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Uniqueness Results and Monotonicity Properties for Strongly Nonlinear Elliptic Variational Inequalities

M. CHIPOT - G. MICHAILLE

1. - Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary Γ. For $p > 1$, let us denote by $W^{1,p}(\Omega)$ the usual Sobolev space of functions in $L^p(\Omega)$ whose derivatives in the distributional sense are in $L^p(\Omega)$. $L^p(\Omega)$ is the space of functions of p^{th} power integrable. We will denote by $|\cdot|_p$ the usual L^p norm. We refer the reader to [1] or [15] for details and notation on Sobolev spaces.

If K is a closed convex set in $W^{1,p}(\Omega)$, let V be the closed subspace in $W^{1,p}(\Omega)$ spanned by

$$K - K = \{k - k' | k, k' \in K\}. \tag{1.1}$$

For $f \in V^*$, the dual of V endowed with the $W^{1,p}(\Omega)$-topology, we would like to study variational inequalities of the type

$$\begin{aligned} &u \in K \\ &< \mathbf{A}(x, u, \nabla u), v - u > \geq < f, v - u > \qquad \text{for all } v \in K. \end{aligned} \tag{1.2}$$

Here $\mathbf{A}$ denotes a nonlinear operator from K into V^* and $< \cdot, \cdot >$ the duality bracket between V^* and V.

Our main interest will be in proving uniqueness results or more generally monotonicity properties for a large class of such variational inequalities. We refer the reader to section 3 of the paper for some applications.

First, we will assume that $\mathbf{A}$ is given (with the summation convention) by

$$\begin{aligned} < \mathbf{A}(x, u, \nabla u), v > \; = & \int_\Omega \left(\mathbf{A}_i(x, u, \nabla u) \cdot \frac{\partial v}{\partial x_i} + a(x, u) \cdot v \right) \, \mathrm{d}x \\ & + \int_\Gamma \gamma(x, u) \cdot v \, \mathrm{d}\sigma, \quad \text{for all } v \in V, \end{aligned} \tag{1.3}$$

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(i.e. $\mathbf{A}_i(x,u,\nabla u)\cdot\frac{\partial v}{\partial x_i}$ stands for $\Sigma_i \mathbf{A}_i(x,u,\nabla u)\cdot\frac{\partial v}{\partial x_i}$, where the summation in i has to be taken from 1 to n. This standard convention will be used throughout the paper. Note that, in the last integral, $d\sigma$ denotes the superficial measure on Γ and u and v stand for their traces on Γ (see [1], [16], [20])). In order for (1.3) to make sense, we will assume that for $i = 1,\dots,n$

$$\mathbf{A}_i(x,u,\nabla u),\quad a(x,u)\in L^{p'}(\Omega),\quad \gamma(x,u)\in L^{p'}(\Gamma)\quad \text{for all } u\in K \tag{1.4}$$

$\left(p' = \frac{p}{p-1}\right.$ is the conjugate exponent of p. Note that (1.4) also guarantees that the operator $\mathbf{A}$ defined by (1.3) is in $\left.V^*\right)$.

A simple way to insure that (1.4) holds is, for instance, to assume that

$$\mathbf{A}_i(x,u,\xi),\ a(x,u),\ \gamma(x,u) \text{ are Caratheodory functions} \tag{1.5}$$

(i.e. measurable in x and continuous in the other variables) and that

$$\begin{gathered}
\text{there exist a constant } C \text{ and functions } C'\in L^{p'}(\Omega), C''\in L^{p'}(\Gamma),\\
C, C', C''\geq 0, \text{ such that}\\
|\mathbf{A}_i(x,u,\xi)|\leq C(|u|^{p-1}+|\xi|^{p-1})+C'(x),\\
|a(x,u)|\leq C|u|^{p-1}+C'(x),\\
\text{for all } u\in\mathbb{R},\ \xi\in\mathbb{R}^n,\ \text{a.e. } x\in\Omega,\\
|\gamma(x,u)|\leq C|u|^{p-1}+C''(x),\ \text{for all } u\in\mathbb{R},\ \text{a.e. } x\in\Gamma.
\end{gathered} \tag{1.6}$$

(Note also that by the Sobolev embedding theorem and the trace theorem, (see [1], [20]), one could make the exponent of $|u|$ larger in the above inequalities).

We will suppose that

$$\begin{gathered}
u\to a(x,u) \text{ is nondecreasing for a.e. } x\in\Omega.\\
u\to \gamma(x,u) \text{ is nondecreasing for a.e. } x\in\Gamma.
\end{gathered} \tag{1.7}$$

The fundamental assumptions on $\mathbf{A}$ are the following.

First, the operator will be assumed to be elliptic, that is to say, for some strictly positive constant ν, we have:

$$\begin{gathered}
[\mathbf{A}_i(x,u,\xi)-\mathbf{A}_i(x,u,\zeta)]\cdot(\xi_i-\zeta_i)\geq \nu\ |\xi-\zeta|^p,\\
\text{for all } \xi,\zeta\in\mathbb{R}^n,\ \text{for all } u\in\mathbb{R},\ \text{a.e. } x\in\Omega.
\end{gathered} \tag{1.8}$$

($\xi=(\xi_1,\xi_2,\dots,\xi_n)$, $\zeta=(\zeta_1,\zeta_2,\dots,\zeta_n)$, $|\xi-\zeta|$ denotes the Euclidean norm of $\xi-\zeta$, $(\ \cdot\ )$ is the usual scalar product).

Moreover, we will assume that there exist a positive, nondecreasing, continuous function ω, a constant C and a function $g \in L^{p'}(\Omega)$ such that:

$$
\begin{aligned}
&|\mathbf{A}(x,u,\xi) - \mathbf{A}(x,v,\xi)| \leq C\omega(|v-u|)(|\xi|^{p-1} + g(x)) \\
&\text{for all } \xi \in \mathbb{R}^n, \text{ for all } u,\ v \in \mathbb{R}, \text{ a.e. } x \in \Omega.
\end{aligned} \tag{1.9}
$$

(Here $\mathbf{A}$ stands for the vector of components $\mathbf{A}_i$, $|\cdot|$ its Euclidean norm).

For ω we will be led to consider the two hypotheses

$$
\int_{0^+} \frac{1}{\omega^{\frac{1}{p-1}}(s)}\, ds = +\infty \tag{1.10}
$$

and

$$
\int_{0^+} \frac{1}{\omega^{p'}(s)}\, ds = +\infty. \tag{1.11}
$$

REMARK 1.1. Clearly (1.10) implies (1.11) since $p' = \frac{p}{p-1} > \frac{1}{p-1}$. Now, when $p \leq 2$, taking into account (1.9), (1.10) holds for $\mathbf{A}_i$'s which are Hölder continuous in u with a Hölder exponent greater or equal to $p-1$ (the Hölder modulus being controlled in ξ - see the section 3 for some convincing applications), and similarly (1.11) holds for $\mathbf{A}_i$'s which are Hölder continuous in u with a Hölder exponent greater or equal to $1/p'$, ω being nothing but the modulus of continuity of $\mathbf{A}_i$ in u. For $p > 2$ the assumption (1.10) does not hold unless the $\mathbf{A}_i$'s do not depend on u.

Note that, most of the time, we will not make any assumption on differentiability on the $\mathbf{A}_i$'s but rely only on the structure assumptions (1.8), (1.9) (as in [15], [23] for the Lipschitz continuous case).

The results introduced below generalize and unify preceding results, among which those of M. Artola [2], L. Boccardo [3], H. Brézis [5], H. Brézis - D. Kinderlehrer - G. Stampacchia [7], J. Carrillo - M. Chipot [8], J. Douglas Jr. - T. Dupont - J. Serrin [12], G. Gagneux [13], N.S. Trudinger [15], [23].

The paper is divided as follows. In section 2 we give a general and abstract result about uniqueness and monotonicity with respect to the data. In section 3 we develop some applications. Section 4 is devoted to a counter-example which shows that our results are optimal as far as certain hypotheses are concerned. In section 5 we investigate some cases of uniqueness which were out of the scope of the preceding sections and finally in section 6 we give an existence result for (1.2).

2. - A general monotonicity property

Let K_1, K_2 be two closed convex sets in $W^{1,p}(\Omega)$. We will say that K_1, K_2 satisfy the hypothesis (H) iff

$$\text{(H)} \qquad \begin{aligned} &u_1 + F(u_2 - u_1) \in K_1, \quad u_2 - F(u_2 - u_1) \in K_2 \\ &\text{for all } u_1 \in K_1, \text{ for all } u_2 \in K_2, \end{aligned}$$

for any nonnegative Lipschitz function F having a Lipschitz modulus less than 1 and such that $F(x) = 0$ for $x \leq 0$.

REMARK 2.1. This property implies in particular that

$$u_1 + (u_2 - u_1)^+ = \operatorname{Max}(u_1, u_2) \in K_1, \quad u_2 - (u_2 - u_1)^+ = \operatorname{Min}(u_1, u_2) \in K_2,$$

for every $u_1 \in K_1$, $u_2 \in K_2$. (See [5] where this was considered). Such a property appears to be a feature of convex sets defined by pointwise constraints.

For $i = 1, 2$, denote by V_i the space spanned by $K_i - K_i$ (see (1.1)) and by V_i^* its dual space. Then our main result is the following.

THEOREM 2.1. *Let K_i $(i = 1, 2)$ be two closed convex sets in $W^{1,p}(\Omega)$ satisfying* (H). *Let $f_i \in V_i^*$ be such that*

$$(2.1) \qquad < f_1, v > \geq < f_2, v > \quad \textit{for every } v \in V_1 \cap V_2, \quad v \geq 0.$$

Assume that (1.3), (1.4), (1.7), (1.8), (1.9) *hold for K_1 and K_2 and let u_i $(i = 1, 2)$ be a solution of*

$$(2.2) \qquad \begin{aligned} &u_i \in K_i \\ &< \mathbf{A}(x, u_i, \nabla u_i), v - u_i > \geq < f_i, v - u_i > \quad \textit{for all } v \in K_i. \end{aligned}$$

Then if:

(i) (1.11) *holds and*

$$(2.3) \qquad u \to a(x, u) \textit{ is increasing} \text{ a.e. } x \in \Omega$$

or

(ii) (1.10) *holds and*:

$$(2.4) \qquad u \to a(x, u) \textit{ is increasing on a set of positive measure of } \Omega$$

or

(2.5) *$u \to \gamma(x,u)$ is increasing on a set of positive measure of Γ*

or

(2.6) *there exists a constant C such that $|v|_p \le C\|\nabla v\|_p$, for all $v \in V_1 \cap V_2$,*

we have:

$$u_2(x) \le u_1(x) \quad \text{a.e. } x \in \Omega. \tag{2.7}$$

REMARK 2.2. Note the simplicity and the generality of the case (i) (see also below Theorem 3.1). The case (ii) is more involved, but we will see in section 4 that the result is somehow optimal as far as the assumption on ω is concerned.

PROOF. Set $(\varepsilon > 0)$

$$I(\varepsilon) = \int_{\varepsilon}^{+\infty} \frac{1}{\omega^{p'}(s)}\, ds; \tag{2.8}$$

taking ω sufficiently large in (1.9), we can assume w.l.o.g. that

$$\int_{1}^{+\infty} \frac{1}{\omega^{p'}(s)}\, ds < +\infty.$$

Then, consider F^ε defined by

$$F^\varepsilon(x) = \begin{cases} \dfrac{1}{I(\varepsilon)} \displaystyle\int_{\varepsilon}^{x} \frac{1}{\omega^{p'}(s)}\, ds & \text{for } x > \varepsilon \\ 0 & \text{for } x \le \varepsilon. \end{cases} \tag{2.9}$$

Clearly F^ε is a nonnegative Lipschitz function which vanishes for $x \le 0$. Thus by (H), for δ small enough, we have:

$$u_1 + \delta F^\varepsilon(u_2 - u_1) \in K_1, \quad u_2 - \delta F^\varepsilon(u_2 - u_1) \in K_2.$$

Substituting these two functions in (2.2) we get:

$$< \mathbf{A}(x, u_1, \nabla u_1),\ \delta F^\varepsilon(u_2 - u_1) > \ge < f_1,\ \delta F^\varepsilon(u_2 - u_1) >$$
$$< \mathbf{A}(x, u_2, \nabla u_2), -\delta F^\varepsilon(u_2 - u_1) > \ge < f_2, -\delta F^\varepsilon(u_2 - u_1) > .$$

Hence, by addition,

$$< \mathbf{A}(x, u_1, \nabla u_1) - \mathbf{A}(x, u_2, \nabla u_2),\ \delta F^\varepsilon(u_2 - u_1) >$$
$$\ge < f_1,\ \delta F^\varepsilon(u_2 - u_1) > \ - \ < f_2,\ \delta F^\varepsilon(u_2 - u_1) > .$$

From (2.1) we obtain:

$$< \mathbf{A}(x, u_1, \nabla u_1) - \mathbf{A}(x, u_2, \nabla u_2),\ F^\varepsilon(u_2 - u_1) > \geq 0$$

(note that, by (H), $F^\varepsilon(u_2 - u_1) \in V_1 \cap V_2$), or

$$\begin{aligned} &< \mathbf{A}(x, u_1, \nabla u_1) - \mathbf{A}(x, u_2, \nabla u_1),\ F^\varepsilon(u_2 - u_1) > \\ &\geq < \mathbf{A}(x, u_2, \nabla u_2) - \mathbf{A}(x, u_2, \nabla u_1),\ F^\varepsilon(u_2 - u_1) > . \end{aligned}$$

Taking into account (1.3), we deduce (for convenience we drop the measures of integration):

$$\begin{aligned} &\int_\Omega \left(\mathbf{A}_i(x, u_2, \nabla u_2) - \mathbf{A}_i(x, u_2, \nabla u_1)\right) \frac{\partial F^\varepsilon(u_2 - u_1)}{\partial x_i} \\ &+ \int_\Omega \left[a(x, u_2) - a(x, u_1)\right]\ F^\varepsilon(u_2 - u_1) \\ &+ \int_\Gamma \left[\gamma(x, u_2) - \gamma(x, u_1)\right]\ F^\varepsilon(u_2 - u_1) \\ &\leq \int_\Omega \left(\mathbf{A}_i(x, u_1, \nabla u_1) - \mathbf{A}_i(x, u_2, \nabla u_1)\right) \frac{\partial F^\varepsilon(u_2 - u_1)}{\partial x_i}. \end{aligned} \tag{2.10}$$

Noting that

$$\frac{\partial F^\varepsilon(u_2 - u_1)}{\partial x_i} = (F^\varepsilon)'(u_2 - u_1) \cdot \frac{\partial (u_2 - u_1)}{\partial x_i}$$

and using (1.8), (1.9), we obtain after replacing $(F^\varepsilon)'(u_2 - u_1)$ by $(F^\varepsilon)'$:

$$\begin{aligned} &\nu \int_\Omega |\nabla(u_2 - u_1)|^p\ (F^\varepsilon)' + \int_\Omega \left[a(x, u_2) - a(x, u_1)\right]\ F^\varepsilon(u_2 - u_1) \\ &+ \int_\Gamma \left[\gamma(x, u_2) - \gamma(x, u_1)\right] F^\varepsilon(u_2 - u_1) \\ &\leq \int_\Omega |\mathbf{A}(x, u_1, \nabla u_1) - \mathbf{A}(x, u_2, \nabla u_1)|\, (F^\varepsilon)'\, |\nabla(u_2 - u_1)| \\ &\leq \int_\Omega C\omega(|u_2 - u_1|)\ \left[|\nabla u_1|^{p-1} + g(x)\right] (F^\varepsilon)'\ |\nabla(u_2 - u_1)|. \end{aligned} \tag{2.11}$$

Using the Young inequality

$$ab \leq \frac{\varepsilon'^p a^p}{p} + \frac{(\varepsilon')^{-p'} b^{p'}}{p'},$$

with $\varepsilon' = \left(\frac{\nu p}{2}\right)^{\frac{1}{p}}$, we get:

$$\int_\Omega C\omega(|u_2-u_1|)\left[|\nabla u_1|^{p-1}+g(x)\right]\ (F^\varepsilon)'|\nabla(u_2-u_1)|$$

$$\leq \frac{\nu}{2}\int_\Omega |\nabla(u_2-u_1)|^p\ (F^\varepsilon)'$$

$$+\frac{C^{p'}}{p'\varepsilon'^{p'}}\int_\Omega \omega^{p'}(|u_2-u_1|)\ \left[|\nabla u_1|^{p-1}+g(x)\right]^{p'}\ (F^\varepsilon)'.$$

Combining this with (2.9) and (2.11), we obtain:

$$\frac{\nu}{2}\int_\Omega |\nabla(u_2-u_1)|^p(F^\varepsilon)' + \int_\Omega \left[a(x,u_2)-a(x,u_1)\right]F^\varepsilon(u_2-u_1)$$

$$+\int_\Gamma \left[\gamma(x,u_2)-\gamma(x,u_1)\right]F^\varepsilon(u_2-u_1) \tag{2.12}$$

$$\leq \frac{C}{I(\varepsilon)}\int_{[u_2-u_1>\varepsilon]}\left[|\nabla u_1|^{p-1}+g(x)\right]^{p'},$$

C is a positive constant and $[u_2-u_1>\varepsilon]$ denotes the set of points of Ω where u_2-u_1 is greater than ε.

Let us first consider the case (i).

By (2.12) and since $(F^\varepsilon)'$ is nonnegative and $u\to\gamma(x,u)$ nondecreasing, we get:

$$\int_\Omega \left[a(x,u_2)-a(x,u_1)\right]\ F^\varepsilon(u_2-u_1)$$

$$\leq \frac{C}{I(\varepsilon)}\int_{[u_2-u_1>\varepsilon]}\left[|\nabla u_1|^{p-1}+g(x)\right]^{p'} \tag{2.13}$$

$$\leq \frac{C}{I(\varepsilon)}\int_\Omega \left[|\nabla u_1|^{p-1}+g(x)\right]^{p'}.$$

(Note that, to use the monotonicity of γ, we need to prove that,. if $\tau: W^{1,p}(\Omega)\to L^p(\Gamma)$ is the trace operator, then $\tau(F^\varepsilon(u_2-u_1)) = F^\varepsilon(\tau(u_2)-\tau(u_1))$. This is easy to establish using approximation by C^1 functions. We will use the fact again in what follows).

Now, when $\varepsilon \to 0$, $I(\varepsilon) \to +\infty$ and

$$F^\varepsilon(x) \to \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Thus, from (2.13) we derive, letting ε go to 0:

$$\int_{[u_2-u_1>0]} [a(x,u_2) - a(x,u_1)] \leq 0$$

which by (2.3) gives (2.7).

Assume now that we are in case (ii).

Recalling (2.12), we get by letting $\varepsilon \to 0$ (recall that (1.10) implies (1.11) and thus $I(\varepsilon) \to +\infty$):

$$\int_{[u_2-u_1>0]} [a(x,u_2) - a(x,u_1)]\ \mathrm{d}x + \int_{[u_2-u_1>0]} [\gamma(x,u_2) - \gamma(x,u_1)]\ \mathrm{d}\sigma \leq 0$$

and so, in the two first cases of (ii), $u_2 - u_1$ is nonpositive on a part of positive measure of Ω or Γ.

Now, by (2.12) we have, also due to (1.7), and after cancellation of $I(\varepsilon)$:

$$\int_{[u_2-u_1>\varepsilon]} \frac{|\nabla(u_2-u_1)|^p}{\omega^{p'}(u_2-u_1)} \leq \frac{2C}{\nu} \int_{[u_2-u_1>\varepsilon]} \left[|\nabla u_1|^{p-1} + g(x)\right]^{p'} \leq C',$$

where C' is independent of ε. (Note that $|\nabla u_1|^{p-1} + g(x) \in L^{p'}(\Omega)$). If we set

$$S^\varepsilon(x) = \begin{cases} \displaystyle\int_\varepsilon^x \frac{1}{\omega^{\frac{1}{p-1}}(s)}\ \mathrm{d}s & \text{for } x > \varepsilon \\ 0 & \text{for } x \leq \varepsilon, \end{cases} \tag{2.14}$$

we have:

$$\int_{[u_2-u_1>\varepsilon]} \frac{|\nabla(u_2-u_1)|^p}{\omega^{p'}(u_2-u_1)} = \int_\Omega |\nabla S^\varepsilon(u_2-u_1)|^p \leq C'.$$

But, since S^ε is a Lipschitz function which vanishes for $x \leq 0$, $S^\varepsilon(u_2-u_1) \in V_1 \cap V_2$ (see (H)) and Poincaré Inequality holds. In cases (2.4), (2.5), this results from the fact that $S^\varepsilon(u_2-u_1)$ is equal to 0 on a set of positive measure of Ω or Γ, in case (2.6) this is part of the assumption. Thus we obtain:

$$\int_\Omega |S^\varepsilon(u_2-u_1)|^p \leq C''$$

and (2.7) follows by letting $\varepsilon \to 0$ due to (1.10).

This completes the proof.

REMARK 2.3. Note that (2.1) holds, for instance, when $f_i \in L^{p'}(\Omega)$ and $f_1 \geq f_2$ a.e. in Ω.

Under the assumption (1.11), it is known that, in general, it is impossible to assume that $u \to a(x,u)$ is only nondecreasing (see [8] example 2.2.1 and section 4 below); however, some results are preserved in this case (see section 5).

We do not know, if $\omega(t)$ is only assumed to tend to 0 as t goes to 0, whether the part (i) of the above result holds. Some progress in this direction are made in [9], [10].

We could have taken different operators for K_1 and K_2. If $\mathbf{A}^i$ denotes the operator corresponding to K_i then the same comparison result holds, provided

$$< \mathbf{A}^2(x,u,\nabla u) - \mathbf{A}^1(x,u,\nabla u), v > \geq 0,$$

$$\text{for all } u \in K_1, \text{ for all } v \in V_1 \cap V_2, \ v \geq 0,$$

with the assumptions on $\mathbf{A}$ transferred to A^2. For instance, the above inequality holds for $a_2 \geq a_1$, $\gamma_2 \geq \gamma_1$, where a_i, γ_i are the functions a, γ corresponding to $\mathbf{A}^i$, the $\mathbf{A}_i$'s being the same in both $\mathbf{A}^1$ and $\mathbf{A}^2$.

3. - Some applications

We would like to show that Theorem 2.1 leads in particular to uniqueness of a solution to any equation or, more generally, to any variational inequality associated to the standard convex sets K with pointwise constraints when the operator is, for instance, quasilinear. So, let us introduce some closed convex sets.

For $i = 1, 2$, let us consider functions

$$\varphi_i : \Gamma \to \overline{\mathbb{R}}, \qquad \psi_i : \Gamma \to \overline{\mathbb{R}} \tag{3.1}$$

$$\Phi_i : \Omega \to \overline{\mathbb{R}}, \qquad \Psi_i : \Omega \to \overline{\mathbb{R}}. \tag{3.2}$$

Set

$$\begin{aligned} K_i = \{ & v \in W^{1,p}(\Omega) \mid \varphi_i(x) \leq v(x) \leq \psi_i(x) \text{ a.e. } x \in \Gamma, \\ & \Phi_i(x) \leq v(x) \leq \Psi_i(x), \ \nabla v(x) \in C(x) \text{ a.e. } x \in \Omega \}, \end{aligned} \tag{3.3}$$

where for a.e. $x \in \Omega$, $C(x)$ is a closed convex set of $\mathbb{R}^n$ and the restriction of v to Γ is taken in the trace sense. (It is easy to show that $K_i, i = 1, 2$, are closed convex sets of $W^{1,p}(\Omega)$).

PROPOSITION 3.1. *Assume that* $(\varphi_2, \psi_2, \Phi_2, \Psi_2) \leq (\varphi_1, \psi_1, \Phi_1, \Psi_1)$, *then* K_1, K_2 *satisfy* (H).

($(\varphi_2, \psi_2, \Phi_2, \Psi_2) \leq (\varphi_1, \psi_1, \Phi_1, \Psi_1)$ means that each component of the first vector is less or equal a.e. on Γ, or a.e. on Ω, to each component of the second one).

PROOF. We assume that F is a nonnegative Lipschitz function having a Lipschitz modulus less than 1 and such that $F(x) = 0$ for $x \leq 0$. Let $u_1 \in K_1$, $u_2 \in K_2$, then (see [15]):

$$u_1 + F(u_2 - u_1), \quad u_2 - F(u_2 - u_1) \in W^{1,p}(\Omega) \tag{3.4}$$

and

$$\begin{aligned} &\varphi_1, \Phi_1 \leq u_1 \leq u_1 + F(u_2 - u_1) \\ &\leq u_1 + (u_2 - u_1)^+ = \mathrm{Max}(u_1, u_2) \leq \psi_1, \Psi_1; \\ &\varphi_2, \Phi_2 \leq \mathrm{Min}(u_1, u_2) = u_2 - (u_2 - u_1)^+ \\ &\leq u_2 - F(u_2 - u_1) \leq u_2 \leq \psi_2, \Psi_2 \end{aligned} \tag{3.5}$$

a.e. $x \in \Gamma$ and a.e. $x \in \Omega$ respectively.

Next,

$$\begin{aligned} &\nabla[u_1 + F(u_2 - u_1)] = \nabla u_1 + (F')\nabla(u_2 - u_1) \in C(x) \text{ a.e. } x \in \Omega \\ &\nabla[u_2 - F(u_2 - u_1)] = \nabla u_2 - (F')\nabla(u_2 - u_1) \in C(x) \text{ a.e. } x \in \Omega, \end{aligned} \tag{3.6}$$

since $F' \in [0, 1]$ and $C(x)$ is convex a.e. $x \in \Omega$. This proves (H).

So, as an obvious consequence of Theorem 2.1, we have:

THEOREM 3.1. *Let* u_i, $i = 1, 2$, *be a solution of*

$$\begin{aligned} &u_i \in K_i \\ &< \mathbf{A}(x, u_i, \nabla u_i), v - u_i > \geq < f_i, v - u_i > \quad \textit{for all } v \in K_i, \end{aligned} \tag{3.7}$$

where K_i *is given by* (3.3) *and* $f_i \in V_i^*$. *Assume that* (1.3), (1.4), (1.7), (1.8), (1.9) *hold for* K_1 *and* K_2 *and*

(i): (1.11) *holds and*

$$u \to a(x, u) \textit{ is increasing} \text{ a.e. } x \in \Omega, \tag{3.8}$$

or

(ii): (1.10) *holds and*

(3.9) $u \to a(x,u)$ *is increasing on a set of positive measure of* Ω

or

(3.10) $u \to \gamma(x,u)$ *is increasing on a set of positive measure of* Γ

or

(3.11) *there exists a constant* C *such that* $|v|_p \le C\|\nabla v\|_p$, *for all* $v \in V_1 \cap V_2$.

Then if $(f_2, \varphi_2, \psi_2, \Phi_2, \Psi_2) \le (f_1, \varphi_1, \psi_1, \Phi_1, \Psi_1)$ *we have*

$$u_2(x) \le u_1(x) \qquad \text{a.e. } x \in \Omega. \tag{3.12}$$

REMARK 3.1. Here $f_2 \le f_1$ in the sense of (2.1). Note that the above result is very natural. Forget for a minute the constraints on the gradient and interpret u as the vertical displacement of a thin elastic membrane under the action of a vertical force of intensity f, ($\varphi, \psi, \Phi, \Psi$ being obstacles preventing the membrane to go up or down). Clearly, the less the force is and the lower the obstacles are, the less the membrane will go up.

COROLLARY 3.1. *Set*

$$\begin{aligned} K = \{v \in W^{1,p}(\Omega) \mid & \varphi(x) \le v(x) \le \psi(x) \text{ a.e. } x \in \Gamma, \\ & \Phi(x) \le v(x) \le \Psi(x), \ \nabla v(x) \in C(x) \text{ a.e. } x \in \Omega\}, \end{aligned} \tag{3.13}$$

where

$$\varphi : \Gamma \to \overline{\mathbb{R}}, \qquad \psi : \Gamma \to \overline{\mathbb{R}} \tag{3.14}$$

$$\Phi : \Omega \to \overline{\mathbb{R}}, \qquad \Psi : \Omega \to \overline{\mathbb{R}} \tag{3.15}$$

are functions from Γ, Ω *into* $\overline{\mathbb{R}}$, *and* $C(x)$ *a closed convex set of* $\mathbb{R}^n$. *Assume* (1.3), (1.4), (1.7), (1.8), (1.9) *and*:

(i): (1.11) *holds and*

$$u \to a(x,u) \text{ is increasing a.e. } x \in \Omega,$$

or

(ii): (1.10) *holds and*

$u \to a(x,u)$ is increasing on a set of positive measure of Ω

or

$u \to \gamma(x,u)$ is increasing on a set of positive measure of Γ

or

there exists a constant C such that $|v|_p \leq C\|\nabla v\|_p$ for all $v \in V$.

Then, for $f \in V^*$ (see (1.1)), there exists at most one u such that:

$$\begin{aligned} &u \in K \\ &< \mathbf{A}(x,u,\nabla u), v-u > \geq < f, v-u > \quad \text{for all } v \in K. \end{aligned} \tag{3.16}$$

PROOF. It is enough to apply Theorem 3.1 with $K_1 = K_2 = K$, $f_1 = f_2 = f$.

The above result gives us uniqueness or, more generally, monotonicity properties for many problems. Let us list a few of them.

1) *Nonlinear elliptic boundary value problems.*

Indeed, choose here $\Phi \equiv -\infty$, $\Psi \equiv +\infty$, $C(x) = \mathbb{R}^n$, $\forall x \in \Omega$. Select some function ϕ in $W^{1,p}(\Omega)$ and choose $\varphi = \psi = \phi$ on a subset Γ_0 of Γ, $\varphi = -\infty$, $\psi \equiv +\infty$ elsewhere. Then one has $K = \phi + V$, where V is defined as the space

$$V = \{v \in W^{1,p}(\Omega) | v = 0 \text{ on } \Gamma_0\}.$$

So, for $f \in V^*$ defined by

$$< f, v >= \int_\Omega f^1 \cdot v + \int_\Gamma f^2 \cdot v,$$

(3.16) is equivalent to

$$\begin{aligned} &u \in K \\ &< \mathbf{A}(x,u,\nabla u), v > \; = \; < f, v > \qquad \text{for all } v \in V \end{aligned} \tag{3.17}$$

and u is the solution of the nonlinear problem

$$-\frac{\partial \mathbf{A}_i(x,u,\nabla u)}{\partial x_i} + a(x,u) = f^1 \text{ in } \Omega$$

$$u = \phi \text{ on } \Gamma_0$$

$$\mathbf{A}_i(x,u,\nabla u) \cdot n_i + \gamma(x,u) = f^2 \text{ on } \Gamma \backslash \Gamma_0.$$

So, in case (i) or (ii), by the previous result, and if a solution is known to exist, then this solution is unique and depends monotonically on the data ϕ, f^1, f^2. In case (ii) and when (3.9), (3.10) do not hold, we need to check (3.11). If Γ_0 has a positive measure, it is well known that the Poincaré inequality holds and we also obtain uniqueness and monotone dependence on the data. In the particular case where $\Gamma_0 = \Gamma$, we have a nonlinear Dirichlet problem. In the case where $\Gamma_0 = \emptyset$ ($\emptyset$ denotes the empty set), the problem is a problem of the Neumann type for which we have uniqueness in case (i), (ii). Now, in case (ii) and when (3.9), (3.10), (3.11) fail, then uniqueness can fail as well as it is well known even in the linear case. For instance, the solution of the linear Neumann problem:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma \end{aligned} \tag{3.18}$$

(in this case $a \equiv 0$) is defined only up to a constant.

2) *Obstacle problems.*

As above take here $\varphi = \psi = \phi$ on Γ_0, where Γ_0 is a subset of $\Gamma, \varphi \equiv -\infty, \psi \equiv +\infty$ elsewhere, $C(x) = \mathbb{R}^n$, $\forall x \in \Omega$.

Then if E, F are two measurable sets in Ω, take

$$\Phi = \Phi \text{ on } E, \quad \Phi = -\infty \quad \text{on } \Omega \backslash E$$
$$\Psi = \Psi \text{ on } F, \quad \Psi = +\infty \quad \text{on } \Omega \backslash F.$$

Then K becomes

$$K = \{v \in W^{1,p}(\Omega) | v(x) = \phi \text{ on } \Gamma_0,$$
$$\Phi(x) \leq v(x) \text{ a.e. } x \in E, \ v(x) \leq \Psi(x) \text{ a.e. } x \in F\}$$

and for such a convex set one gets uniqueness of the solution to (3.16) as well as monotone dependence with respect to the data f, ϕ, Φ, Ψ. Note that when $E = \Omega$, $F = \emptyset$ we have the usual one obstacle problem and when $E = F = \Omega$ the double obstacle problem.

3) *Signorini's problems or thin obstacle problems.*

Take $\Phi \equiv -\infty$, $\Psi \equiv +\infty$, $C(x) = \mathbb{R}^n$ for all $x \in \Omega$ and if E, F are two measurable sets included in Γ

$$\varphi = \varphi \text{ on } E, \quad \varphi = -\infty \quad \text{on } \Gamma \backslash E$$
$$\psi = \psi \text{ on } F, \quad \psi = +\infty \quad \text{on } \Gamma \backslash F.$$

Then K becomes

$$K = \{v \in W^{1,p}(\Omega) | \varphi(x) \leq v(x) \text{ a.e. } x \in E,\ v(x) \leq \psi(x) \text{ a.e. } x \in F\}$$

and for such a convex set, and provided that we are in case (i) or (ii), one has uniqueness and monotonicity in f, ϕ, ψ for the solution of (3.16).

Note that we choose our thin obstacles on Γ but they could have been chosen on any other thin part of $\overline{\Omega}$ for which we can define a trace.

4) *Problems with constraints on the derivatives.*

Take for instance $\varphi = \psi = \phi$ on Γ, $\Phi \equiv -\infty$, $\Psi \equiv +\infty$, then K becomes

$$K = \{v \in W^{1,p}(\Omega) | v = \phi \text{ on } \Gamma,\ \nabla v(x) \in C(x) \text{ a.e. } x \in \Omega\}.$$

In the case $p = 2$, $\phi \equiv 0$, $C(x) = B_1$, where B_1 is the unit ball of $\mathbb{R}^n$, we get the convex of the elastic-plastic torsion problem:

$$K = \{v \in W^{1,2}(\Omega) | v = 0 \text{ on } \Gamma,\ |\nabla v(x)| \leq 1 \text{ a.e. } x \in \Omega\}.$$

If now $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^q$ is a linear map and C a closed convex set of $\mathbb{R}^q$, $\mathbf{A}^{-1}C = \{\xi | \mathbf{A}\xi \in C\}$ is a closed convex set of $\mathbb{R}^n$. So, if $\mathbf{A}(x)$ is any matrix defined on Ω, uniqueness holds for convex sets of the type

$$K = \{v \in W^{1,p}(\Omega) | v = \phi \text{ on } \Gamma,\ \mathbf{A}(x)\nabla v(x) \in C(x) \text{ a.e. } x \in \Omega\}.$$

Taking for instance $\mathbf{A}(x) = (a_{ij}(x))$, $i = 1, \ldots, q$, $j = 1, \ldots, n$ and $C(x) = \Pi_i [c_{1,i}(x),\ c_{2,i}(x)]$, where $c_{1,i}$, $c_{2,i}$ are functions from Ω into $\mathbb{R}$, K becomes:

$$K = \{v \in W^{1,p}(\Omega) | v = \phi \text{ on } \Gamma,$$
$$c_{1,i}(x) \leq \sum_j a_{ij}(x) \frac{\partial v}{\partial x_j} \leq c_{2,i}(x) \text{ a.e. } x \in \Omega,\ i = 1, \ldots, q\}.$$

For these convex sets, in case (i) and (ii), we have uniqueness and monotone dependence in terms of the data. Thus, roughly speaking, uniqueness holds for every convex set defined by pointwise constraints on lower operators.

Let us now examine what kind of operator satisfies the assumptions which are useful to us - i.e. the hypotheses (1.4), (1.8), (1.9). For simplicity, and to illustrate the purpose of the next section, let us restrict ourselves to the important case $p = 2$. Let us denote by $a_{ij}(x, u)$, $\beta_i(x, u)$ Caratheodory functions, where there exist a positive constant C and a positive function $C' \in L^2(\Omega)$ such that

$$|a_{ij}(x, u)| \leq C, \quad |\beta_i(x, u)| \leq C|u| + C'(x)$$
$$\text{for all } u \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \quad i, j = 1, \ldots, n. \tag{3.19}$$

Assume that for some positive constant ν

$$a_{ij}(x,u)\xi_i\xi_j \geq \nu\ |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n. \tag{3.20}$$

Then, if we set

$$\mathbf{A}_i(x,u,\xi) = a_{ij}(x,u)\ \xi_j + \beta_i(x,u) \tag{3.21}$$

and if the assumptions on $a(x,u)$, $\gamma(x,u)$ are those of the preceding section (i.e. if (1.5), (1.6) hold), the quasilinear operator $\mathbf{A}$

$$\begin{aligned} < \mathbf{A}(x,u,\nabla u), v > \ = \int_\Omega \Big[& a_{ij}(x,u)\ \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_i} \\ & + \beta_i(x,u) \cdot \frac{\partial v}{\partial x_i} + a(x,u) \cdot v \Big]\ \mathrm{d}x \\ & + \int_\Gamma \gamma(x,u) \cdot v\ \mathrm{d}\sigma \end{aligned}$$

is well defined for every $v \in V$ and the $\mathbf{A}_i$'s satisfy (1.8). Now, if there exists a positive, increasing continuous function ω such that:

$$\begin{aligned} & |a_{ij}(x,u) - a_{ij}(x,v)| \leq C\omega(|v-u|) \quad \text{for all } u \in \mathbb{R}^n, \text{ a.e. } x \in \Omega \\ & |\beta_i(x,u) - \beta_i(x,v)| \leq C\omega(|v-u|)\ (g(x)) \quad \text{for all } u \in \mathbb{R}^n, \text{ a.e. } x \in \Omega \end{aligned} \tag{3.22}$$

for some $g \in L^2(\Omega)$, we have (1.9). So, such an operator satisfies all the assumptions of the preceding sections and Theorem 3.1 and Corollary 3.1 apply.

Now in the case $p = 2$, (1.10) and (1.11) read

$$\int_{0+} \frac{1}{\omega(s)}\ \mathrm{d}s = +\infty \tag{3.23}$$

and

$$\int_{0+} \frac{1}{\omega^2(s)}\ \mathrm{d}s = +\infty. \tag{3.24}$$

So, in particular, if the $\mathbf{A}_i(x,u,\xi)$'s are Lipschitz continuous in u (with a Lipschitz modulus controlled in ξ), (1.10) holds. (This case, with no transport term, was studied by Artola [2] in the particular case of the one obstacle problem and with different test functions than ours).

If the $\mathbf{A}_i(x,u,\xi)$'s are Hölder continuous in u with exponent greater or equal to $1/2$, (1.11) holds. Under the assumption (3.22), the two cases

correspond to when the a_{ij}, β_i are respectively Lipschitz continuous in u and Hölder continous in u with exponent greater or equal to $1/2$. As we mentioned it earlier, when (3.23) fails, uniqueness can fail as well, even if (3.9), (3.10), (3.11) hold. Let us now give an example of this.

4. - A counter-example

Consider a function $\beta = \beta(r)$ which satisfies for $0 < \alpha < 1$:

$$\beta(r) = 0 \text{ if } r \leq 0 \text{ or } r \geq 1, \quad \beta(r) > 0 \text{ if } 0 < r < 1,$$

$$\beta(r) \sim r^{\alpha} \text{ for } r > 0, \ r \to 0, \ \beta(r) \sim (1-r)^{\alpha} \text{ for } r < 1, \ r \to 1,$$

$$\int_0^1 \beta(s)\, ds = 1.$$

Then, define $U(r)$ by

$$\text{(4.1)} \qquad \begin{cases} U(r) = 0 & \text{if } r \leq 0 \\ r = \displaystyle\int_0^{U(r)} \frac{ds}{\beta(s)} & \text{if } 0 \leq r \leq 1 \\ U(r) = 1 & \text{if } r \geq 1 \end{cases}$$

Let Ω be the ball of center 0 and radius 2 in $\mathbb{R}^2$ and denote by (r, θ) the polar coordinates of a point $x = (x_1, x_2)$ in $\Omega \backslash (0,0)$, $(r = |x|)$. Let λ be a smooth nonnegative function defined on Ω and which satisfies:

$$\lambda(x) = \lambda(r) = 1 \text{ if } r \leq \frac{1}{4}, \ \lambda(x) = \lambda(r) = 0 \text{ if } r \geq \frac{1}{2}.$$

Set

$$\beta_1(x,u) = -\cos\ \theta \cdot \beta(u), \quad \beta_2(x,u) = -\sin\ \theta \cdot \beta(u),$$

$$a(x,u) = \lambda(x) \cdot u, \ \gamma(x,u) = u.$$

(If one wishes to have β_i smooth in x, it is enough to replace $\cos\theta$ and $\sin\theta$ by smooth functions which agree with them for $r \geq 1/2$). Then, for $1/2 \leq r_0 \leq 1$, set

$$u(x) = U(r - r_0), \quad (r = |x|).$$

We claim that u satisfies:

$$\int_\Omega \left\{ \left[\frac{\partial u}{\partial x_i} + \beta_i(x,u) \right] \cdot \frac{\partial v}{\partial x_i} + a(x,u) \cdot v \right\} \, dx + \int_\Gamma \gamma(x,u) \cdot v \, d\sigma$$

(4.2)

$$= \int_\Gamma 1 \cdot v \, d\sigma \text{ for all } v \in W^{1.2}(\Omega).$$

Indeed,

$$a(x,u) \equiv 0 \text{ on } \Omega, \quad \gamma(x,u) \equiv 1 \text{ on } \Gamma$$

and by (4.1)

$$\int_\Omega \left\{ \frac{\partial u}{\partial x_i} + \beta_i(x,u) \right\} \cdot \frac{\partial v}{\partial x_i}$$

$$= \int_\Omega \left[\nabla u \cdot \nabla v - \cos\theta \cdot \beta(u) \cdot \frac{\partial v}{\partial x_1} - \sin\theta \cdot \beta(u) \cdot \frac{\partial v}{\partial x_2} \right]$$

$$= \int_{r \geq 1/2} \left(\frac{\partial u}{\partial r} \cdot \frac{\partial v}{\partial r} - \beta(u) \cdot \frac{\partial v}{\partial r} \right) = 0.$$

So, since r_0 can take any value between $1/2$ and 1, the problem (4.2) has infinitely many solutions although

$u \to a(x,u)$ is increasing on a set of positive measure of Ω

$u \to \gamma(x,u)$ is increasing on a set of positive measure of Γ

and, if we consider

$$K = \{v \in W^{1.2}(\Omega) | v = 1 \text{ on } \Gamma\}$$

$$V = K - K = W_0^{1.2}(\Omega),$$

then

there exists a constant C such that $|v|_2 \leq C\|\nabla v\|_2$ for all $v \in V$

and u is an element in K satisfying (4.2) for every v in V!

Note also that u is the solution of the nonlinear Neumann problem

$$-\frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} + \beta_i(x,u) \right) = 0 \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma.$$

This shows that, in the nonlinear case, two solutions do not necessarily differ by a constant.

Now, we would like to show that the lack of uniqueness is due to the presence of the transport term (i.e. the β_i's).

5. - Some extensions

In this section, we restrict ourselves to the case $p = 2$ and to operators $\mathbf{A}$ where $\mathbf{A}_i$ is given by

$$\mathbf{A}_i(x, u, \xi) = a_{ij}(x, u)\ \xi_j + \beta_i(x, u). \tag{5.1}$$

As we saw in the preceding section, when $a(x, u)$ fails to be increasing a.e. on Ω, then uniqueness can fail even if (1.11), (3.9), (3.10), (3.11) hold. We would like to show now, that, in the case of obstacle problems, and thus also in the particular case of equations (see section 3 paragraph 1)) when $a(x, u)$ is assumed to be only nondecreasing, one can have uniqueness and monotonicity with respect to the data in the same condition that case (i), i.e. when (1.11) holds, but at the expense of further assumptions on the coefficients $\beta_i(x, u)$. More precisely, if there exist constants a_i, $i = 1, \ldots, n$, not all of them equal to 0, such that

$$u \to \sum_{i=1}^{n} a_i \beta_i(x, u) \tag{5.2}$$

is nondecreasing or nonincreasing, then uniqueness can be restored under a weaker assumption than (1.10), namely (1.11), (i.e.

$$\mathbf{A}_i(x, u, \xi) = a_{ij}(x, u)\ \xi_j + \beta_i(x, u)$$

satisfies (1.9) with ω satisfying (1.11)).

This case has some applications, since for (5.2) to hold, it is enough that one of the β_i's does not depend on u or is monotone in u. Indeed if $\beta_1(x, u)$ is monotone in u, then for $a_1 = 1$, $a_i = 0$ if $i \neq 1$, (5.2) is monotone in u.

For $i = 1, 2$, let $\phi_i \in W^{1,2}(\Omega)$, and $\varphi_i, \psi_i, \Phi_i, \Psi_i$ be functions as in (3.1), (3.2). For $i = 1, 2$ set

$$\begin{aligned} K_i = \{ & v \in W^{1,2}(\Omega) \mid v = \phi_i \text{ on } \Gamma_0,\ \varphi_i(x) \leq v(x) \leq \psi_i(x) \text{ a.e. } x \in \Gamma, \\ & \Phi_i(x) \leq v(x) \leq \Psi_i(x) \text{ a.e. } x \in \Omega \}, \end{aligned} \tag{5.3}$$

where Γ_0 is a subset of Γ.

Then one has:

THEOREM 5.1. *Let* u_i $(i = 1, 2)$ *be a solution of*

$$\begin{aligned} &u_i \in K_i \\ &< \mathbf{A}(x, u_i, \nabla u_i), v - u_i > \geq < f_i, v - u_i > \quad \text{for all } v \in K_i, \end{aligned} \tag{5.4}$$

where K_i *is given by* (5.3) *and* $f_i \in V_i^*$. *Assume that*:

$\mathbf{A}$ *is defined by* (1.3), (5.1), *and*

(1.4), (1.7), (1.8), (1.9) *hold with* $p = 2$ *for* K_1 *and* K_2.

There exist constants a_i, $i = 1, \ldots, n$, *not all of them equal to* 0, *such that*:

$$u \to \sum_{i=1}^{n} a_i \beta_i(x, u)$$

is monotone (*nondecreasing or nonincreasing*).

For every u *in* $\mathbb{R}$, $a_{ij}(x, u)$ *belong to* $W^{1,\infty}(\Omega)$ *and there exists a constant* C *such that*

$$\left| \frac{\partial a_{ij}(x, u)}{\partial x_k} \right| \leq C \text{ a.e. } x \in \Omega, \text{ for all } u \in \mathbb{R}, \quad i, j, k = 1, \ldots, n. \tag{5.5}$$

$$u \to \gamma(x, u) \text{ is increasing a.e. on } \Gamma \backslash \Gamma_0. \tag{5.6}$$

Then if (1.11) *holds and if* $(f_2, \phi_2, \varphi_2, \psi_2, \Phi_2, \Psi_2) \leq (f_1, \phi_1, \varphi_1, \psi_1, \Phi_1, \Psi_1)$, *one has*

$$u_2(x) \leq u_1(x) \quad \text{a.e. } x \in \Omega. \tag{5.7}$$

($\phi_2 \leq \phi_1$ means $\phi_2 \leq \phi_1$ a.e. on Γ_0).

COROLLARY 5.1. *Under the assumption of Theorem* 5.1, *there exists at most one solution of*

$$\begin{aligned} &u \in K \\ &< \mathbf{A}(x, u, \nabla u), v - u > \geq < f, v - u > \quad \text{for all } v \in K, \end{aligned} \tag{5.8}$$

where K *is a convex of type* K_i.

Since the corollary is an immediate consequence of the theorem, let us prove Theorem 5.1.

PROOF OF THEOREM 5.1. One uses a technique of [8].

STEP 1. First we claim:

$$\int_{[u_2-u_1>0]} [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\cdot\frac{\partial\xi}{\partial x_i}\,dx=0, \tag{5.9}$$

for all $\xi\in C^1(\overline{\Omega})$.

Consider $\xi\in C^1(\overline{\Omega})$, $\xi\geq 0$. Let $F=F^\varepsilon$ be defined by (2.9). Then for δ small enough one has:

$$u_1+\delta\xi F^\varepsilon(u_2-u_1)\in K_1,\quad u_2-\delta\xi F^\varepsilon(u_2-u_1)\in K_2.$$

(Choose δ such that $\delta\xi\leq 1$, then the proof is identical to (3.5)). Substituting these two functions in (5.4), we derive (see the proof of Theorem 2.1):

$$<\mathbf{A}(x,u_1,\nabla u_1)-\mathbf{A}(x,u_2,\nabla u_2),\ \xi F^\varepsilon(u_2-u_1)>\geq 0$$

or

$$\begin{aligned}&\int_\Omega [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\ \frac{\partial[\xi F^\varepsilon(u_2-u_1)]}{\partial x_i}\\ &+\int_\Omega [a(x,u_2)-a(x,u_1)]\ \xi F^\varepsilon(u_2-u_1)\\ &+\int_\Gamma [\gamma(x,u_2)-\gamma(x,u_1)]\ \xi F^\varepsilon(u_2-u_1)\leq 0.\end{aligned} \tag{5.10}$$

Consequently,

$$\begin{aligned}&\int_\Omega [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\ \frac{\partial\xi}{\partial x_i}F^\varepsilon(u_2-u_1)\\ &+\int_\Omega [a(x,u_2)-a(x,u_1)]\ \xi F^\varepsilon(u_2-u_1)\\ &+\int_\Gamma [\gamma(x,u_2)-\gamma(x,u_1)]\ \xi F^\varepsilon(u_2-u_1)\\ &\leq -\int_\Omega [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\ \xi\frac{\partial F^\varepsilon(u_2-u_1)}{\partial x_i}.\end{aligned} \tag{5.11}$$

Let us estimate RH, the right hand side of (5.11). One has:

$$RH = -\int_\Omega [\mathbf{A}_i(x, u_2, \nabla u_2) - \mathbf{A}_i(x, u_2, \nabla u_1)] \ \xi \frac{\partial F^\varepsilon(u_2 - u_1)}{\partial x_i}$$
$$+ \int_\Omega [\mathbf{A}_i(x, u_1, \nabla u_1) - \mathbf{A}_i(x, u_2, \nabla u_1)] \ \xi \frac{\partial F^\varepsilon(u_2 - u_1)}{\partial x_i}.$$

Hence by (1.8) and (1.9)

$$RH \leq -\nu \int_\Omega |\nabla(u_2 - u_1)|^2 \xi (F^\varepsilon)'$$
$$+ \int_\Omega |\mathbf{A}(x, u_1, \nabla u_1) - \mathbf{A}(x, u_2, \nabla u_1)| \xi (F^\varepsilon)' |\nabla(u_2 - u_1)|$$
$$\leq -\nu \int_\Omega |\nabla(u_2 - u_1)|^2 \xi (F^\varepsilon)'$$
$$+ \int_\Omega C\omega(|u_2 - u_1|)(|\nabla u_1| + g(x)) \xi (F^\varepsilon)' |\nabla(u_2 - u_1)|.$$

Using the Young inequality we obtain, as we did in (2.12),

$$RH \leq -\frac{\nu}{2} \int_\Omega |\nabla(u_2 - u_1)|^2 \xi (F^\varepsilon)' + \frac{C}{I(\varepsilon)} \int_\Omega (|\nabla u_1| + g(x))^2 \xi$$
$$\leq \frac{C}{I(\varepsilon)} \int_\Omega (|\nabla u_1| + g(x))^2 \xi$$

for some positive constant C.

Recalling (5.11), we have

$$\int_\Omega [\mathbf{A}_i(x, u_2, \nabla u_2) - \mathbf{A}_i(x, u_1, \nabla u_1)] \ \frac{\partial \xi}{\partial x_i} F^\varepsilon(u_2 - u_1)$$
$$+ \int_\Omega [a(x, u_2) - a(x, u_1)] \ \xi F^\varepsilon(u_2 - u_1)$$
$$+ \int_\Gamma [\gamma(x, u_2) - \gamma(x, u_1)] \ \xi F^\varepsilon(u_2 - u_1) \tag{5.12}$$
$$\leq \frac{C}{I(\varepsilon)} \int_\Omega (|\nabla u_1| + g(x))^2 \xi.$$

Letting ε go to 0 (recall (1.11) and the definition of $I(\varepsilon)$), we obtain:

$$\int_{[u_2-u_1>0]} [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\cdot\frac{\partial\xi}{\partial x_i}$$

$$+\int_{[u_2-u_1>0]} [a(x,u_2)-a(x,u_1)]\cdot\xi \tag{5.13}$$

$$+\int_{[u_2-u_1>0]} [\gamma(x,u_2)-\gamma(x,u_1)]\cdot\xi\le 0,$$

for every $\xi\in C^1(\overline{\Omega})$, $\xi\ge 0$. By (1.7) we have

$$\int_{[u_2-u_1>0]} [\mathbf{A}_i(x,u_2,\nabla u_2)-\mathbf{A}_i(x,u_1,\nabla u_1)]\cdot\frac{\partial\xi}{\partial x_i}\le 0$$

$$\text{for all } \xi\in C^1(\overline{\Omega}),\ \xi\ge 0.$$

Now changing ξ to $M-\xi$, where M is a constant greater than ξ, in the above formula, leads to (5.9).

STEP 2. We prove that $(u_2-u_1)^+\in W_0^{1,2}(\Omega)$.
Due to (5.9) and (1.7), (5.13) becomes

$$\int_{[u_2-u_1>0]} [\gamma(x,u_2)-\gamma(x\cdot u_1)]\cdot\xi\le 0 \quad \text{for all } \xi\in C^1(\overline{\Omega}),\ \xi\ge 0.$$

By (5.6) we deduce that $(u_2-u_1)^+=0$ on $\Gamma\backslash\Gamma_0$. Now on Γ_0 one has $(u_2-u_1)^+=(\phi_2-\phi_1)^+=0$. This proves that $(u_2-u_1)^+\in W_0^{1,2}(\Omega)$.

STEP 3. End of the proof.
Taking into account (5.1), (5.9) can be written as

$$\int_{[u_2-u_1>0]} \left\{a_{ij}(x,u_2)\,\frac{\partial u_2}{\partial x_j}-a_{ij}(x,u_1)\,\frac{\partial u_1}{\partial x_j}\right\}\cdot\frac{\partial\xi}{\partial x_i}$$

$$+\int_{[u_2-u_1>0]} [\beta_i(x,u_2)-\beta_i(x,u_1)]\cdot\frac{\partial\xi}{\partial x_i}=0, \tag{5.14}$$

$$\text{for all } \xi\in C^1(\overline{\Omega}).$$

In (5.14) choose $\xi=e^{\alpha(a\cdot x)}$, where $(a\cdot x)$ denotes the usual scalar product between the vector $a=(a_1,\dots,a_n)$ and $x=(x_1,\dots,x_n)$, α is a constant which

is positive, if (5.2) is nonincreasing, and negative, if (5.2) is nondecreasing. We then obtain:

$$\int_{[u_2-u_1>0]} \left\{ a_{ij}(x,u_2)\ \frac{\partial u_2}{\partial x_j} - a_{ij}(x,u_1)\ \frac{\partial u_1}{\partial x_j} \right\} \cdot \alpha\ a_i\ e^{\alpha(a\cdot x)} \geq 0. \tag{5.15}$$

Define

$$\mathbf{A}_{ij}(x,u) = \int_0^u a_{ij}(x,s)\ ds. \tag{5.16}$$

Then, for $k = 1, 2, \mathbf{A}_{ij}(x,u_k) \in W^{1,2}(\Omega)$ and

$$\frac{\partial \mathbf{A}_{ij}(x,u_k)}{\partial x_j} = a_{ij}(x,u_k)\ \frac{\partial u_k}{\partial x_j} + \int_0^{u_k} \frac{\partial a_{ij}(x,s)}{\partial x_j}\ ds.$$

So that (5.15) becomes

$$\int_{[u_2-u_1>0]} \left\{ \frac{\partial}{\partial x_j} [\mathbf{A}_{ij}(x,u_2) - \mathbf{A}_{ij}(x,u_1)] - \int_{u_1}^{u_2} \frac{\partial a_{ij}(x,s)}{\partial x_j}\ ds \right\} \cdot \alpha a_i e^{\alpha(a\cdot x)} \geq 0. \tag{5.17}$$

Setting $w = (u_2 - u_1)^+$, we obtain:

$$\int_\Omega \frac{\partial}{\partial x_j} [\mathbf{A}_{ij}(x,u_1+w) - \mathbf{A}_{ij}(x,u_1)] \cdot \alpha a_i e^{\alpha(a\cdot x)}$$

$$- \int_\Omega \int_{u_1}^{u_1+w} \frac{\partial a_{ij}(x,s)}{\partial x_j}\ ds \cdot \alpha a_i e^{\alpha(a\cdot x)} \geq 0.$$

Now $A_{ij}(x,u_1+w) - A_{ij}(x,u_1) \in W_0^{1,2}(\Omega)$. Integrating by parts, taking into account (5.16), (1.8), we obtain

$$\int_\Omega \int_{u_1}^{u_1+w} \left[-\alpha^2 \nu |a|^2 + \alpha a_i\ \frac{\partial a_{ij}(x,s)}{\partial x_j} \right] ds\ e^{\alpha(a\cdot x)}$$

$$\geq \int_\Omega \int_{u_1}^{u_1+w} \left[-\alpha^2 a_{ij}(x,s) a_i a_j + \alpha a_i\ \frac{\partial a_{ij}(x,s)}{\partial x_j} \right] ds\ e^{\alpha(a\cdot x)} \geq 0.$$

Taking α large enough (see (5.5)), this leads to a contradiction unless $w \equiv 0$. This completes the proof.

REMARK 5.1. It is possible to relax slightly the assumption (5.5). We refer the reader to [19] for details. One can also show that such problem can develop a free boundary (see [11], [19]).

6. - An existence result

For the sake of completeness, we would like to conclude this paper by a very elementary existence result. For other results, with different assumptions, we refer to [4], [6], [14], [17], [18], [21], [22].

Let K be a closed convex set of $W^{1,p}(\Omega)$ and $\mathbf{A}(x,u,\nabla u)$ an operator from K into V^* defined by (1.3). (V is the closed subspace of $W^{1,p}(\Omega)$ spanned by $K-K$).

Assume that

(6.1) $\quad \mathbf{A}_i(x,u,\xi)$, $a(x,u)$, $\gamma(x,u)$ are Caratheodory functions

and there exist constants C_1 and C_2, functions $C_3 \in L^{p'}(\Omega)$, $C_4 \in L^{p'}(\Gamma)$, $C_i \geq 0$, such that

$$
\begin{aligned}
&|\mathbf{A}(x,u,\xi)| \leq C_1|u|^{p-1} + C_2|\xi|^{p-1} + C_3(x), \\
&\text{for all } u \in \mathbb{R}, \text{ for all } \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,
\end{aligned} \tag{6.2}
$$

$$
|a(x,u)| \leq C_2|u|^{p-1} + C_3(x), \text{ for all } u \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{6.3}
$$

$$
|\gamma(x,u)| \leq C_2|u|^{p-1} + C_4(x), \text{ for all } u \in \mathbb{R}, \text{ a.e. } x \in \Gamma. \tag{6.4}
$$

($\mathbf{A}$ denotes the vector $(\mathbf{A}_1,\ldots,\mathbf{A}_n)$, $|\cdot|$ its Euclidean norm).

Assume also that

$$
\begin{aligned}
&u \to a(x,u) \text{ is nondecreasing a.e. } x \in \Omega, \\
&u \to \gamma(x,u) \text{ is nondecreasing a.e. } x \in \Gamma.
\end{aligned} \tag{6.5}
$$

For $u \in K$, $w \in \overline{K}$ ($\overline{K}$ denotes the closure of K in $L^p(\Omega)$) define the operator $\mathbf{A}(x,w,\nabla u)$ by

$$
\begin{aligned}
< \mathbf{A}(x,w,\nabla u), v > = &\int_\Omega \left[\mathbf{A}_i(x,w,\nabla u) \cdot \frac{\partial v}{\partial x_i} + a(x,u) \cdot v \right] dx \\
&+ \int_\Gamma \gamma(x,u) \cdot v \, d\sigma, \quad \text{for all } v \in V,
\end{aligned} \tag{6.6}
$$

and assume that

$$< \mathbf{A}(x, w, \nabla v) - \mathbf{A}(x, w, \nabla u), v - u > \geq \nu(|v - u|_{1,p})^p \quad \text{for all } u, v \in K, \text{ for all } w \in \overline{K} \tag{6.7}$$

where ν is a positive constant and $|\cdot|_{1,p} = |\cdot|_p + \|\nabla \cdot\|_p$ denotes the usual norm in $W^{1,p}(\Omega)$.

Then we have:

THEOREM 6.1. *Let K be a closed convex set in $W^{1,p}(\Omega)$ and $\mathbf{A}(x, u, \nabla u)$ an operator from K into V^* satisfying* (6.1)-(6.7). *Then if*:

(6.8) (i) *K is bounded in $L^p(\Omega)$*

or

(6.9) (ii) $C_1 < \nu$,

then for $f \in V^*$ there exists a solution to

$$\begin{aligned} &u \in K \\ &< \mathbf{A}(x, u, \nabla u), v - u > \geq < f, v - u > \quad \textit{for all } v \in K. \end{aligned} \tag{6.10}$$

REMARK 6.1. First note that if there exist constants C_1', C_2, and a function C_3 such that, for $\varepsilon > 0$,

$$|\mathbf{A}(x, u, \xi)| \leq C_1'|u|^{p-1-\varepsilon} + C_2|\xi|^{p-1} + C_3(x), \quad \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \tag{6.11}$$

then (6.9) is automatically fulfilled for C_1 as small as we wish and thus existence holds in this case.

For the assumption (6.7), note that, if (1.8) holds, then one has

$$\begin{aligned} < \mathbf{A}(x, w, \nabla v) - \mathbf{A}(x, w, \nabla u), v - u > \geq \nu(|\nabla(v - u)|_p)^p \\ + \int_\Omega [a(x, v) - a(x, u)] \cdot (v - u) \, \mathrm{d}x \\ + \int_\Gamma [\gamma(x, v) - \gamma(x, u)] (v - u) \, \mathrm{d}\sigma, \\ \forall \ u, v \in K, \ \forall \ w \in \overline{K}, \end{aligned}$$

and (6.7) holds, for instance, (see (6.5)) if for some constant c

$$[a(x,v)-a(x,u)]\cdot(v-u)>c|v-u|^p$$

a.e. on a part of positive measure of Ω

or

$$[\gamma(x,v)-\gamma(x,u)](v-u)>c|v-u|^p$$

a.e. on a part of positive measure of Γ

or

there exists a constant C such that $|v|_p\le C\|\nabla v\|_p$, for all $v\in V$.

PROOF OF THEOREM 6.1.

STEP 1. Let $w\in\overline{K}\cap B(0,R)$, where $B(0,R)$ is the ball of center 0 and radius R in $L^p(\Omega)$. Consider $u=T(w)$ the solution of:

$$\begin{aligned}&u\in K\\&<\mathbf{A}(x,w,\nabla u),v-u>\ \ge\ <f,v-u>\qquad\text{for all } v\in K.\end{aligned}\tag{6.12}$$

Such a solution exists and is unique. Indeed, due to our growth assumptions

$$\nabla u\to\mathbf{A}_i(x,w,\nabla u),\ \ u\to a(x,u),\ \ u\to\gamma(x,u)$$

are Nemyckii operators and thus continuous from $L^p(\Omega)$ into $L^{p'}(\Omega)$ (see [18] p. 184 or [20] p. 37). Thus $u\to\mathbf{A}(x,w,\nabla u)$ is continuous from K into V^* and one can solve (6.12) in K intersected with any finite dimensional subspace (see [16]). Using (6.7) together with Minty's lemma, it is easy to conclude the existence of u (we refer the reader to [16], [18] for details on the technique).

STEP 2. T maps $\overline{K}\cap B(0,R)$ into itself for R large enough.
In case (i) there is nothing to prove if R is large enough.
In case (ii), taking $v_0\in K$, from (6.12) we derive

$$\begin{aligned}&<\mathbf{A}(x,w,\nabla u),v_0-u>\ \ge\ <f,v_0-u>\qquad\text{iff}\\&<\mathbf{A}(x,w,\nabla v_0)-\mathbf{A}(x,w,\nabla u),v_0-u>\\&\le\ <\mathbf{A}(x,w,\nabla v_0),v_0-u>\ -\ <f,v_0-u>\ .\end{aligned}$$

By (6.7) we deduce

$$\nu(|v_0-u|_{1,p})^p\le|<\mathbf{A}(x,w,\nabla v_0),v_0-u>|+|f|_{V^*}\cdot|v_0-u|_{1,p}.\tag{6.13}$$

Let us estimate the first term of the right hand side of (6.13).

$$
\begin{aligned}
&| < \mathbf{A}(x, w, \nabla v_0), v_0 - u > | \\
&= \left| \int_\Omega \left\{ \mathbf{A}_i(x, w, \nabla v_0) \cdot \frac{\partial (v_0 - u)}{\partial x_i} + a(x, v_0) \cdot (v_0 - u) \right\} \, \mathrm{d}x \right. \\
&\left. + \int_\Gamma \gamma(x, v_0) \cdot (v_0 - u) \mathrm{d}\sigma \right| \\
&\leq |v_0 - u|_{1,p} \{ \|\mathbf{A}(x, w, \nabla v_0)\|_{p'} + |a(x, v_0)|_{p'} + |\gamma(x, v_0)|_{p'} \} \\
&\leq |v_0 - u|_{1,p} \ \{ C_1 (|w|_p)^{p-1} + C \},
\end{aligned}
$$

where C is some fixed constant (depending on v_0).

[This results easily from (6.2), since

$$
\begin{aligned}
\|\mathbf{A}(x, w, \nabla v_0)\|_{p'} &\leq \left| C_1 |w|^{p-1} + C_2 |\nabla v_0|^{p-1} + C_3(x) \right|_{p'} \\
&\leq C_1 \left| \ |w|^{p-1} \right|_{p'} + C_2 \left| \ |\nabla v_0|^{p-1} \right|_{p'} + |C_3(x)|_{p'}].
\end{aligned}
$$

Recalling (6.13) we deduce

$$
\nu \{ |v_0 - u|_p \}^{p-1} \leq \nu \ (|v_0 - u|_{1,p})^{p-1} \leq C_1 (|w|_p)^{p-1} + C. \tag{6.14}
$$

This implies

$$
|v_0 - u|_p \leq \left(\frac{C_1}{\nu} \ |w|_p^{p-1} + \frac{C}{\nu} \right)^{\frac{1}{p-1}} \leq \left(\frac{C_1}{\nu} \right)^{\frac{1}{p-1}} \ |w|_p + \left(\frac{C}{\nu} \right)^{\frac{1}{p-1}}.
$$

Hence

$$
|u|_p \leq \left(\frac{C_1}{\nu} \right)^{\frac{1}{p-1}} \ |w|_p + C'.
$$

If we assume $\alpha = \left(\frac{C_1}{\nu} \right)^{\frac{1}{p-1}} < 1$ then, for $R > \frac{C'}{1-\alpha}$, the above inequality shows that T maps $\overline{K} \cap B(0, R)$ into itself. This completes the proof of this step.

STEP 3. End of the proof.

From (6.14) and the fact that $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, it is clear that $T(\overline{K} \cap B(0, R))$ is relatively compact in $\overline{K} \cap B(0, R)$. To conclude the proof by Schauder fixed point theorem (see [15]), it is enough to show that T is continuous.

Let $w_k \in \overline{K} \cap B(0, R)$ be such that $w_k \to w$ in $L^p(\Omega)$, and $u_k = T(w_k)$ be the solution of (6.12) corresponding to $w = w_k$.

From (6.14) we deduce that $|u_k|_{1,p}$ is bounded independently of k and we can extract a subsequence - still denoted by u_k - such that u_k converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ toward u_∞, when $k \to \infty$.

Now, from Minty's lemma, (6.12) is equivalent to

$$\begin{aligned} &u_k \in K \\ &< \mathbf{A}(x, w_k, \nabla v), v - u_k > \geq < f, v - u_k >, \quad \text{for all } v \in K. \end{aligned} \tag{6.15}$$

The second relation of (6.15) can be written:

$$\int_\Omega \left[\mathbf{A}_i(x, w_k, \nabla v) \frac{\partial (v - u_k)}{\partial x_i} + a(x, v)\ (v - u_k) \right]$$

$$+ \int_\Gamma \gamma(x, v)(v - u_k) \geq < f, v - u_k >$$

and, letting $k \to \infty$, we see that u_∞ satisfies

$$\begin{aligned} &u_\infty \in K \\ &< \mathbf{A}(x, w, \nabla v), v - u_\infty > \geq < f, v - u_\infty >, \quad \text{for all } v \in K, \end{aligned} \tag{6.16}$$

provided that we can show that

$$\mathbf{A}_i(x, w_k, \nabla v) \to \mathbf{A}_i(x, w, \nabla v) \quad \text{in } L^{p'}(\Omega). \tag{6.17}$$

But, due to our growth assumptions $w \to \mathbf{A}_i(x, w, \nabla v)$ is a Nemyckii operator and (6.17) follows. Applying Minty's lemma again we obtain $u_\infty = u$ and T is continuous. This completes the proof.

REMARK 6.2. Note that, when K is bounded in $L^p(\Omega)$, our assumption (ii) is unnecessary. This is also the case when we can get an a priori estimate on u (see [4], [21], [22]).

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