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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 15, nº 4 (1988), p. 603-614

<http://www.numdam.org/item?id=ASNSP_1988_4_15_4_603_0>

An Elementary Treatment of a General Diophantine Problem

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1. - Introduction

This paper is concerned with the order of magnitude of $I_8(H,\Delta)$, the number of integer solutions $(h_1,\cdots,h_4,k_1,\cdots,k_4)$ of

$$\sum_{j=1}^{4}(h_j^2-k_j^2)=\sum_{j=1}^{4}(h_j-k_j)=0 \tag{1}$$

and

$$\left|\sum_{j=1}^{4}\left[g\left(\frac{h_j}{H}\right)-g\left(\frac{k_j}{H}\right)\right]\right|<\Delta, \tag{2}$$

with

$$H\le h_j,k_j<2H,\ \text{ for } j=1,\cdots,4, \tag{3}$$

where H is a positive integer, $\Delta>0$ and g is a fixed complex-valued function defined on the interval $[1,2)$. The following result is obtained.

THEOREM. *If g is analytic on some open subset V of $\mathbb{C}$ containing the interval $[1,2]$ and*

$$g^{(3)}(x)g^{(5)}(x)\neq(g^{(4)}(x))^2,\ \textit{for}\ 1\le x\le 2, \tag{4}$$

then

$$I_8(H,\Delta)<<H^4+\Delta H^5\log^3 H. \tag{5}$$

Interest in $I_8(H,\Delta)$ goes back to 1985, when E. Bombieri and H. Iwaniec showed, in [2], that if $g^{(1)}(x)=x^\beta$, for $1\le x\le 2$, where β is some real

Pervenuto alla Redazione il 10 Giugno 1988.

constant other than 0 or 1, then

$$I_8(H,\Delta) <<_\varepsilon (1+\Delta H)H^{4+\varepsilon}, \text{ for } \varepsilon > 0. \tag{6}$$

Later I obtained the bound (5) for $\beta = \frac{1}{2}$ (see [5, Theorem 1]).

The bounds for $\beta = \frac{1}{2}$ are important in applications to exponential sums. In [1] Bombieri and Iwaniec used the bound (6), for $\beta = \frac{1}{2}$, to show that $\zeta\left(\frac{1}{2}+it\right) <<_\varepsilon t^{\varepsilon+9/56}$, for $t > \epsilon > 0$. The bound (5), for $\beta = \frac{1}{2}$, has been applied in [3] and [4].

To prove (6) Bombieri and Iwaniec treated $I_8(H,\Delta)$ as a mean-value of an exponential sum and used an ingenious argument involving Poisson summation. In this paper the more elementary techniques of [5] are adapted to generalize [5, Theorem 1]. The resulting Theorem (above) does not generalize the bound (6) of Bombieri and Iwaniec, as the condition (4) is not satisfied for $\beta = 2$.

My thanks are due to the Science and Engineering Research Council, University College Cardiff and my supervisor, Dr M. N. Huxley.

2. - Preliminary results and definitions

The positive number $(2^{\frac{1}{4}}-1)^{-1}$, which we shall henceforth refer to as α, occurs naturally in the next Lemma. It satisfies the equation

$$\left(1+\frac{1}{\alpha}\right)^4 = 2.$$

Now

$$\left(1+\frac{1}{6}\right)^4 < 2 < \left(1+\frac{4}{21}\right)^4,$$

so

$$\frac{21}{4} < \alpha < 6. \tag{7}$$

Also

$$a^4 = 4\alpha^3 + 6\alpha^2 + 4\alpha + 1,$$

so the sequence $\{\alpha^n\}$ is generated by the recurrence formula,

$$\alpha^r = 4\alpha^{r-1} + 6\alpha^{r-2} + 4\alpha^{r-3} + \alpha^{r-4}. \tag{8}$$

Starting from the lower bounds $0, \frac{1}{6}, 1$ and $\frac{21}{4}$ for $\alpha^{-2}, \alpha^{-1}, \alpha^0$ and α (respectively), we use the recurrence formula to produce the lower bounds

$$\frac{83}{3}, \frac{439}{3}, \frac{2320}{3}, \frac{16349}{4}, 21602, \frac{685027}{6}, 603419 \text{ and } \frac{12756793}{4} \tag{9}$$

for $\alpha^2, \cdots, \alpha^9$ (respectively).

LEMMA 1. *Let $\delta \geq 0$ and $K > 0$. For $r = 1, \cdots, 4$, let*

$$|x_r| \leq K, |y_r| \leq K \text{ and } \left|\sum_{j=1}^{4}(x_j^r - y_j^r)\right| \leq \delta K^r.$$

Then, for $i = 1, \cdots, 4$,

$$\left|\prod_{j=1}^{4}(x_i - y_j)\right| \leq \frac{147}{4}\delta K^4 \tag{10}$$

and, for $r = 5, 6, 7, \cdots$,

$$\left|\sum_{j=1}^{4}(x_j^r - y_j^r)\right| \leq \frac{1}{25}\delta(\alpha K)^r. \tag{11}$$

PROOF. It is sufficient to prove these results for $\delta > 0$, since the case $\delta = 0$ then follows by letting δ tend to zero from above.

For $r = 1, 2, 3, \cdots$, let

$$S_r = \sum_{j=1}^{4} x_j^r \qquad \text{and} \qquad T_r = \sum_{j=1}^{4} y_j^r.$$

Let $E_0, \cdots, E_4$ be given by the polynomial expansion,

$$(X - x_1)\cdots(X - x_4) = E_0X^4 + E_1X^3 + \cdots + E_4.$$

Define $F_0, \cdots, F_4$ similarly, but in terms of $y_1, \cdots, y_4$. Then

$$S_r + S_{r-1}E_1 + \cdots + S_1E_{r-1} + rE_r = 0 = T_r + T_{r-1}F_1 + \cdots + T_1F_{r-1} + rF_r,$$

for $r = 1, \cdots, 4$. Therefore, for $R = 1, \cdots, 4$,

$$R|E_R - F_R| \leq \sum_{r=1}^{R} |S_rE_{R-r} - T_rF_{R-r}|$$

$$\leq \sum_{r=1}^{R}[|S_r(E_{R-r} - F_{R-r})| + |F_{R-r}(S_r - T_r)|]$$

$$\leq \sum_{r=1}^{R}\left[4K^r|E_{R-r} - F_{R-r}| + \binom{4}{R-r}K^{R-r}\delta K^r\right].$$

Starting from the fact that $E_0 = F_0 = 1$, we deduce that

$$|E_1 - F_1| \le \delta K,$$
$$|E_2 - F_2| \le \frac{9}{2}\delta K^2,$$
$$|E_3 - F_3| \le 11\delta K^3$$

and

$$|E_4 - F_4| \le \frac{81}{4}\delta K^4.$$

Let i be an integer with $1 \le i \le 4$. Then

$$(x_i - y_1)\cdots(x_i - y_4) = x_i^4 + F_1 x_i^3 + \cdots + F_4$$

and

$$0 = (x_i - x_1)\cdots(x_i - x_4) = x_i^4 + E_1 x_i^3 + \cdots + E_4.$$

Therefore

$$|(x_i - y_1)\cdots(x_i - y_4)| \le |F_1 - E_1|K^3 + \cdots + |F_4 - E_4|$$
$$\le \left(1 + \frac{9}{2} + 11 + \frac{81}{4}\right)\delta K^4 = \frac{147}{4}\delta K^4,$$

which is the result (10).

For $r = 5, 6, 7, \cdots$,

$$S_r + S_{r-1}E_1 + \cdots + S_{r-4}E_4 = 0 = T_r + T_{r-1}F_1 + \cdots + T_{r-4}F_4.$$

Therefore, for $R = 5, 6, 7, \cdots$,

$$|S_R - T_R| \le \sum_{r=1}^{4}[|S_{R-r}(E_r - F_r)| + |F_r(S_{R-r} - T_{R-r})|]$$
$$\le 147\delta K^R + \sum_{r=1}^{4}\binom{4}{r}K^r|S_{R-r} - T_{R-r}|.$$

Now, for $r = 1, 2, 3, \cdots$, let U_r be given by

$$|S_r - T_r| = \left(U_r - \frac{21}{2}\right)\delta K^r.$$

Then, for $R = 5, 6, 7, \cdots$,

$$U_R \le \sum_{r=1}^{4}\binom{4}{r}U_{R-r}.$$

Starting from the upper bound of $\frac{23}{2}$ for U_1, U_2, U_3 and U_4, we apply this inequality to produce the upper bounds

$$\frac{345}{2}, \frac{1633}{2}, \frac{8717}{2}, \frac{46069}{2} \text{ and } \frac{243455}{2}$$

for $U_5, \cdots, U_9$ (respectively). Using (9) we can easily check that

$$26U_r \leq \alpha^r, \text{ for } r = 6, \cdots, 9.$$

Now suppose that R is an integer greater than 9 and such that

$$U_r \leq \frac{1}{26}\alpha^r, \text{ for } r = 6, \cdots, R-1.$$

Then

$$U_R \leq \frac{1}{26}\sum_{r=1}^{4}\binom{4}{r}\alpha^{R-r} = \frac{1}{26}\alpha^R,$$

by (8). The result (11), for $r > 5$, follows by induction. To complete the proof note that $25\left(U_5 - \frac{21}{2}\right) \leq 25 \times 162 < \alpha^5$, by (9).

Let $(h_1, \cdots, k_4)$ be an integer solution of (1). Then so is $(h_1+t, \cdots, k_4+t)$, for any integer t. As in [5] we call this set of integer solutions of (1) a family f (say). The integer solution $(h_1, \cdots, k_4)$ of (1) is called trivial if and only if $(h_1, \cdots, h_4)$ is a permutation of $(k_1, \cdots, k_4)$. The family f is called trivial if and only if

$$(h_i - k_1)\cdots(h_i - k_4) = (k_i - h_1)\cdots(k_i - h_4) = 0,$$

for $i = 1, \cdots, 4$.

LEMMA 2. *An integer solution of* (1) *is trivial if and only if it is a member of a trivial family. The number of trivial integer solutions of* (1) *and* (3) *is*

$$A(H) = 24H^4 - 72H^3 + 82H^2 - 33H.$$

PROOF. This Lemma is almost a restatement of [5, Lemma 4], since there is a one-to-one correspondence between the trivial integer solutions of (1) and (3) and the trivial integer solutions of (1) with $0 \leq h_i, k_i < H$, for $i = 1, \cdots 4$. The correspondence is given by

$$(h_1, \cdots, k_4) \longrightarrow (h_1 - H, \cdots, k_4 - H).$$

LEMMA 3. *The integer solutions of* (1) *and* (3) *fall into* $O(H^4)$ *families.*

PROOF. This Lemma follows directly from [5, Lemma 6].

LEMMA 4. *Let $\delta > 0$ and $K > 0$. Let F be the number of non-trivial families which contain a member $(a_1, \cdots, a_4, b_1, \cdots, b_4)$ with*

$$\left|\sum_{j=1}^{4}(a_j^r - b_j^r)\right| \le \delta K^r, \text{ for } r = 3, 4,$$

and

$$|a_i| \le K, \ |b_i| \le K, \text{ for } i = 1, \cdots, 4.$$

Then

$$F << \delta K^4 \log^2 K.$$

PROOF. If $K < 2$, then

$$a_i \in \{-1, 0, 1\} \text{ and } b_i \in \{-1, 0, 1\}, \text{ for } i = 1, \cdots, 4,$$

and it follows from (1) that there are no such non-trivial families. The rest of the proof is almost the same as that of [5, Lemma 9], except that the appeals to [5, Lemma 8] are made with $r = 1$, $x_j = a_j$ and $y_j = b_j$, for $j = 1, \cdots, 4$.

For $J \subset [H, 2H)$, let $S(J)$ be the set of integer solutions of (1) and (2) with $h_i \in J$ and $k_i \in J$, for $i = 1, \cdots 4$.

LEMMA 5. *If $[H, 2H) = J_1 \cup \cdots \cup J_Q$, then*

$$I_8(H, \Delta) << Q^7 \sum_{q=1}^{Q} |S(J_q)|.$$

PROOF. The condition (2) implies that

$$\left|\sum_{j=1}^{4}\left[\text{Re } g\left(\frac{h_j}{H}\right) - \text{Re } g\left(\frac{k_j}{H}\right)\right]\right| < \Delta$$

and

$$\left|\sum_{j=1}^{4}\left[\text{Im } g\left(\frac{h_j}{H}\right) - \text{Im } g\left(\frac{k_j}{H}\right)\right]\right| < \Delta.$$

Conversely we note that if $-\Delta < x, y < \Delta$, then $|x + iy| < \sqrt{2}\Delta$. Therefore the Lemma follows by [4, Lemma 2.2 and Lemma 2.3].

3. - Proof of the Theorem

For $c \epsilon \mathbb{C}$ and $r \geq 0$, let

$$D(c, r) = \{z \epsilon \mathbb{C} \,||z - c| \leq r\}.$$

For $1 \leq x \leq 2$, let

$$R(x) = \sup\{r \epsilon [0, 1] | D(x, r) \subset V\}.$$

Then R is a continuous positive-valued function on $[1, 2]$, so there exists a positive constant ρ such that

$$R(x) > \rho, \text{ for } 1 \leq x \leq 2. \tag{12}$$

Let A be the union of the sets $D(x, \rho)$ with $1 \leq x \leq 2$. Then A is a compact subset of $\mathbb{C}$. Let B be the maximun value attained by $|g(z)|$ for $z \epsilon A$. By (4), $B > 0$. If $1 \leq x \leq 2$, then by Cauchy's Integral Formulae,

$$|g^{(n)}(x)| \leq n! \rho^{-n} B, \text{ for } n = 0, 1, 2, \cdots. \tag{13}$$

For $1 \leq x \leq 2$, let

$$E(x) = |g^{(3)}(x) g^{(5)}(x) - (g^{(4)}(x))^2|.$$

By (4), E is a positive-valued continuous function on $[1, 2]$, so there exists a positive constant ε such that

$$E(x) \geq \frac{3!4!B^2}{\rho^8} \varepsilon, \text{ for } 1 \leq x \leq 2. \tag{14}$$

By (13),

$$\varepsilon \leq 5 + 4 = 9. \tag{15}$$

Let Q be the least integer with

$$\varepsilon \rho Q \geq \alpha^5. \tag{16}$$

We divide $[H, 2H)$ into disjoint intervals $J_1, \cdots, J_Q$ of equal length. Let J be one of these intervals. Let $(h_1, \cdots, k_4) \epsilon f \cap S(J)$, where f is a family. Let $(a_1, \cdots, b_4)$ be the member of f with $b_4 = 0$. Then, for $i = 1, \cdots, 4$,

$$a_i = h_i - k_4 \text{ and } b_i = k_i - k_4,$$

so that

$$|a_i| \leq \frac{H}{Q} \text{ and } |b_i| \leq \frac{H}{Q}. \tag{17}$$

For $r = 0, 1, 2, \cdots$, let

$$d_r = \sum_{j=1}^{4}(a_j^r - b_j^r).$$

Let

(18) $$\delta = \max_{r=3,4} \left[\frac{Q}{H}\right]^r |d_r|.$$

By (1), (17) and Lemma 1 (11),

(19) $$d_0 = d_1 = d_2 = 0$$

and

(20) $$|d_r| \leq \frac{1}{25}\delta \left[\frac{\alpha H}{Q}\right]^r, \quad \text{for } r = 5, 6, 7, \cdots.$$

To estimate $|f \cap S(J)|$ we consider the function

$$G(z) = \sum_{j=1}^{4}\left[g\left(\frac{h_j}{H} + z\right) - g\left(\frac{k_j}{H} + z\right)\right],$$

which (by (12)) is certainly analytic for $|z| \leq \rho$. The members of $f \cap S(J)$ are in one-to-one correspondence with those integer solutions t of

(21) $$\left|G\left(\frac{t}{H}\right)\right| < \Delta$$

which have

(22) $$h_j + t\epsilon J \text{ and } k_j + t\epsilon J, \text{ for } j = 1, \cdots, 4.$$

Now, by (7), (15) and (16),

(23) $$\rho Q \geq \frac{\alpha^5}{9} > \left(\frac{25}{3}\right)^2 \alpha > 64\alpha > 256,$$

so that $\frac{h_j}{H}\epsilon D\left(\frac{k_4}{H}, \rho\right)$, for $j = 1, \cdots, 4$. Therefore, for $n = 0, 1, 2, \cdots$,

$$g^{(n)}\left(\frac{h_j}{H}\right) = \sum_{r=0}^{\infty}\frac{1}{r!}c_{n+r}\left(\frac{a_i}{H}\right)^r,$$

where $c_r = g^{(r)}\left(\frac{k_4}{H}\right)$, for $r = 0, 1, 2, \cdots$. The various $g^{(n)}\left(\frac{k_j}{H}\right)$ have similar Taylor series. Combining these results we find that

$$G^{(n)}(0) = \sum_{r=0}^{\infty} \frac{1}{r!} c_{n+r} d_r H^{-r}, \text{ for } n = 0, 1, 2, \cdots.$$

Hence, by (13), (19), (20) and (23),

$$\left| G^{(n)}(0) - \sum_{r=3}^{4} \frac{c_{n+r} d_r}{r! H^r} \right| \leq \frac{n! B\delta}{25\rho^n} \sum_{r=5}^{\infty} \binom{n+r}{n} \left[\frac{\alpha}{\rho Q}\right]^r, \text{ for } n = 0, 1, 2, \cdots.$$

Now, if n and s are non-negative integers and $0 \leq x < 1$, then by Taylor's Theorem there exists $\eta \epsilon (0, 1)$ with

$$\sum_{r=s}^{\infty} \binom{n+r}{n} x^r = \binom{n+s}{n} x^s (1 - \eta x)^{-(n+s+1)}. \tag{24}$$

Therefore, if we let $\tau = \frac{1}{1 - \frac{\alpha}{\rho Q}}$, then, for $n = 0, 1, 2, \cdots$,

$$\begin{aligned} \left| G^{(n)}(0) - \sum_{r=3}^{4} \frac{c_{n+r} d_r}{r! H^r} \right| &\leq \frac{n! B\delta}{25\rho^n} \binom{n+5}{5} \left[\frac{\alpha}{\rho Q}\right]^5 \tau^{n+6} \\ &\leq \frac{(n+5)!}{5!} \left[\frac{\tau}{\rho}\right]^n \frac{\tau^6}{25} \frac{B\epsilon\delta}{(\rho Q)^4}, \end{aligned} \tag{25}$$

by (16).

Now, by (23), $1 < \tau < \frac{64}{64 - 1}$,

so

$$1 < \tau^7 < \frac{64}{64 - 7} < \frac{25}{22}. \tag{26}$$

Therefore, since

$$\begin{bmatrix} c_5 & -c_4 \\ -c_4 & c_3 \end{bmatrix} \begin{bmatrix} c_3 & c_4 \\ c_4 & c_5 \end{bmatrix} = E\left(\frac{k_4}{H}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

it follows from (25) that, for $r = 3, 4$,

$$\left| \frac{E(\frac{k_4}{H}) d_r}{r! H^r} + \sum_{n=0}^{1} (-1)^{n+r} c_{8-n-r} G^{(n)}(0) \right| < \frac{1}{22} \left[\sum_{n=0}^{1} |c_{8-n-r}| \frac{(n+5)!}{5! \rho^n}\right] \frac{B\varepsilon\delta}{(\rho Q)^4}.$$

Hence, by (13) and (14),

$$\left[\frac{Q}{H}\right]^r |d_r| \le \frac{(Q\rho)^r}{B\varepsilon}[(8-r)|G(0)| + \rho|G^{(1)}(0)|] + \frac{(14-r)}{2}(\rho Q)^{r-4}\delta$$
$$\le \frac{(Q\rho)^r}{B\varepsilon}\Gamma + \frac{1}{2}(\rho Q)^{r-4}\delta, \text{ for } r = 3, 4,$$

where $$\Gamma = 5|G(0)| + \rho|G^{(1)}(0)|.$$

By (23), $\rho Q > 1$, so it now follows from (18) that

$$\left[\frac{Q}{H}\right]^4 |d_4| \le \delta \le \frac{2(Q\rho)^4}{B\varepsilon}\Gamma \tag{27}$$

and

$$\left[\frac{Q}{H}\right]^3 |d_3| \le \frac{2(Q\rho)^3}{B\varepsilon}\Gamma. \tag{28}$$

We now return to (25) and apply the results (13), (15), (26), (27) and (28) to show that, for $n = 0, 1, 2, \cdots$,

$$\begin{aligned}|G^{(n)}(0)| &\le 2\frac{(n+3)!\Gamma}{3!\rho^n\varepsilon} + 2\frac{(n+4)!\Gamma}{4!\rho^n\varepsilon} + \frac{1}{11}\frac{(n+5)!\tau^n\Gamma}{5!\rho^n} \\ &\le \frac{5\Gamma}{\varepsilon}\frac{(n+5)!}{5!}\left(\frac{\tau}{\rho}\right)^n.\end{aligned} \tag{29}$$

Now suppose that (21) and (22) hold. Then, by (23),

$$\left|\frac{t}{H}\right| \le \frac{1}{Q} \le \rho, \tag{30}$$

so

$$G\left(\frac{t}{H}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}G^{(n)}(0)\left(\frac{t}{H}\right)^n. \tag{31}$$

By (7), (16), (23), (24), (26), (29) and (30).

$$\left|\sum_{n=2}^{\infty}\frac{1}{n!}G^{(n)}(0)\left(\frac{t}{H}\right)^n\right| \le \frac{5\Gamma}{\epsilon}\sum_{n=2}^{\infty}\binom{n+5}{5}\left|\frac{\tau t}{\rho H}\right|^n \le \frac{5\Gamma}{\epsilon}\binom{7}{5}\left|\frac{\tau t}{\rho H}\right|^2\left[1-\frac{\tau}{\rho H}\right]^{-8}$$

$$\le \left|\frac{105\tau^2\Gamma t}{\varepsilon\rho^2 QH}\left[1-\frac{8\times 2}{256}\right]^{-1}\right| < \left|\frac{256\Gamma t}{\alpha^5\rho H}\left[1-\frac{1}{2}\right]^{-1}\right| < \left|\frac{2^9\Gamma t}{2^{10}\rho H}\right|.$$

Therefore, by (31),

$$\left|G^{(1)}(0)\frac{t}{H}\right| \leq \left|G\left(\frac{t}{H}\right) - G(0)\right| + \frac{\Gamma|t|}{2\rho H}$$

and

$$\left|\Gamma\frac{t}{H}\right| \leq \left|5G(0)\frac{t}{H}\right| + \rho\left|G\left(\frac{t}{H}\right) - G(0)\right| + \left|\Gamma\frac{t}{2H}\right|.$$

Hence, by (2), (21), (23) and (30),

$$\left|\Gamma\frac{t}{H}\right| \leq \left|10G(0)\frac{t}{H}\right| + 2\rho\left|G\left(\frac{t}{H}\right) - G(0)\right| \leq 5\rho\Delta.$$

Therefore, by (27).

$$\left|\delta\frac{t}{H}\right| \leq \left(\frac{10Q^4\rho^5}{B\varepsilon}\right)\Delta,$$

so

$$|\delta t| << \Delta H. \tag{32}$$

We can now bound $|S(J)|$. First note that, by (17) and (18),

$$0 \leq \delta \leq 8.$$

Therefore, it is sufficient to consider the following three cases.

Case 1: $\delta = 0.$

By Lemma 1 (10), f is a trivial family. By Lemma 2, $(h_1, \cdots, k_4)$ is a trivial integer solution of (1) and $S(J)$ contains at most $24H^4$ such solutions.

Case 2: $\Delta H < \delta.$

By (32),

$$|f \cap S(J)| << 1.$$

Hence, by Lemma 3, at most $O(H^4)$ members of $S(J)$ fall into Case 2.

Case 3: $0 < \delta \leq \Delta H.$

By Lemma 2, f is not a trivial family. Since $a_1, \cdots, b_4$ are integers and $Q \geq 1$, it follows from (17) and (18) that

$$\frac{1}{H^4} \leq \delta \leq 8.$$

We now consider the $O(\log H)$ non-empty subcases of the form,

$$\Delta_2 < \delta \leq 2\Delta_2,$$

where Δ_2 is an integer power of 2. By (32),

$$|f \cap S(J)| << \left(\frac{\Delta}{\Delta_2}\right) H.$$

By (17), (18) and Lemma 4, at most $O(\Delta_2 H^4 \log^2 H)$ families fall into any one such subcase. Hence, at most $O(\Delta H^5 \log^2 H)$ members of $S(J)$ fall into each subcase and the total number falling into Case 3 is therefore $O(\Delta H^5 \log^3 H)$.

Collecting the results from the three cases we find that

$$|S(J)| << H^4 + \Delta H^5 \log^3 H.$$

The Theorem now follows by Lemma 5.

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