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Growth Properties of Subharmonic Functions in the Unit Disk

SHINJI YAMASHITA

1. - Introduction

We shall consider growth properties of specified subharmonic functions defined in $D = \{|z| < 1\}$ in terms of their p-th integral means on circles $|z| = r < 1$.

Let $\mathrm{d}m \equiv \mathrm{d}m(t) = (2\pi)^{-1}\mathrm{d}t$ be the normalized Lebesgue measure on the right-open interval $T = [0,\ 2\pi)$. For $u \geq 0$ subharmonic in D, $0 < p \leq +\infty$ and $0 \leq r < 1$, we write

$$L_p(r, u) \begin{cases} = \left\{ \displaystyle\int_T u\left(re^{it}\right)^p \mathrm{d}m \right\}^{\frac{1}{p}}, & \text{if } p \neq +\infty, \\ = \displaystyle\sup_{t \in T}\ u\left(re^{it}\right), & \text{if } p = +\infty, \end{cases}$$

$$L_p(u) = \lim_{r \to 1}\ \sup\ L_p(r, u) \leq +\infty.$$

Let PL be the family of functions $u \geq 0$ in D such that $\log u$ is subharmonic in D; we regard $0 \in PL$. Each $u = \exp(\log u) \in PL$ is subharmonic in D. See [R] for references of class PL; in particular, PL has some relations to the differential geometry [R, pp. 23-24].

A typical and important example is $|f| \in PL$, where f is holomorphic in D. More generally, if f_j is holomorphic in D, and $\beta > 0$, $\alpha_j > 0$, $1 \leq j \leq n$, then

$$\left(\sum_j |f_j|^{\alpha_j}\right)^{\beta} \in PL.$$

For the proof we remember that $u^p \in PL$ and $u + v \in PL$, if u, $v \in PL$ and $0 < p < +\infty$.

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We begin with comparative growth of mean values, the main result in our paper.

THEOREM 1. *The following three propositions hold for* $u \in PL$ *and* $0 < p < q \leq +\infty$.

(I) *If for* $\alpha \geq 0$ *and* $C \geq 0$,

(Ia) $$L_p(r, u) \leq C(1-r)^{-\alpha}, \qquad 0 \leq r < 1,$$

then

(Ib) $$L_q(r, u) \leq 2^{\alpha\left(1-\frac{p}{q}\right)} C(1-r)^{-\alpha+\frac{1}{q}-\frac{1}{p}}, \quad 0 \leq r < 1.$$

(II) *If for* $\alpha > 0$,

(IIa) $$L_p(r, u) = o\left((1-r)^{-\alpha}\right) \quad as \quad r \to 1,$$

then

(IIb) $$L_q(r, u) = o\left((1-r)^{-\alpha+\frac{1}{q}-\frac{1}{p}}\right) \quad as \quad r \to 1.$$

(III) *If* $\alpha = 0$ *in* (Ia), *that is,*

(IIIa) $$L_p(u) < +\infty,$$

then

(IIIb) $$L_q(r, u) = o\left((1-r)^{\frac{1}{q}-\frac{1}{p}}\right) \quad as \quad r \to 1.$$

All the results are sharp, in the sense that we cannot add any positive constant $\varepsilon > 0$ to the exponents of $(1-r)$ in (Ib), (IIb) and (IIIb).

An application of (I) and (III) of Theorem 1 to $u = |f|$, f holomorphic in D, yields the classical G.H. Hardy and J.E. Littlewood theorem [HL1, p. 623], [HL2, p. 406]; see [D, Theorem 5.9, p. 84], where

$$M_p(r, f) = L_p(r, |f|), \quad \|f\|_p = L_p(|f|)$$

and

$$H^p = \{f; \ \|f\|_p < +\infty\}$$

is the Hardy class, $0 < p \leq +\infty$.

Our proof of Theorem 1, even in the specified case, $u = |f|$, is different from the original for $|f|$ at least in two points. First we do not make use of an estimate of the specified integral (see (2.5) for $A > 1$) in the proof except for the sharpness. Our method yields directly the constant

$$2^{\alpha\left(1-\frac{p}{q}\right)} \leq 2^{\alpha}$$

in (Ib); the constant is not explicit in the Hardy-Littlewood theorem. Secondly, we do not need a decomposition of $f \in H^p$ into the sum of two zero-free members of H^p. Although Hardy and Littlewood published a revised proof in [HL3, p. 227], there still remains the use of the cited decomposition.

We shall improve the case $\alpha = 0$ and $C = L_p(u)$ in (I) later in (IV) of Section 2.

For the proof of the case $q = +\infty$ in (III), we shall prove a general theorem, Theorem 2, in Section 3, on the growth condition of a harmonic function expressed as the Poisson integral of an integrable function on the circle. The result can be extended to higher dimensional cases. Use is made of the C. De la Vallée Poussin Lemma (Lemma 3.2) on integrable functions.

2. - Proof of Theorem 1

Let

$$P(z,w) = \frac{1-|z|^2}{|w-z|^2}, \qquad \text{for } z \in D,\ w \in \partial D,$$

and g be a complex-valued integrable function on T. Then the Poisson integral of g

$$\Pi(z,g) = \int_T P\left(z, e^{it}\right) g(t) \mathrm{d}m, \qquad z \in D,$$

is a complex harmonic function.

LEMMA 2.1. *For $u \in PL$,*

$$u(z) \leq R \int_T |Re^{it} - z|^{-1} u(Re^{it}) \mathrm{d}m(t), \qquad |z| < R < 1. \tag{2.1}$$

PROOF. We may suppose that $u \not\equiv 0$. We may further suppose that $R = 1$ and $\log u$ is subharmonic in $\{|z| < 1+\varepsilon\}$, $\varepsilon > 0$. Otherwise we consider $u(Rz)$, for $|z| < \frac{1}{R}$.

Since $\log u$ is subharmonic, we have

$$\log u(z) \leq h(z) \equiv \Pi(z, \log u), \qquad z \in D,$$

where h is harmonic and bounded, $h \leq K$, in D because $\log u\left(e^{it}\right)$ is bounded, $\log u \leq K$ on T. Therefore, h has the radial limit

$$\lim_{r \to 1-0} h\left(re^{it}\right) = \log u\left(e^{it}\right) \neq \pm\infty,$$

for a.e. $t \in T$. Let h^* be a harmonic conjugate of h, so that $f = \exp(h + ih^*)$ is holomorphic and bounded, $|f| \leq e^K$, in D. Then, the radial limit $f\left(e^{it}\right) \neq \infty$ exists and $|f\left(e^{it}\right)| = u\left(e^{it}\right)$ for a.e. $t \in T$. Since $u \leq |f|$ we obtain (2.1) for $R = 1$ by the Cauchy formula for f,

$$f(z) = \int_T \frac{f\left(e^{it}\right) e^{it}}{e^{it} - z} \mathrm{d}m, \qquad z \in D;$$

see [D, Theorem 3.6, p. 40].

Before the proof of Theorem 1 we prepare two estimates:

(A) $$L_Q(r,v) \leq L_\infty(r,v)^{1-\frac{1}{Q}}\ L_1(r,v)^{\frac{1}{Q}},$$

for $v \geq 0$ *subharmonic in* D, $1 < Q < \infty$, *and* $0 \leq r < 1$.

(B) $$L_\infty(r,v) \leq R^2\left(R^2-r^2\right)^{-1} L_1(R,v),$$

for $v \in PL$, *and* $0 \leq r < R < 1$.

In fact, for (A), we have

$$L_Q(r,v)^Q = \int_T v(re^{it})^{Q-1}\ v(re^{it})\mathrm{d}m \leq L_\infty(r,v)^{Q-1}L_1(r,v).$$

For (B), we set $|z| = r$ and apply Lemma 2.1 to $v^{1/2} \in PL$. The Cauchy-Schwarz inequality yields that

$$v(z)^{1/2} \leq R\int_T |Re^{it}-z|^{-1}v(Re^{it})^{1/2}\mathrm{d}m \leq REL_1(R,v)^{1/2},$$

where

$$E^2 = \int_T |Re^{it}-z|^{-2}\mathrm{d}m = \left(R^2-r^2\right)^{-1},$$

whence follows (B).

PROOF OF THEOREM 1. Throughout the proof we set $v = u^p$, so that

$$L_1(r,v) = L_p(r,u)^p \quad \text{and} \quad L_\infty(r,v) = L_\infty(r,u)^p.$$

Furthermore, we set $Q = \frac{q}{p}$, so that $1 < Q \leq +\infty$ and

$$L_Q(r,v) = L_q(r,u)^p, \qquad 0 < p < q \leq +\infty.$$

Proof of (I). By (Ia),

(2.2) $$L_1(r,v) \leq C^p(1-r)^{-\alpha p},$$

so that (B), for $R = \frac{1+r}{2}$, reads that

(2.3) $$L_\infty(r,v) \leq 2^{\alpha p}C^pG(r)(1-r)^{-\alpha p-1} \leq 2^{\alpha p}C^p(1-r)^{-\alpha p-1},$$

because

$$G(r) = \frac{(1+r)^2}{1+3r} \leq 1.$$

We therefore have

$$L_\infty(r,u) \le 2^\alpha C\,(1-r)^{-\alpha-\frac{1}{p}},$$

the case $q=+\infty$ in (Ib). For $q<+\infty$, it follows from (A), together with (2.2) and (2.3), that

$$L_q(r,u) = L_Q(r,v)^{1/p} \le 2^{\alpha\left(1-\frac{p}{q}\right)} C(1-r)^{-\alpha-\frac{1}{p}+\frac{1}{q}}.$$

Proof of (II). It is a small modification of that one of (I).

Proof of (III). It suffices to prove that

$$L_Q(r,v) = o\left((1-r)^{-1+\frac{1}{Q}}\right), \qquad \text{as } r\to 1, \tag{2.4}$$

if $L_1(v) = L_p(u)^p < +\infty$. The case $Q=+\infty$ in (2.4) is a consequence of Corollary 3.1 of Theorem 2 in the next section. The case $Q<+\infty$ in (2.4) is a consequence of (A), with $L_\infty(r,v) = o\left((1-r)^{-1}\right)$, just explained.

For the sharpness we set

$$F_A(r) = \int_T |1-re^{it}|^{-A}\mathrm{d}m, \qquad 0\le r<1,\ A>0. \tag{2.5}$$

If $A>1$, then $(1-r)^{A-1}F_A(r)$ is bounded away from zero and from infinity as $r\to 1$. See [D, p. 65] for the “infinity” part. For the “zero” part, we observe that

$$|1-re^{i\theta}|^2 \le (1-r)^2+\theta^2, \qquad \text{for } -\frac{\pi}{2}<\theta<\frac{\pi}{2},$$

which is applied to the proof of

$$\liminf_{r\to 1}\,(1-r)^{A-1}F_A(r) > 0.$$

It is well known that if $0<A<1$, then $F_A(r)$ is bounded.

Examples showing the sharpness described in Section 1 are of the form $u(z) = |1-z|^{-\gamma}$, $\gamma>0$, the moduli of holomorphic functions considered in the classical case [D, pp. 86 and 91], except for (II), so that we only list up γ suited for each case and leave the details as exercises with the aid of the estimates of (2.5).

(I): *In case* $\alpha>0$, *set* $\gamma=\alpha+\frac{1}{p}$.
In case $\alpha=0$, *set* $\gamma=\frac{1}{p}-\eta$, *where*

$$0<\eta<\frac{1}{p}-\frac{1}{q} \qquad \textit{and} \qquad \eta<\varepsilon.$$

(II): *Set* $\gamma=\alpha+\frac{1}{p}-\eta$, *where* $0<\eta<\alpha$ *and* $\eta<\varepsilon$.

(III): *The same as the case* $\alpha=0$ *in* (I).

See [DT] and [T] also for the sharpness of (IIIb) for $u = |f|$.

The case $\alpha = 0$ in (I), with $C = L_p(u)$, reads:

If (IIIa) *holds, then*

$$L_q(r, u) \le (1-r)^{\frac{1}{q}-\frac{1}{p}} L_p(u), \qquad 0 \le r < 1. \tag{2.6}$$

We can actually show much more:

(IV) *If* (IIIa) *holds, then for* $0 < p < q \le \infty$,

(IVb) $$L_q(r, u) \le (1-r^2)^{\frac{1}{q}-\frac{1}{p}} L_p(u), \qquad 0 \le r < 1.$$

The estimate is better than (2.6). The case $q = +\infty$, for $u = |f|$, $f \in H^p$, in (IVb) is well known [D, p. 144]. The equality may hold, in this case, for each r, $0 \le r < 1$; we choose z_0, $|z_0| = r$, and we set

$$f(z) = \left\{ \frac{1-r^2}{(1-\bar{z}_0 z)^2} \right\}^{\frac{1}{p}}, \qquad z \in D,$$

as the function $(f(0) = 1)$. The case $q = 1$, for $u = |f|$, $f \in H^p$, where $0 < p < 1$, in (IVb), improves the familiar estimate [P, p. 58]:

$$M_1(r, f) \le \left(\frac{1+r}{1-r} \right)^{\frac{1}{p}-1} \|f\|_p, \qquad 0 \le r < 1.$$

For the proof of (IV) we set $v = u^p$. In case $q = +\infty$, we let $R \to 1$ in (B) to have

$$L_\infty(r, v) \le (1-r^2)^{-1} L_1(v), \tag{2.7}$$

or

$$L_\infty(r, u) \le (1-r^2)^{-\frac{1}{p}} L_p(u).$$

In case $q < +\infty$, we consider (A) for $Q = \frac{q}{p}$. Then

$$L_q(r, u)^p = L_Q(r, v) \le \left(\frac{L_1(v)}{1-r^2} \right)^{1-\frac{1}{Q}} L_1(v)^{\frac{1}{Q}},$$

whence (IVb).

See [Y4, Theorem 1, (1.1)] for a generalization of (IVb) for $q = +\infty$, $u = |f|$, $f \in H^p$, to the other direction.

It is known that if

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \in H^p, \qquad 0 < p \leq 1,$$

then $|a_0| \leq \|f\|_p$ and [besides $a_n = o(n^{\frac{1}{p}-1})$],

$$|a_n| \leq C_p\, n^{\frac{1}{p}-1} \|f\|_p, \qquad n \geq 1,$$

where $C_p > 0$ is a constant; see [D, Theorem 6.4, p. 98] and [P, p. 109]. With the aid of (IV), we can prove

$$(2.8) \qquad |a_n| \leq C_{p,n} \|f\|_p, \qquad n \geq 1,$$

where

$$C_{1,n} = 1$$

and for $0 < p < 1$,

$$C_{p,n} = \left\{2\left(\frac{1}{p} - 1\right)\right\}^{1-\frac{1}{p}} n^{-\frac{n}{2}} \left\{n + 2\left(\frac{1}{p} - 1\right)\right\}^{\frac{n}{2}+\frac{1}{p}-1}$$
$$< \left\{\frac{\left(\frac{2}{p} - 1\right) e}{2\left(\frac{1}{p} - 1\right)}\right\}^{\frac{1}{p}-1} n^{\frac{1}{p}-1}.$$

For the proof we first have $C_{1,n} = 1$ by letting $r \to 1$ in

$$(2.9) \qquad |a_n| \leq r^{-n} M_1(r, f), \qquad 0 < r < 1.$$

Next, for $0 < p < 1$, we have, by (IVb), $q = 1$, with (2.9), that

$$|a_n| \leq F(r)\ \|f\|_p, \qquad 0 < r < 1,$$

where the function

$$F(r) = r^{-n}(1 - r^2)^{1-1/p}$$

attains its minimum $C_{p,n}$ at

$$r = \left[\frac{n}{n + 2\left(\frac{1}{p} - 1\right)}\right]^{1/2},$$

so that we have (2.8).

3. - Poisson integral

Let Φ be the family of convex and strictly increasing (and therefore, continuous) functions $\phi \geq 0$ on $[0,+\infty)$ such that $\frac{\phi(x)}{x} \to +\infty$ as $x \to +\infty$; we always set $\phi(+\infty) = +\infty$.

THEOREM 2. *If h is the Poisson integral of a complex integrable function on T, then there exists $\phi \in \Phi$ such that*

$$L_\infty(r, \phi \circ |h|) = o((1-r)^{-1}), \qquad \text{as } r \to 1. \tag{3.1}$$

The composed function $\phi \circ |h| = \phi(|h|)$ is subharmonic in D.

COROLLARY 3.1. *If $u \in PL$ and $L_1(u) < +\infty$, then there exists $\phi \in \Phi$ such that*

$$L_\infty(r, \phi \circ u) = o((1-r)^{-1}), \qquad \text{as } r \to 1.$$

In particular, if $u \in PL$ with (IIIa), then

$$L_\infty(r, u) = o((1-r)^{-1/p}), \qquad \text{as } r \to 1;$$

this is the case $q = +\infty$ in (III), whose proof is promised in Section 2.

LEMMA 3.2. (De la Vallée Poussin). *For g a complex, integrable function on T, there exists $\phi \in \Phi$ such that $\phi \circ |g|$ is integrable on T.*

PROOF. This is a modification of the argument due to De la Vallée Poussin [Va, p. 452]. There exists a sequence $\{M_k\}$ such that $0 < 2M_k < M_{k+1}$ and

$$\int_{E_k} |g(t)| \mathrm{d}m < 4^{-k},$$

where $E_k = \{t \in T;\ |g(t)| \geq M_k\}$, $k \geq 1$. Setting $M_0 = 0$ and starting with $\phi(0) = 0$, we define ϕ in $[0,+\infty)$ inductively as follows:

$$\phi(x) = 2^k(x - M_k) + \phi(M_k), \qquad M_k \leq x \leq M_{k+1}, \qquad k \geq 0.$$

Then, we have $\phi \in \Phi$ because

$$2^{k-2} \leq 2^{k-1}\left(1 - \frac{M_{k-1}}{x}\right) + \frac{\phi(M_{k-1})}{x} \leq \frac{\phi(x)}{x}, \quad M_k \leq x \leq M_{k+1},$$

for $k \geq 2$. Furthermore,

$$\phi(x) \leq 2^k x, \qquad \text{for } M_k \leq x \leq M_{k+1}, \qquad k \geq 0,$$

because $\phi(M_k) \leq 2^k M_k$ by induction. We thus obtain

$$\int_T \phi(|g(t)|)\mathrm{d}m = \int_{T\setminus E_1} \phi(|g(t)|)\mathrm{d}m + \sum_{k\geq 1} \int_{E_k\setminus E_{k+1}} \phi(|g(t)|)\mathrm{d}m$$
$$\leq M_1 + \sum_{k\geq 1} 2^k \int_{E_k} |g(t)|\mathrm{d}m \leq M_1 + 1.$$

REMARK. The converse of Lemma 3.2 is obvious for measurable g.

PROOF OF THEOREM 2. Let $h(z) = \Pi(z, g)$. Then there exists $\phi \in \Phi$ such that $G = \phi \circ |g|$ is integrable on T. Since $|h(z)| \leq \Pi(z, |g|)$, then it follows from Jensen's inequality [J, p. 186] that $\phi \circ |h|(z) \leq \Pi(z, G)$. Now, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_{E_\delta} G(t)\mathrm{d}m < \varepsilon,$$

for $E_\delta = \{t \in T;\ G(t) > \delta\}$. Since

$$P(z, e^{it}) \leq \frac{1+r}{1-r},$$

for $|z| = r < 1$, and since

$$\int_{I\setminus E_\delta} P(z, e^{it})G(t)\mathrm{d}m \leq \delta\Pi(z, 1) = \delta,$$

it follows that

$$\phi \circ |h|(z) \leq \Pi(z, G) \leq \frac{1+r}{1-r} \int_{E_\delta} G(t)\mathrm{d}m + \delta$$
$$\leq \frac{2\varepsilon}{1-r} + \delta,$$

whence

$$(1-r)L_\infty(r, \phi \circ |h|) \leq 2\varepsilon + \delta(1-r). \tag{3.2}$$

Letting $r \to 1$, we observe that the left-hand side of (3.2) in the upper limit is less than 2ε. Since ε is arbitrary, this completes the proof.

PROOF OF COROLLARY 3.1. It suffices to show that there exists an integrable function $g_1 \geq 0$ on T such that

$$u(z) \leq h(z) \equiv \Pi(z, g_1) \qquad \text{in } D.$$

Actually, $v = \log u$ is subharmonic and $u = \psi(v)$, $\psi(x) = e^x$, $-\infty \le x < +\infty$. The Solomentsev-Gårding-Hörmander's theorem [S], [GH] (see [GH, Theorem] for the details), now yields the requested $g_1 \ge 0$,

$$g_1(t) = \lim_{r \to 1-0} u(re^{it}), \text{ for a.e. } t \in T.$$

As another application of Corollary 3.1, we show that if $f \in H^p$, $0 < p < +\infty$, then there exists $\phi \in \Phi$ such that

$$L_\infty(r, \phi(|f|^p)) = o((1-r)^{-1}), \text{ as } r \to 1.$$

Actually, $|f|^p \in PL$ and $L_1(|f|^p) < +\infty$. In particular, the case $q = +\infty$ for (III), with $u = |f|$, follows.

4. - Concluding remarks

(i). We can extend Theorem 2 to the Euclidean space $\mathbb{R}^n$, $n \ge 2$. See [HK] for subharmonic functions in $\mathbb{R}^n$. Let $|x-y|$ be the Euclidean distance, $|x| = |x - 0|$, and let

$$B = \{x \in \mathbb{R}^n;\ |x| < 1\}, \quad \partial B = \{y \in \mathbb{R}^n;\ |y| = 1\}.$$

Let $\mathrm{d}m(y)$ be the Lebesgue measure on ∂B divided by the total measure of ∂B, so that $m(\partial B) = 1$. Let

$$P(x,y) = \frac{1 - |x|^2}{|x-y|^n}, \qquad x \in B, \quad y \in \partial B,$$

so that

$$P(x,y) \le \frac{1+|x|}{(1-|x|)^{n-1}}.$$

If h is the Poisson integral of a complex, integrable function g on ∂B, namely,

$$h(x) = \int_{\partial B} P(x,y) g(y) \mathrm{d}m(y), \qquad x \in B,$$

then there exists $\phi \in \Phi$ such that

$$\lim_{|x| \to 1} (1 - |x|)^{n-1}\ \phi(|h(x)|) = 0.$$

There is no problem in extending Lemma 3.2 from T to ∂B.

(ii). The PL-version of another Hardy-Littlewood's theorem (see [D, Theorem 5.11, p. 87]), which has a relation to Theorem 1, is seen in [Y5, p. 243].

(iii). If f is holomorphic and bounded, $|f| < 1$, in D, then

$$\sigma(f) \equiv \sigma(f,0) = \tanh^{-1} |f| \in PL,$$

where $\sigma(z,w)$ is the non-Euclidean hyperbolic distance in D. This fact is found independently in [Y1, p. 263] and [Ve1]; see [Ve2] and further [Ve3, Lemma 1.1, p. 212].

As we know, the theory of PL functions works effectively in the study of hyperbolic Hardy classes: [Y2], [Y3], [Y5], [Y6]. Therefore, one might expect that Theorem 1 yields some significant information on the growth properties of $\sigma(f)$. However, unfortunately, this does not appear to be the case for Theorem 1. The Schwarz-Pick Lemma yields

$$L_\infty(r,\sigma(f)) \le \tanh^{-1} r + \sigma(f(0),0),$$

whence

$$L_p(r,\sigma(f)) = O(-\log(1-r)),$$

for each $p,\ 0 < p \le +\infty$. Therefore, in particular,

$$L_p(r,\sigma(f)) = o((1-r)^{-\alpha}),$$

for all $p,\ 0 < p \le +\infty$, and $\alpha > 0$.

(iv). Suppose that $f \in H^p$ and the boundary value function $f(e^{it}) \in \Lambda^p_\alpha$, where $1 \le p < +\infty,\ 0 < \alpha \le 1$; see [D, p. 72] for Λ_α and Λ^p_α. We remember the proof [HL1, p. 627] of the following:

if $p > \frac{1}{\alpha}$, *then* f *is continuous on* $\overline{D}$ *and* $f(e^{it}) \in \Lambda_{\alpha-\frac{1}{p}}$,
while
if $p \le \frac{1}{\alpha}$, *then* $f \in H^q$ *and* $f(e^{it}) \in \Lambda^q_{\alpha+\frac{1}{q}-\frac{1}{p}}$ *for each* q *satisfying*

$$p < q < \frac{p}{1-\alpha p}. \tag{4.1}$$

By [D, Theorem 5.4, p. 78], which we shall call Theorem A, we have

$$M_p(r,f') = O((1-r)^{\alpha-1}).$$

Then, (I) of Theorem 1 yields that

$$M_q(r,f') = O((1-r)^{\alpha-1+\frac{1}{q}-\frac{1}{p}}).$$

The first half follows from [D, Theorem 5.1, p. 74], with $q = +\infty$, which we shall call Theorem B, while the second one follows from Theorem A.

If f is holomorphic and bounded, $|f| < 1$, in D, then $f^* = \dfrac{|f'|}{1-|f|^2} \in PL$ and

$$L_\infty(r, f^*) \leq \frac{1}{1-r^2}, \qquad 0 \leq r < 1.$$

In contrast with (iii), we may consider an application of Theorem 1 to $u = f^*$. Since there are hyperbolic analogues of Theorems A and B (see [Y7, Theorems 2 and 1]), we now have the following; see [Y7] for the definition of H^p_σ and $\sigma\Lambda^p_\alpha$.

Suppose that $f \in H^p_\sigma$ *and* $f(e^{it}) \in \sigma\Lambda^p_\alpha$, *where* $1 \leq p < +\infty$ *and* $0 < \alpha \leq 1$. *If* $p > \frac{1}{\alpha}$, *then* f *is continuous on* $\overline{D}$ *and* $f(e^{it}) \in \sigma\Lambda_{\alpha - \frac{1}{p}}$, *while if* $p \leq \frac{1}{\alpha}$, *then* $f \in H^q_\sigma$ *and* $f(e^{it}) \in \sigma\Lambda^q_{\alpha + \frac{1}{q} - \frac{1}{p}}$, *for each* q *satisfying* (4.1).

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