^s(Q^m, \tilde{E}^s) \to \mathcal{H}^{\infty}(Q^m, E^{\infty}),$$

for all $s \in \mathbb{R}$. In an analogous way, we define $ML^{\mu}(Q^m; \mathcal{E}, \tilde{\mathcal{E}})$, cf. 2.1. Definition 9.

3. THEOREM. *Every* $\mathbf{A} \in ML^{\mu}(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})$ *induces continuous operators*

$$\mathbf{A} : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^{s-\mu}(Q^m, \tilde{E}^{s-\mu}),$$

and every $\mathbf{A} \in ML^{\mu}(Q^m; \mathcal{E}, \tilde{\mathcal{E}})$

$$\mathbf{A} : \mathcal{H}^s_{\text{comp}}(Q^m, E^s) \to \mathcal{H}^{s-\mu}_{\text{loc}}(Q^m; \tilde{E}^{s-\mu}),$$

for all $s \in \mathbb{R}$.

This is an immediate consequence of 2.1. Theorem 8.

4. THEOREM. *Let* $a_j \in S^{\mu-j}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, $j \in \mathbb{N}$, *be a sequence,* $p = 2m$ *or* $p = m$. *Then, there exists an* $a \in S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ *such that, for every* $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N} a_j \in S^{\mu-(N+1)}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}).$$

If $\tilde{a}$ *is another such amplitude function, then* $a - \tilde{a} \in S^{-\infty}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$.

An analogous statement holds over Q^p.

PROOF. Denote by $T^{\mu}(\overline{Q^p} \times Q^m; \mathcal{E}, \tilde{\mathcal{E}})$ the space of all $K(r, \rho)$ belonging to amplitude functions in $S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, equipped with the Fréchet space structure from the bijection $a \leftrightarrow K$. Then, it suffices to justify the method of the proof of 2.1. Theorem 14 for the kernels in the present situation.

The point is that we have to check the convergence of the sum over $\psi_1(c_j\rho)K(\tilde{a}_j)$ and $j \geq j_1(\pi)$, for any choice of a semi-norm π on T^{μ}, $c_j = c_j(\pi)$. In addition, we also have to discuss the amplitude functions $a_{j;0}$ which cannot be neglected now. For simplicity, let us discuss again the case of constant coefficients. The generalization to variable coefficients is trivial.

The proof of the convergence of $\sum \psi_1(c_j\rho)K(\mathrm{D}^{\beta}_{\tau}\tilde{a}_j)$, with respect to the semi-norms of $S^{\mu-|\beta|}(\overline{Q^p} \times \mathbb{R}^m; E^s, \tilde{E}^{s-\mu+|\beta|})$, is practically the same as that in 2.1. Theorem 14. The only change concerns the space L which is now to be replaced by $\mathcal{L}(E^s, \tilde{E}^{s-\mu+|\beta|})$, according to the order of differentiation. Here we may deal again with relaxed smoothness along the fibre spaces by removing first finite sums. Moreover, it suffices to consider a countable set of numbers s, because of the condition (v) for the scales. Then, a diagonal argument applies again.

For interpreting the Mellin image $\tilde{a}(\tau)$ of the resulting kernel

$$\sum_{j=0}^{\infty} \psi_1(c_j\rho)K(\tilde{a}_j) = K(\tilde{a}),$$

we have to be careful, since the differentiation of $\tilde{a}(\tau)$, with respect to τ, does not necessarily lead to the required gain of regularity along the fibre spaces.

This got lost in the modification $a_j \to \tilde{a}_j$, since the remainders $a_{j,0}(\tau)$ are not of this sort. On the other hand, we have $K(a_{j,0}) \in C_0^\infty(Q^m, \mathcal{A}^{\mu-j}(\mathcal{E}, \tilde{\mathcal{E}}))$, and we can choose these kernels in such a way that supp $K(a_{j,0})$ is in a fixed compact set containing $\rho = \{1, \dots, 1\}$, for all j.

Then, it suffices to show that we can choose another sequence $\tilde{a}_{j,0} \in C_0^\infty(Q^m, \mathcal{A}^{\mu-j}(\mathcal{E}, \tilde{\mathcal{E}}))$, with $a_{j,0} - \tilde{a}_{j,0} \in C_0^\infty(Q^m, \mathcal{A}^{-\infty}(\mathcal{E}, \tilde{\mathcal{E}}))$, for all j, and an analogous support property, such that

$$\sum \psi_1(c_j\rho)\,\{K(\tilde{a}_j) + K(\tilde{a}_{j,0})\}$$

also converges in the system of norms of $\bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu})$, as well as

$$\sum \left(\rho\,\frac{\partial}{\partial\rho}\right)^\beta \log^\gamma \rho\ \psi_1(c_j\rho)\,\{K(\tilde{a}_j) + K(\tilde{a}_{j,0})\}$$

in the system of norms of $\bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu+|\gamma|})$, for all multi-indices β, γ, with analogous relations to each other as in the proof of 2.1. Theorem 14. The kernels $\psi_1(c_j\rho)\,\{K(\tilde{a}_j) + K(\tilde{a}_{j,0})\}$ have certainly the property to become smoother along the fibre spaces, after multiplying by $\log^\gamma \rho$, because of the scale axiom, and since the change $a_{j,0} \to \tilde{a}_{j,0}$ only contributes a smoothing kernel.

Inspite of the fact that the kernels $K(\tilde{a}_{j,0})$ are not of this sort, it suffices to choose them in such a way that

$$\sum \left(\rho\,\frac{\partial}{\partial\rho}\right)^\beta \log^\gamma \rho\ \psi_1(c_j\rho) K(\tilde{a}_{j,0})$$

converges in $\bigcap \mathcal{L}(E^s, \tilde{E}^{s-\mu+|\gamma|})$, since we may remove again finite partial sums and the remaining terms have any fixed smoothing property that we want. But then, it suffices to employ the axiom (f_1). Summing up, we get a sequence $a_j'(\tau) \in S^{\mu-j}(\dots; \mathcal{E}, \tilde{\mathcal{E}})$ with $a_j(\tau) - a'_j(\tau) \in S^{-\infty}(\dots; \mathcal{E}, \tilde{\mathcal{E}})$, and $\sum a'_j$ converges, with respect to those semi-norms in $S^\mu(\dots; \mathcal{E}, \tilde{\mathcal{E}})$ that rely on the operator norms along the fibre spaces.

Now we see that we get even convergence in the space $C^\infty(\overline{Q^p} \times \mathbb{R}^m, \mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \tilde{\mathcal{E}}))$, since the contributions to the topology of $\mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \tilde{\mathcal{E}})$ from the symbol spaces (apart from that of $\bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu+|\beta|})$) are relevant only for finitely many summands, on every symbolic level, cf. the remarks after (f_1). The convergence of $(1 - \psi(\rho)) \sum \psi_1(c_j\rho) K(a_j')$ in the topology of the space in (1) is also clear, since $(1 - \psi(\rho))\psi_1(c_j\rho) = 0$ for all sufficiently large j. $\square$

5. REMARK. 2.1. Proposition 10 and 2.1. Proposition 12 have obvious analogues in the case of the scale axiom.

6. THEOREM. *Every* $\mathbf{A} = op_M(a) \in ML^\mu(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})$ *has a complete symbol* $\sigma_{\mathbf{A}}(t, \tau) \in S^\mu(\overline{Q^m} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ *which admits the asymptotic expansion* 2.2.(4).

The mapping $\mathbf{A} \to \sigma_{\mathbf{A}}$ *induces an isomorphism*

$$\begin{aligned} &ML^{\mu}(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})/ML^{-\infty}(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}}) \\ &\to S^{\mu}(\overline{Q^m} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}(\overline{Q^m} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \end{aligned}$$

An analogous assertion holds over Q^m.

This follows in an analogous manner as the corresponding statements in Section 2.2.

7. REMARK. 2.2. Theorems 7 and 8 have obvious analogues for the classes in the present section.

2.4. *The Formalism with Reductions of Orders*

Let us now introduce another variant of a calculus of Mellin ψDO's with operator-valued symbols which is based on families of reductions or orders. In order to motivate the definitions below, we assume for a moment that, for the scale $\mathcal{E}$ and every $\mu \in \mathbb{R}$, there exist symbols with 'constant coefficients'

$$b^{\mu}(\tau) \in S^{\mu}(Q^m \times \mathbb{R}^m; \mathcal{E}, \mathcal{E}), \;\; b^{-\mu}(\tau) \in S^{-\mu}(Q^m \times \mathbb{R}^m; \mathcal{E}, \mathcal{E}),$$

for which $b^{\mu}(\tau) : E^s \to E^{s-\mu}$ is an isomorphism and $b^{-\mu}(\tau) : E^{s-\mu} \to E^s$ the inverse, for every fixed $s \in \mathbb{R}$, $\tau \in \mathbb{R}^m$.

We then talk about order reducing symbols for the scale $\mathcal{E}$. Let us mention some simple properties which follow immediately from the definition.

The point-wise composition of amplitude functions induces an isomorphism

$$\tilde{b}^{\nu} : S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \to S^{\mu+\nu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}),$$

$\tilde{b}^{\nu}$ being an order reducing symbol of order ν, for the scale $\tilde{\mathcal{E}}$, and similarly with composition from the right side by b^{ν}. In particular,

$$\begin{aligned} S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) &= \left\{ \tilde{b}^{\mu} a_0 : a_0 \in S^0(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \right\} \\ &= \left\{ a_0 b^{\mu} : a_0 \in S^0(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \right\}. \end{aligned} \tag{1}$$

An analogous statement holds over Q^p.

The symbols $b^{\mu}(\tau)$ can always be chosen to be formally self-adjoint, i.e. $(b^{\mu}(\tau))^{(*)} = b^{\mu}(\tau)$, for all $\tau \in \mathbb{R}^m$ and all $\mu \in \mathbb{R}$.

Indeed, if we are given a system $b^{\mu}(\tau)$, for all $\mu \in \mathbb{R}$, then

$$b_1^{\mu}(\tau) := b^{\frac{\mu}{2}}(\tau)(b^{\frac{\mu}{2}}(\tau))^{(*)}$$

is formally self-adjoint, and it is again of the desired sort.

1. PROPOSITION. *Let $b^\mu(\tau)$ be an order-reducing symbol of order μ, for the scale $\mathcal{E}$. Then, $\mathbf{B}^\mu = op_M(b^\mu)$ induces isomorphisms*

$$\mathbf{B}^\mu : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^{s-\mu}(Q^m, E^{s-\mu}),$$

for all $s \in \mathbb{R}$. Moreover,

$$\|u\|_{\mathcal{H}^s(Q^m, E^s)} \sim \left\{ \int_{\mathbb{R}^m} \|b^s(\tau)(M_{t\to\tau}u)(\tau)\|^2_{E^0} \, d\tau \right\}^{\frac{1}{2}} \tag{2}$$

($\sim$ *means equivalence of norms*).

PROOF. By 2.1. Theorem 8, the operators $\mathbf{B}^\mu$ are continuous, for all $s \in \mathbb{R}$. Moreover,

$$1 = op_M(b^\mu b^{-\mu}) = op_M(b^\mu)\ op_M(b^{-\mu}),$$

i.e. $\mathbf{B}^\mu$ is invertible with the inverse $\mathbf{B}^{-\mu}$. In particular,

$$\mathbf{B}^s : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^0(Q^m, E^0)$$

is an isomorphism with the inverse $\mathbf{B}^{-s}$. Thus $\|\mathbf{B}^s u\|_{\mathcal{H}^0(Q^m, E^0)}$ is a norm on the space $\mathcal{H}^s(Q^m, E^s)$ which is equivalent to the original one. □

The existence of order reducing symbols $b^\mu(\tau)$ is not an immediate consequence of the general calculus of the preceding sections.

On the other hand, we may use the system of $b^\mu(\tau)$ to define the spaces independently and to perform a calculus that does not refer at all to $\varkappa_\lambda$.

In order to suggest the identification between corresponding objects, in the cases when we have group actions and order reductions at the same time, we will use the same notations, both for the symbol classes and the distribution spaces, although it may happen that the group actions do not exist.

The scales of spaces $\mathcal{E} = \{E^s\}$ are assumed to satisfy the conditions (i) - (iii) and (v), formulated in the beginning of Section 2.3. The condition (iv) will be dropped from now on.

Moreover, we assume that we are given the operator and symbol spaces $\mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}})$, $\mathrm{Symb}^\mu(\mathcal{E}, \tilde{\mathcal{E}})$, with the properties (a)-(f), (a_1)-(f_1) from the preceding section. Set

$$p(\mu, \nu, \tau) = \begin{cases} [\tau]^\mu, & \text{for } \nu \geq 0, \\ [\tau]^{\mu-\nu}, & \text{for } \nu \leq 0, \end{cases} \tag{3}$$

$\mu, \nu \in \mathbb{R}, \nu \geq \mu,\ \ \tau \in \mathbb{R}^m.$

2. DEFINITION. A family of operator functions

$$b^\mu(\tau) \in C^\infty(\mathbb{R}^m, \mathcal{A}^\mu(\mathcal{E}, \mathcal{E}))$$

is called a system of order reducing symbols, for the scale $\mathcal{E} = \{E^s\}$, if it has the following properties

(i) $D_\tau^\beta b(\tau) \in C^\infty(\mathbb{R}^m, \mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \mathcal{E}))$, for all $\beta \in \mathbb{N}^m$,

(ii) $b^\mu(\tau) : E^s \to E^{s-\mu}$ is an isomorphism, for every $s \in \mathbb{R}$, $\tau \in \mathbb{R}^m$, $\mu \in \mathbb{R}$, and $b^{-\mu}(\tau) = (b^\mu(\tau))^{-1}$, $b^0(\tau) = 1$,

(iii) for every $\gamma, \delta, \nu, s \in \mathbb{R}$, $\beta \in \mathbb{N}^m$, $\nu \geq \mu := \gamma + \delta - |\beta|$, we have

$$\left\| (D_\tau^\beta b^\gamma(\tau)) b^\delta(\tau) \right\|_{\mathcal{L}(E^s, E^{s-\nu})} \leq c \; p(\mu, \nu, \tau),$$

$$\left\| b^\gamma(\tau) D_\tau^\beta b(\tau) \right\|_{\mathcal{L}(E^s, E^{s-\nu})} \leq c \; p(\mu, \nu, \tau),$$

$c = c(s, \mu, \nu) > 0$ a constant,

(iv) $b^\mu(\tau)$ satisfies the scale axiom (cf. 2.3. Definition 1, (ii)) for the kernel $K(b^\mu)$

(v) let $\mathcal{E}, \tilde{\mathcal{E}}$ be two scales and $\{b^\mu\}$, $\{\tilde{b}^\mu\}$ be associated families, with the corresponding properties (i)-(iv). Then, the parameter-depending norms $\|u\|_{E^s_\tau} := \|b^s(\tau)u\|_{E^0}$, $\|\tilde{u}\|_{E^s} := \|\tilde{b}^s(\tau)u\|_{E^0}$ satisfy, for all $a \in \bigcap_s \mathcal{L}(E^s, \tilde{E}^{s-\mu})$,

$$\|a\|_{\mathcal{L}(E^s_\tau, \tilde{E}^{s-\mu}_\tau)} \leq c \; \max \left\{ \|a\|_{\mathcal{L}(E^{s'}_\tau, \tilde{E}^{s'-\mu}_\tau)}, \; \|a\|_{\mathcal{L}(E^{s''}_\tau, \tilde{E}^{s''-\mu}_\tau)} \right\},$$

for all $s, s', s'' \in \mathbb{R}$ and $s' \leq s \leq s''$, for a constant $c = c(s', s'')$ independent of $\tau \in \mathbb{R}$; here the operator norms are taken parameter-depending (cf. also Remark 5 below).

The existence of $b^\mu(\tau)$ is required for every occurring scale $\mathcal{E} = \{E^s\}_{s\in\mathbb{R}}$. Note that (iii) implies, in particular,

$$\|b^s(\tau)\|_{\mathcal{L}(E^s, E^0)} \leq c[\tau]^s \text{ for } s \geq 0, \; \leq c \text{ for } s \leq 0, \tag{4}$$

$$\|b^{-s}(\tau)\|_{\mathcal{L}(E^0, E^s)} \leq c \text{ for } s \geq 0, \; \leq c[\tau]^{-s} \text{ for } s \leq 0. \tag{5}$$

3. Definition. $S^\mu(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, $p = m$ or $p = 2m$, denotes the space of all $a(r, \tau) \in C^\infty(\overline{Q^p} \times \mathbb{R}^m, \mathcal{A}^\mu(\mathcal{E}, \tilde{\mathcal{E}}))$ with

(i) $D_r^\alpha D_\tau^\beta a(r, \tau) \in C^\infty(\overline{Q^p} \times \mathbb{R}^m, \mathcal{A}^{\mu-|\beta|}(\mathcal{E}, \tilde{\mathcal{E}}))$, for all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^m$, and the same for the functions a_j, as in (ii) of 2.1. Definition 4,

(ii) for all $s \in \mathbb{R}$, we have

$$\|\tilde{b}^{s-\mu+|\beta|}(\tau)(D_r^\alpha D_\tau^\beta a(r, \tau)) \; b^{-s}(\tau)\|_{\mathcal{L}(E^0, \tilde{E}^0)} \leq c, \tag{6}$$

for all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^m$, and all $(r, \tau) \in K \times \mathbb{R}^m$, for every compact $K \subset\subset \overline{Q^p}$, with a constant $c = c(\alpha, \beta, K) > 0$, and the same for the a_j,

(iii) $a(r,\tau)$ satisfies the scale axiom, in the sense of 2.3. Definition 1, (ii).

Define $S^{\mu}(Q^p \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ by analogous properties over Q^p.

4. REMARK. From (6) and Definition 2, (iii), follows

$$\|\tilde{b}^{s-\mu+|\beta|-\lambda}(\tau)(\mathrm{D}_r^{\alpha}\mathrm{D}_\tau^{\beta}a(r,\tau))\; b^{-s}(\tau)\|_{\mathcal{L}(E^0,\tilde{E}^0)} \leq c[\tau]^{-\lambda},$$

for all $\lambda \geq 0$. The estimates (6) may be replaced by

$$\left\|\mathrm{D}_r^{\alpha}\mathrm{D}_\tau^{\beta}(\tilde{b}^{s-\mu+|\beta|}(\tau)a(r,\tau)\; b^{-s}(\tau))\right\|_{\mathcal{L}(E^0,\tilde{E}^0)} \leq c,$$

without changing the symbol spaces. It also follows that

$$b^{\mu}(\tau) \in S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}), \text{ for all } \mu \in \mathbb{R}.$$

5. REMARK. The parameter-depending norms on the spaces E^s and $\tilde{E}^s$, respectively, give rise to parameter-depending operator norms $c(a,s,r,\tau) = \left\|\tilde{b}^{s-\mu}(\tau)\; a(r,\tau)\; b^{-s}(\tau)\right\|_{\mathcal{L}(E^0,\tilde{E}^0)}$. Then,

$$\begin{aligned}
&\|a(r,\tau)u\|_{\tilde{E}_\tau^{s-\mu}} \leq \|\tilde{b}^{s-\mu}(\tau)a(r,\tau)u\|_{\tilde{E}^0}\\
&= \left\|\tilde{b}^{s-\mu}(\tau)a(r,\tau)\; b^{-s}(\tau)b^{s}(\tau)\right\|_{\tilde{E}^0}\\
&\leq c(a,s,r,\tau)\; \|b^{s}(\tau)u\|_{E^0} = c(a,s,r,\tau)\; \|u\|_{E_\tau^s}.
\end{aligned}$$

Thus the symbol estimates in (6) are estimates of the parameter-depending operator norms.

The best constants, in the estimates (6), form a semi-norm system on $S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$. Together with those from (i) and (iii) in Definition 3, the space $S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ is Fréchet. Here we have used Definition 2, (v).

6. PROPOSITION. $a_1 \in S^{\mu}(\overline{Q^p}\times\mathbb{R}^m; \mathcal{E}', \tilde{\mathcal{E}})$, $a_2 \in S^{\nu}(\overline{Q^p}\times\mathbb{R}^m; \mathcal{E}, \mathcal{E}')$ *implies*

$$a_1 a_2 \in S^{\mu+\nu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}). \tag{7}$$

In particular, we have the identities (1). *Moreover,*

$$S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \subseteq S^{\nu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \text{ for } \nu \geq \mu, \tag{8}$$

$$\mathrm{D}_r^{\alpha}\mathrm{D}_\tau^{\beta}S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}) \subseteq S^{\mu-|\beta|}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}}), \tag{9}$$

for all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^m$. *Analogous assertions hold over* Q^p.

The proof is straightforward and left to the reader.

7. DEFINITION. $\mathcal{H}^s(Q^m, E^s)$, $s \in \mathbb{R}$, denotes the closure of $C_0^{\infty}(Q^m, E^{\infty})$ with respect to the norm $\|B^s u\|_{\mathcal{H}^0(Q^m,E^0)}$, $B^s = op_M(b^s)$, where $\mathcal{H}^0(Q^m, E^0)$

is defined as the closure of $C_0^\infty(Q^m, E^\infty)$ with respect to

$$\|u\|_{\mathcal{H}^0(Q^m,E^0)} = \left\{ \int_{\mathbb{R}^m} \|M_{t\to\tau} u(\tau)\|^2_{E^0} \, d\tau \right\}^{\frac{1}{2}}.$$

The spaces $\mathcal{H}^s(Q^m, E^s)$ admit analogous constructions as those in Section 2.1., $\mathcal{H}^s(Q^m, E^s)$ is a Hilbert space with a natural scalar product. We have continuous embeddings

$$\mathcal{H}^{s'}(Q^m, E^{s'}) \hookrightarrow \mathcal{H}^s(Q^m, E^s), \text{ for } s' \geq s.$$

$(\cdot,\cdot)_{\mathcal{H}^0(Q^m,E^0)}$ extends to a non-degenerate pairing

$$(\cdot,\cdot)_{\mathcal{H}^0(Q^m,E^0)} : \mathcal{H}^s(Q^m, E^s) \times \mathcal{H}^{-s}(Q^m, E^{-s}) \to \mathbb{C}$$

such that $\mathcal{H}^{-s}(Q^m, E^{-s}) = (\mathcal{H}^s(Q^m, E^s))'$.

8. THEOREM. *$\mathcal{H}^s(Q^m, E^s)$ is a $C_0^\infty(\overline{Q^m})$-module. Moreover,*

$$\mathcal{M}_\varphi \to 0, \text{ in } \mathcal{L}(\mathcal{H}^s(Q^m; E^s)),$$

for $\varphi \to 0$ in $C_0^\infty(\overline{Q^m})$.

The proof is simple, so it will only be sketched. First, it is rather elementary to see that $\mathcal{H}^s(Q^m, E^s)$ is closed under multiplication by $\omega(t_i)$, $i = 1, \ldots, m$, where $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ is a cut-off function with respect to the variable t_i. We also can multiply by functions in $\varphi \in C_0^\infty(\overline{\mathbb{R}})$, $\varphi(0) = 0$. The arguments are analogous, as in the proof that ψDO's, with symbols that have variable coefficients, act continuously in the Sobolev spaces, here applied to symbols independent of the covariable, where the proof is based on the Mellin transform. More details may be found in [S6], Section 1.1.2., in particular, that $\mathcal{M}_\varphi \to 0$, as $\varphi \to 0$. Then a tensor product argument yields the case in m-variables.

The spaces $\mathcal{H}^s_{\text{comp(loc)}}(Q^m, E^s)$ will be defined in an analogous manner, as in the $\mathcal{X}_\lambda$ setting.

An immediate consequence of Theorem 8 and of the continuity of

$$op_M(a) : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^{s-\mu}(Q^m, \tilde{E}^{s-\mu}), \tag{10}$$

for $a(\tau) \in S^\mu(Q^m \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, with constant coefficients (where the operator norm tends to zero as $a \to 0$), is the continuity of (10) in general, i.e. when $a \in S^\mu(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$.

For $a \in S^\mu(Q^{2m} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, we get continuity between the corresponding comp, loc spaces.

9. DEFINITION. $ML^\mu(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})$ is the space of all operators $\mathbf{A}+\mathbf{G}$, where $\mathbf{A} = op_M(a)$ is a properly supported Mellin ψDO, with an amplitude function

$a \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$, and $\mathbf{G}$ an operator which induces continuous mappings

$$\mathbf{G} : \mathcal{H}^s(Q^m, E^s) \to \mathcal{H}^\infty(Q^m, \tilde{E}^\infty),$$
$$\mathbf{G}^* : \mathcal{H}^s(Q^m, \tilde{E}^s) \to \mathcal{H}^\infty(Q^m, E^\infty),$$

for every $s \in \mathbb{R}$. In an analogous manner we define $ML^{\mu}(Q^m; \mathcal{E}, \tilde{\mathcal{E}})$.

Now we want to check the analogue of 2.1. Proposition 10 for the classes defined with reductions of orders. We mainly consider the closed quadrants. The open case follows in an analogous manner.

10. LEMMA. *Let $a(t, r, \tau) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$ and $K(t, r, \rho)$ be associated with a by the formula 2.1.(16). Then, $K(t, r, \rho)$ restricts to a function $\in C^\infty(\overline{Q^{2m}} \times (Q^m \setminus \{1, \dots, 1\}), \mathcal{L}(E^s, \tilde{E}^\infty))$, for all $s \in \mathbb{R}$.*

PROOF. First, remind of the identity

$$K(t, r, \rho) = v(\rho)^{-N} (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2} - i\tau} \Delta_\tau^N a(t, r, \tau) \, d\tau, \tag{11}$$

for every $N \in \mathbb{N}$, cf. Section 2.1.. From (9), we obtain $\Delta_\tau^N a(t, r, \tau) \in C^\infty(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{L}(E^s, \tilde{E}^{s-\mu+N}))$, for all $s \in \mathbb{R}$. Moreover, for every ν with $s + \nu \geq 0$, there exists an N such that

$$K_N(t, r, \rho) = (2\pi)^{-m} \int_{-\infty}^{\infty} \rho^{-\frac{1}{2} - i\tau} \Delta_\tau^N a(t, r, \tau) \, d\tau$$

converges in the usual sense and

$$K_N(t, r, \rho) \in C^\nu(\overline{Q^{2m}} \times Q^m, \mathcal{L}(E^s, \tilde{E}^{s+\nu})), \tag{12}$$

for all $s \in \mathbb{R}$. Indeed, let us choose τ-independent isomorphisms $b_1^s : E^s \to E^0$, $\tilde{b}_1^s : \tilde{E}^s \to \tilde{E}^0$, $s \in \mathbb{R}$. Then,

$$\begin{aligned}
\|\Delta_\tau^N a\|_{s, s+\nu} &= \|\tilde{b}_1^{s+\nu} \; \Delta_\tau^N a \; b_1^{-s}\|_{0,0} \\
&= \|\tilde{b}_1^{s+\nu} \; \tilde{b}^{-s+\mu-N} \; \tilde{b}^{s-\mu+N} \; \Delta_\tau^N a \; b^{-s} b^s b_1^{-s}\|_{0,0} \\
&\leq c \; \|\tilde{b}_1^{s+\nu} \; \tilde{b}^{-s+\mu-N}\|_{0,0} \; \|b^s b_1^{-s}\|_{0,0},
\end{aligned}$$

with a constant c from the symbol estimates. Using (4), we get

$$\|b^s b_1^{-s}\|_{0,0} \leq c[\tau]^s \text{ for } s \geq 0, \quad \geq c \text{ for } s \leq 0,$$

and (5) implies, for $0 \leq s + \nu \leq s - \mu + N$,

$$\begin{aligned}
\|\tilde{b}_1^{s+\nu} \; \tilde{b}^{-s+\mu-N}\|_{0,0} &\leq c \; \|\tilde{b}^{-s+\mu-N}\|_{0, s+\nu} \\
&\leq c[\tau]^{s+\nu+(-s+\mu-N)} = c[\tau]^{\nu+\mu-N},
\end{aligned}$$

with different constants c. Choosing N large enough, we obviously obtain (12). The formula (11) then implies the assertion. $\square$

11. LEMMA. *$a(\tau) \in S^{-\infty}(\ldots;\mathcal{E},\tilde{\mathcal{E}})$ implies*

$$K(\rho) \in \mathcal{H}^{\infty}(Q^m, \mathcal{L}(E^s, \tilde{E}^{\infty}))^{\infty},$$

for every $s \in \mathbb{R}$, cf. the notations in Section 2.1. for $L = \mathcal{L}(E^s, \tilde{E}^{s'})$, $s' \in \mathbb{R}$ arbitrary.

PROOF. By analogous arguments as in the proof of the preceding Lemma, we can show that $\|\tau^{\alpha} \mathrm{D}_{\tau}^{\beta} a(\tau)\|_L \leq c_{\alpha\beta}$ with constants $c_{\alpha,\beta}$, for every couple α, β. $\square$

Combined with the proof of 2.4. Proposition 10, we easily obtain

12. PROPOSITION. *$a(t,r,\tau) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$ implies*

$$(1-\psi(\rho))K(t,r,\rho) \in C^{\infty}(\overline{Q^{2m}}, \mathcal{H}^{\infty}(Q^m, \mathcal{L}(E^s, \tilde{E}^{s'}))^{\infty}),$$

for every $\psi \in C_0^{\infty}(Q^m)$ with $\psi(\rho) \equiv 1$ close to $\rho\{1,\ldots,1\}$, and all $s, s' \in \mathbb{R}$. Moreover, $a(t,r,\tau) \in S^{-\infty}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$ is equivalent to

$$K(t,r,\rho) \in C^{\infty}(\overline{Q^{2m}},\ \mathcal{H}^{\infty}(Q^m, \mathcal{L}(E^s, \tilde{E}^{s'}))^{\infty}),$$

for all $s, s' \in \mathbb{R}$. An analogous statement holds for Q^{2m} instead of $\overline{Q^{2m}}$.

13. COROLLARY. *Let $K(t,r,\rho)$ be associated with some $a(t,r,\tau) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$ and $\varphi(\rho) \in C_0^{\infty}(Q^{2m})$. Then, $\tilde{K}(t,r,\rho) = \varphi(\rho)K(t,r,\rho)$ is associated with some $\tilde{a}(t,r,\rho) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$.*

14. COROLLARY. *Every $\mathbf{A} = op_M(a)$, with $a \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$, can be written as $\mathbf{A} = \mathbf{A}_0 + \mathbf{G}$, where $\mathbf{A}_0$ is properly supported and $\mathbf{G} \in ML^{-\infty}(\overline{Q^m};\mathcal{E},\tilde{\mathcal{E}})$. An analogous statement holds over Q^m.*

As an immediate analogue of 2.4. Proposition 12 we also obtain

15. PROPOSITION. *For every $a(t,r,\frac{1}{2}+iy) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$, there exists a function $a_1(t,r,z) \in \bigcap_s C^{\infty}(\overline{Q^{2m}} \times \mathbb{C}^m, \mathcal{L}(E^s, \tilde{E}^{s-\mu}))$ which is holomorphic in $z \in \mathbb{C}^m$, for every fixed t,r and $s \in \mathbb{R}$, such that, for every $\gamma \in \mathbb{R}$,*

$$a_1\left(t,r,\frac{1}{2}-\gamma+iy\right) \in S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}})$$

and

$$a\left(t,r,\frac{1}{2}+iy\right) - a_1\left(t,r,\frac{1}{2}-\gamma+iy\right) \in S^{\mu-1}(\overline{Q^{2m}} \times \mathbb{R}^m;\mathcal{E},\tilde{\mathcal{E}}).$$

An analogous statement holds over Q^{2m}.

The formula 2.1.(11) defines a space of distributional kernels $T^{\mu}(\overline{Q^{2m}} \times Q^m; \mathcal{E}, \tilde{\mathcal{E}})$, when $a(t, r, \tau)$ runs over $S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$. In other words, the Mellin transform $M_{\rho\to\tau}$ induces an isomorphism

$$M : T^{\mu}(\overline{Q^{2m}} \times Q^m; \mathcal{E}, \tilde{\mathcal{E}}) \to S^{\mu}(\overline{Q^{2m}} \times \mathbb{R}^m; \mathcal{E}, \tilde{\mathcal{E}})$$

and $T^{\mu}(\ldots)$ will be considered in the induced Fréchet space topology.

16. THEOREM 2.3. *Theorem* 4 *holds in the analogous form also in the present setting with reductions of orders.*

PROOF. To simplify notations, let us set again $\mathcal{E} = \tilde{\mathcal{E}}$. As in the proof of 2.3. Theorem 4, the main point are the arguments for amplitude functions with constant coefficients. The general case is then a trivial generalization and may be dropped.

It is obvious that also in the present situation the semi-norms of the amplitude functions, that refer to the symbolic structure of $\mathcal{A}^{\mu}(\mathcal{E}, \mathcal{E})$, can be neglected. In other words, we have to look at the symbol estimates and the kernel cut-off.

Remember that an argument, for proving 2.1. Theorem 14 and 2.3. Theorem 4, was to drop finite sums and to consider remaining sums of terms of very negative orders. Also here we can proceed in this way, for every fixed semi-norm, by starting the sum with the terms of sufficiently negative order. It may be necessary to fix the order of the starting terms, lower and lower, with increasing number of semi-norms that are involved. But then, we can apply again a diagonal argument and obtain convergence for all semi-norms. In other words, it suffices to consider a fixed semi-norm for $S^{\mu}(\overline{Q^p} \times \mathbb{R}^m; \mathcal{E}, \mathcal{E})$ and to assume that then $\operatorname{ord} a_j \le \mu - N$, for all j and every fixed $N \in \mathbb{N}$. Denote the norm in $\mathcal{L}(E^s, E^{s-\gamma})$ by $\|\cdot\|_{s,s-\gamma}$.

Set for a moment $p = D_{\tau}^{\beta} a$. Let us choose τ-independent isomorphisms $b_1^s : E^s \to E^0$, for all $s \in \mathbb{R}$. For every $\gamma, \nu, \mu - N \le \gamma \le \nu \le \mu$, we have

$$\begin{aligned}
&\|b^{s-\mu+|\beta|}\; p b^{-s}\|_{0,0} \le \|b^{\nu-\mu}\|_{0,0}\; \|b^{s-\nu+|\beta|}\; p b^{-s}\|_{0,0} \\
&\le c[\tau]^{\nu-\mu}\; \|b^{s-\nu+|\beta|}\; p b^{-s}\|_{0,0}, \\
&\|b^{s-\nu+|\beta|}\; p b^{-s}\|_{0,0} = \|b^{s-\nu+|\beta|}\; b_1^{-s+\gamma}\; b_1^{s-\gamma} p b_1^{-s} b_1^{s} b^{-s}\|_{0,0} \\
&\le \|b^{s-\nu+|\beta|}\; b_1^{-s+\gamma}\|_{0,0}\; \|b_1^{-s+\gamma}\; p b_1^{-s}\|_{0,0}\; \|b_1^{s} b^{-s}\|_{0,0}.
\end{aligned}$$

Note that $\|b_1^{-s+\gamma}\; p b_1^{-s}\|_{0,0} = \|p\|_{s,s-\gamma}$. Since we can choose γ so negative as we want, we may fix $\nu = \nu(s, |\beta|)$ in such a way that $s - \nu + |\beta| \le 0$ and

$$\|b^{s-\nu+|\beta|}\; b_1^{-s+\gamma}\|_{0,0}\; \|b_1^{s} b^{-s}\|_{0,0} \le c, \text{ for all } \tau.$$

Thus $\|b^{s-\nu+|\beta|} p b^{-s}\|_{0,0} \le c\|p\|_{s,s-\gamma}$, and consequently

$$\|b^{s-\mu+|\beta|}\; p b^{-s}\| \le c[\tau]^{\nu-\mu}\|p\|_{s,s-\gamma}.$$

Conversely, the estimates of the type, as in the proof of Lemma 10, show that

$$[\tau]^{\rho}\|p\|_{s,s-\gamma} \leq c\|b^{s-\lambda+|\beta|}\, pb^{-s}\|_{0,0},$$

for any given $\rho > 0$, γ negative enough as well as λ.

Now analogous conclusions, as in the proof of 2.1. Theorem 14, show that it is allowed to replace the considered semi-norm by

$$a \longrightarrow \left\{\int \left\|\tau^{\delta} \mathrm{D}_{\tau}^{\beta} a\right\|_{L}^{2} d\tau\right\}^{\frac{1}{2}},$$

$L = \mathcal{L}(E^{s}, E^{s-\gamma})$, where δ runs over all multi-indices, with $|\beta| - |\delta| + k \geq 0$, for some $k \in \mathbb{N}$. This reduces the proof to the scheme of that of 2.1. Theorem 14 with the extra arguments of 2.3. Theorem 4. It is then clear again that the semi-norms of Definition 3, (iii), do not affect the procedure. □

Our next objective is to extend the considerations of Section 2.2. to the classes based on reductions of orders. First, remark that the approach of the present section has a straightforward generalization to the Fourier transform, instead of the Mellin transform. In other words, we have an obvious definition of the symbol spaces $S^{\mu}(\Omega \times \mathbb{R}^{m}; \mathcal{E}, \tilde{\mathcal{E}})$, $\Omega \subseteq \mathbb{R}^{p}$ open, satisfying the scale axiom. For every $a(x, x', \xi) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^{m}; \mathcal{E}, \tilde{\mathcal{E}})$, $\Omega \subseteq \mathbb{R}^{m}$ open, we then have $op_{F}(a) = F^{-1}aF$. This leads to the spaces of 'standard' ψDO's, $L^{\mu}(\Omega; \mathcal{E}, \tilde{\mathcal{E}})$. In particular, $L^{-\infty}(\Omega; \mathcal{E}, \tilde{\mathcal{E}})$ consists of all operators $\mathbf{G}$ for which

$$\mathbf{G} : \mathcal{W}^{s}_{\text{comp}}(\Omega, E^{s}) \rightarrow \mathcal{W}^{\infty}_{\text{loc}}(\Omega, \tilde{E}^{\infty}),$$
$$\mathbf{G}^{*} : \mathcal{W}^{s}_{\text{comp}}(\Omega, \tilde{E}^{s}) \rightarrow \mathcal{W}^{\infty}_{\text{loc}}(\Omega, E^{\infty}),$$

are continuous, for all $s \in \mathbb{R}$. Here the spaces $\mathcal{W}^{s}_{\text{comp(loc)}}(\Omega, E^{s})$ are the obvious analogues of the $\mathcal{H}^{s}_{\text{comp(loc)}}(Q^{m}, E^{s})$ spaces, now based on the Fourier transform. We then have in particular

$$ML^{\mu}(Q^{m}; \mathcal{E}, \tilde{\mathcal{E}}) = L^{\mu}(Q^{m}; \mathcal{E}, \tilde{\mathcal{E}})$$

and an analogous result as 2.2. Proposition 1. $\sigma_{\mathbf{A}}(t, \tau) \in S^{\mu}(Q^{m} \times \mathbb{R}^{m}; \mathcal{E}, \tilde{\mathcal{E}})$ is called a complete symbol of $\mathbf{A} \in ML^{\mu}(Q^{m}; \mathcal{E}, \tilde{\mathcal{E}})$, if $\mathbf{A} - op_{M}(\sigma_{A}) \in ML^{-\infty}(Q^{m}; \mathcal{E}, \tilde{\mathcal{E}})$. Similarly, we define complete symbols in $S^{\mu}(\overline{Q^{m}} \times \mathbb{R}^{m}; \mathcal{E}, \tilde{\mathcal{E}})$, for $\mathbf{A} \in ML^{\mu}(\overline{Q^{m}}; \mathcal{E}, \tilde{\mathcal{E}})$, namely that $\mathbf{A} - op_{M}(\sigma_{A}) \in ML^{-\infty}(\overline{Q^{m}} \times \mathbb{R}^{m}; \mathcal{E}, \tilde{\mathcal{E}})$. The method of proving 2.2. Theorem 5 can obviously be applied also in the present situation.

This yields an immediate analogue of the theorem. Note, in particular, that Theorem 16 is very essential again. From this we get

17. THEOREM. $\mathbf{A} \in ML^{\mu}(\overline{Q^{m}}; \mathcal{E}_{2}, \mathcal{E}_{3})$, $\mathbf{B} \in ML^{\nu}(\overline{Q^{m}}; \mathcal{E}_{1}, \mathcal{E}_{2})$ *imply* $\mathbf{AB} \in ML^{\mu+\nu}(\overline{Q^{m}}; \mathcal{E}_{1}, \mathcal{E}_{3})$ *and, for the complete symbols, the asymptotic*

formula as in 2.2. Theorem 8. An analogous result is true over Q^m. Moreover, $\mathbf{A} \in ML^\mu(\overline{Q^m}; \mathcal{E}, \tilde{\mathcal{E}})$ implies $\mathbf{A}^* \in ML^\mu(\overline{Q^m}; \tilde{\mathcal{E}}, \mathcal{E})$ and, for the complete symbols, the expansion as in 2.2. Theorem 7.

3. - The Cone Algebra with Parameters

3.1. *Cone Operators with Continuous Asymptotics*

This section deals with the concept of parameter-depending cone operators, where a parameter is involved in such a way that it can be considered in the corner calculus as an additional Mellin covariable. The parameter-depending theory is useful also to treat spectral problems for operators on manifolds with conical singularities and we have a natural notion of parameter-depending ellipticity, cf. Section 3.2.

First, we want to remind of the parameter-depending ψDO's $L^\mu(X; \Lambda) \ni \mathbf{A}(\lambda)$ on a closed compact manifold X, depending on the parameter $\lambda \in \Lambda$. Here Λ is a closed subset in a finite-dimensional vector space with metric $|\cdot|$. The operators may be generated locally in coordinate neighbourhoods U, by

$$F^{-1}_{\xi \to x}\ F_{x' \to \xi}\ a_U(x, x', \xi, \lambda) u(x') = \mathbf{A}_U(\lambda) u,$$

with amplitude functions $a = a_U$, satisfying the symbol estimates

$$|\mathrm{D}^\alpha_{x,x'}\ \mathrm{D}^\beta_{\xi,\lambda}\ a(x, x', \xi, \lambda)| \le c(1 + |\xi| + |\lambda|)^{\mu - |\beta|}, \tag{1}$$

for all multi-indices α, β and $x, x' \in K \subset\subset U$, $(\xi, \lambda) \in \mathbb{R}^n \times \Lambda$, ($n = \dim X$). The differentiation, with respect to λ, refers to extensions of $\mathrm{D}^\alpha_{x,x'}\ \mathrm{D}^\beta_\xi a$ to an open neighbourhood $\tilde{\Lambda}$ of Λ (the extensions are always assumed to be smooth in λ). The derivatives are then to be restricted again to Λ. This is the interpretation of (1). A partition of unity argument, then yields the operators $\mathbf{A}(\lambda)$ globally. In [S1] it was studied a variant without differentiations in λ. The role of the differentiations here is that, in the applications below, λ plays the role of an extra covariable. The main conclusions of [S1] may be carried out also here, with obvious modifications. The smoothing parameter-depending operators in $L^\infty(X; \Lambda)$ form the subclass $L^{-\infty}(X; \Lambda)$.

They consist of operators with kernels $g(x, x', \lambda)$ in $C^\infty(X \times X \times \Lambda)$, with a strong decrease of $\mathrm{D}^\gamma_\lambda g(x, x', \lambda)$, for $|\lambda| \to \infty$, in the $C^\infty(X \times X)$-topology. Then, as in the standard ψDO calculus, $\mathbf{A}(\lambda) \in L^\mu(X; \Lambda)$ can be described by x'-independent amplitude functions modulo $L^{-\infty}(X; \Lambda)$. Let us give an equivalent description of the negligible operator class $L^{-\infty}(X; \Lambda)$.

An operator $\mathbf{G}$ has a kernel in $C^\infty(X \times X)$ iff it induces continuous operators

$$\mathbf{G}, \mathbf{G}^* : H^s(X) \to H^t(X), \text{ for all } s, t \in \mathbb{R}.$$

Here $\mathbf{G}^*$ denotes the formal adjoint of $\mathbf{G}$ with respect to a fixed $L^2(X)$-scalar product; $H^s(X)$ is the standard Sobolev space over X of order s. $\mathcal{L}(H^s(X), H^t(X))$ is a Banach space in the operator norm $\|\cdot\|_{s,t}$ and

$$\mathcal{L}(H^s(X),\ H^\infty(X)) = \bigcap_{t\in\mathbb{R}} \mathcal{L}(H^s(X),\ H^t(X))$$

a Fréchet space with a corresponding countable norm system.

$\mathcal{G}(X) := \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s(X),\ H^\infty(X))$ also is a Fréchet space with a countable norm system $\{\eta^j\}_{j\in\mathbb{Z}}$. Then, $L^{-\infty}(X;\Lambda)$ consists of the set of operator functions $\mathbf{G}(\lambda)$ for which

$$\mathbf{G}(\lambda),\ \mathbf{G}^*(\lambda) \in C^\infty(\Lambda, \mathcal{G}(X))$$

and, for every $N\in\mathbb{N}$, $j\in\mathbb{Z}$, γ, there exists a constant $c = c(N, j, \gamma)$ such that

$$\eta^j(\mathrm{D}_\lambda^\gamma G(\lambda)),\ \eta^j(\mathrm{D}_\lambda^\gamma G^*(\lambda)) \le c(1+|\lambda|)^{-N},$$

for all $\lambda\in\Lambda$.

1. REMARK. Incidentally, we also have the situation that Λ is not closed. In that case, we require the estimates in every closed subset. This yields the definition of $L^\mu(X;\Lambda)$ also in this case.

Similarly, we proceed for parameter-depending cone operators below.

Another well-known property of the parameter-depending class $L^\mu(X;\Lambda)$ is that every $\mathbf{A}(\lambda)\in L^\mu(X;\Lambda)$ satisfies the estimates

$$\|\mathbf{A}(\lambda)\|_{s,s-\nu} \le c(s,\nu)p(\mu,\nu,\lambda),$$

for every $\nu\ge\mu$, $s\in\mathbb{R}$, with a constant $c(s,\nu)$ (also depending on $\mathbf{A}$) and p the function of 2.4. Definition 2. A proof of this result may be found in [S1] (Theorem 2.1.). Remark that we also can define the class $L^\mu_{cl}(X;\Lambda)$ of those operator families for which the amplitude functions in the local expressions admit asymptotic expansions into functions that are homogeneous of order $\mu - j$ in the variable (ξ,λ), for $|\xi|^2+|\lambda|^2 \ge$ const. Here we tacitly assume that $\lambda\in\Lambda$ implies $r\lambda\in\Lambda$, for all $r\ge 1$. For every $\mu\in\mathbb{R}$, there exists an operator family $b^\mu(\lambda)\in L^\mu_{cl}(X;\Lambda)$ (with parameter-depending homogeneous principal symbol $(|\xi|^2+|\lambda|^2)^{\frac{\mu}{2}}$) which is order reducing for the scale $H^s(X)$, in the sense of 2.4. Definition 2 (with λ instead of r).

Thus, we can apply the results of Section 2.4. in the special case of the scale $\mathcal{E} = \{H^s(X)\}$ of Sobolev spaces over X and ψDO symbols over $\overline{Q^m}$ with values in $L^\mu(X)$. For our applications, we are interested in the case $m = 1$. In other words, we have the class of Mellin ψDO's $ML^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E})$.

Remember that the latter class of operators is an essential step in the definition of the space of cone operators $\mathcal{N}^\mu(X^\wedge)$, $X^\wedge := \mathbb{R}_+\times X$. We denote by $\mathcal{N}^\mu_F(X^\wedge)_\theta$, $\theta = (k,k)$, $k\in\mathbb{N}$, the subclass of all $\mathbf{A}\in ML^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E})$ which have (locally) classical symbols as ψDO's over $\mathbb{R}_+ \times U$, for any coordinate neighbourhood U on X, and which induce continuous operators

$$\mathbf{A},\mathbf{A}^* : \mathcal{H}^s(X^\wedge) \to \mathcal{H}^{s-\mu;\gamma}(X^\wedge), \tag{2}$$

for all $\gamma \in \mathbb{R}$, $-k \le \gamma \le k$. Here $\mathcal{H}^s(X^\wedge) := \mathcal{H}^s(\mathbb{R}_+, H^s(X))$ in the sense of the definitions in Section 2.4., and $\mathcal{H}^{s;\gamma}(X^\wedge) := t^\gamma \mathcal{H}^s(X^\wedge)$, t being the variable on $\mathbb{R}_+$. The subscript F indicates flatness of order k (at $t = 0$ and $t = \infty$). Note that, for $k = 0$, the condition (2) is automatically satisfied.

The operator-valued symbols of corner Mellin operators have values in the space of cone operators. The calculus of the preceding sections shows that they are needed in the adequate parameter-depending form. In particular, we have to introduce the class of parameter-depending flat operators $\mathcal{N}_F^\mu(X^\wedge;\Lambda)_\theta$. They will be a subspace of the corresponding parameter-depending class $ML^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$, Λ a parameter set as above. The remarks, in the beginning on the definition of $L^\mu(X;\Lambda)$, give a scheme for the analogous definitions here.

We already have mentioned an order reducing system $b^\mu(\lambda)$ for the scale $\mathcal{E} = \{H^s(X)\}$. Now we replace Λ by $\Lambda' = \mathbb{R}_\tau \times \Lambda$ which is also an admissible choice of a parameter set. This gives rise to an order reducing system $b^\mu(\tau,\lambda)$, depending on the parameter $\lambda' = (\tau,\lambda)$.

Remark that the essential aspect for us is the behaviour for $|\lambda| \to \infty$. Since λ will play, later on, the role of a covariable (may be of higher dimension), we suppose once and for all

$$\Lambda = \mathbb{R}^l, \ l \ge 1. \tag{3}$$

Define the parameter-depending classes of amplitude functions

$$S^\mu(\overline{Q^2} \times \mathbb{R};\mathcal{E},\mathcal{E};\Lambda) = S^\mu(\overline{Q^2} \times (\mathbb{R} \times \Lambda);\mathcal{E},\mathcal{E}),$$

$\mathcal{E} = \{H^s(X)\}_{s\in\mathbb{R}}$, as the space of those

$$a(r,\tau,\lambda) \in C^\infty\left(\overline{Q^2} \times \mathbb{R} \times \Lambda, \bigcap_s \mathcal{L}(H^s(X), H^{s-\mu}(X))\right)$$

for which the symbol estimates hold, now for the λ-depending system of order reductions. In other words, the conditions are

$$\left\| b^{s-\mu+|\beta|}(\tau,\lambda)(\mathrm{D}_r^\alpha \mathrm{D}_{\tau,\lambda}^\beta a)(r,\tau,\lambda)\ b^{-s}(\tau,\lambda)\right\|_{0,0} \le c, \tag{4}$$

for all multi-indices $\alpha \in \mathbb{N}^2$, $\beta \in \mathbb{N}^{1+l}$ ($l = \dim\ \lambda$) and all $(r,\tau,\lambda) \in K \times \mathbb{R} \times \Lambda$, for every compact $K \subset\subset (\overline{\mathbb{R}_+})^2$, $c = c(\alpha,\beta,K) > 0$ a constant, and the same for the functions with r_j replaced by r_j^{-1}, $j = 1,2$. Moreover, we assume the scale axiom to be satisfied, now including λ which is treated as a Mellin covariable.

From the latter definition, we can easily read off the adequate definition of the class $ML^{-\infty}(\overline{\mathbb{R}_+}; \mathcal{E}, \mathcal{E}; \Lambda)$ of parameter-depending smoothing operators. First, we have the parameter-depending kernels

$$K(r, \rho, \lambda) = M^{-1}_{\tau \to \rho}\ a(r, \tau, \lambda). \tag{5}$$

They also satisfy the scale axiom for every fixed λ. Then, a cut-off with a function $\psi(\rho) \in C_0^\infty(\mathbb{R}_+)$, $\psi(\rho) \equiv 1$ close to $\rho = 1$, gives rise to a family

$$K_0(r, \rho, \lambda) := (1 - \psi(\rho))K(r, \rho, \lambda) \in$$
$$C^\infty\left(\overline{Q^2} \times \mathbb{R}_+ \times \Lambda;\ \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s(X), H^\infty(X))\right)$$

or, more precisely,

$$K_0(r, \rho, \lambda) \in C^\infty\left[\overline{Q^2} \times \Lambda, \mathcal{H}^\infty\ (Q, \mathcal{L}(H^s(X), H^{s'}(X))^\infty)\right],$$

for all $s, s' \in \mathbb{R}$, cf. 2.1. Proposition 10. We can even say that

$$M^{-1}_{\lambda \to \pi}\ K_0(r, \rho, \lambda) \in C^\infty\left[\overline{Q^2},\ \mathcal{H}^\infty(Q^{1+l}, \mathcal{L}(H^s(X), H^{s'}(X))^\infty)\right], \tag{6}$$

for all $s, s' \in \mathbb{R}$. Here $\pi \in Q^1$ is the Mellin preimage variable, $\lambda \in \mathbb{R}^l = \Lambda$ being the dual variable (cf. (3)). (6) follows by a simple reinterpretation of 2.1. Proposition 10.

Let

$$\mathcal{G}(X^\wedge) = \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s(X^\wedge),\ \mathcal{H}^\infty(X^\wedge))$$

be equipped with a countable system of norms $\{\eta^j\}_{j\in\mathbb{Z}}$, similarly as for $\mathcal{G}(X)$ above.

If $\mathbf{G}(\lambda)$ is the operator family belonging to $K_0(r, \rho, \lambda)$, then

$$\mathbf{G}(\lambda), \mathbf{G}^*(\lambda) \in C^\infty(\Lambda, \mathcal{G}(X^\wedge)) \tag{7}$$

and

$$\eta^j(\mathrm{D}_\lambda^\beta \mathbf{G}(\lambda)),\ \eta^j(\mathrm{D}_\lambda^\beta \mathbf{G}^*(\lambda)) \le c(1 + |\lambda|)^{-N}, \tag{8}$$

for all $N \in \mathbb{N}$, $j \in \mathbb{Z}$, multi-indices β, with constants $c = c(N, j, \beta) > 0$, for all $\lambda \in \Lambda$. This a consequence of (6).

Now the class $ML^{-\infty}(\overline{\mathbb{R}_+}; \mathcal{E}, \mathcal{E}; \Lambda)$ is just defined as the space of all operator families $\mathbf{G}(\lambda)$ for which (7) and (8) holds. Furthermore $ML^\mu(\overline{R_+}; \mathcal{E}, \mathcal{E}; \Lambda)$ is the space of all

$$\mathbf{A}(\lambda) = op_M(a)(\lambda) + \mathbf{G}(\lambda), \tag{9}$$

where $a(t,t',\tau,\lambda) \in S^\mu(\overline{Q^2} \times \mathbb{R}; \mathcal{E},\mathcal{E};\Lambda)$ and $\mathbf{G}(\lambda) \in ML^{-\infty}(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$. Remark that the theorem on the existence of a complete symbol (cf. 2.2. Theorem 5) has a straightforward generalization to the parameter-depending class. In other words, every $\mathbf{A}(\lambda)$ has a complete symbol

$$\sigma_{\mathbf{A}}(t,\tau,\lambda) \in S^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda),$$

in the sense that

$$\mathbf{A}(\lambda) - op_M(\sigma_{\mathbf{A}})(\lambda) \in ML^{-\infty}(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda).$$

2. DEFINITION. $\mathcal{N}_F^\mu(X^\wedge;\Lambda)_\theta$, $\theta = (k,k)$, $k \in \mathbb{N}$, is the subspace of all $\mathbf{A}(\lambda) \in ML^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$ such that

(i) $\mathbf{A}(\lambda)$, $\mathbf{A}^*(\lambda)$ induce continuous operators

$$\mathcal{H}^s(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma}(X^\wedge), \tag{10}$$

for all $s \in \mathbb{R}$ and $-k \le \gamma \le k$;

(ii) $\mathbf{A}(\lambda)|_{\mathbb{R}_+} \in L^\mu_{cl}(\mathbb{R}_+ \times X;\Lambda)$, i.e. $\mathbf{A}(\lambda)$ is a classical parameter-depending ψDO over $\mathbb{R}_+ \times X$;

(iii) $\mathbf{A}(\lambda)$, $\mathbf{A}^*(\lambda)$ have complete symbols

$$a(t,\tau,\lambda),\ a^*(t,\tau,\lambda) \in S^\mu(\overline{\mathbb{R}_+} \times \mathbb{R};\mathcal{E},\mathcal{E};\Lambda)$$

with values in $L^\mu_{cl}(X;\Lambda)$, for every $(t,\tau) \in \overline{\mathbb{R}_+} \times \mathbb{R}$, and

$$\omega(t)t^{-k}a(t,\tau,\lambda),\ (1-\omega(t))t^k a(t,\tau,\lambda),$$
$$\omega(t)t^{-k}a^*(t,\tau,\lambda),\ (1-\omega(t))t^k a^*(t,\tau,\lambda),$$

are of the same sort (i.e. in $S^\mu(\overline{\mathbb{R}_+} \times \mathbb{R};\mathcal{E},\mathcal{E};\Lambda)$ and $L^\mu_{cl}(X;\Lambda)$-valued).

In (ii), we have used the obvious definition of parameter-depending standard ψDO's over non-compact manifolds, cf. the analogous notations in the beginning of this section.

Let us explicitly describe the negligible operators $\mathbf{G}(\lambda) \in \mathcal{N}_F^{-\infty}(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$. Set

$$\mathcal{G}(X^\wedge)_\theta = \bigcap_{\substack{s\in\mathbb{R}\\ -k\le\gamma\le k}} \mathcal{L}(\mathcal{H}^s(X^\wedge),\ \mathcal{H}^{\infty,\gamma}(X^\wedge)),$$

$\theta = (k,k)$, $k \in \mathbb{N}$, equipped with the natural norm system $\{\eta^{j,\gamma}\}_{j\in\mathbb{Z},-k\le\gamma\le k}$. Then, there is a countable subsystem that also defines the topology. Thus $\mathcal{G}(X^\wedge)_\theta$ is a Fréchet space.

Then, $\mathcal{N}_F^{-\infty}(X^\wedge;\Lambda)_\theta$, $\theta=(k,k)$, $k\in\mathbb{N}$, is the subspace of all $\mathbf{G}(\lambda)\in ML^{-\infty}(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$, with

$$\mathbf{G}(\lambda),\ \mathbf{G}^*(\lambda)\in C^\infty(\Lambda,\mathcal{G}(X^\wedge)_\theta)$$

such that

$$\eta^{j,\gamma}(\mathrm{D}_\lambda^\beta\mathbf{G}(\lambda)),\ \eta^{j,\gamma}(\mathrm{D}_\lambda^\beta\mathbf{G}^*(\lambda))\le c(1+|\lambda|)^{-N},$$

for all $N\in\mathbb{N}$, $j\in\mathbb{Z}$, $-k\le\gamma\le k$, β, with constants $c=c(N,j,\gamma,\beta)>0$.

3. Remark. Every $\mathbf{A}(\lambda)\in\mathcal{N}_F^\mu(X^\wedge;\Lambda)_\theta$ can be written in the form (9), where $a_1\left(t,\frac{1}{2}+i\tau,\lambda\right)\in S^\mu(\overline{\mathbb{R}_+}\times\mathbb{R};\mathcal{E},\mathcal{E};\Lambda)$ is a complete symbol, with the properties as in (iii), and a_1 extends to a holomorphic function $a(t,z,\lambda)$ in z, with

$$a_1\left(t,\frac{1}{2}-\gamma+i\tau,\lambda\right)\in S^\mu(\overline{\mathbb{R}_+}\times\mathbb{R};\mathcal{E},\mathcal{E};\Lambda),$$

for all $\gamma\in\mathbb{R}$, further $\mathbf{G}(\lambda)\in\mathcal{N}_F^{-\infty}(X^\wedge;\Lambda)_\theta$.

This follows by the same constructions as in the proof of 2.1 Proposition 12, where we start with a complete symbol $a\left(t,\frac{1}{2}+i\tau,\lambda\right)$, as in (iii). We then preserve the mapping property with the weights in the image, since the complete symbol a has the form

$$t^k\omega(t)\tilde{a}\left(t,\frac{1}{2}+i\tau,\lambda\right)+t^{-k}(1-\omega(t))\ \tilde{a}'\left(t,\frac{1}{2}+i\tau,\lambda\right),$$

with $\tilde{a},\tilde{a}'\in S^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$, which is unchanged under the manipulations with the ρ-variable in the Mellin preimage. In other words,

$$a_1(t,z,\lambda)=t^k\omega(t)\tilde{a}_1(t,z,\lambda)+t^{-k}(1-\omega(t))\tilde{a}'(t,z,\lambda).$$

Note that, for any cut-off function ω,

$$\omega(t)t^k op_M(\tilde{a}_1)(\lambda)\ u=\omega(t)op_M(\tilde{a}_1)(\lambda)t^k u,$$

for all u with bounded support in t, and analogously for $\tilde{a}_1'$. This is a consequence of Cauchy's integral formula and the holomorphy of $\tilde{a}_1,\tilde{a}_1'$ in z.

4. Remark. The same constructions may be performed, of course, not only over the 'infinite cone' $X^\wedge$, but also for any stretched manifold C, associated with a 'manifold' with conical singularities. That means C is compact, with boundary $\partial C=X$, and if $V\doteq[0,1)\times X$ is a collar neighbourhood of ∂C, then $V/(\{0\}\times X)$ is locally the model of the given conical singularity. For more details, cf. e.g. [S2].

If $\omega\in C_0^\infty(V)$ is a function with $\omega\equiv 1$, close to ∂C, we get the class $\mathcal{N}_F^\mu(C;\Lambda)_\theta$, by the condition

$$\omega\mathcal{N}_F^\mu(C;\Lambda)_\theta\ \omega=\omega\mathcal{N}_F^\mu(X^\wedge;\Lambda)_\theta\ \omega,$$

$(1-\omega)\mathcal{N}_F^\mu(C;\Lambda)_\theta(1-\omega) = (1-\omega)L_{cl}^\mu(\text{int } C;\Lambda)(1-\omega)$, with a fixed identification $V \triangleq [0,1) \times X$.

The parameter-depending flat operators form subclasses of the parameter-depending cone operators. Let us return now to $X^\wedge$ and pass to the definition of the parameter-depending Green and Mellin operators over $X^\wedge$. Then we easily get the operators also over C.

For reasons that become clear below, it is natural to deal with the cone operators with continuous asymptotics. The non-parameter-depending theory was elaborated in [S2].

First of all, we need the scales of spaces $\mathcal{H}_V^s(X^\wedge)_\Delta$, $s \in \mathbb{R}$, with continuous asymptotics.

Let us briefly remind of the definition. If $f(z)$ is an $H^s(X)$-valued function, defined on $\Gamma_\rho = \{z \in \mathbb{C} : \text{Re } z = \text{Re } \rho\}$, $\rho \in \mathbb{C}$ fixed, we set

$$\|f\|_{s,\rho} = \left\{ \int_{\Gamma_\rho} \|b^s(z)f(z)\|_{L^2(X)}^2 |dz| \right\}^{\frac{1}{2}} .$$

Here $b^s(z)$ is a parameter-depending classical ψDO on X of order s, with the parameter-depending homogeneous principal symbol $(|\xi|^2 + |\text{Im } z|^2)^{\frac{s}{2}}$, and $b^s(z) : H^s(X) \to H^0(X) = L^2(X)$ an isomorphism for every $z \in \Gamma_\rho$.

Remember that $\mathcal{H}^s(X^\wedge)$ is the closure of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\|u\|_{\mathcal{H}^s(X^\wedge)} = \|Mu\|_{s,\frac{1}{2}},$$

and $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma\ \mathcal{H}^s(X^\wedge)$ the closure of $C_0^\infty(X^\wedge)$ with respect to $\|Mu\|_{s,\frac{1}{2}-\gamma}$.

For $V \subseteq \mathbb{C}$, we define $V^c = \mathbb{C}\backslash\Omega$, where Ω is the union of all unbounded connected components of $\mathbb{C}\backslash\overline{V}$. For any system of subsets $V_j \subseteq \mathbb{C}$, $j \in J$, we set $\sum_j V_j = \left(\bigcup_j V_j\right)^c$. Moreover, we use the notation

$$\sigma(V) = \sup\ \{\text{Re } z : z \in V\},\ \tau(V) = \inf\{\text{Re } z : z \in V\}, \tag{11}$$

$V \subseteq \mathbb{C}$.

A function $\chi_V \in C^\infty(\mathbb{C})$ is called a V-excision function if $\chi_V \equiv 0$ in a neighbourhood of $\overline{V}$, $\chi_V \equiv 1$ outside another neighbourhood of $\overline{V}$, $|\chi_V| \leq$ const.

Let $\mathcal{V} = \{V \subset \mathbb{C} : V \cap \{\alpha \leq \text{Re } z \leq \beta\}$ compact for all $\alpha, \beta \in \mathbb{R}$, $V^c = V$, and

$$\mathcal{V}^\gamma = \left\{V \in \mathcal{V} : V \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset\right\}.$$

Denote by $\mathcal{I}$ the set of all couples $\Delta = (\delta, \delta')$, where $0 < \delta \leq \infty$, $0 < \delta' \leq \infty$

or $\delta = \delta' = 0$. Set

$$S_\Delta = \left\{\frac{1}{2} - \delta < \operatorname{Re} z < \frac{1}{2} + \delta'\right\}, \quad \text{for } \delta, \delta' > 0,$$

$$S_\Delta^\gamma = T^{-\gamma} S_\Delta = \{z - \gamma : z \in S_\Delta\}, \quad \gamma \in \mathbb{R}.$$

5. DEFINITION. Let $V \subseteq \mathbb{C}$, $\Delta \in I$ finite, $\Delta \neq (0,0)$, $V \cap \partial S_\Delta = \emptyset$, $V \cap S_\Delta \in \mathcal{V}$. Then, $\mathcal{A}_V^s(X)_\Delta$ is the subspace of all $h \in \mathcal{A}(S_\Delta \backslash V,\ H^s(X))$ for which

(i) $\|\chi h\|_{s,\rho} < \infty$, for every V-excision function χ uniformly for all $\rho \in S_{\Delta'}$ for all Δ' with $\overline{S}_{\Delta'} \subset S_\Delta$,

(ii) for every $f \in \mathcal{A}(\mathbb{C})$, we have

$$< \varsigma[h], f > := \frac{1}{2\pi i} \int_C h(z) f(z) \mathrm{d}z \in C^\infty(X),$$

for every curve $C \subset S_\Delta \backslash V$, surrounding $V_\Delta := V \cap S_\Delta$ clock-wise.

ς defines a linear operator

$$\varsigma : \mathcal{A}_V^s(X)_\Delta \to \mathcal{A}'(V_\Delta) \otimes_\pi C^\infty(X). \tag{12}$$

$\mathcal{A}_V^s(X)_\Delta$ is a Fréchet space with respect to the norms $\|\chi h\|_{s,\rho}$, together with those from (12). Let us describe an adequate choice of a countable system of norms on $\mathcal{A}'(K)$, $K \subset \mathbb{C}$ being a compact set with $K^c = K$. Let $C_j \subset \{z : 2^{-(j+1)} < \operatorname{dist}(z, K) < 2^{-j}\}$ be a smooth curve surrounding K and $L^2(C_j)$ the space of square integrable functions on C_j with respect to the measure $|\mathrm{d}z|$. Then, we have $< \varsigma, (w - z)^{-1} > \in L^2(C_j)$ as a function of z (pairing of ς with respect to w). Further

$$\varsigma \to \| < \varsigma,\ (w - z)^{-1} > \|_{L^2(C_j)} \tag{13}$$

is a norm on $\mathcal{A}'(K)$. If j runs over $\mathbb{N}$, then we get the Fréchet space topology of $\mathcal{A}'(K)$.

$\mathcal{A}_V^\infty(X)_\Delta$ is a nuclear Fréchet space. Moreover,

$$\mathcal{A}_V^s(X)_\Delta = \mathcal{A}_0^s(X)_\Delta + \mathcal{A}_V^\infty(X)_\Delta, \tag{14}$$

in the sense of sums of Fréchet spaces. In general, this means that, when E_1, E_2 are locally convex vector spaces, with the systems of semi-norms $(\alpha_\iota)_{\iota \in I}$ and $(\beta_x)_{x \in K}$, respectively, and if E_1, E_2 are vector subspaces of another vector space E, then $E_1 + E_2 = \{u_1 + u_2 : u_i \in E_i,\ i = 1, 2\}$ is a locally convex vector space, with the system of semi-norms

$$\gamma_{\iota x}(u) = \inf_{u = u_1 + u_2} (\alpha_\iota(u_1) + \beta_x(u_2)), \quad \iota \in I,\ x \in K.$$

In our applications, we mainly have Fréchet spaces with countable systems of norms.

Incidentally, we also need intersections $E_1 \cap E_2$ with the corresponding semi-norm system, namely

$$\delta_{\iota x}(u) = \max(\alpha_\iota(u), \beta_x(u)), \ \iota \in I, \ x \in K.$$

The inverse Mellin transform defines an injective operator

$$M^{-1} : \mathcal{A}_V^s(X)_\Delta \to \mathcal{H}^s(X^\wedge), \text{ for } V \cap \Gamma_{\frac{1}{2}} = \emptyset. \tag{15}$$

The space $\mathcal{H}_V^s(X^\wedge)_\Delta$ is defined as the image of $\mathcal{A}_V^s(X)_\Delta$ under M^{-1}. It is a Fréchet space with the norm system induced by that of $\mathcal{A}_V^s(X)_\Delta$. For $\Delta = (0,0)$, we set $\mathcal{H}^s(X^\wedge) = \mathcal{H}_V^s(X^\wedge)_\Delta$, this is then independent of V.

Now we want to introduce the spaces $\mathcal{H}_V^s(X^\wedge)_\Delta$, for arbitrary $V \in \mathcal{V}^0$. We use the following 'decomposition method'. First, observe that $V_\Delta = V \cap \overline{S_\Delta}$ can be written as

$$V_\Delta = V_1 + V_2,$$

where $V_i = \bigcup_{j\in\mathbb{Z}} V_{ij}$ with compact sets $V_{ij} \in \mathcal{V}^0$, $V_{ij} \subset S_\Delta$, $\delta(V_{ij}) < \tau(V_{i.j+1})$, and $\sigma(V_{ij}) \to \frac{1}{2} - \delta$ for $j \to -\infty$, $\tau(V_{ij}) \to \frac{1}{2} + \delta'$ for $j \to +\infty$, $i = 1,2$ (cf. (11)). It is then obvious that there exist sequences $\Delta_{ik} \in I$, Δ_{ik} finite, $k \in \mathbb{N}$, $i = 1,2$, with $S_\Delta = \bigcup_{k\in\mathbb{N}} S_{\Delta_{ik}}$, $V_i \cap \partial S_{\Delta_{ik}} = \emptyset$, for all $k \in \mathbb{N}$, $i = 1,2$. The above construction yields the spaces $\mathcal{H}_{V_i}^s(X^\wedge)_{\Delta_{ik}}$. Then,

$$\mathcal{H}_{V_i}^s(X^\wedge)_\Delta := \varprojlim_{k\in\mathbb{N}} \mathcal{H}_{V_i}^s(X^\wedge)_{\Delta_{ik}}$$

is a Fréchet space which is independent of the concrete choice of the sequence Δ_{ik}. The sum

$$\mathcal{H}_V^s(X^\wedge)_\Delta := \mathcal{H}_{V_1}^s(X^\wedge)_\Delta + \mathcal{H}_{V_2}^s(X^\wedge)_\Delta \tag{16}$$

is also a Fréchet space, and it only depends on V, but not on the concrete choice of the V_{ij}, Δ_{ik}. Then,

$$\mathcal{H}_V^s(X^\wedge)_\Delta = \mathcal{H}_0^s(X^\wedge)_\Delta + \mathcal{H}_V^\infty(X^\wedge)_\Delta. \tag{17}$$

We can also define the space $\mathcal{H}_V^s(X^\wedge)_\Delta$, with respect to closed (or half open) weight intervals $\Delta = [\delta, \delta']$, $\delta, \delta' \in \overline{\mathbb{R}_+}$ ($\Delta = [\delta; \delta')$, $\delta \in \overline{\mathbb{R}_+}$, $\delta' \in \mathbb{R}_+$; $\Delta = (\delta, \delta']$, $\delta \in \mathbb{R}_+$, $\delta' \in \overline{\mathbb{R}_+}$). For instance, $\mathcal{H}_0^s(X^\wedge)_\Delta$ for $\Delta = [\delta, \delta']$ is the subspace of all $u \in \mathcal{H}^s(X^\wedge)$ for which $Mu(z)$ extends to a function $h(z) \in \mathcal{A}_0^s(X)_{\tilde{\Delta}}$ $\tilde{\Delta} = (\delta, \delta')$, with $\lim_{\gamma\to\frac{1}{2}-\delta} \|h(z)\|_{s,\gamma} < \infty$, $\lim_{\gamma\to\frac{1}{2}+\delta'} \|h(z)\|_{s,\gamma} < \infty$.

For $\Delta = [\delta, \delta')$ $(\Delta = (\delta, \delta'])$, only the first (second) limit has to be finite. For non-empty $V \in \mathcal{V}^0$, we only need the variants

$$\mathcal{H}_V^s(X^\wedge)_{[0,\delta')} := \mathcal{H}_0^s(X^\wedge)_{[0,\delta')} + \mathcal{H}_V^\infty(X^\wedge)_{(\varepsilon,\delta')},$$
$$\mathcal{H}_V^s(X^\wedge)_{(\delta,0]} := \mathcal{H}_0^s(X^\wedge)_{(\delta,0]} + \mathcal{H}_V^\infty(X^\wedge)_{(\delta,\varepsilon)},$$

for arbitrary $\varepsilon > 0$ (these spaces are then independent of ε).

We also can define spaces with asymptotics and weight $\gamma \in \mathbb{R}$, namely

$$\mathcal{H}_V^{s;\gamma}(X^\wedge)_\Delta := t^\gamma \; \mathcal{H}_{T^{-\gamma}V}^s(X^\wedge)_\Delta,$$

$V \in \mathcal{V}^\gamma$ (then $T^{-\gamma} \; V \in \mathcal{V}^0$).

Set

$$\mathcal{G}_V(X^\wedge)_\Delta = \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s(X^\wedge), \; \mathcal{H}_V^\infty(X^\wedge)_\Delta), \tag{18}$$

$V \in \mathcal{V}^0$, $\Delta \in \mathcal{I}$, considered in the natural Fréchet space structure, defined as follows. First, $\mathcal{L}(\mathcal{H}^s(X^\wedge), \; \mathcal{H}_V^{s'}(X^\wedge)_\Delta)$ is a Fréchet space, for every $s, s' \in \mathbb{R}$. Then,

$$\bigcap_{s'} \mathcal{L}(\mathcal{H}^s(X^\wedge), \; \mathcal{H}_V^{s'}(X^\wedge)_\Delta) = \mathcal{L}(\mathcal{H}^s(X^\wedge), \; \mathcal{H}_V^\infty(X^\wedge)_\Delta)$$

is also a Fréchet space, since a countable subset of s' suffices.

For a similar reason, the space (18) is Fréchet with a system of norms that we denote by $\{\eta_V^j\}_{j\in\mathbb{Z}}$, Δ being fixed.

6. Definition. A family of operators $\mathbf{G}(\lambda) \in ML^{-\infty}(\overline{\mathbb{R}_+}; \mathcal{E}, \mathcal{E}; \Lambda)$ is called of the class $\mathcal{N}_G(X^\wedge; \Lambda)_\theta$, $\theta = (k, k)$, $k \in \mathbb{N}$, if

(i) $\mathbf{G}(\lambda) \in C^\infty(\Lambda, \mathcal{G}_V(X^\wedge)_\theta)$, $\mathbf{G}^*(\lambda) \in C^\infty(\Lambda, \mathcal{G}_W(X^\wedge)_\theta)$, for certain $V, W \in \mathcal{V}^0$ depending on $\mathbf{G}$,

(ii) $\eta_V^j(\mathrm{D}_\lambda^\beta \mathbf{G}(\lambda))$, $\eta_W^j(\mathrm{D}_\lambda^\beta \mathbf{G}^*(\lambda)) \leq c(1 + |\lambda|)^{-N}$, for all $N \in \mathbb{N}$, $j \in \mathbb{Z}$, multi-indices β, with constants $c = c(N, j, \beta) > 0$.

$\mathcal{N}_G(X^\wedge; \Lambda)_\theta$ is just the class of parameter-depending Green operators over $X^\wedge$ with respect to the weight in terval θ. Note that, for $k = 0$, we get $ML^{-\infty}(\overline{\mathbb{R}_+}; \mathcal{E}, \mathcal{E}; \Lambda)$. If C is the stretchted manifold, as in Remark 4, we define

$$\mathcal{N}_G(C; \Lambda)_\theta = \omega \mathcal{N}_G(X^\wedge; \Lambda)_\theta \omega + (1 - \omega) \; L_{cl}^{-\infty}(\text{int } C; \Lambda)(1 - \omega)$$

with the canonical identification of objects on $X^\wedge$ and on a collar neighbourhood of ∂C.

Similarly, we obtain the Sobolev spaces over C and those with asymptotics. We set

$$\mathcal{H}^s(C) = \omega \mathcal{H}^s(X^\wedge) + (1 - \omega) \; H_{\text{loc}}^s(\text{int } C), \tag{19}$$

with an obvious identification of a collar neighbourhood of ∂C with $[0,1) \times X$,

$$\mathcal{H}^{s,\gamma}(C) = g^{\gamma}\mathcal{H}^{s}(C), \tag{20}$$

g^{γ} being a non-vanishing C^{∞} function on int C, with $g^{\gamma} = t^{\gamma}$ close to ∂C, $\gamma \in \mathbb{R}$. The spaces with asymptotics are defined as the sums

$$\mathcal{H}_V^{s,\gamma}(C)_{\Delta} = \omega\mathcal{H}_V^{s,\gamma}(X^{\wedge})_{\Delta} + (1-\omega)H^{s}_{\mathrm{loc}}(\mathrm{int}\ C), \tag{21}$$

where here Δ runs over the set of all half-open weight intervals

$$\mathcal{I} = \{(\delta, 0] : \delta \geq 0\}. \tag{22}$$

Then V belongs to

$$\mathcal{B}^{\gamma} = \left\{V \in \mathcal{V}^{\gamma} : V \subseteq \left\{\mathrm{Re}\ z < \frac{1}{2} - \gamma\right\}\right\}. \tag{23}$$

The topology of the sums is as usual. We might first pass to the closure of the spaces $\omega\mathcal{H}^{s,\gamma}$ in $\mathcal{H}^{s,\gamma}(X^{\wedge})(\mathcal{H}_V^{s,\gamma}(X^{\wedge})_{\Delta})$, and $(1-\omega)H^{s}_{\mathrm{loc}}(\ldots)$ in $H^{s}_{\mathrm{loc}}(\ldots)$ and then take the sums of the corresponding Fréchet spaces.

Let us set

$$\mathcal{H}_{as}^{s,\gamma}(C)_{\Delta} = \varprojlim_{v \in \mathcal{B}^{\gamma}} \mathcal{H}_V^{s,\gamma}(C)_{\Delta} \tag{24}$$

and $\mathcal{H}_{as}^{s}(C)_{\Delta} = \mathcal{H}_{as}^{s,0}(C)_{\Delta}$. Note that $\mathcal{H}_{as}^{\infty,\gamma}(C)_{\Delta}$, as an inductive limit of the nuclear Fréchet spaces $\mathcal{H}_V^{\infty,\gamma}(C)_{\Delta}$, is nuclear.

Incidentally, we use the fact that $\mathcal{H}_V^{s,\gamma}(C)_{\Delta}$ can be written as a projective limit

$$\mathcal{H}_V^{s,\gamma}(C)_{\Delta} = \varinjlim_{j \in \mathbb{N}} \mathcal{H}_V^{s,\gamma}(C)_{\Delta}^{(j)}$$

of Hilbert spaces. We choose this system in such a way that there are continuous embeddings

$$\mathcal{H}_V^{s,\gamma}(C)_{\Delta}^{(j+1)} \hookrightarrow \mathcal{H}_V^{s,\gamma}(C)_{\Delta}^{(j)}$$

for every j.

7. REMARK. It can be proved that the spaces with asymptotics $\mathcal{H}_V^{s,\gamma}(C)_{\Delta}$ are invariant under pull back with respect to diffeomorphisms $C \to C$, provided $V \in \mathcal{V}^{\gamma}$ satisfies the 'shadow condition' $T^{-j}\ V \subseteq V$, for all $j \in \mathbb{N}$ (for details, cf. [S5]).

Now we want to introduce the parameter-depending Mellin operators. They are based on the parameter-depending Mellin symbols that may be introduced in an analogous manner as the above Sobolev spaces with continuous asymptotics.

Let $V \in \mathcal{V}$, $\Delta = (\delta, \delta') \in \mathcal{I}$ be finite, $\Delta \neq (0,0)$, $V \cap \partial S_\Delta = \emptyset$, and $\mu \in \mathbb{R}$. Then, $\mathcal{M}_V^\mu(X;\Lambda)_\Delta$ denotes the subspace of all

$$a(z,\lambda) \in \mathcal{A}(S_\Delta \backslash V, L^\mu(X;\Lambda)) \cap L^\mu_{cl}(X;(S_\Delta \backslash V) \times \Lambda) \tag{25}$$

belonging uniformly to these spaces for all Δ' with $\overline{S}_{\Delta'} \subset S_\Delta$, for which

$$< \varsigma[a], f > = \frac{1}{2\pi i} \int_B a(z,\lambda) f(z) \mathrm{d}z \in L^{-\infty}(X;\Lambda), \tag{26}$$

for all $f \in \mathcal{A}(\mathbb{C})$, $B \subset S_\Delta \backslash V$ being a curve surrounding $V \cap S_\Delta$.

The spaces on the right side of (25) are interpreted as follows.

$L^\mu(X,\Lambda)$ is a Fréchet space in a natural way. We also can endow $L^\mu_{cl}(X;T(V,\varepsilon) \times \Lambda)$,

$$T(V,\varepsilon) = \left\{ z : \frac{1}{2} + \varepsilon - \delta \leq \operatorname{Re}\, z \leq \frac{1}{2} - \varepsilon + \delta' \right\} \backslash \{ z : \operatorname{dist}(z,V) < \varepsilon \},$$

$\varepsilon > 0$ sufficiently small, with a natural Fréchet space structure (analogous constructions in the non-parameter-depending case were given in [R1]). Then,

$$L^\mu_{cl}(X;(S_\Delta \backslash V) \times \Lambda) := \bigcap_{\varepsilon > 0} L^\mu_{cl}(X;T(V,\varepsilon) \times \Lambda)$$

is also a Fréchet space. (25), as an intersection of two Fréchet spaces, is again a Fréchet space. (25) defines an embedding of $\mathcal{M}_V^\mu(X;\Lambda)_\Delta$ into the latter space. Moreover, (26) defines a linear mapping

$$\varsigma : \mathcal{M}_V^\mu(X;\Lambda)_\Delta \to \mathcal{A}'(V_\Delta) \otimes_\pi L^{-\infty}(X;\Lambda), \tag{27}$$

$V_\Delta = V \cap S_\Delta$. We consider $\mathcal{M}_V^\mu(X;\Lambda)_\Delta$ in the Fréchet space structure defined by these mappings. For $\Delta = (0,0)$, we set

$$\mathcal{M}^\mu(X;\Lambda)_\Delta = \mathcal{M}^{\mu,0}(X;\Lambda) := L^\mu_{cl}\left[X; \left(\operatorname{Re}\, z = \frac{1}{2}\right) \times \Lambda\right].$$

This is then independent of any V.

The decomposition method, applied above to the $\mathcal{H}_V^s(X^\wedge)_\Delta$ spaces, may easily be adapted to Mellin symbols. In other words, we get the space $\mathcal{M}_V^\mu(X;\Lambda)_\Delta$, for arbitrary $V \in \mathcal{V}$, $\Delta \in \mathcal{I}$ finite. Then, (similarly as in the non-parameter-depending version, cf. [S2])

$$\mathcal{M}_{V_1}^\mu(X;\Lambda)_\Delta + \mathcal{M}_{V_2}^\mu(X;\Lambda)_\Delta = \mathcal{M}_{V_1+V_2}^\mu(X;\Lambda)_\Delta, \tag{28}$$

$$\mathcal{M}_V^\mu(X;\Lambda)_\Delta = \mathcal{M}_0^\mu(X;\Lambda)_\Delta + \mathcal{M}_V^{-\infty}(X;\Lambda)_\Delta, \tag{29}$$

where 0 indicates $V = \emptyset$. For $\Delta = (\infty,\infty)$, we set $\mathcal{M}_V^\mu(X;\Lambda)_{(\infty,\infty)} :=$

$\mathcal{M}_V^\mu(X;\Lambda) := \varprojlim \mathcal{M}_V^\mu(X;\Lambda)_\Delta$, where the projective limit is taken over all finite $\Delta \in \mathcal{I}$. The formulas (28), (29) are then true also for $\Delta = (\infty, \infty)$. Incidentally, we use the notations

$$\mathcal{M}_{as}^\mu(X;\Lambda) = \varinjlim_{V \in \mathcal{V}} \mathcal{M}_V^\mu(X;\Lambda).$$

Moreover, $\mathcal{M}_{as}^{\mu;\gamma}(X;\Lambda)$, $\gamma \in \mathbb{R}$, denotes the subspace of all $a(z,\lambda)$ for which the corresponding $V = V(a)$ belongs to $\mathcal{V}^\gamma$.

For every $a \in \mathcal{M}_{as}^{\mu;\gamma}(X;\Lambda)$, we define

$$op_M^\gamma(a)(\lambda) = t^\gamma op_M(T^{-\gamma}a)(\lambda)t^{-\gamma},$$

where $(T^\delta a)(z,\lambda) = a(z+\delta,\lambda)$. Further, set $I\ u(t,x) = t^{-1}u(t^{-1},x)$.

8. DEFINITION. $\mathcal{N}^\mu(X^\wedge;\Lambda)_\theta$, $\theta = (k,k)$, $k \in \mathbb{N}\backslash\{0\}$, consists of all operator families of the form

$$\mathbf{A}(\lambda) = \mathbf{S}(\lambda) + \mathbf{S}'(\lambda) + \mathbf{F}(\lambda) + \mathbf{G}(\lambda),$$

where $\mathbf{F}(\lambda) \in \mathcal{N}_F^\mu(X^\wedge;\Lambda)_\theta$, $\mathbf{G}(\lambda) \in \mathcal{N}_G(X^\wedge;\Lambda)_\theta$, and

$$\mathbf{S}(\lambda) = \sum_{i=1,2} \sum_{j=0}^{k-1} \omega(t)t^j\ op_M^{\gamma_{ij}}(a_{ij})(\lambda)\ \omega(t),$$

with arbitrary $a_{ij} \in \mathcal{M}_{as}^{\mu;\gamma_{ij}}(X;\Lambda)$, $-j \le \gamma_{ij} \le 0$, and $\mathbf{S}'(\lambda) = I^{-1}\ \tilde{\mathbf{S}}(\lambda)I$, with some $\tilde{\mathbf{S}}(\lambda)$ of analogous structure as $\mathbf{S}(\lambda)$. For $k = 0$, we set $\mathcal{N}^\mu(X^\wedge;\Lambda)_\theta = ML_{cl}^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E};\Lambda)$. Moreover, for C as in Remark 4, we set

$$\mathcal{N}^\mu(C;\Lambda)_\theta = \omega\mathcal{N}^\mu(X^\wedge;\Lambda)_\theta\ \omega + (1-\omega)L_{cl}^\mu(\text{int } C;\Lambda)(1-\omega) \tag{30}$$

(cf. the analogous definitions of $\mathcal{N}_F^\mu(C;\Lambda)_\theta$ and $\mathcal{N}_G(C;\Lambda)_\theta$).

For $\Lambda = \emptyset$, we use the notations $\mathcal{N}^\mu(C)_\theta$, $\mathcal{N}_F^\mu(C)_\theta, \ldots$.

9. REMARK. $\mathcal{N}^\mu(X^\wedge)_\theta \subseteq ML^\mu(\overline{\mathbb{R}_+};\mathcal{E},\mathcal{E})$, $\theta = (k,k)$, consists of the subspace of those operators $\mathbf{A}$ which have a complete Mellin symbol $a(t,z)$, $z = \frac{1}{2} + i\tau$, (cf. Section 2.2., Theorem 5) with $\mathbf{A} = op_M(a) + \mathbf{G}$, $\mathbf{G} \in \mathcal{N}_G(X^\wedge)_\theta$, the Taylor expansion of $a(t,z)$, at $t = 0$, is

$$a(t,z) = \sum_{j=0}^{k-1} \sigma_M^{-j}(\mathbf{A})(z)t^j + a_F(t,z),$$

i.e. $\left.\frac{1}{j!}\frac{\partial^j}{\partial t^j}\ a(t,z)\right|_{t=0}$ extends to an operator function in $\mathcal{M}_{as}^\mu(X)$, for $j = 0,\ldots,k-1$, $a_F(t,z)$ being flat of order k at $t = 0$, and an analogous property holds for $t \to \infty$.

Note that $X^\wedge$ corresponds to a special manifold C_1, namely $C_1 = [0,1] \times X$. Then, we have a canonical isomorphism

$$\mathcal{N}^\mu(\overset{\circ}{C_1}; \Lambda)_\theta \,\hat{=}\, \mathcal{N}^\mu(X^\wedge; \Lambda)_\theta. \tag{31}$$

We mainly consider in the following the case of general C.

It is clear that we have a canonical embedding

$$\mathcal{N}^\mu(C; \Lambda)_{\theta_1} \hookrightarrow \mathcal{N}^\mu(C; \Lambda)_{\theta_2}, \text{ for } \theta_1 \geq \theta_2$$

(i.e. $k_1 \geq k_2$). Set

$$\mathcal{N}^\mu(C; \Lambda) = \bigcap_\theta \mathcal{N}^\mu(C; \Lambda)_\theta,$$

where the intersection is taken over all $\theta = (k, k)$, $k \in \mathbb{N}$.

The calculus of cone operators with continuous asymptotics of [S2] extends, in a natural way, to the present parameter-depending version. We do not repeat here everything. Let us restrict ourselves to some typical elements that are needed for references below.

Assume $\theta = (k, k)$ and $k \in \mathbb{N} \setminus \{0\}$; the case $k = 0$ is trivial and may be added by the reader.

First, it is clear that $\mathrm{A}(\lambda) \in \mathcal{N}^\mu(C; \Lambda)_\theta$ induces a family of continuous operators

$$\mathrm{A}(\lambda) : \mathcal{H}_V^s(C)_\theta \to \mathcal{H}_W^{s-\mu}(C)_\theta, \tag{32}$$

for every $V \in \mathcal{B}^0$, with some $W \in \mathcal{B}^0$ depending on V and A.

For every $\mathrm{A}(\lambda) \in \mathcal{N}^\mu(C; \Lambda)_\theta$, we have a well-defined sequence of Mellin symbols

$$\sigma_M^{-j}(\mathrm{A})(z, \lambda) \in \mathcal{M}_{as}^\mu(X; \Lambda), \ j = 0, \dots, k-1,$$

namely $\sigma_M^{-j}(\mathrm{A})(z, \lambda) = a_{1j}(z, \lambda) + a_{2j}(z, \lambda)$. In particular, $\sigma_M^0(A)(z, \lambda) \in \mathcal{M}_{as}^{\mu;0}(X; \Lambda)$. This gives rise to the space of Mellin symbols

$$\mathcal{Y}_{M,\theta}^\mu = \mathcal{M}_{as}^{\mu;0}(X; \Lambda) \times \prod_{j=1}^{k-1} \mathcal{M}_{as}^\mu(X; \Lambda)$$

and a Mellin symbol map

$$\sigma_{M,\theta} : \mathcal{N}^\mu(C; \Lambda)_\theta \to \mathcal{Y}_{M,\theta}^\mu.$$

Here

$$\ker \sigma_{M,\theta} = \mathcal{N}_{F+G}^\mu(C; \Lambda)_\theta := \mathcal{N}_F^\mu(C; \Lambda)_\theta + \mathcal{N}_G(C; \lambda)_\theta.$$

Furthermore, we have

$$\mathcal{N}^\mu(C; \Lambda)_{\theta|\text{int } C} \subset L_{cl}^\mu(\text{int } C; \Lambda),$$

i.e. a map σ_ψ^μ to the parameter-depending homogeneous principal symbol of order μ. This induces a map

$$\sigma_\psi^\mu : \mathcal{N}^\mu(C;\Lambda)_\theta \to S_b^{(\mu)}((T^*C \times \Lambda)\backslash 0). \tag{33}$$

Here $S_b^{(\mu)}((T^*C\times\Lambda)\backslash 0)$ denotes the space of all $p(v,\chi,\lambda) \in C^\infty((T^*C\times\Lambda)\backslash 0)$, which are positively homogeneous in (χ,λ) of order μ ((v,χ) being local coordinates on T^*C) and induce locally close to ∂C in the coordinates $v = (t,x) \in [0,1) \times X = V$, $\chi = (\tau,\xi)$, functions $p_b(t,x,\tau,\xi,\lambda) := p(t,x,t^{-1}\tau,\xi,\lambda)$ which are smooth up to $t=0$. (33) is surjective, and ker $\sigma_\psi^\mu = \mathcal{N}^{\mu-1}(X;\Lambda)_\theta$. Denote by

$$\mathcal{Y}_\theta^\mu \subset S_b^{(\mu)}((T^*C \times \Lambda)\backslash 0) \times \mathcal{Y}_{M.\theta}^\mu$$

the subspace of those couple (p,h), $h = \{h_{0,\dots,}h_{k-1}\}$, for which $\sigma_{M,\theta}(\mathbf{A}) = h$, $\sigma_\psi^\mu(\mathbf{A}) = p$, for a certain $\mathbf{A} \in \mathcal{N}^\mu(C;\Lambda)_\theta$. The space $\mathcal{Y}_\theta^\mu$ can be characterized by a compatibility condition between p,h (cf. [S2] in the analogous non-parameter-depending case).

The kernel of the symbol map

$$(\sigma_\psi^\mu, \sigma_{M,\theta}) : \mathcal{N}^\mu(C;\Lambda)_\theta \to \mathcal{Y}_\theta^\mu$$

consists of $\mathcal{N}_{F+G}^{\mu-1}(C;\Lambda)_\theta$. .

10. THEOREM. $\mathbf{A} \in \mathcal{N}^\mu(C;\Lambda)_\theta$, $\mathbf{B} \in \mathcal{N}^\nu(C;\Lambda)_\theta$, $\mu,\nu \in \mathbb{R}$, *imply* $\mathbf{AB} \in \mathcal{N}^{\mu+\nu}(C;\Lambda)_\theta$ *and*

$$\sigma_M^{-j}(\mathbf{AB})(z,\lambda) = \sum_{j=p+q} \sigma_M^{-p}(\mathbf{A})(z-q,\lambda)\ \sigma_M^{-q}(\mathbf{B})(z,\lambda),$$

$j = 0,\dots,k-1$,

$$\sigma_\psi^{\mu+\nu}(\mathbf{AB})(v,\chi,\lambda) = \sigma_\psi^\mu(\mathbf{A})(v,\chi,\lambda)\ \sigma_\psi^\nu(\mathbf{B})(v,\chi,\lambda).$$

Moreover, the formal adjoint $\mathbf{A}^*$ of $\mathbf{A}$ belongs to $\mathcal{N}^\mu(C;\Lambda)_\theta$, the symbolic rules are analogous to those without parameters, cf. [S2], §5, Theorem 10.

The obvious modifications of the proof, compared with the case without parameters, are left to the reader.

It is necessary also to have a locally convex topology in the spaces $\mathcal{N}^\mu(C)_\theta$, $\mathcal{N}^\mu(C;\Lambda)_\theta$. First, remark that $\mathcal{N}_G(C)_\theta$ has a natural topology, by the identification

$$\mathcal{N}_G(C)_\theta = \left\{\mathcal{H}^0(C) \otimes_\pi \mathcal{H}_{as}^\infty(C)_\theta\right\} \cap \left\{\mathcal{H}_{as}^\infty(C)_\theta \otimes_\pi \mathcal{H}^0(C)\right\}, \tag{34}$$

cf. the notation (24). Moreover, $\mathcal{N}_F^\mu(C)_\theta$ can be equipped with a natural Fréchet space structure. The procedure is completely analogous to the construction for

$L^{\mu}_{cl}(X)$, given in [R1], Section 2. So it may be dropped. Then, we also get a locally convex topology on the sum

$$\mathcal{N}^{\mu}_{F+G}(C)_{\theta} = \mathcal{N}^{\mu}_{F}(C)_{\theta} + \mathcal{N}_{G}(C)_{\theta}.$$

Denote by $\mathcal{N}^{\mu}(C)^{0}_{\theta}(\mathcal{N}^{\mu}(C)^{\beta}_{\theta})$ the subspace of those $\mathbf{A} \in \mathcal{N}^{\mu}(C)_{\theta}$ for which $\delta^{-j}_{M}(\mathbf{A})(z) \in \mathcal{M}^{\mu;0}_{as}(X)\,(\mathcal{M}^{\mu;\beta}_{as}(X))$, for all $j = 0, \ldots, k-1$ (remember that $\theta = (k, 0]$ and, without loss of generality, $k \in \mathbb{N}\backslash\{0\}$). We may set, for instance, $\beta = -\frac{1}{2}$. Then,

$$\mathcal{N}^{\mu}(C)_{\theta} = \mathcal{N}^{\mu}(C)^{0}_{\theta} + \mathcal{N}^{\mu}(C)^{\beta}_{\theta}. \tag{35}$$

Set $\beta_0 = 0, \beta_1 = \beta$ and $\mathcal{Y}^{\mu,\beta_j}_{M,\theta} = \sigma_{M.\theta}\ \mathcal{N}^{\mu}(C)^{\beta_j}_{\theta}$, $j = 0, 1$. Then,

$$0 \to \mathcal{N}^{\mu}_{F+G}(C)_{\theta} \to \mathcal{N}^{\mu}(C)^{\beta_j}_{\theta} \xrightarrow{\sigma_{M.\theta}} \mathcal{Y}^{\mu;\beta_j}_{M;\theta} \to 0$$

is exact and splits. Thus, we have algebraic isomorphisms

$$\mathcal{N}^{\mu}(C)^{\beta_j}_{\theta} \doteq \mathcal{N}^{\mu}_{F+G}(C)_{\theta} \oplus \mathcal{Y}^{\mu;\beta_j}_{M;\theta}, \quad j = 0, 1. \tag{36}$$

$\mathcal{Y}^{\mu;\beta_j}_{M;\theta}$ has a locally convex topology from the spaces of Mellin symbols. Thus, we also have a topology on the right hand side of (36) and hence also on the left side. (35) then yields, as desired, a natural locally convex topology on the space $\mathcal{N}^{\mu}(C)_{\theta}$. Remark that we consider

$$\mathcal{N}^{\mu}(C) = \varprojlim_{\theta} \mathcal{N}^{\mu}(C)_{\theta},$$

in the topology of the projective limit over all $k \in \mathbb{N}$ $(\theta = (k, 0])$.

For $\mathcal{N}^{\mu}(C; \Lambda)_{\theta}$, $\mathcal{N}^{\mu}(C; \Lambda)$, we can proceed in an analogous manner.

11. Remark. $\mathbf{A}(\lambda) \in \mathcal{N}^{\mu}(C; \Lambda)_{\theta}$ implies $\mathbf{A}(\lambda) \in C^{\infty}(\Lambda; \mathcal{N}^{\mu}(C)_{\theta})$ and

$$(\mathrm{D}^{\beta}_{\lambda}\mathbf{A})(\lambda) \in \mathcal{N}^{\mu-|\beta|}(C; \lambda)_{\theta},$$

for every multi-index β.

This is an immediate consequence of the definitions.

12. Remark. $\mathbf{A}(\lambda) \in \mathcal{N}^{0}(C; \Lambda)_{\theta}$ represents a bounded set $\{\mathbf{A}(\lambda) : \lambda \in \Lambda\} \subset \mathcal{N}^{0}(C)_{\theta}$. In particular, the norm of $\mathbf{A}(\lambda)$ in $\mathcal{L}(\mathcal{H}^{0}(C), \mathcal{H}^{0}(C))$ is uniformly bounded for all $\lambda \in \Lambda$.

3.2. *Cone Operators with Point-Wise Discrete Asymptotics*

This section gives some more comment on the notion of asymptotics in the cone distribution spaces and cone operator classes.

It is a standard observation, in the context of differential operators on manifolds with conical singularities, that the solutions have 'discrete conormal asymptotics'

(1) $$u(t,x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} \varsigma_{jk}(x)\ t^{-p_j}\ \log^k\ t,\ \text{as } t \to 0.$$

Here we talk about differential operators on C (cf. 3.1. Remark 4) that are in the coordinates (t,x) of a collar neighbourhood of ∂C of the form

(2) $$\mathbf{A} = \sum_{j=0}^{\mu} A_j(t)\ \left(-t\ \frac{\partial}{\partial t}\right)^j,$$

$A_j(t) \in C^\infty([0,1),\ \mathrm{Diff}^{\mu-j}(X))$. Under the natural ellipticity condition, for totally characteristic operators (cf., for instance, [K1], [M1], [R1], [S2]), the solutions u admit asymptotic expansions (1), where $p_j \in C$, $\mathrm{Re}\ p_j \to -\infty$, as $j \to \infty$, $m_j \in \mathbb{N}$, and the ς_{jk} belong to a finite-dimensional subspace $L_j \subset C^\infty(X)$, $0 \le k \le m_j$.

It is convenient to talk about discrete asymptotic types

$$P = \{(p_j, m_j, L_j)\}_{j\in\mathbb{N}}.$$

Let $\mathcal{P}^\gamma(X)$ denote the system of all those P with $\mathrm{Re}\ p_j < \frac{1}{2} - \gamma$, for all j. If we set $\pi_{\mathbb{C}} P = \bigcup_j \{p_j\}$, then we have the space $\mathcal{H}^s_{\pi_{\mathbb{C}} P}(C)_\Delta$, for every $\Delta \in \mathcal{I}$, with $\pi_{\mathbb{C}} P \cap \partial S_\Delta = \emptyset$. It is natural to speak about the subspace $\mathcal{H}^s_P(C)$ of those $u \in \mathcal{H}^s_{\pi_{\mathbb{C}} P}(C)_\Delta$ for which $M\omega u$ is meromorphic in S_Δ, with poles at p_j of multiplicities $m_j + 1$ and coefficients of the Laurent expansion near p_j at $(z - p_j)^{-(k+1)}$ in L_j, $0 \le k \le m_j$. Then, we also have the projective limits

$$\mathcal{H}^s_P(C) = \varprojlim \mathcal{H}^s_P(C)_\Delta$$

over a system of weight intervals with length tending to infinity. In an analogous manner, we can introduce discrete asymptotic types for the Mellin symbols. Denote by $\mathcal{R}(X \times X)$ the set of all sequences

$$R = \{(r_j, n_j, M_j)\}_{j\in\mathbb{Z}},$$

with $r_j \in C$, $\mathrm{Re}\ r_j \to \pm\infty$, as $j \to \pm\infty$, $n_j \in \mathbb{N}$, $M_j \subset C^\infty(X \times X)$ a finite-dimensional subspace. Further, let $\mathcal{R}^\gamma(X \times X)$ be the subset of all R with $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, $\pi_{\mathbb{C}} R := \bigcup \{r_j\}$.

Then $\mathcal{M}^\mu_R(X)$, $\mu \in \mathbb{R}$, $R \in \mathcal{R}(X \times X)$, denotes the subspace of all $a \in \mathcal{M}^\mu_{\pi_{\mathbb{C}} R}(X)$ which are meromorphic, with poles at r_j of multiplicities

n_j+1 and coefficients of the Laurent expansion near r_j at $(z-r_j)^{-(k+1)}$ in M_j, $0 \le k \le n_j$.

If we are given a parameter-depending cone operator $\mathbf{A} = \mathbf{A}(\lambda)$, $\lambda \in \Lambda$, like (1), then we may expect a parameter dependence of the discrete asymptotic types for the solutions. But there is, in general, so smooth behaviour of the p_j, m_j, L_j, with varying λ. For every fixed $\lambda_0 \in \Lambda$, we have a $P(\lambda_0)$ with individual numeration of the triplet $p_j(\lambda_0), m_j(\lambda_0), L_j(\lambda_0)$. In [S4], this phenomenon was studied in detail for analyzing the 'branching of asymptotics' for solutions of elliptic operators on manifolds with edges, where the variable on the edge plays the role of a parameter.

One of the main motivations of the notion of continuous asymptotics is to give a precise description of the nature of the branching behaviour of the asymptotics. Let us explain this in the case of the spaces $\mathcal{M}_V^\mu(X;\Lambda)$ introduced in Section 3.1. A consequence of the definition is that we have

$$\mathcal{M}_V^\mu(X;\Lambda) \subset C^\infty(\Lambda, \mathcal{M}_V^\mu(X)).$$

The parameter-depending Mellin symbols appear in the symbolic structure of corner operators. As announced in the beginning, they occur in the form 1.(2). Let us write down this operator more explicitly. We have, close to the corner $r = t = 0$,

$$\mathbf{A} = \sum_{j+k=0}^{\mu} A_{jk}(t,r) \left(-t\,\frac{\partial}{\partial t}\right)^j \left(-r\,\frac{\partial}{\partial r}\right)^k,$$

with $A_{jk} \in C^\infty\left([0,1) \times [0,1),\ \mathrm{Diff}^{\mu-(j+k)}(X)\right)$. Assume, for the moment, that A_{jk} is independent of t, r.

Consider the Mellin symbol of two complex variables z, w, namely

$$a(z,w) = \sum_{j+k=0}^{\mu} A_{jk} z^j w^k.$$

In the applications for the corner calculus, the ellipticity condition implies that

$$a(x+iy,\ \nu+i\lambda) : H^s(X) \to H^{s-\mu}(X) \tag{3}$$

is bijective for $|y|^2 + |\lambda|^2 \ge c$, with a constant $c = c(x,\nu) > 0$ sufficiently large. $c(x,\nu)$ can be chosen uniformly in finite x and ν intervals. The points $z(\nu,\lambda)$, where (3) is not bijective, for given ν, λ, belong to a set $V \in \mathcal{V}$ for all $\lambda \in \mathbb{R}$ and ν varying in a finite interval. That leads in the parametrix construction to a (ν,λ)-depending family $h(z,\nu+i\lambda)$ of cone symbols, where

$$h(z,\nu+i\lambda) \in \mathcal{M}_V^{-\mu}(X;\Lambda),$$

for every fixed ν, and moreover

$$h(z,\nu+i\lambda) \in \mathcal{M}_{R(\nu,\lambda)}^{-\mu}(X),$$

for every fixed $\nu, \lambda, R(\nu, \lambda) \in \mathcal{R}(X \times X)$.

In the case of proper r, t dependence, we have to discuss this behaviour for the Taylor coefficients at $r = t = 0$. The complete discussion, including the global effects from the base C of the corner $\mathbb{R}_+ \times C = C^\wedge$, will be given in [S5].

For the moment, we have explained why it is justified to deal with the parameter-depending classes of cone operators, as introduced in the preceding section. Furthermore, we have observed that we are in fact in subclasses with point-wise discrete asymptotics, with respect to λ (and also ν). In other words, we may expect that our parametrix constructions, for the corner that employ the operator families $\mathcal{N}^\mu(C; \Lambda)_\theta$, lead automatically to the subclass

$$\mathcal{N}^\mu(C; \Lambda)_\theta^\bullet,$$

consisting of such operator families which belong, for every fixed λ, to the cone operator classes based on spaces, Green operators and Mellin symbols with discrete asymptotics. This can be described locally in open neighbourhoods of given points $\lambda = \lambda_0$. The precise definitions of C^∞ families of cone operators, with point-wise discrete asymptotics, were given in [S4].

Remember that the typical behaviour comes from C^∞ functions of the parameter with values in the analytic functionals that are point-wise discrete and of finite order. In particular, from 3.1. (27), we can extract a mapping

$$\varsigma : \mathcal{M}_V^\mu(X; \Lambda)_\Delta \to C^\infty\left(\Lambda, \mathcal{A}'(V_\Delta) \otimes_\pi L^{-\infty}(X)\right)$$

which has the mentioned property for the Mellin symbols in the dotted subclass. In [S5], we return to this discussion once again and give a simpler version of the dotted subspaces of cone operator families.

Let us conclude this section with a brief description of the parameter-depending ellipticity of cone operators.

1. Definition. An operator $\mathbf{A}(\lambda) \in \mathcal{N}^\mu(C; \Lambda)_\theta$, $\theta = (k, 0]$; $k \in \mathbb{N} \setminus \{0\}$ is called parameter-depending elliptic if

(i) $\sigma_\psi^\mu(\mathbf{A}) \in S_b^{(\mu)}((T^*C \times \Lambda) \setminus 0)$ does not vanish for all $|\chi, \lambda| \neq 0$ and $|t^{-1}\tau, \xi, \lambda| \neq 0$, including $t = 0$ (here $\chi = (\tau, \xi)$ close to $t = 0$, cf. the notations after 3.1.(33)),

(ii) $\sigma_M^0(\mathbf{A})(z, \lambda) : H^s(X) \to H^{s-\mu}(X)$ is bijective for all $z \in \Gamma_{\frac{1}{2}}$ and all $\lambda \in \Lambda$.

Parameter-depending elliptic operators may appear, for instance, as operators depending on a spectral parameter. Consider as an example

$$\mathbf{A}(\lambda) = \mathbf{A} - \lambda^\mu,$$

where $\mathbf{A}$ is a differential operator over C of order μ which is totally characteristic, in the sense of the form (2), close to ∂C.

We can easily construct examples, where the condition (i) of Definition 1 is satisfied, for instance, for $\mu = 2$ and $\mathbf{A} = g^2\Delta$ (g^μ being a non-vanishing C^∞ function on int C, with $g = t^\mu$ close to $t = 0$), Δ the Laplacian with respect to a Riemannian metric, associated with the conical structure. The bijectivity of (ii) is satisfied for $s \in \Gamma_{\frac{1}{2}+\rho}$, for almost all $\rho \in \mathbb{R}$, i.e. except of a countable set $\{\rho_j\}_{j\in\mathbb{Z}}$ with $\rho_j \to \pm\infty$, as $j \to \pm\infty$.

Now a small shift in the operator which turns $\sigma_M^0(\mathbf{A})(z,\lambda)$ to $\sigma_M^0(\mathbf{A})(z+\varepsilon,\lambda)$, $\varepsilon > 0$, yields an operator which satisfies both (i) and (ii) of Definition 1.

Other examples of parameter-depending elliptic cone operators will be obtained in the following section, cf. 3.3. Remark 3.

2. THEOREM. *Let $\mathbf{A}(\lambda) \in \mathcal{N}^\mu(C;\Lambda)_\theta$ be parameter-depending elliptic. Then, there is a $\mathbf{B}(\lambda) \in \mathcal{N}^{-\mu}(C;\Lambda)_\theta$ which is also parameter-depending elliptic and a parametrix of $\mathbf{A}(\lambda)$, in the sense $\mathbf{A}(\lambda)\mathbf{B}(\lambda) - \mathbf{I}$, $\mathbf{B}(\lambda)\mathbf{A}(\lambda) - \mathbf{I} \in \mathcal{N}_{\mathbf{G}}(C;\Lambda)_\theta$. Moreover, there is a $c_1 > 0$ such that*

$$\mathbf{A}(\lambda) : \mathcal{H}^s(C) \to \mathcal{H}^{s-\mu}(C)$$

defines an isomorphism for all $|\lambda| \geq c_1$ and all $s \in \mathbb{R}$.

PROOF. The proof of the first statement is straightforward. So we only sketch the idea. The parameter-depending symbolic structure admits to pass to a parameter-depending operator $\mathbf{B}_1(\lambda) \in \mathcal{N}^{-\mu}(C;\Lambda)_\theta$ such that $\mathbf{R}_1(\lambda) = \mathbf{A}(\lambda)\mathbf{B}_1(\lambda) - \mathbf{I} \in \mathcal{N}^{-1}(C;\Lambda)_\theta$ and the conormal order of $\mathbf{R}_1(\lambda)$ is ≤ -1. Then, the power $\mathbf{R}_1(\lambda)^j$ belongs to $\mathcal{N}^{-j}(C;\Lambda)_\theta$ with the conormal order $\leq \max\{-k,-j\}$. Now there exists an operator

$$\mathbf{T}(\lambda) \in \mathcal{N}^0(C;\Lambda)_\theta$$

such that

$$\mathbf{T}(\lambda) - \sum_{j=0}^{N} (-1)^j\, \mathbf{R}_1^j(\lambda) \in \mathcal{N}^{-(N+1)}(C;\Lambda)_\theta,$$

for all $N \in \mathbb{N}$. For N large enough, we have even that the difference belongs to $\mathcal{N}_{F+G}^{-(N+1)}(C;\Lambda)_\theta$, since the first $k-1$ Mellin symbols then necessarily vanish. This gives us

$$\mathbf{G}(\lambda) = \mathbf{A}(\lambda)\mathbf{B}_1(\lambda)\mathbf{T}(\lambda) - \mathbf{I} \in \bigcap_N \mathcal{N}_{F+G}^{-(N+1)}(C;\Lambda)_\theta \subset \mathcal{N}_G(C;\Lambda)_\theta,$$

i.e. $\mathbf{B}_1(\lambda)\mathbf{T}(\lambda) =: \mathbf{B}(\lambda)$ is as desired. Now by analogous arguments as in [S1], §9, we see that $\mathbf{I} + \mathbf{G}(\lambda)$ is invertible in $\mathcal{H}^0(C)$, for $|\lambda| \geq \tilde{c}$ with some $\tilde{c} > 0$, and that $\chi(\lambda)(\mathbf{I} + \mathbf{G}(\lambda))^{-1} = \chi(\lambda)\,(\mathbf{I} + \mathbf{G}_1(\lambda))$ for another operator family $\mathbf{G}_1(\lambda) \in \mathcal{N}_G(C;\Lambda)_\theta$, χ being an excision function which equals 1 for $|\lambda| \geq 2\tilde{c}$, and vanishes for $|\lambda| \leq \tilde{c}$.

Thus $\mathbf{B}(\lambda)(\mathbf{I}+\mathbf{G}(\lambda))^{-1}\ \chi(\lambda) \in \mathcal{N}^{-\mu}(C;\Lambda)_\theta$ is a right inverse of $\mathbf{A}(\lambda)$, for $|\lambda| \geq c_1 = 2\tilde{c}$. In an analogous manner, we can argue from the left. This proves the second statement. $\square$

3.3. *Order Reduction for the Cone*

The abstract approach of Section 2.4., for a Mellin ψDO calculus, suggests the following application. We insert the scale

$$\mathcal{E}(C) = \{\mathcal{H}^s(C)\}_{s\in\mathbb{R}}, \tag{1}$$

C being as in 3.1. Remark 4, and look for a system of order reducing symbols for (1) with the properties of 2.4. Definition 2. For the corner calculus, we also need $\mathcal{N}^\mu(C)$-valued order reducing symbols. If we want to apply the constructions of Section 3.1. for a 'cone' $C^\wedge = \mathbb{R}_+ \times C$, with the base C, we also need symbols that depend on a complex parameter w outside some carrier of asymptotics, where the growth properties, with respect to the parameter, refer to parallels to the imaginary axis. We shall see that it is also essential to ensure holomorphy in a strip $S_\Delta = \left\{\frac{1}{2} - \delta < \operatorname{Re} w < \frac{1}{2} + \delta'\right\}$ that can be chosen so large as we want.

1. THEOREM. *For every $\mu \in \mathbb{R}$ and $K = (x, x') \in I$ finite, there exists a family*

$$b^\mu(w) \in \mathcal{A}(\mathbb{C}, \mathcal{N}^\mu(C)),$$

with $b^\mu(\beta + i\eta) \in \mathcal{N}^\mu(C;\mathbb{R}_\eta)$, for all $\beta \in \mathbb{R}$, such that $b^\mu(\beta+i\eta)$ has the properties (i), (iii), (iv) *of* 2.4. *Definition* 2 *with respect to the scale* (1), *for every fixed ν, $\frac{1}{2} - x \leq \beta \leq \frac{1}{2} + x'$ (here η plays the role of τ).*

PROOF. First, remember that the mapping 3.1.(33) is surjective.

Let (v, χ) be local coordinates on T^*C, χ being the covariable of v. Close to ∂C, we have a splitting $v = (t, x)$, $t \in \overline{\mathbb{R}_+}$, $x \in X$, and $\chi = (\tau, \xi)$. Let $\omega \in C_0^\infty(C)$ be supported by a collar neighbourhood $V \doteq [0,1) \times X$ of ∂C, $\omega \equiv 1$ close to $t = 0$. We then look for a system of order reductions for which the corresponding homogeneous principal symbol, in the sense of 3.1.(33), equals

$$\omega(v)(|t\tau|^2 + |\xi|^2 + |\eta|^2)^{\frac{\mu}{2}} + (1-\omega)(v))(|\chi|^2 + |\eta|^2)^{\frac{\mu}{2}}.$$

In a first step, we construct the order reducing family over $\mathbb{R}_+ \times X$ which corresponds to the case C_1 in the notations of 3.1.(31). In the second step, we get an analogous family over int C in the setting of standard ψDO's. Then, by the partition of unity $1 = \omega + (1-\omega)$, we shall obtain the desired order reductions over C. Let $\lambda \in [1, \infty)$ be an additional parameter and consider the amplitude function

$$h(\tau, \xi, \eta, \lambda) = (|\tau|^2 + |\xi|^2 + |\eta|^2 + \lambda^2)^{\mu/2}. \tag{2}$$

For any coordinate neighbourhood U on X with the local coordinates x, we have a ψDO a_U, depending on the parameters τ, η, λ, namely

$$a_U(\tau,\eta,\lambda)u(x) = (2\pi)^{-m} \int e^{i(x-x')\xi}\ h(\tau,\xi,\eta,\lambda)u(x')\mathrm{d}x'\mathrm{d}\xi.$$

Let $\{U_1,\ldots,U_N\}$ be an open covering of X and $\{\varphi_1,\ldots,\varphi_N\}$ a subordinated partition of unity. Further, let $\psi_j \in C_0^\infty(U_j)$, with $\psi_j\varphi_j = \varphi_j$ for $j = 1,\ldots,N$. Then, we get an operator family

$$a\left(\frac{1}{2}+i\tau,\ \frac{1}{2}+i\eta,\lambda\right) = \sum_{j=1}^{N} \varphi_j a_{U_j}(\tau,\eta,\lambda)\psi_j \in L^\mu_{cl}\left(X;\mathbb{R}_\tau \times \mathbb{R}_\eta \times [1,\infty)\right)$$

(the notation $\frac{1}{2}+i\tau,\ \frac{1}{2}+i\eta$ has only technical reasons).

With $a(\tau,\eta,\lambda)$, we associate a family of distributional kernels

$$K(\rho,\pi,\lambda) = (2\pi)^{-2}\int \rho^{-\frac{1}{2}-i\tau}\pi^{-\frac{1}{2}-i\eta} a\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\eta,\lambda\right)\mathrm{d}\tau\ \mathrm{d}\eta,$$

cf. 2.1.(11), here depending on λ. Applying the constructions in the proof of 2.1. Proposition 12 (in the obvious parameter-depending version), we get another operator family $a_1(z,w,\lambda)$ which is holomorphic in $(z,w) \in \mathbb{C}^2$ and

$$(3) \qquad a_1(\sigma+i\tau,\ \beta+i\eta,\lambda) \in L^\mu_{cl}(X;\mathbb{R}_\tau \times \mathbb{R}_\eta \times [1,\infty)),$$

for all $\sigma = \mathrm{Re}\ z,\ \beta = \mathrm{Re}\ w$, where a_1 has the same parameter-depending homogeneous principal symbol of order μ as $a\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\eta,\lambda\right)$ namely (2), independently of σ,β.

Remember that the operators $g(\tau,\eta,\lambda)$ in $L^\mu(X;\mathbb{R}^2_{\tau,\eta}\times[1,\infty))$ satisfy the estimates

$$\|g(\tau,\eta,\lambda)\|_{\mathcal{L}(H^s(X),H^{s-\nu}(X))} \leq c\ p(\mu,\nu,\tau),$$

for all $s\in\mathbb{R},\ \nu\geq\mu$, with p as in 2.4. Definition 2. For the derivatives in τ,η,λ, we get analogous estimates with the corresponding lower orders ($g \in L^\mu(\ldots)$ implies $\mathrm{D}^\alpha_{\tau,\eta,\lambda}\ g \in L^{\mu-|\alpha|}(\ldots)$).

Moreover, $L^\mu(\ldots)$ is closed under compositions.

Our next observation is that $a_1(\sigma+i\tau,\beta+i\eta,\lambda)$ is parameter-depending elliptic for all σ,β, i.e. the homogeneous principal symbol (2) does not vanish on the sphere $|\tau|^2+|\xi|^2+|\eta|^2+\lambda^2=1$.

A standard result on parameter-depending elliptic ψDO's says that there is a constant c_1 such that

$$(4) \qquad a_1(\sigma+i\tau,\beta+i\eta,\lambda) : H^s(X) \to H^{s-\mu}(X)$$

is an isomorphism for all $s\in\mathbb{R}$ and all $|\tau,\eta,\lambda|\geq c_1$. For every $c',c''>0$, we can choose $c_1 = c_1(c',c'')$ in such a way that this is true for all σ,β, with

$|\sigma| \leq c'$, $|\beta| \leq c''$. In particular, for $\lambda \geq c_1(c', c'')$, we obtain that (4) is an isomorphism for all $\sigma, \tau, \beta, \eta$ and $|\sigma| \leq c'$, $|\beta| \leq c''$. Let us set

$$b_1^{\mu}(\sigma + i\tau, \ \beta + i\eta) := a_1(\sigma + i\tau, \ \beta + i\eta, \lambda_1), \ \lambda_1 \geq c_1(c', c'').$$

From the definition of the parameter-depending Mellin symbols for the cone in Section 3.1, we see that $b_1^{\mu}(z, w) \in \mathcal{M}_0^{\mu}(X; \mathbb{R}_{\eta})$, $\eta = \text{Im } w$, for every $\beta = \text{Re } w \in \mathbb{R}$. Now remember that

$$op_M(b_1^{\mu})(w) : \mathcal{H}^s(X^\wedge) \to \mathcal{H}^{s-\nu}(X^\wedge)$$

is continuous, for every $\nu \geq \mu$ and every fixed w.

Let us derive an estimate of $\|op_M(b_1^{\mu})(w)\|_{s,s-\nu}$, where $\| \cdot \|_{s,s'}$ denotes the norm in $\mathcal{L}(\mathcal{H}^s(X^\wedge), \ \mathcal{H}^{s'}(X^\wedge))$. To this end, we fix a parameter-depending reduction of order $b^s(z)$ for the scale $\mathcal{E} = \{H^s(X)\}_{s \in \mathbb{R}}$, with the parameter-depending homogeneous principal symbol $(|\xi|^2 + |\tau|^2)^{\frac{s}{2}}$, $\tau = \text{Im } z$. Then,

$$\|u\|^2_{\mathcal{H}^s(X^\wedge)} = \int_{\Gamma_{\frac{1}{2}}} \|b^s(z) Mu(z)\|^2_{L^2(X)} |dz|,$$

$$\text{(5)} \qquad \begin{aligned} \|op_M(b_1^{\mu})(w)u\|^2_{\mathcal{H}^{s-\nu}(X^\wedge)} &= \int \|b^{s-\nu}(z) b_1^{\mu}(z, w) Mu(z)\|^2_{L^2(X)} |dz| \\ &= \int \|b^s(z) r(z, w) Mu(z)\|^2_{L^2(X)} |dz|, \end{aligned}$$

with $r(z, w) = b^{-\nu}(z) b_1^{\mu}(z, w)$. We have

$$\text{(6)} \qquad \|r(z, w)\|_{\mathcal{L}(L^2(X).L^2(X))} \leq c \ p(\mu, \nu, \eta),$$

where c is a constant which only depends on μ, ν and the bound c'' for $|\text{Re } w|$. This follows by the same arguments as in Šubin's monograph [S1], §9.2, in particular, from the inequality

$$\sup_{\varsigma \geq 0} (1 + \varsigma + \gamma)^{\mu} (1 + \varsigma)^{-\nu} \leq \begin{cases} (1 + \gamma)^{\mu}, & \text{for } \nu \geq 0, \\ c_{\mu\nu} \gamma^{\mu - \nu}, & \text{for } \nu \leq 0, \ \gamma \geq \gamma_0(\mu, \nu), \end{cases}$$

for arbitrary reals μ, ν, with $\mu \leq \nu$.

From (5), (6), we get

$$\|op_M(b_1^{\mu})(w)\|_{s,s-\nu} \leq c \ p(\mu, \nu, \eta),$$

with a constant c as mentioned. It is obvious that we also have the estimates

$$\begin{aligned} &\|(D_{\eta}^{\delta} op_M(b_1^{\gamma})(w)) \ op_M(b_1^{\epsilon})(w)\|_{s,s-\nu}, \\ &\|op_M(b_1^{\gamma})(w) D_{\eta}^{\delta} \ op_M(b_1^{\epsilon})(w)\|_{s.s-\nu} \leq c \ p(\mu, \nu, \eta), \end{aligned}$$

for every $\gamma, \varepsilon,\ s \in \mathbb{R},\ \delta \in \mathbb{N},\ \nu \geq \mu := \gamma - \delta + \varepsilon$, cf. 2.4. Definition 2.

Now we can perform an analogous construction over int C. To this end, we first take the double $2C$ and construct an operator family

$$b_2^\mu(w) \in \mathcal{A}(\mathbb{C}, L^\mu(2C)),$$

with $b_2^\mu(\beta + i\eta) \in L_{cl}^\mu(2C; \mathbb{R}_\eta)$, for all β, by starting with the homogeneous function (2). The kernel construction is precisely as before and we also can apply the arguments on the parameter-depending ellipticity which yield that

$$b_2^\mu(\beta + i\eta) : H^s(2C) \to H^{s-\mu}(2C)$$

is an isomorphism for all $|\beta| \leq c''$. Moreover,

$$\|b_2^\mu(w)\|_{\mathcal{L}(H^s(2C),\ H^{s-\nu}(2C))} \leq c\ p(\mu, \nu, \eta).$$

Then, we can set

$$b^\mu(w) = \omega op_M(b_1^\mu)(w)\omega' + (1-\omega)b_2^\mu(w)|_{\text{int } C}(1-\omega''), \tag{7}$$

where ω', ω'' are also cut-off functions supported by a collar neighbourhood of ∂C, with $\omega'\omega = \omega,\ \omega''\omega = \omega''$. It is a simple exercise to check the corresponding assertions for (7) and the spaces $\mathcal{H}^s(C)$, cf. 3.1.(19). □

The family of reductions of orders consists, by definition, of a sum where the part which refers to a collar neighbourhood of ∂C is just a Mellin operator $B_1^\mu = \omega op_M(b_1^\mu)(w)\omega$. For every $\gamma \in \mathbb{R}$, we can choose Δ so large that it also induces an order reducing family for the scale

$$\mathcal{E}^\gamma(C) := \{\mathcal{H}^{s,\gamma}(C)\}_{s\in\mathbb{R}}\,. \tag{8}$$

We simply have to replace B_1^μ by $\omega t^\gamma op_M(T^{-\gamma} b_1^\mu)(w)t^{-\gamma}\omega$. Applying Cauchy's integral formula and the holomorphy of b_1^μ in z, we obtain that the latter operator is only an extension of B_1^μ from $C_0^\infty(\text{int } C)$, by continuity, to $\mathcal{H}^{s,\gamma}(C)$. In other words, from Theorem 1, we get the following

2. COROLLARY. *There is an order reducing family $b^{\mu,\gamma}(\beta + i\eta)$, in the sense of Theorem* 1, *also with respect to the scale* (8), *for every fixed β in a strip $\frac{1}{2} - x \leq \beta \leq \frac{1}{2} + x'$ and x, x' fixed as large as we want.*

3. REMARK. Let $b^\mu(w)$ be as in Theorem 1. Then $b^\mu(\beta_0 + i\lambda)$, $\frac{1}{2} - x \leq \beta_0 \leq \frac{1}{2} + x'$, is parameter-depending elliptic, in the sense of 3.2. Definition 1.

4. PROPOSITION. *Let $a(\eta) \in \mathcal{N}^\mu(C; \Lambda)_\theta,\ \eta \in \Lambda = \mathbb{R}$, and consider η as a Mellin covariable. Then, $a(\eta)$ represents an element in $S^\mu(\overline{\mathbb{R}_+} \times \mathbb{R}; \mathcal{E}(C), \mathcal{E}(C))$ (with constant coefficients, since it does not depend on the variable on $\overline{\mathbb{R}_+}$).*

PROOF. Denote by $b^\mu(\eta)$ the order reducing family for the scale $\mathcal{E}(C)$ which follows from $b^\mu(w)$ in Theorem 1, by replacing w by $\frac{1}{2}+i\eta$. From 3.1. Remark 11 follows $D_\eta^j a(\eta) \in \mathcal{N}^{\mu-j}(C;\Lambda)_\theta$, for all j. Moreover, Remark 3 and 3.1. Theorem 10 show that

$$b^{s-\mu+j}(\eta)(D_\eta^j a(\eta))\ b^{-s}(\eta) \in \mathcal{N}^0(C;\Lambda)_\theta.$$

Then the symbol estimates follow from 3.1. Remark 12. □

For purposes below, it is useful to mention a slight modification of the notion of parameter-depending cone operators. If we replace Λ by $K \times \Lambda$, where K is a compact set with a C^∞ structure, then we also can introduce $\mathcal{N}^\mu(C;K\times\Lambda)_\theta$, by analogous definitions as before. The only minor novelty is that for the definition of homogeneous or classical symbols, we impose the condition that the parameter space is closed under homotheties only for Λ, i.e. as earlier $\lambda \in \Lambda$ implies $\rho\lambda \in \Lambda$, for all $\rho > 0$.

In particular, we can set $K = \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}$, where $\overline{\mathbb{R}_+} = \{$compactification of $\mathbb{R}_+$ by 0 and $\infty\}$, $\Lambda = \mathbb{R}$. Then, we get operator families $a(r,r',\eta)$ which can be interpreted as amplitude functions in $S^\mu(\overline{Q^2}\times\mathbb{R};\mathcal{E}(C),\mathcal{E}(C))$ based on the order reducing operators constructed above.

5. DEFINITION. An amplitude function $a(r,r',\eta) \in S^\mu(\overline{Q^2}\times\mathbb{R};\mathcal{E}(C),\mathcal{E}(C))$ is called $\mathcal{N}^\mu(C)_\theta$-valued if it may be interpreted as an element in $\mathcal{N}^\mu(C;\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+}\times\mathbb{R})_\theta$ in the mentioned sense.

Let us give some more interpretation of the properties of $\mathcal{N}^\mu(C)_\theta$-valued amplitude functions.

First, we have the abstract theory of the space $ML^\mu(\overline{\mathbb{R}_+};\mathcal{E}(C),\mathcal{E}(C))$ of Mellin operators for the scale 3.3.(1), based on the reductions of orders. There is an immediate extension of this concept to the more complicated scales of the sort

$$\mathcal{E}_B(C)_\theta = \{\mathcal{H}_B^s(C)_\theta^{(j)}\}_{s\in\mathbb{R},j\in\mathbb{N}},$$

$B \in \mathcal{B}^0$ (cf. the notations in Section 3.1. before Remark 7). The dropped subscript γ means $\gamma = 0$. The amplitude functions $a(r,r',\eta)$ of order μ belong to

$$\bigcap_s \mathcal{L}(\mathcal{H}_B^s(C)_\theta^{(j)},\ \mathcal{H}_D^{s-\mu}(C)_\theta^{(m(j))}),$$

$B, D \in \mathcal{B}^0$, where $m(j) \to \infty$, as $j \to \infty$, and the correspondence $j \to m(j)$ may depend on a. The symbol estimates, based on the reductions of orders, as in Section 2.4., refer to the operator families

$$(D_{r,r'}^\alpha\ D_\eta^\beta a)(r,r',\eta) : \mathcal{H}_B^s(C)_\theta^{(j)} \to \mathcal{H}_D^{s-\mu}(C)_\theta^{(m(j))}.$$

It is clear that we may fix the reductions of orders in Theorem 1, and the Hilbert spaces $\mathcal{H}_B^s(C)_\theta^{(j)}$ in such a way that they induce isomorphisms

$$b^\nu(\beta+i\eta) : \mathcal{H}_B^s(C)_\theta^{(j)} \to \mathcal{H}_B^{s-\nu}(C)_\theta^{(j)},$$

for every $B \in \mathcal{B}^0$, $s \in \mathbb{R}$, any fixed β in the weight strip, as indicated in Theorem 1. If necessary we change the Hilbert spaces $\mathcal{H}^s_B(C)^{(j)}_\theta$ in such a way that it becomes the image of $\mathcal{H}^0_B(C)^{(j)}_\theta$ under b^{-s}.

Now let $\mathcal{M}$ be the set of all $j \to m(j)$, with $m(j) \to \infty$, as $j \to \infty$ and

$$\mathcal{E}^{(j)} = \{\mathcal{H}^s_B(C)^{(j)}_\theta\}_{s\in\mathbb{R}}, \quad \tilde{\mathcal{E}}^{(m(j))} = \{\mathcal{H}^s_D(C)^{(m(j))}_\theta\}_{s\in\mathbb{R}}.$$

Then, we have, by the abstract calculus, the Mellin operator space

$$\begin{aligned} &ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_B(C)_\theta, \mathcal{E}_D(C)_\theta)_m \\ &= \bigcap_{j\in\mathbb{N}} ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}^{(j)}, \tilde{\mathcal{E}}^{m(j)}), \end{aligned} \tag{9}$$

$m \in \mathcal{M}$ fixed, and we define

$$ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_B(C)_\theta, \mathcal{E}_D(C)_\theta) = \bigcup_{m\in\mathcal{M}} ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_B(C)_\theta, \mathcal{E}_D(C)_\theta)_m,$$
$$ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_{as}(C)_\theta, \mathcal{E}_{as}(C)_\theta) = \bigcup_{B,D\in\mathcal{B}^0} ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_B(C)_\theta, \mathcal{E}_D(C)_\theta).$$

In an analogous sense, we use the notations

$$S^\mu((\overline{\mathbb{R}_+})^l \times \mathbb{R}; \mathcal{E}_B(C)_\theta, \mathcal{E}_D(C)_\theta), \quad S^\mu\left((\overline{\mathbb{R}_+})^l \times \mathbb{R}; \mathcal{E}_{as}(C)_\theta, \mathcal{E}_{as}(C)_\theta\right)$$

$l = 1, 2$.

Then, we obtain that the $\mathcal{N}^\mu(C)_\theta$-valued amplitude functions belong to the latter symbol spaces and define Mellin ψDO's in $ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}_{as}(C)_\theta, \mathcal{E}_{as}(C)_\theta)$.

4. - Mellin Operators for the Corner

4.1. *The Corner Sobolev Spaces*

The results of Chapter 3 enable us to perform an iteration of the conification, based on the Sobolev spaces and operators over C.

In [S5], we shall present the analogue of $\mathcal{N}^\mu(C)_\theta$, for $C^\wedge = \mathbb{R}_+ \times C$, which is a program on its own. Here we study the elements of the Mellin operator calculus.

1. DEFINITION. Let C be the stretched object associated with a manifold with conical singularities, cf. 3.1. Remark 4. Let $s \in \mathbb{R}$, $\gamma = (\rho, \sigma) \in \mathbb{R}^2$, and $b^{s,\sigma}(w)$ be the family of order reductions of 3.3. Corollary 2, $w = \nu + i\eta \in \mathbb{C}$, with $K = (x, x') \in \mathcal{I}$ fixed and sufficiently large (depending on ρ). Then,

$\mathcal{H}^{s,\gamma}(C^\wedge)$ denotes the closure of $C_0^\infty(\text{int } C^\wedge)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(C^\wedge)} = \left\{ \int_{\Gamma_{\frac{1}{2}-\rho}} \|b^{s,\sigma}(w) M_{r\to w} u(w)\|^2_{\mathcal{H}^{0,\sigma}(C)} |dw| \right\}^{\frac{1}{2}},$$

M being the Mellin transform on the real r axis with the dual variable w.

Clearly the definition is independent of the concrete choice of the order reducing family.

Let us state a number of simple conclusions from the definition.

Write $\mathcal{H}^s(C^\wedge) = \mathcal{H}^{s,(0,0)}(C^\wedge)$. Then, $\mathcal{H}^0(C^\wedge)$ is a Hilbert space with the scalar product

$$(1) \qquad (u, v)_{\mathcal{H}^0(C^\wedge)} = \int_{\Gamma_{\frac{1}{2}}} (M_{r\to w} u, M_{r\to w} v)_{\mathcal{H}^0(C)} |dw|.$$

It extends to a non-degenerate pairing

$$\mathcal{H}^{s,\gamma}(C^\wedge) \times \mathcal{H}^{-s,-\gamma}(C^\wedge) \to \mathbb{C},$$

for all $s \in \mathbb{R}$, $\gamma = (\rho, \sigma) \in \mathbb{R}^2 (-\gamma = (-\rho, -\sigma))$. This admits the identification $(\mathcal{H}^{s,\gamma}(C^\wedge))' = \mathcal{H}^{-s,-\gamma}(C^\wedge)$. Moreover, we obviously have

$$\mathcal{H}^{s,(\rho,\sigma)}(C^\wedge) = r^\rho\ \mathcal{H}^{s,(0,\sigma)}(C^\wedge),$$

for every $\rho \in \mathbb{R}$.

Remember that, when g^σ denotes a function in $C^\infty(\text{int } C)$ which is non-vanishing and equals t^σ in a collar neighbourhood of ∂C (in the coordinates (t, x)), then $\mathcal{H}^{s,\sigma}(C) = g^\sigma \mathcal{H}^s(C)$. From this, we obtain

$$\mathcal{H}^{s,(\rho,\sigma)}(C^\wedge) = r^\rho g^\sigma \mathcal{H}^s(C^\wedge).$$

In fact, we have (up to equivalence of norms, uniformly in w)

$$\begin{aligned}\|b^{s,\sigma}(w) M\ \ u(w)\|_{\mathcal{H}^{0,\sigma}(C)} &= \|g^{-\sigma} b^{s,\sigma}(w) M\ \ u(w)\|_{\mathcal{H}^0(C)} \\ &= \|g^{-\sigma} b^{s,\sigma}(w) g^\sigma M g^{-\sigma} u(w)\|_{\mathcal{H}^0(C)}.\end{aligned}$$

Since $g^{-\sigma} b^{s,\sigma}(w) g^\sigma$ may be used as an order reducing family, for the definition of $\mathcal{H}^0(C^\wedge)$, we obtain $u \in \mathcal{H}^{s,(0,\sigma)}(C^\wedge)$ implies $g^{-\sigma} u \in \mathcal{H}^s(C^\wedge)$. The converse follows in the same way.

Thus, we can pass by weight shifts to the spaces $\mathcal{H}^s(C^\wedge)$.

The scalar product (1) will be used below to define formal adjoint operators. If we are given an operator $\mathbf{A} \in \bigcap_s \mathcal{L}(\mathcal{H}^s(C^\wedge),\ \mathcal{H}^{s-\mu}(C^\wedge))$ for some $\mu \in \mathbb{R}$, then the formal adjoint $\mathbf{A}^*$ also belongs to $\bigcap_s \mathcal{L}(\mathcal{H}^s(C^\wedge), \mathcal{H}^{s-\mu}(C^\wedge))$.

In Section 3.1, we gave the definition of the spaces $\mathcal{H}_B^{s;\sigma}(C)_\Sigma$ over C, associated with a weight interval $\Sigma = (\lambda, 0] \in \mathcal{I}$ and a carrier of asymptotics $B \in \mathcal{B}^\sigma$. That are Fréchet spaces, with a norm system that we denote by $\|\cdot\|_{j;\mathcal{H}_B^{s;\sigma}(C)_\Sigma}$, $j \in \mathbb{Z}$.

These norms are rather complex objects. They may be defined via sums of the sort

$$\mathcal{H}_B^{s;\sigma}(C)_\Sigma = \mathcal{H}_0^{s;\sigma}(C)_\Sigma + \mathcal{H}_{B_1}^{\infty;\sigma}(C)_\Sigma + \mathcal{H}_{B_2}^{\infty;\sigma}(C)_\Sigma,$$

where $B = B_1 + B_2$ is as in the decomposition method, cf. 3.1.(16).

Remember that the key contributions come from $\mathcal{H}_{B_j}^{s;\sigma}(X^\wedge)_\Sigma$ (cf. 3.1.(21)) which are projective limits of the spaces of the type

$$\mathcal{H}_{B_j}^{s;\sigma}(X^\wedge)_{\Sigma_k} \triangleq \mathcal{A}_0^{s;\sigma}(X)_{\Sigma_k} \oplus \mathcal{A}'(B_{jk})$$

with an increasing sequence of weight intervals $\Sigma_k \nearrow \Sigma$ and $B_{jk} = B_j \cap S^\sigma_{\Sigma_k}$, $B_j \cap \partial S^\sigma_{\Sigma_k} = \emptyset$ for all k. The topology of $\mathcal{A}_0^{s;\sigma}(X)_{\Sigma_k}$ was given in 3.1. Definition 5 and that of $\mathcal{A}'(B_{jk})$ in 3.1. formula (13).

For every fixed $\Sigma \in \mathcal{I}$, the order reducing isomorphisms $b^{\mu;\sigma}(w)$, of Section 3.3, can be chosen in such a way that they induce isomorphisms

$$b^{\mu;\sigma}(w) : \mathcal{H}_B^{s;\sigma}(C)_\Sigma \to \mathcal{H}_B^{s-\mu;\sigma}(C)_\Sigma,$$

for all w in an arbitrary fixed weight strip K and all $\mathbf{B} \in \mathcal{B}^\sigma$, $s \in \mathbb{R}$. This is an immediate consequence of the holomorphy of the Mellin symbol in the z-variable.

2. DEFINITION. Let $s \in \mathbb{R}$, $\gamma = (\rho, \sigma) \in \mathbb{R}^2$, $\Delta = (K, \Sigma) \in \mathcal{I} \times \mathcal{I}$, $W = (V, B) \in \mathcal{V}^\rho \times \mathcal{B}^\sigma$, $V \cap \partial S_K^\rho = \emptyset$ ($S_K^\rho := T^{-\rho} S_K$). Then $\mathcal{A}_W^{s;\gamma}(C)_\Delta = \mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta$ denotes the subspace of all $h \in \mathcal{A}(S_K^\rho \setminus V, \mathcal{H}^{s;\sigma}(C))$ such that

(i) for every V-excision function χ, we have

$$\left\{ \int_{\Gamma_\eta} \|b^{s;\sigma}(w)\ \chi(w) h(w)\|^2_{\mathcal{H}^{0;\sigma}(C)} |\mathrm{d}w| \right\}^{\frac{1}{2}} < \infty, \tag{2}$$

for all $\eta \in S_K^\rho$, uniformly in every closed substrip,

(ii)

$$< \varsigma[h], f > = \frac{1}{2\pi i} \int_L h(z) f(z) \mathrm{d}z \in \mathcal{H}_B^{\infty;\sigma}(C)_\Sigma, \tag{3}$$

for every curve $L \subset S_K^\rho \setminus V$ surrounding $V_K = V \cap S_K^\rho$ clock-wise, and every $f \in \mathcal{A}(\mathbb{C})$.

Note that, for $V = \emptyset$, the condition (ii) disappears. In this case, we write $0 = (\emptyset, *)$ where $*$ stands for any B and denote the corresponding space by $\mathcal{A}_0^{s;\gamma}(C)_\Delta$, $\Delta = (K, \Sigma)$ with any $\Sigma \in \mathcal{I}$. Different B, Σ lead to the same space.

ς defines a linear operator

$$\varsigma : \mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta \to \mathcal{A}'(V_K) \otimes_\pi \mathcal{H}_B^{\infty;\sigma}(C)_\Sigma. \tag{4}$$

$\mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta$ is a Fréchet space with respect to the norms (2) together with those induced from (4) (in the sense of the projective limit with respect to the mapping (4)).

Let us set

$$\mathcal{A}_{V,as}^{s;\gamma}(C)_\Delta = \varinjlim_{B \in \mathcal{B}^\sigma} \mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta. \tag{5}$$

For $\gamma = (0,0)$, we omit the corresponding subscript. Similarly, as in Section 3.1, we have

$$\mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta = \mathcal{A}_0^{s;\gamma}(C)_\Delta + \mathcal{A}_{V,B}^{\infty;\gamma}(C)_\Delta, \tag{6}$$

in the sense of sums of Fréchet spaces, where the 0 stands for the empty set.

The inverse Mellin transform $M_{w \to r}^{-1}$ defines an injective operator

$$M^{-1} : \mathcal{A}_{V,B}^{s;\gamma}(C)_\Delta \to \mathcal{H}^{s;\gamma}(C^\wedge). \tag{7}$$

3. Definition. Let $s \in \mathbb{R}$, $\gamma = (\rho, \sigma) \in \mathbb{R}^2$, $W = (V, B) \in \mathcal{V}^\rho \times \mathcal{B}^\sigma$, $\Delta = (K, \Sigma) \in \mathcal{I} \times \mathcal{I}$, $V \cap \partial S_K^\rho = \emptyset$. Then, $\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta = \mathcal{H}_{V,B}^{s;\gamma}(C^\wedge)_\Delta$ denotes the image of $\mathcal{A}_W^{s;\gamma}(C)_\Delta$ under (7). For general $V \in \mathcal{V}^\rho$, the space $\mathcal{H}_{V,B}^{s;\gamma}(C^\wedge)_\Delta$ is defined by means of the direct analogue of the decomposition method of Section 3.1. For $V = \emptyset$, we denote the corresponding space by $\mathcal{H}_0^{s;\gamma}(C^\wedge)_\Delta$.

4. Proposition. *Let $W = (V, B) \in \mathcal{V}^\rho \times \mathcal{B}^\sigma$, then*

$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta = \mathcal{H}_0^{s;\gamma}(C^\wedge)_\Delta + \mathcal{H}_W^{\infty;\gamma}(C^\wedge)_\Delta \tag{8}$$

and for $V_1, V_2 \in \mathcal{V}^\rho$, with $V = V_1 + V_2$,

$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta = \mathcal{H}_{V_1;B}^{s;\gamma}(C^\wedge)_\Delta + \mathcal{H}_{V_2;B}^{s;\gamma}(C^\wedge)_\Delta. \tag{9}$$

$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta$ can be written as a projective limit

$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta = \varprojlim_{j \in \mathbb{Z}} \mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta^{(j)}$$

of Hilbert spaces $\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta^{(j)}$. The choice of such a system of Hilbert spaces is not canonical, but we keep it fixed in every concrete case. This can be done in such a way that we have natural continuous embeddings

$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta^{(j+1)} \hookrightarrow \mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta^{(j)},$$

for every $j \in \mathbb{N}$.

A slight modification of the definitions yields spaces of the type $\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta$ also for half open or closed weight intervals as components in Δ (cf. the corresponding definitions for the cone). Let us mention, in particular, the variants

$$\Delta = ((x, 0], \Sigma), \quad \Delta' = ([0, x'), \Sigma).$$

Let $\varepsilon > 0$ and $\Delta_\varepsilon = ((x, \varepsilon), \Sigma)$, $\Delta'_\varepsilon = (\varepsilon, x'), \Sigma)$. Choose ε so small that $V \cap \overline{S^\rho_{(\varepsilon,\varepsilon)}} = \emptyset$. Then, we define

$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_\Delta = \mathcal{H}_0^{s;\gamma}(C^\wedge)_\Delta + \mathcal{H}_W^{\infty;\gamma}(C^\wedge)_{\Delta_\varepsilon},$$
$$\mathcal{H}_W^{s;\gamma}(C^\wedge)_{\Delta'} = \mathcal{H}_0^{s;\gamma}(C^\wedge)_{\Delta'} + \mathcal{H}_W^{\infty;\gamma}(C^\wedge)_{\Delta'_\varepsilon}$$

as sums of Fréchet spaces. They are then independent of the choice of ε. We also need the case when $K = [x, x']$ and $V = \emptyset$, i.e.

(10) $$\mathcal{H}_0^{s;\gamma}(C^\wedge)_{([x,x'],\Sigma)}$$

which is defined in the Mellin image by the conditions that the norm expressions (2) are finite also for $\eta \in \partial S^\rho_K$. Remember that Σ is meaningless for $V = \emptyset$. It may happen that $\Delta = (K, [0, 0])$.

In that case, we get the spaces that we denote by

$$\mathcal{H}_{V,1}^{s;\gamma}(C^\wedge)_\Delta.$$

For $\Delta = ([0, 0], \Sigma)$, it follows simply $\mathcal{H}^{s;\gamma}(C^\wedge)$.

It is convenient to introduce the abbreviations

$$\mathcal{H}_{V,as}^{s;\gamma}(C^\wedge)_\Delta = \varinjlim_{B \in \mathcal{B}^\sigma} \mathcal{H}_{V,B}^{s;\gamma}(C^\wedge)_\Delta,$$
$$\mathcal{H}_{as}^{s;\gamma}(C^\wedge)_\Delta = \varinjlim_{V \in \mathcal{V}^\rho} \mathcal{H}_{V,as}^{s;\gamma}(C^\wedge)_\Delta,$$

$\Delta = (K, \Sigma)$, $\gamma = (\rho, \sigma)$ and

$$\mathcal{H}_{V,B}^{s}(C^\wedge)_\Delta = \mathcal{H}_{V,B}^{s;\gamma_0}(C^\wedge)_\Delta, \; \mathcal{H}_{V,as}^{s}(C^\wedge)_\Delta = \mathcal{H}_{V,as}^{s;\gamma_0}(C^\wedge)_\Delta,$$
$$\mathcal{H}_{as}^{s}(C^\wedge)_\Delta = \mathcal{H}_{as}^{s;\gamma_0}(C^\wedge)_\Delta \text{ for } \gamma_0 := (0, 0).$$

For simplicity, we mainly shall discuss the case of vanishing weights. We often will assume that $K = (k,k)$, $\Sigma = (l,0]$, $k,l \in \mathbb{N}$ (where, by definition, $[0,0] = (0,0) = (0,0])$. In that case, we write $\theta = (K,\Sigma)$.

6. DEFINITION. An operator $\mathbf{A} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s(C^\wedge), \mathcal{H}^\infty(C^\wedge))$ is called a Green operator of the class $\mathcal{N}_G(C^\wedge)_\Delta$, $\Delta = (K,\Sigma) \in \mathcal{I} \times \mathcal{I}$, if it induces continuous operators

$$\mathbf{A},\mathbf{A}^* : \mathcal{H}^s(C^\wedge) \to \mathcal{H}^\infty_{as}(C^\wedge)_\Delta,$$

for all $s \in \mathbb{R}$ (where the resulting asymptotic types depend on the operator but not on s).

4.2. *Corner Mellin Operators*

Now we come to the investigation of the Mellin operators in the 'corner algebra' $\mathcal{N}^\infty(C^\wedge)_\theta$. They are related to the subclass $\mathcal{N}_F(C^\wedge)_\theta$ of flat operators. Remember once again that we always use the notation θ for the couple K,Σ of weight intervals, when $K = (k,k)$, $\Sigma = (l,0]$, $k,l \in \mathbb{N}$.

1. DEFINITION. $\mathcal{N}^\mu_F(C^\wedge)_\theta$, $\theta = (K,\Sigma)$, $\mu \in \mathbb{R}$, is the subspace of all $\mathbf{A} \in ML^\mu(\overline{\mathbb{R}_+}; \mathcal{E}(C), \mathcal{E}(C))$ with the following properties

(i) $\mathbf{A},\mathbf{A}^*$ induce continuous operators

$$\mathcal{H}^s(C^\wedge) \to \mathcal{H}^{s-\mu}_0(C^\wedge)_{\theta'}$$

for all $s \in \mathbb{R}$ and $\theta' := ([k,k],*)$ (that is flatness of order k at $r = 0$ and $r = \infty$), $*$ stands for any weight interval in the t-variable, cf. 4.1.(10),

(ii) $\mathbf{A},\mathbf{A}^*$ may be defined by $\mathcal{N}^\mu(C)_\Sigma$-valued amplitude functions, in the sense of 3.3. Definition 5,

(iii) $\mathbf{A},\mathbf{A}^*$ have complete symbols $a(r,\eta)$, $a^*(r,\eta)$ which are $\mathcal{N}^\mu(C)_\Sigma$-valued as well as $\omega(r)r^{-k}a(r,\eta)$, $(1-\omega(r))r^k a(r,\eta)$, $\omega(r)r^{-k}a^*(r,\eta)$, $(1-\omega(r))r^k a^*(r,\eta)$.

The condition (ii) means that the operators may always be written as $op_M(h)$, $h(r,r',\eta)$ being $\mathcal{N}^\mu(C)_\Sigma$-valued. We do not necessarily require flatness of h in r,r' of order k at $r = 0$, $r' = 0$ and $r = \infty$, $r' = \infty$. The flatness of the operator is expressed by (i). Note, in particular, that also the operators in $\mathcal{N}^{-\infty}_F(C^\wedge)_\theta$ have $\mathcal{N}^{-\infty}(C)_\Sigma$-valued amplitude functions ($\mathcal{N}^{-\infty}(C)_\Sigma$ is not equal to $\mathcal{N}_G(C)_\Sigma$!). Further, observe that the conditions (iii) mean, in particular, smoothness of $\omega(r)r^{-k}a(r,\eta),\ldots$ up to $r = 0$ and $r = \infty$, respectively.

Examples of operators in $\mathcal{N}^\mu_F(C^\wedge)_\theta$ will be obtained below in calculations with Mellin operators.

2. REMARK. $\mathbf{A} \in \mathcal{N}^\mu_F(C^\wedge)_\theta$, $\mathbf{B} \in \mathcal{N}^\nu_F(C^\wedge)_\theta$ imply $\mathbf{AB} \in \mathcal{N}^{\mu+\nu}_F(C^\wedge)_\theta$,

$\mathbf{A}^* \in \mathcal{N}_F^\mu(C^\wedge)_\theta$. An analogous property have the classes

$$\mathcal{N}_{F+G}^\mu(C^\wedge)_\theta := \mathcal{N}_F^\mu(C^\wedge)_\theta + \mathcal{N}_G(C^\wedge)_\theta, \ \mu \in \mathbb{R},$$

where $\mathbf{AB} \in \mathcal{N}_G(C^\wedge)_\theta$, when $\mathbf{A}$ or $\mathbf{B}$ belong to $\mathcal{N}_G(C^\wedge)_\theta$.

3. DEFINITION. Let $\mu \in \mathbb{R}$, $\Delta = (K, \Sigma) \in \mathcal{I} \times \mathcal{I}$, $K = (x, x')$ finite and non-trivial (i.e. $x, x' > 0$), $\Sigma = (l, 0]$, $l \in \mathbb{N}$. Further, let $V \in \mathcal{V}$, $V \cap \partial S_K = \emptyset$. Then, $\mathcal{M}_V^\mu(C)_\Delta$ denotes the subspace of all

$$a(w) \in \mathcal{A}(S_K \backslash V, \ \mathcal{N}^\mu(C)_\Sigma) \cap \mathcal{N}^\mu(C; S_K \backslash V)_\Sigma \tag{1}$$

for which

$$< \varsigma[a], f > \ = \ \frac{1}{2\pi i} \int\limits_L a(w) f(w) dw \in \mathcal{N}_{\mathbf{G}}(C)_\Sigma, \tag{2}$$

for every $f \in \mathcal{A}(\mathbb{C})$, $L \subset S_K \backslash V$ being a curve surrounding $V_K = V \cap S_K$.

For $V \in \mathcal{V}$ arbitrary, we define $\mathcal{M}_V^\mu(C)_\Delta$ by the obvious analogue of the decomposition method (cf. the constructions of Section 3.1.).

In particular, we also get the space

$$\mathcal{M}_V^\mu(C)_\Sigma = \varprojlim_{K \in \mathcal{I}} \mathcal{M}_V^\mu(C)_{(K, \Sigma)}. \tag{3}$$

They will play a major role here, whereas the Mellin symbol spaces for finite K are only of auxiliary character. (Σ on the left side of (3) is an abbreviation for $((\infty, \infty), \Sigma)$).

Remember that in Definition 3 we have employed the natural locally convex topology of the spaces $\mathcal{N}^\mu(C)_\Sigma$ and $\mathcal{N}^\mu(C; S_K \backslash V)_\Sigma$, cf. the end of Section 3.1. This gives rise to a locally convex topology of the space on the right side of (1), namely the intersection topology. Moreover, (2) leads to a linear operator

$$\varsigma : \mathcal{M}_V^\mu(C)_\Delta \to \mathcal{A}'(V_K) \otimes_\pi \mathcal{N}_G(C)_\Sigma. \tag{4}$$

The space $\mathcal{M}_V^\mu(C)_\Delta$ is considered in the projective limit topology, with respect to (4), together with the embedding into the space on the right side of (1). Then, the definition extends to arbitrary V by taking sums.

By the usual methods, we prove that then

$$\mathcal{M}_V^\mu(C)_\Delta = \mathcal{M}_{V_1}^\mu(C)_\Delta + \mathcal{M}_{V_2}^\mu(C)_\Delta, \tag{5}$$

for arbitrary $V_1, V_2 \in \mathcal{V}$, with $V = V_1 + V_2$.

By definition, we have for $V \in \mathcal{V}^0$

$$\mathcal{M}_V^\mu(C)_\Delta \subset S^\mu(\overline{\mathbb{R}_+} \times \mathbb{R}; \mathcal{E}(C), \mathcal{E}(C)), \tag{6}$$

i.e. we talk about special Mellin symbols in the setting of Mellin ψDO's. They have constant coefficients.

Remark that the corner Mellin symbols $a(w) \in \mathcal{M}_V^\mu(C)_\Delta$ have a rich internal structure which is induced from the structure of the operator spaces over C (cf. 3.1. Definition 8 and formula 3.1.(30)). In particular, we can talk about the subclasses

$$\mathcal{M}_{V;F}^\mu(C)_\Delta = \mathcal{M}_V^\mu(C)_\Delta \cap \mathcal{N}_F^\mu(C; S_K \backslash V)_\Sigma,$$
$$\mathcal{M}_{V;G}^\mu(C)_\Delta = \mathcal{M}_V^\mu(C)_\Delta \cap \mathcal{N}_G(C; S_K \backslash V)_\Sigma.$$

It is clear that

$$\mathcal{M}_{V;G}^\mu(C)_\Delta = \mathcal{M}_{V;G}^{-\infty}(C)_\Delta.$$

The property (2) of Definition 3 implies

$$\mathcal{M}_V^\mu(C)_\Delta = \mathcal{M}_0^\mu(C)_\Delta + \mathcal{M}_{V;G}^{-\infty}(C)_\Delta \tag{7}$$

which is then true also for arbitrary V and $K = (\infty, \infty)$. Remark that, as a consequence of (5),

$$\mathcal{M}_{V;G}^{-\infty}(C)_\Delta = \mathcal{M}_{V_1;G}^{-\infty}(C)_\Delta + \mathcal{M}_{V_2;G}^{-\infty}(C)_\Delta, \tag{8}$$

$V = V_1 + V_2$.

Let us write, for abbreviation,

$$\mathcal{M}_{as}^\mu(C)_\Sigma = \varinjlim_{V \in \mathcal{V}} \mathcal{M}_V^\mu(C)_\Sigma.$$

4. REMARK. The operator family $b^\mu(w)$ of 3.3. Theorem 1. belongs to $\mathcal{M}_0^\mu(C)_\Sigma$, for every $\Sigma \in \mathcal{I}$.

5. PROPOSITION. *Let $\theta = (K, \Sigma)$ and $\mu, \nu \in \mathbb{R}$, then $g(w) \in \mathcal{M}_V^\mu(C)_\theta$, $h(w) \in \mathcal{M}_W^\nu(C)_\theta$ imply $g(w)h(w) \in \mathcal{M}_{V+W}^{\mu+\nu}(C)_\theta$, $g(1-\overline{w})^* \in \mathcal{M}_{V^*}^\mu(C)_\theta$, where the $*$ at g denotes the point-wise formal adjoint in the class $\mathcal{N}^\mu(C)_\Sigma$, $V^* = \{1 - \overline{w} : w \in V\}$. For every finite K, Σ, the order-reducing symbol $b^\mu(w)$ can be chosen in such a way that the point-wise composition induces isomorphisms*

$$b^\mu(w) : \mathcal{M}_W^\nu(C)_\theta \to \mathcal{M}_W^{\nu+\mu}(C)_\theta,$$

for all $\nu \in \mathbb{R}$, $W \in \mathcal{V}$, and the inverse is given by $(b^\mu(w))^{-1}$.

The proof is a straightforward generalization of that of [S2], Proposition 7, in Section 3.

For every $a(w) \in \mathcal{M}_V^\mu(C)_\Delta$, with $V \in \mathcal{V}^0$, we can define the associated Mellin ψDO $op_M(a)$ (cf. the formula (6)). We now assume that the couple of weight intervals is of the type θ (the case of more general first components may easily be added by the reader).

6. THEOREM. *Let* $a \in \mathcal{M}_V^\mu(C)_\theta$, $\mu \in \mathbb{R}$, $V \in \mathcal{V}^0$. *Then* $op_M(a)$ *induces continuous operators*

$$op_M(a) : \mathcal{H}^s_{B,as}(C^\wedge)_\theta \to \mathcal{H}^{s-\mu}_{D,as}(C^\wedge)_\theta,$$

for all $s \in \mathbb{R}$ *and every* $B \in \mathcal{V}^0$, *with some* $D \in \mathcal{V}^0$ *depending on* a *and* B.

PROOF. Applying (7), we can write $a = a_0 + a_1$, $a_0 \in \mathcal{M}_0^\mu(C)_\theta$, $a_1 \in \mathcal{M}_{V;G}^{-\infty}(C)_\theta$. In view of the formula 4.1.(8), it suffices that $op_M(a_i)$ has over the spaces

$$E_0 = \mathcal{H}_0^s(C^\wedge)_\theta, \ E_1 = \mathcal{H}^\infty_{B,U}(C^\wedge)_\theta$$

the desired mapping properties, $U \in \mathcal{B}^0$. In virtue of the decomposition arguments in the definition of the spaces, we may assume that $V, B \subset S_K$ (K is the first component of θ). The action of $op_M(a_i)$ means, in the Mellin image with respect to w, that we have to apply the action 'along' C, point-wise, for every w. Clearly for $u_i \in E_i$, $f_i = M_{r\to w} u_i$,

$$\begin{aligned}
&a_0(w)f_0(w) \in \mathcal{A}(S_K, \ \mathcal{H}^{s-\mu}(C)),\\
&a_0(w)f_1(w) \in \mathcal{A}(S_K \backslash B, \ \mathcal{H}^\infty(C)),\\
&a_1(w)f_0(w) \in \mathcal{A}(S_K \backslash V, \ \mathcal{H}^\infty(C)),\\
&a_1(w)f_1(w) \in \mathcal{A}(S_K \backslash (V+B), \ \mathcal{H}^\infty(C)).
\end{aligned}$$

We have to show that even

$$\begin{aligned}
&a_0 f_0 \in \mathcal{A}_0^{s-\mu}(C)_\Delta, \ a_0 f_1 \in \mathcal{A}^\infty_{B,as}(C)_\Delta,\\
&a_1 f_0 \in \mathcal{A}_{V,as}(C)_\Delta, \ a_1 f_1 \in \mathcal{A}^\infty_{V+B,as}(C)_\Delta,
\end{aligned}$$

(cf. the notation 4.1.(5)). For $a_0 f_0$, we have to check the norms in 4.1. Definition 2(i), i.e. to derive the norm estimates for the continuity. To this end, we fix order reducing symbols $b^q(w)$ related to the given couple θ of weight intervals, $q \in \mathbb{R}$. Write

$$\begin{aligned}
b^{s-\mu}(w) \ a_0(w)f_0(w) &= \tilde{a}_0(w) \ b^s(w)f_0(w),\\
\tilde{a}_0(w) &= b^{s-\mu}(w) \ a_0(w) b^{-s}(w).
\end{aligned}$$

Among the norms that have to be checked are those for the continuity

$\mathcal{H}^s(C^\wedge) \to \mathcal{H}^{s-\mu}(C^\wedge)$. This follows from

$$\int_{\Gamma_{\frac{1}{2}}} \|b^{s-\mu}(w)a_0(w)f_0(w)\|^2_{\mathcal{H}^0(C)} |dw| = \int_{\Gamma_{\frac{1}{2}}} \|\tilde{a}_0(w)b^s(w)f_0(w)\|^2_{\mathcal{H}^0(C)} |dw|$$

$$\leq c(a_0)^2 \int_{\Gamma_{\frac{1}{2}}} \|b^s(w)f_0(w)\|^2_{\mathcal{H}^0(C)} |dw|,$$

$c(a_0) = \sup\left\{\|\tilde{a}_0(w)\|_{\mathcal{L}(\mathcal{H}^0(C))} : w \in \Gamma_{\frac{1}{2}}\right\}$. For the other weight lines in the complex w plane, we can argue in an analogous manner.

The norm expressions in the other combinations $a_i f_j$, $i + j \geq 1$, can be estimated in exactly the same way. In virtue of the strong decrease of one factor for $|\text{Im } w| \to \infty$, we obtain the same for the product. It remains to verify the relations 4.1.(3) for the functions in the products. For the combinations with f_1 this is obvious. For $a_1 f_0$ it follows, since a_1 is $\mathbb{N}_G(C)_\Sigma$-valued, cf. (7). □

7. THEOREM. *Let* $a \in \mathcal{M}^\mu_V(C)_\Sigma$, $\mu \in \mathbb{R}$, $\Sigma = (l, 0]$, $l \in \mathbb{N}$, *and* $V \in \mathcal{V}^0 \cap \mathcal{V}^{-\beta}$, *for some* $\beta \geq 0$. *Then, for arbitrary cut-off functions* $\omega(r)$, $\omega_1(r)$,

$$\omega r^\beta op_M(a)\omega_1 - \omega op_M(T^\beta a)r^\beta \omega_1 \in \mathcal{N}_G(C^\wedge)_{\tilde{\theta}}, \tag{9}$$

with $\tilde{\theta} = ((\infty,\infty),\Sigma)$, $(T^\beta a)(w) = a(w+\beta)$. *Moreover,* $a \in \mathcal{M}^\mu_V(C)_\Sigma$, $V \in \mathcal{V}^0 \cap \mathcal{V}^{-\beta}$, $\beta \geq 0$, $k \in \mathbb{N}\backslash\{0\}$, $\alpha = k - \beta \geq 0$, *implies*

$$\omega r^\alpha op_M(T^\beta a)r^\beta \omega_1 \in \mathcal{N}^\mu_F(C^\wedge)_\theta + \mathcal{N}_G(C^\wedge)_\theta, \tag{10}$$

for $\theta = ((k,k),\Sigma)$.

PROOF. Applying (7), we can write $a = a_0 + a_1$, where $a_0 \in \mathcal{M}^\mu_0(C)_\Sigma$, $a_1 \in \mathcal{M}^{-\infty}_{V:G}(C)_\Sigma$. Denote by $\mathbf{A}$ the operator in (9). Then, $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, where $\mathbf{A}_i$ is associated with a_i. Now let $u \in C_0^\infty(\mathbb{R}_+ \times (\text{int } C))$ and set $v = (2\pi i)^{-1}\omega_1 u$. Then,

$$\mathbf{A}_i u = \omega r^\beta \int_{\Gamma_{\frac{1}{2}}} r^{-w} a_i(w)\ Mv(w)dw - \omega \int_{\Gamma_{\frac{1}{2}}} r^{-w} a_i(w+\beta) Mv(w+\beta)dw$$

$$= \omega r^\beta \left\{ \int_{\Gamma_{\frac{1}{2}}} r^{-w} a_i(w) Mv(w)dw - \int_{\Gamma_{\frac{1}{2}+\beta}} r^{-w} a_i(w) Mv(w)dw \right\}.$$

Applying Cauchy's integral formula, we obtain

$$\mathbf{A}_i u = \omega r^\beta \int_L r^{-w} a_i(w) Mv(w)dw, \tag{11}$$

where L is a smooth curve surrounding $V \cap \left\{\frac{1}{2} < \operatorname{Re} w < \frac{1}{2} + \beta\right\}$.

Here we have used that $a_i(w)Mv(w)$ is holomorphic in $\left\{\frac{1}{2} < \operatorname{Re} z < \frac{1}{2} + \beta\right\} \setminus V$. Since $a_0(w)Mv(w)$ is holomorphic in the whole strip, we obtain $\mathbf{A}_0 u = 0$. Since $C_0^\infty(\mathbb{R}_+ \times (\operatorname{int} C))$ is dense in $\mathcal{H}^s(C^\wedge)$, it follows $\mathbf{A}_0 \equiv 0$. The function $\mathbf{A}_1 u$ obviously belongs to $\mathcal{H}^\infty_{U,as}(C^\wedge)_{\tilde{\theta}}$, $U = T^{-\beta}V \cap \left\{\frac{1}{2} - \beta < \operatorname{Re} w < \frac{1}{2}\right\}$. Of course, it extends to a continuous operator $\mathbf{A}_1 : \mathcal{H}^s(C^\wedge) \to \mathcal{H}^\infty_{U.as}(C^\wedge)_{\tilde{\theta}}$, for every $s \in \mathbb{R}$.

For the adjoint, we can argue in the same way; in other words, we have obtained (9), cf. 4.1. Definition 6.

For proving (10), we write again $a = a_0 + a_1$, as above, and show that

$$\mathbf{A}_F := \omega r^\alpha op_M(T^\beta a_0)\ r^\beta \omega_1 \in \mathcal{N}_F^\mu(C^\wedge)_\theta, \tag{12}$$

$$\mathbf{A}_G := \omega r^\alpha op_M(T^\beta a_1)\ r^\beta \omega_1 \in \mathcal{N}_G(C^\wedge)_\theta. \tag{13}$$

By the arguments that lead to (11), we get

$$\mathbf{A}_F = \omega r^k op_M(a_0)\omega_1.$$

For the adjoint, we can do the same and it is then obvious that (12) holds, cf. Definition 1. The Mellin symbol a_1 belongs to $\mathcal{M}^{-\infty}_{V;G}(C)_\Sigma$. Choose ε, δ, $0 < \varepsilon < \delta < \beta$, and write $V = V(\varepsilon) + V(\delta)$, for certain $V(\varepsilon) \in \mathcal{V}^{-\varepsilon} \cap \mathcal{V}^{-\beta}$, $V(\delta) \in \mathcal{V}^{-\delta} \cap \mathcal{V}^{-\beta}$. In virtue of (8), we can write $a_1 = f + g$, $f \in \mathcal{M}^{-\infty}_{V(\varepsilon);G}(C)_\Sigma$, $g \in \mathcal{M}^{-\infty}_{V(\delta);G}(C)_\Sigma$ and then,

$$\mathbf{A}_G = \omega r^\alpha op_M(T^\beta f) r^\beta \omega_1 + \omega r^\alpha op_M(T^\beta g) r^\beta \omega_1.$$

Similarly, as in the proof of (9), we can write $\mathbf{A}_G = \mathbf{A}_\varepsilon + \mathbf{A}_\delta + G$, with some $G \in \mathcal{N}_G(C^\wedge)_\theta$ and

$$\mathbf{A}_\varepsilon = \omega r^{k-\varepsilon} op_M(T^\varepsilon f)\ r^\varepsilon \omega_1,$$
$$\mathbf{A}_\delta = \omega r^{k-\delta} op_M(T^\delta g)\ r^\delta \omega_1.$$

In view of Theorem 6, we get continuous operators

$$\mathbf{A}_\varepsilon : \mathcal{H}^s(C^\wedge) \to \mathcal{H}_0^\infty(C^\wedge)_{\theta_\varepsilon},$$
$$\mathbf{A}_\delta : \mathcal{H}^s(C^\wedge) \to \mathcal{H}_0^\infty(C^\wedge)_{\theta_\delta},$$

where $\theta_\lambda = ((k - \lambda, \infty), \Sigma)$.

Here we also have used that f and g are $\mathcal{N}_G(C)_\Sigma$-valued. Since $\varepsilon < \delta$, we can apply the continuous embedding

$$\mathcal{H}_0^\infty(C^\wedge)_{\theta_\varepsilon} \to \mathcal{H}_0^\infty(C^\wedge)_{\theta_\delta}$$

and thus

$$\mathbf{A}_G : \mathcal{H}^s(C^\wedge) \to \mathcal{H}^\infty_{as}(C^\wedge)_{\theta_\delta}$$

is continuous. Since $\delta,\ 0 < \delta < \beta$, is arbitrary, we also get the continuity

$$\mathbf{A}_G : \mathcal{H}^s(C^\wedge) \to \mathcal{H}^\infty_{as}(C^\wedge)_{\theta_0},\quad s \in \mathbb{R}.$$

We have used that $\mathcal{H}^\infty_{as}(C^\wedge)_{\theta_0}$ is the projective limit of all $\mathcal{H}^\infty_{as}(C^\wedge)_{\theta_\delta}$, for $\delta \to 0$. For the adjoint, we can do the same.

Thus we obtain (13) as desired. □

8. REMARK. Let $\omega, \omega_1, \tilde{\omega}, \tilde{\omega}_1$ be arbitrary cut-off functions and $a \in \mathcal{M}^\mu_V(C)_\Sigma,\ V \in \mathcal{V}^0$. Then,

$$\omega op_M(a)\omega_1 - \tilde{\omega} op_M(a)\tilde{\omega}_1 \in \mathcal{N}^\mu_F(C^\wedge)_\theta,$$

with $\theta = ((\infty, \infty), \Sigma)$.

The proof is obvious.

9. PROPOSITION. *Let ω, ω_1 be arbitrary cut-off functions and $a \in \mathcal{M}^\mu_V(C)_\Sigma,\ V \in \mathcal{V}^0$. Then,*

$$\omega op_M(a)(1 - \omega_1) \in \mathcal{N}^\mu_F(C^\wedge)_\theta + \mathcal{N}_G(C^\wedge)_\theta,$$

$\theta = ((k, k), \Sigma)$, where $k \in \mathbb{N}$ is arbitrary.

PROOF. Using (7), we can write $a = a_0 + a_1,\ a_0 \in \mathcal{M}^\mu_0(C)_\Sigma,\ a_1 \in \mathcal{M}^{-\infty}_{V;G}(C)_\Sigma$. Let $u \in \mathcal{H}^s(C^\wedge)$ and supp u bounded. Then, $(1 - \omega_1)u \in \mathcal{H}^s_0(C^\wedge)$. From the definition of a_1, it follows that

$$op_M(a_1) : \mathcal{H}^s_0(C^\wedge) \to \mathcal{H}^\infty_{V,as}(C^\wedge)_\theta,$$

$\theta = ((\infty, \infty), \Sigma)$; in other words, $\omega op_M(a_1)(1 - \omega_1)u \in \mathcal{H}^\infty_{V,as}(C^\wedge)_\theta$.

If we drop the condition that supp u is bounded, we only obtain that $(1 - \omega_1)u \in \mathcal{H}^s_0(C^\wedge)_{(\infty,0]}$, where $(\infty, 0]$ refers to the asymptotics in the variable r. Now for the same reason as above,

$$op_M(a_1) : \mathcal{H}^s_0(C^\wedge)_{(\infty,0]} \to \mathcal{H}^s_{V,as}(C^\wedge)_{((\infty,0],\Sigma)}$$

and

$$\mathcal{M}_\omega : \mathcal{H}^s_{V,as}(C^\wedge)_{((\infty,0],\Sigma)} \to \mathcal{H}^\infty_{V,as}(C^\wedge)_{((\infty,\infty),\Sigma)}.$$

Thus, we get a continous operator $\mathbf{A}_G := \omega op_M(a_1)(1 - \omega_1) : \mathcal{H}^s(C^\wedge) \to \mathcal{H}^\infty_{V,as}(C^\wedge)_\theta$. For the formal adjoint, we may argue in an analogous manner. From 4.1. Definition 6, we then obtain that $\mathbf{A}_G \in \mathcal{N}_G(C^\wedge)_\theta$. For $\mathbf{A}_F = \omega op_M(a_0)(1 - \omega_1)$, we can argue as follows.

Since a_0 is holomorphic in the complex w-plane, $op_M(a_0)$ preserves the asymptotic behaviour of the argument function, separately, for $r \to 0,\ r \to \infty$.

For $u \in \mathcal{H}^s(C^\wedge)$, we have $(1-\omega_1)u \in \mathcal{H}_0^s(C^\wedge)_\Pi$, $\Pi = ((\infty, 0], *)$, i.e. $op_M(a_0)(1-\omega_1)u \in \mathcal{H}_0^{s-\mu}(C^\wedge)_\Pi$.

From $\mathcal{M}_\omega : \mathcal{H}_0^{s-\mu}(C^\wedge)_\Pi \to \mathcal{H}_0^{s-\mu}(C^\wedge)_{\Pi_0}$, with $\Pi_0 = ((\infty, \infty), *)$, it follows that $\mathbf{A}_F$ induces continuous operators

$$\mathbf{A}_F : \mathcal{H}^s(C^\wedge) \to \mathcal{H}_0^{s-\mu}(C^\wedge)_{\Pi_0}.$$

The same can be done for the formal adjoint, cf. (i) in Definition 1. Since

$$\omega(r)r^{-k}u, \ (1-\omega(r))r^k u \in \mathcal{H}^{s-\mu}(C^\wedge),$$

for $u \in \mathcal{H}_0^{s-\mu}(C^\wedge)$ and every $k \in \mathbb{N}$, the complete symbols of $\mathbf{A}$ and $\mathbf{A}^*$ satisfy the conditions (iii) of Definition 1. The condition (ii) is obviously also fulfilled. Thus $\mathbf{A}_F \in \mathcal{N}_F^\mu(C^\wedge)_\theta$. □

10. THEOREM. *Let $a(w) \in \mathcal{M}_V^\mu(C)_\Sigma$, $V \in \mathcal{V}^0$, and assume that $a(w)$ is a Fredholm operator $\mathcal{H}^s(C) \to \mathcal{H}^{s-\mu}(C)$, for every $w \in \mathbb{C}\backslash V$, $s \in \mathbb{R}$, and induces an isomorphism $a(w) : \mathcal{H}^s(C) \to \mathcal{H}^{s-\mu}(C)$, for every $w \in \Gamma_{\frac{1}{2}}$ and $s \in \mathbb{R}$. Then, $a^{-1}(w)$ extends to an element $a^{-1}(w) \in \mathcal{M}_{V_1}^{-\mu}(C)_\Sigma$, for another $V_1 \in \mathcal{V}^0$.*

PROOF. The operator-valued Mellin symbol $a(w)$ is a parameter-depending elliptic family of cone operators, cf. 3.2. Definition 1, first on $\Gamma_{\frac{1}{2}}$ and then on every Γ_ρ, $\rho \in \mathbb{R}$ for $|\mathrm{Im}\ w| > c$, where c may be chosen uniformly in every strip $c_1 \leq \mathrm{Re}\ w \leq c_2$.

Thus, for $|\mathrm{Im}\ w|$ sufficiently large, we can form $a^{-1}(w)$. It belongs to $\mathcal{N}^{-\mu}(C)_\Sigma$, for every w, cf. [S2], 6. Proposition 4, and it is holomorphic in w, by standard arguments on vector-valued holomorphic functions, for $|\mathrm{Im}\ w|$ sufficiently large. Thus, there is a $V_1 \in \mathcal{V}^0$ for which

$$a^{-1}(w) \in \mathcal{A}(S_K\backslash V_1, \ \mathcal{N}^{-\mu}(C)_\Sigma) \cap \mathcal{N}^{-\mu}(C; S_K\backslash V_1)_\Sigma,$$

for every weight interval K. It remains to check (2) for a^{-1}, when $V_1 \cap \partial S_K = \emptyset$, and in general that $a^{-1} = h_1 + h_2$, where h_1, h_2 are of this sort for appropriate weight intervals K_1, K_2.

From (7), it follows that $a = a_0 + a_1$ for certain $a_0 \in \mathcal{M}_0^\mu(C)_\Sigma$, $a_1 \in \mathcal{M}_{V;G}^{-\infty}(C)_\Sigma$. Since the order of a_1 is $-\infty$, we obtain that a_0 alone also is parameter-depending elliptic in the desired sense. We want first to invert a_0. By abstract functional analysis of Fredholm families (cf. e.g. [S4]; Section 3.3.), we obtain that $a_0(w) : \mathcal{H}^s(C) \to \mathcal{H}^{s-\mu}(C)$ is bijective for all $w \in \mathbb{C}$, except for a countable subset $\{w_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$ with $\mathrm{Re}\ w_j \to \pm\infty$ as $j \to \pm\infty$. The arguments of [S4], Section 3.3, also show that $a_0^{-1}(w)$ is meromorphic with poles at the w_j of finite orders, where the Laurent coefficients at $(w-w_j)^{-(k+1)}$, $k = 0, 1, 2, \ldots$, are finite-dimensional operators in $\mathcal{N}_G(C)_\Sigma$. Here we employ the elliptic regularity for cone operators which asserts that the kernels of elliptic operators are finite-dimensional subspaces of $\mathcal{H}_{as}^\infty(C)_\Sigma$, and the same for the adjoints (cf. [S2], 6. Theorem 3). Thus, $a_0^{-1}(w)$ is already of the sort that we want to establish for $a^{-1}(w)$ in general. Now let us multiply $a = a_0 + a_1$ from the right by a_0^{-1}. Then, $aa_0^{-1} = 1 + a_1 a_0^{-1} = 1 + h$. By Proposition 5, we know that h is a Mellin symbol in our class of order $-\infty$. It even belongs to $\mathcal{M}_{W;G}^{-\infty}(C)_\Sigma$, for some $W \in \mathcal{V}^0$, since the Green operator-valued Mellin symbols form an ideal under compositions. If we show that $(1+h)^{-1} = 1 + g$ for some $g \in \mathcal{M}_{W_1;G}^{-\infty}(C)_\Sigma$, then $a_0^{-1}(1+g) = a^{-1}$ is as desired. But g is certainly what we want, for the point-wise inverse of $1 + h(w)$ gives us Green operator-valued g (cf. [S2], 6. Proposition 4) and a vector-valued Cousin problem argument gives the decomposition according to (8). $\square$

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