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Subharmonic Solutions for Hamiltonian Systems via a $\mathbb{Z}_p$ Pseudoindex Theory

GABRIELLA TARANTELLO

0. - Introduction

Let the Hamiltonian $\hat{H} \in C^1$ be of the form:

$$(0.1) \qquad \hat{H}(z,t) = \frac{1}{2}Qz \cdot z + H(z,t)$$

where $z = (z_1, \cdots, z_{2N}) \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$, Q is a $2N \times 2N$ symmetric matrix and H is T-periodic in the t-variable ($\cdot$ denotes the usual scalar product in $\mathbb{R}^{2N}$).

We call *subharmonics* the periodic solutions of the Hamiltonian system:

$$(0.2) \qquad J\dot{z}(t) = Qz(t) + H_z(z(t),t) \quad J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

with period an integral multiple of T.

One of the first results on subharmonic solutions for Hamiltonian systems was obtained by Birkhoff-Lewis [5]. They show that if zero is an equilibrium and suitable assumptions are satisfied, then near zero there exists a sequence of subharmonics with arbitrarily large minimal period. See also [14]. Subsequently, by means of variational methods the problem has been studied from a global point of view. Hence somewhat global versions of the Birkhoff-Lewis result were obtained in various situations. See [6], [7], [13] and [16].

Here we ask the more precise question whether for any given integer $p > 1$ there exist subharmonics with minimal period pT, and how many of them it is reasonable to expect as $p \to +\infty$.

A partial answer in this direction was given in [13] where the matrix $Q \equiv 0$ and H is convex (in the z-variable) with subquadratic growth both at the origin and at infinity.

Using the $\mathbb{Z}_p$-symmetry of the problem (with respect to jT-phase shift, $j = 0, 1, \cdots, p-1$), it is shown in [13] that for any prime integer p sufficiently large, there exists an integer $k_p \geq 1$, such that the Hamiltonian system (0.2) admits at least $k_p N$ distinct subharmonics with minimal period pT, and $k_p \to +\infty$ as $p \to +\infty$.

This result is here extended for Hamiltonians as in (0.1), where H might have *superquadratic* or *subquadratic* growth at infinity and in the first case need not be convex.

Similar results are obtained concerning subharmonics for second order systems of O.D.E.

Again, the idea is to exploit the $\mathbb{Z}_p$ group symmetry of the problem. We introduce an appropriate $\mathbb{Z}_p$ pseudoindex theory, partly inspired by the one described in [4] for the group S^1, and more generally by [2].

Needless to say, it is because of the "huge" $\mathbb{Z}_p$-fixed point set that the theories in [2] and [4] fail to apply directly. In order to handle this, one needs a more appropriate use for the $\mathbb{Z}_p$-Borsuk-Ulam theorem (cf. [12]). It would be interesting to see the pseudoindex theory introduced here applied to other situations.

While this work was being completed, the author learned from Benci that he and Fortunato [19] have recently obtained a Birkhoff-Lewis type result for a rather general class of Hamiltonian systems.

Acknowledgment

1. - Statement of the results

We seek solutions of the following problem:

$$(1)_p \quad \begin{cases} J\dot{z}(t) = Qz(t) + H_z(z(t), t) \\ z(0) = z(pT) \end{cases}$$

where $p > 1$ is integer.

As already discussed in [5] and subsequently in [16] (in the variational framework), there is a natural hypothesis that the matrix Q needs to satisfy so that $(1)_p$ admits solutions.

Namely, we shall require that the eigenvalue problem:

$$(Q_1) \qquad\qquad Q\xi = \lambda J\xi \qquad\qquad \xi \in \mathbb{C}^{2N}$$

admits $2N$ purely imaginary eigenvalues including possibly zero.

Notice that if (λ, ξ) is an eigenpair for (Q_1) so it is $(\overline{\lambda}, \overline{\xi})$. Thus if we denote by (λ_j, ξ_j) $j = 1, -, 2N$ the eigenpairs of (Q_1), there exist real numbers

$$0 \le \omega_1 \le \cdots \le \omega_N$$

such that

$$
\begin{aligned}
\lambda_j &= i\omega_j & j = 1, \cdots, N \\
\lambda_{j+N} &= -i\omega_j & j = 1, \cdots, N
\end{aligned}
$$

and

$$\xi_{j+N} = \overline{\xi}_j \qquad j = 1, \cdots, N.$$

Moreover we may assume that the eigenvectors $\xi_1, \cdots, \xi_{2N}$ are normalized as follows:

(1.2) $$(iJ\xi_j, \xi_k) = \epsilon_j \delta_{j,k} \quad j, k = 1, \cdots, 2N$$

with

$$
\epsilon_j = \begin{cases} 1 & j = 1, \cdots, N \\ -1 & j = N+1, \cdots, 2N \end{cases}
$$

and

(1.3) $$(\xi_j, \xi_k) = \mu_j \delta_{j,k} \qquad j, k = 1, \cdots, 2N$$

for some $\mu_j > 0$ with $\mu_j = \mu_{j+N}$ $j = 1, \cdots, N$. Here $(\cdot, \cdot)$ denotes the usual hermitian product on $\mathbb{C}^{2N}$.

We shall start by discussing solutions of $(1)_p$ is case H has superquadratic growth at infinity. More precisely, let H satisfy the following:

(H_1) $\qquad H(z, t) = H(z, t + T) \quad \forall z \in \mathbb{R}^{2N}, \ \forall t \in \mathbb{R}; \ H \in C^1(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R})$

(H_2) $\qquad$ there exists $\alpha > 2$ such that:

$\qquad$ a) $0 \le H(z, t) \le \frac{1}{\alpha} H_z(z, t) \cdot z \qquad \forall z \in \mathbb{R}^{2N}, \ \forall t,$

$\qquad$ b) there exists $R > 0$ such that $H(z, t) > 0$ for every $|z| > R$.

(H_3) $\qquad$ there exist $\gamma \in (\frac{1}{2}, 1)$ and $C_1 > 0$ such that

$$|H_z(z, t)| \le C_1 (H(z, t))^\gamma \qquad \forall z \in \mathbb{R}^{2N} \forall t;$$

(H_4) $\qquad$ there exists $C_2 > 0$ such that

$$|H_t(z, t)| \le C_2 H(z, t).$$

(H_0) $\qquad$ If $z(t)$ is periodic with minimal period qT, q rational, and $H_z(z(t), t)$ is periodic with minimal period qT, then necessarily $q \in \mathbb{N}$.

REMARKS. (1) The hypothesis (H_2) guarantees superquadratic growth of H at ∞. In fact one can show that for any $\tau > 0$ there exists $a(\tau) > 0$ such that

$$(1.4) \quad H(z,t) \geq \frac{a(\tau)}{\alpha} |z|^\alpha - \left[\frac{1}{2} - \frac{1}{\alpha}\right] \tau \qquad \forall z \in \mathbb{R}^{2N}, t \in \mathbb{R}$$

and $a(\tau) \to 0$ as $\tau \to 0$, provided $\alpha < \dfrac{1}{1-\gamma}$ (see (1.5) below).

We shall specify later the value of τ suitable for our purposes.

(2) The hypothesis (H_3) together with (H_2) give an upper bound on the growth of H, that is

$$(1.5) \qquad H(z,t) \leq \left[\frac{C_1}{\alpha}\right]^{\frac{1}{1-\gamma}} |z|^{\frac{1}{1-\gamma}} \qquad \forall z \in \mathbb{R}^{2N}, \ \forall t;$$

and $\dfrac{1}{1-\gamma} > 2$.

Notice that comparing (1.4) and (1.5), one necessarily has $2 < \alpha \leq \dfrac{1}{1-\gamma}$.

(3) The hypothesis (H_4) shall be needed to estimate (from below) the L^1-norm of $H(z(t),t)$ in terms of its L^∞-norm whenever $z = z(t)$ is a solution of $(1)_p$.

Notice that this is always the case when $H(z,t) = a(t)H(z)$. At first the author was only able to treat this situation.

L. Niremberg suggested the more general hypothesis (H_4).

(4) The hypothesis (H_0) already introduced in [13] is a generic one, and emphasizes the essential time dependence of H.

In order to state the results we need to introduce some notation. Set:

$$(1.6) \qquad \mu = \min_{j=1,\cdots,N} \mu_j, \quad \overline{\mu} = \sum_{j=1}^{N} \mu_j \qquad (\mu_j \text{ as in (1.3)})$$

$$(1.7) \qquad \sigma_T = \min_{\substack{m \neq 0 \\ j=1,\cdots,N}} \left|1 - \frac{T\omega_j}{2\pi \, m}\right|, \qquad 0 \leq \sigma_T \leq 1$$

$$(1.8) \qquad \rho_T = \min_{\substack{\frac{2\pi}{T} m - \omega_j \neq 0 \\ m \in \mathbb{Z}, j=1,\cdots,N}} \left|\frac{2\pi m}{T} - \omega_j\right| \qquad 0 < \rho_T \leq \frac{2\pi}{T}$$

Further assume that:

$$(Q_2) \qquad \frac{T\omega_j}{2\pi} \notin \mathbb{N} \backslash \{0\} \qquad \forall j = 1, \cdots, N.$$

This readily implies that

$$\sigma_T > 0.$$

Let us now fix the constant τ_0 so that:

$$0 < \tau_0 < \left[\frac{\alpha \rho_T}{C_1^2} \right]^{\frac{1}{2\gamma - 1}} \frac{\alpha \sigma_T}{\sigma_T(1 + TC_2) + T\alpha \|Q\|}$$

($\|Q\|$ denotes the norm of the matrix Q in the usual sense); and set

$$a_0 = a(\tau_0) \qquad (a(\tau) \text{ defined by (1.4)}).$$

Define

$$(1.9) \qquad \tilde{c}_T = \left[\left[\frac{\alpha \rho_T}{C_1^2} \right]^{\frac{1}{2\gamma - 1}} \frac{\alpha \sigma_T}{\sigma_T(1 + TC_2) + T\alpha \|Q\|} - \tau_0 \right]^{\frac{\alpha - 2}{\alpha}} \mu a_0^{2/\alpha} \frac{T}{2\pi}$$

and set

$$(1.10) \qquad c_T = \min_{j=1,\cdots,N} \{\tilde{c}_T, 1 - r_j\}, \quad r_T = \min_{j=1,\cdots,N} \{1 - r_j\}$$

where $r_j = \frac{T}{2\pi} \omega_j - \left[\frac{T}{2\pi} \omega_j \right]$ ($[\beta]$ = biggest integer $\leq \beta$).

Notice that (Q_2) implies $0 < r_j < 1$ whenever $\omega_j \neq 0$, while $r_j = 0$ if $\omega_j = 0$.

In this case we prove:

THEOREM 1. *Let Q satisfy (Q_1), (Q_2) and H satisfy $(H_0) - (H_4)$. For any prime integer $p > 1$ satisfying*

$$\frac{k}{p} < c_T \text{ for some integer } k \geq 1$$

there exist at least kN distinct solutions of $(1)_p$ with minimal period pT.   □

From Theorem 1 immediately follows:

COROLLARY 1. *Let $Q \equiv 0$ and H satisfy $(H_0) - (H_4)$. For any prime integer $p > 1$ satisfying:*

$$\frac{k}{p} < \left[\left[\frac{2\pi}{T} \frac{\alpha}{C_1^2} \right]^{\frac{1}{2\gamma - 1}} \frac{\alpha}{1 + TC_2} - \tau_0 \right]^{\frac{\alpha - 2}{\alpha}} \frac{T}{2\pi} a_0^{2/\alpha}$$

for some integer $k \geq 1$, there exist at least kN distinct solutions of $(1)_p$ with minimal period pT.   □

One can even deduce a slight generalization of the result in [10] concerning autonomous Hamiltonian systems.

COROLLARY 2. *Let* $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ *satisfy:*

(i) $0 \le H(z) \le \frac{1}{\alpha} H_z(z) \cdot z$ *for some* $\alpha > 2$, $\exists R > 0$ *such that* $H(z) > 0 \; \forall z$ *with* $|z| > R$;

(ii) $\exists \gamma \in (1/2, 1)$ *and* $C_1 > 0$ *such that*

$$|H_z(z)| \le C_1 (H(z))^\gamma \qquad \forall z \in \mathbb{R}^{2N}.$$

Let $T > 0$, *satisfy*

$$\frac{1}{2} < \left[\left[\frac{2\pi}{T} \frac{\alpha}{C_1^2} \right]^{\frac{1}{2\gamma - 1}} \alpha - \tau_0 \right]^{\frac{\alpha - 2}{\alpha}} \frac{T}{2\pi} a_0^{2/\alpha}$$

for some τ_0 *and* $a_0 > 0$ *satisfying*

$$0 < \tau_0 < \alpha \left[\frac{2\pi}{T} \frac{\alpha}{C_1^2} \right]^{\frac{1}{2\gamma - 1}}$$

and

$$H(z) \ge \frac{a_0}{\alpha} |z|^\alpha - \left[\frac{1}{2} - \frac{1}{\alpha} \right] \tau_0.$$

Then the Hamiltonian system $J\dot{z}(t) = H_z(z(t))$ *admits at least* N *distinct periodic solutions with minimal period* T.   □

More on the minimal period problem for autonomous Hamiltonian systems can be found in [8] and [11].

Now consider the case where the Hamiltonian $\hat{H}$ is given by $\hat{H}(z,t) = \frac{1}{2} Qz \cdot z + H(z,t)$ and H has subquadratic growth at infinity.

More precisely assume:

$(H_1)^*$ $H(\cdot, t)$ is convex, $H(z, \cdot)$ is T-periodic, $H_z \in C(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R}^{2N})$, and $H(z,t) \ge 0 \; \forall z \in \mathbb{R}^{2N} \; \forall t \in \mathbb{R}$, $H(0,t) = 0 \; \forall t \in \mathbb{R}$.

$(H_2)^*$ There exist positive constants a_j, b_j, $j = 1, 2$ and $1 < \alpha \le \gamma < 2$ such that:

$$\frac{a_1}{\alpha} |z|^\alpha - b_1 \le H(z,t) \le \frac{a_2}{\gamma} |z|^\gamma + b_2$$

$$\forall z \in \mathbb{R}^{2N} \text{ and } \forall t \in \mathbb{R}.$$

Set

$$(1.11) \quad d_T = \frac{\pi\overline{\mu}}{T} \left[\frac{1}{a_1}\right]^{2/\alpha} \left\{\frac{2\alpha}{2-\alpha} \left[\frac{2-\gamma}{2\gamma} a_2^{\frac{2}{2-\gamma}} \left[\frac{T}{2\pi r_T \mu}\right]^{\frac{\gamma}{2-\gamma}} + b_1 + b_2\right]\right\}^{\frac{2-\alpha}{\alpha}}$$

In this case we prove the following generalization of Theorem 1 in [13].

THEOREM 2. *Let Q satisfy (Q_1) and H satisfy $(H_1)^*$, $(H_2)^*$, and (H_0). For any prime integer $p > 1$ satisfying:*

$$(1.12) \qquad\qquad \frac{p}{k(k+1)} > d_T \;\; and \;\; \frac{k}{p} < r_T$$

for some integer $k \geq 1$, there exist at least kN distinct solutions of $(1)_p$ with minimal period pT. ☐

Next we shall be concerned with subharmonic solutions for nonautonomous *second order systems* of O.D.E.

Namely let the potential $\hat{V}$ be of the form:

$$\hat{V}(x,t) = \frac{1}{2} Qx \cdot x + V(x,t)$$

with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, Q $n \times n$ symmetric matrix and

$$(V_1) \qquad \left[\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right] = V_x \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n), V(x,t) = V(x, t+T)$$

for every $x \in \mathbb{R}^n$, $\forall t \in \mathbb{R}$.

We seek solutions for the problem:

$$(2)_p \qquad \begin{cases} \ddot{x} + Qx + V_x(x,t) = 0 \\ x \;\; pT\text{-periodic} \qquad\qquad p > 1 \text{ integer.} \end{cases}$$

Since the quadratic term $\frac{1}{2} Qx \cdot x$ is most naturally thought of as the kinetic energy for the given mechanical system, we shall assume:

(Q_3) $\qquad Q$ is a $n \times n$ positive semidefinite symmetric matrix.

Denote by $0 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2$ the eigenvalues of Q.

We start by discussing the case where V has *superquadratic* growth at infinity. As in the Hamiltonian case assume:

(V_2) $\exists \alpha > 2$ such that $0 \leq V(x,t) \leq \frac{1}{\alpha} V_x(x,t) \cdot x \; \forall x, \; \forall t; \; \exists R > 0: \; V(x,t) > 0$
$\forall x$ such that $|x| > R$.

(V_3) $\exists \gamma \in \left[\frac{1}{2}, 1\right]$ and $C_1 > 0$ such that

$$|V_x(x,t)| \leq C_1 (V(x,t))^\gamma \; \forall x, \; \forall t$$

(V_0) The same as (H_0) with H_z replaced by V_x.

Set

(1.13) $\beta_T = \sum_{j=1}^n \sum_{|m| > \frac{T\omega_j}{2\pi}} \frac{T}{(2\pi m)^2 - (T\omega_j)^2}$

and fix $\tau_0 \in \mathbb{R}$ such that:

$$0 < \tau_0 < \left[\frac{\alpha^{2\gamma}}{\beta_T C_1^2 T}\right]^{\frac{1}{2\gamma - 1}}.$$

Hence by (V_2) there exists $a_0 \in \mathbb{R}^+$ such that

(1.14) $V(x,t) \geq \frac{a_0}{\alpha} |x|^\alpha - \left[\frac{1}{2} - \frac{1}{\alpha}\right] \tau_0.$

Finally set

(1.15) $s_T = \max_{j=1,\cdots n} \left\{1 - \tilde{r}_j + \frac{T\omega_j}{\pi}\right\} > 0$ with $\tilde{r}_j = \frac{T\omega_j}{2\pi} - \left[\frac{T\omega_j}{2\pi}\right]$

$\tilde{r}_T = \min_{j=1,\cdots,n} \{1 - \tilde{r}_j\};$

(1.16) $e_T = \min \left\{ \frac{a_0^{2/\alpha}}{s_T} \left[\frac{T}{2\pi}\right]^2 \left[\left[\frac{\alpha^{2\gamma}}{\beta_T C_1^2 T}\right]^{\frac{1}{2\gamma-1}} - \tau_0\right]^{\frac{\alpha-2}{\alpha}}, \; \tilde{r}_T \right\}.$

In this case we prove:

THEOREM 3. *Let Q satisfy (Q_3) and V satisfy $(V_0) - (V_3)$. For any prime integer $p > 1$ such that*

$$\frac{k}{p} < e_T \; \textit{for some} \; k \in \mathbb{N}^+,$$

there exist at least kn distinct solutions of $(2)_p$ with minimal period pT. $\square$

REMARK. As above, from Theorem 3, one can derive corollaries concerning the case $Q \equiv 0$ (i.e. $\beta_T = \frac{nT}{12}$, $s_T = 1$, $r_j = 0$) and the autonomous case.

To our knowledge, in the superquadratic case the only result concerning subharmonics (with prescribed minimal period) for second order systems of

O.D.E. is given in [3]. However, the point of view in [3] is rather different from ours; there a solution for $(2)_p$ with minimal period pT is obtained only provided $\frac{T\omega_j}{2\pi}$ is sufficiently small.

We conclude this paragraph by describing in the subquadratic case the analogue of Theorem 2.

To this end set:

$$(1.17) \qquad \lambda_T = \min \left\{ m^2 - \left[\frac{T\omega_j}{2\pi} \right]^2, \ |m| > \frac{T\omega_j}{2\pi} \ j = 1, \cdots, m \right\} \quad \text{and}$$

$$(1.18) \qquad \qquad \eta_T = \sum_{j=1}^{n} \left[\frac{T\omega_j}{2\pi} + \frac{1}{2} \left(1 - \tilde{r}_j \right) \right].$$

Moreover denote by $(V_1)^*$ and $(V_2)^*$ the corresponding assumptions $(H_1)^*$ and $(H_2)^*$ for V and define:

(1.19)

$$\tilde{d}_T = \eta_T \left[\frac{2\pi}{T} \right]^2 \left[\frac{1}{a_1} \right]^{\frac{2}{\alpha}} \left\{ \frac{2\alpha}{2-\alpha} \left[a^{\frac{2}{2-\gamma}} \frac{2\gamma}{2-\gamma} \left[\frac{1}{\lambda_T} \left[\frac{T}{2\pi} \right]^2 \right]^{\frac{\gamma}{2-\gamma}} + b_1 + b_2 \right] \right\}^{\frac{2-\alpha}{2}}.$$

We have:

THEOREM 4. *Let Q satisfy (Q_3) and V satisfy (V_0), $(V_1)^*$ and $(V_2)^*$. For any prime integer $p > 1$ such that*

$$\frac{k}{p} < \tilde{r}_T \quad \text{and} \quad \frac{p}{k(k+1)} > \tilde{d}_T \text{ for some } k \in \mathbb{N}^+;$$

problem $(2)_p$ admits at least kn distinct solutions with minimal period pT. □

In particular Theorem 4 generalizes Theorem 2 in [13].

We have already noticed how problems $(1)_p$ and $(2)_p$ are $\mathbb{Z}_p$ symmetric. Namely, for any solution $z(t)$ of $(1)_p$ (or $(2)_p$), we have that $z(t + jT)$, $j = 1, \cdots, p - 1$ is a solution as well.

We specify however that the solutions claimed in Theorems 1-2-3-4 are "$\mathbb{Z}_p$-distinct" in the sense that one can not be obtained from the other by a jT-phase shift.

Although highly non-generic, it might still happen that, in any case, such solutions describe the same geometric orbit.

But one can easily check that this is certainly not the case if, for example, H is of the form:

$$H(z, t) = H_1(z) + H_2(z'', t) \text{ with } z = (z', z'') \in \mathbb{R}^{2N}$$

$$z', z'' \in \mathbb{R}^N,$$

and satisfies:

$$\frac{\partial H_1}{\partial z'}(z', z'') \cdot z' > 0 \quad \forall z' \neq 0 \quad \forall z'';$$

$(H_0)^+$ If $z \neq$ const. is pT periodic and $H_z(z(t), t + c) = H_z(z(t), t)$ for some
$c \in \mathbb{R}$ then necessarily $c = jT$, $j \in \mathbb{Z}$.

In particular this guarantees that if we replace the assumption (H_0) with $(H_0)^+$ in theorem 3, and 4, then the solutions there claimed are indeed geometrically distinct.

As is well known, solutions of $(1)_p$ and $(2)_p$ are the critical points of some suitable functional in a suitable space. Naturally such a functional inherits the $\mathbb{Z}_p$ symmetry. This motivates the next section where we prove existence of multiple critical points for functionals invariant under a $\mathbb{Z}_p$ group action.

2. - A critical point theory for $\mathbb{Z}_p$ invariant functionals

The Ljusternik-Schnirelman theory for even functionals (cf. [17]) has started a series of theories intended to obtain multiple critical points for symmetric functionals. We only mention the case of S^1-symmetric functionals (under time shift) commonly arising in autonomous oscillation problems.

However, while the L.S.-theory applies naturally to bounded functionals (from above or below), one usually deals with unbounded ones. Hence in recent years a great effort has been devoted to combine symmetry and unboundedness. In this direction one has the relative index introduced in [9] and [4], and the pseudoindex of [2]. But such theories all require strong assumptions on the fixed point set F for the given group action. So for example in [4] one needs F finite dimensional, while in [2] F has to be contained in a "good" subspace.

Neither of these results can be applied to our situation where the fixed point set is "big", consisting namely of all the T-periodic functions.

It is our task here to define appropriate index theories under no restrictions on the size of F, which are suitable for our purposes. Although our discussion will be restricted to the group $\mathbb{Z}_p$ (needed for the applications), we emphasize that by minor changes one can treat more general groups with a Borsuk-Ulam type theorem available. To start let us recall the definition for the $\mathbb{Z}_p$ index map (cf. [13]).

Let E be a complex Hilbert space and

$$T : E \to E$$

be a norm preserving operator that generates a $\mathbb{Z}_p$ group action on E; i.e. there exists an orthonormal basis $\{u_k, \ k \in \mathbb{N}\}$ of E and integers m_k such that $T u_k = e^{i \frac{2\pi}{p} m_k} u_k$.

DEFINITION. (i) The subset $A \subset E$ is called *T-invariant* if $TA = A$,

(ii) the continuous map $h : E \to E$ is called *T-equivariant* if $h(Tu) = Th(u)$.

Set

$$\Sigma = \{A \subset E : A \text{ is invariant and closed}\}$$

and define the $\mathbb{Z}_p$ index map:

$$i_p : \Sigma \to \mathbb{N} \cup \{\infty\}$$

as follows: for $A \in \Sigma$, $i_p(A) = k$ if k is the smallest nonnegative integer such that there exists a continuous map

$$h : A \to \mathbb{C}^k \backslash \{0\}, \quad h = (h_1, \cdots, h_k)$$

and integer $m_j \neq 0$ relatively prime to p, $1 \leq j \leq k$, such that

$$h_j(Tu) = e^{i \frac{2\pi}{p} m_j} h_j(u) \qquad \forall u \in A.$$

Set

$$i_p(A) = +\infty \text{ if no such map exists}$$

and

$$i_p(\phi) = 0.$$

The index map i_p satisfies the following properties:

(2.1) if $\eta : A \to B$ is a continuous equivariant map then:

$$i_p(A) \leq i_p(B);$$

(2.2) $$i_p(A \cup B) \leq i_p(A) + i_p(B);$$

(2.3) if $G \in \Sigma$ is compact and $\delta > 0$ is small enough then

$$N_\delta(G) = \{u \in E : \text{dist}(u, G) \leq \delta\} \in \Sigma$$

and

$$i_p(N_\delta(G)) = i_p(G)$$

(cf. [12] and [13]).

Furthermore let $f \in C^1(E, \mathbb{R})$ be a *T-invariant* functional, i.e.

$$f(Tu) = f(u) \qquad \forall u \in E$$

and assume that f satisfies the Palais-Smale (P.S.) condition. Recall that f is said to satisfy the (P.S.) condition in the interval $[a, b] \subset \mathbb{R}$, if any sequence satisfying:

$$f(u_n) \overset{n \to +\infty}{\longrightarrow} c \text{ with } c \in [a, b]$$

and

$$\|f'(u_n)\|_{E^*} \overset{n \to +\infty}{\longrightarrow} 0,$$

admits a convergent subsequence.

Set

$$A_c = \{u \in E : f(u) \leq c\},$$

and

$$K_c = \{u \in E : f(u) = c \text{ and } f'(u) = 0\}, \quad c \in \mathbb{R}.$$

We have $A_c, K_c \in \Sigma$ and K_c is compact.

The link between index theory and critical values of invariant functionals is contained in the following:

PROPOSITION 2.1. *Let* $f \in C^1(E, \mathbb{R})$ *be a T-invariant functional and let* $a < b \in \mathbb{R}$. *If* f *satisfies the* (P.S.) *condition in* $[a, b]$ *and* $\forall c \in [a, b]$ *we have* $K_c = \phi$ *then*

$$i_p(A_a) = i_p(A_b).$$

$\square$

PROOF. Set $c_0 = \max\{c \in [a, b] : i_p(A_c) = i_p(A_a)\}$. If $c_0 < b$ then by the deformation lemma (cf. [17] and [13] for the $\mathbb{Z}_p$-version of it) we have $K_{c_0} \neq \phi$. $\square$

Furthermore, Proposition 2.1 even indicates how to get critical values for bounded functionals. Indeed, if for example f is bounded from below, then

(2.4) $c_k = \inf \{c : i_p(A_c) \geq k\}$ $k = 1.2 \cdots$

is a critical value for f. If f is *not* bounded from below, (2.4) might only give $c_k = -\infty$.

The idea, in such situation, is to define a more suitable (relative, pseudo) index map (depending on the functional), such that when replaced in (2.4) gives finite c_k.

From now on, we shall take E to be a complex Hilbert space with hermitian product denoted by $<,>$.

In addition we take the functional f of the form:

$$(*) \quad \begin{cases} \quad f(u) = \dfrac{1}{2} <Lu, u> - \Psi(u) \qquad u \in E \\ \text{where:} \\ \qquad L \text{ is a bounded, selfadjoint, } T\text{-equivariant operator,} \\ \qquad \text{and } \Psi' : E \to E \text{ is compact.} \end{cases}$$

Denote by F the *fixed point set* for the given $\mathbb{Z}_p$ action on E, i.e.

$$F = \{u \in E : Tu = u\};$$

so F is a closed subspace of E.

We start by defining a $\mathbb{Z}_p$-index relative to a given subspace of E. Inspired by [4] we proceed as follows.

Given E^- a closed T-invariant subspace of E with $LE^- = E^-$, set $F^- = E^- \cap F$.

DEFINITION. We call the *relative* $\mathbb{Z}_p$-index with respect to E^- the map:

$$\tau_p : \Sigma \to \mathbb{N} \cup \{\infty\} \qquad \tau_p(\cdot) = \tau_p(\cdot, E^-)$$

defined as follows:

$\tau_p(A) = k$, if k is the smallest nonnegative integer such that there exists a continuous map:

$$h : A \to E^- \times \mathbb{C}^k \qquad h = (h_1, h_2)$$

satisfying:

(2.5.1) $$(0,0) \notin h(A);$$

(2.5.2) $$h_1(u) = p_{E^-} e^{\alpha(u)L} u + K(u)$$

where

$p_{E^-} : E \to E^-$ is the usual orthogonal projection:

$\alpha : E \to \mathbb{R}$ is a continuous bounded, T-invariant functional (i.e. $\exists M > 0 :$ $|\alpha(u)| \le M \ \forall u \in E$ and $\alpha(Tu) = \alpha(u) \ \forall u \in E$);

$K : E \to E^-$ is compact, T-equivariant map.

(2.5.3) (Equivariancy) (a) $h_1(Tu) = Th_1(u)$

(b) There exist integers $m_j \ne 0 \ j = 1, \cdots, k$ relatively prime to p such that:

$$h_2(Tu) = (h_{2,1}(Tu), \cdots, h_{2,k}(Tu))$$
$$= \left[e^{\frac{i2\pi}{p} m_1} h_{2,1}(u), \cdots, e^{\frac{i2\pi}{p} m_k} h_{2,k}(u) \right].$$

(2.5.4) For every $u \in A \cap F^-$, $\alpha(u) = 0$, $K(u) = 0$ (so $h_1(u) = u$ and by (2.5.3)-(b) necessarily $h_2(u) = 0$).

REMARK. Intuitively $\tau_p(A)$ measures how "big" is the component of A in $E^+ = (E^-)^\perp$. Moreover $\tau_p = i_p$ if $E^- = \{0\}$.

Next, we define the class of maps $y : E \to E$ under which the relative index map is invariant (i.e. $\tau_p \circ y = \tau_p$).

DEFINITION. A continuous map $y : E \to E$ is called *admissible* if

(2.6.1) y is a diffeomorphism;

(2.6.2) $y(u) = e^{\alpha(u)L}u + K(u)$ where:

$\quad \alpha : E \to \mathbb{R}$ is a continuous T-invariant bounded functional;

$\quad K : E \to E$ is a continuous T-equivariant compact map;

(2.6.3) both y and y^{-1} map bounded sets in bounded sets and

(2.6.4) for every $u \in F^-$, $\alpha(u) = 0$, $K(u) = 0$.

REMARK. Since L is T-equivariant, by (2.6.2) it follows that y is T-equivariant.

Set

$$\mathcal{F} = \{y : E \to E : y \text{ is admissible}\}.$$

Hence $Id \in \mathcal{F}$ and $y \in \mathcal{F}$ if and only if $y^{-1} \in \mathcal{F}$. The main properties of τ_p are summarized in the following:

PROPOSITION 2.2. (a) *(Monotonicity)*: $\forall y \in \mathcal{F}$, $\forall A \in \Sigma$ *we have*:

$$\tau_p(y(A)) = \tau_p(A).$$

(b) *(Subadditivity)*: $\forall A, B \in \Sigma$

$$\tau_p(A \cup B) \leq \tau_p(A) + i_p(B).$$

(c) *If* $\tau_p(A) \geq 1$ *then* $A \cap (E^-)^\perp \neq \phi$.

PROOF. (a) We just need to show $\tau_p(A) \leq \tau_p(y(A))$. Assume $\tau_p(y(A)) = k < +\infty$ (if $k = +\infty$ the inequality is trivial). Hence there exists a continuous map

$$h : y(A) \to E^- \times \mathbb{C}^k$$

satisfying (2.5.1)-(2.5.4).

Define

$$\hat{h} : A \to E^- \times \mathbb{C}^k \qquad \hat{h} = (\hat{h}_1, \hat{h}_2)$$

by

$$\hat{h} = h \circ y.$$

We claim that $\hat{h}$ satisfies (2.5.1)-(2.5.4). Indeed one easily checks (2.5.1) while (2.5.3) follows from the fact that y is T-equivariant. In order to prove (2.5.2) and (2.5.4) notice that

$$\hat{h}(u) = p_{E^-}e^{\alpha(y(u))L}y(u) + K(y(u))$$

and $y(u) = e^{\beta(u)L}u + K^*(u)$ with α and β T-invariant, bounded functionals and K and K^* compact operators into E^- and E respectively. Therefore

$$\hat{h}_1(u) = p_{E^-}\,e^{[\alpha(y(u))+\beta(u)]L}u + p_{E^-}\,e^{\alpha(y(u))L}K^*(u) + K(y(u))$$

that is

$$\hat{h}_1(u) = p_{E^-}\,e^{\gamma(u)L}u + \hat{K}(u)$$

where $\gamma = \alpha \circ y + \beta$ is a T-invariant bounded functional and

$$\hat{K}(u) = p_{E^-}\,e^{\alpha(y(u))L}K^*(u) + K(y(u)) \text{ is a compact map.}$$

(b) Since the claim is trivial for $\tau_p(A) = +\infty$ or $i_p(B) = +\infty$, let us assume that $\tau_p(A) = k$ and $i_p(B) = \ell$. By definition, there exist continuous maps:

$$h : A \to E^- \times \mathbb{C}^k, \qquad h = (h_1, h_2)$$

satisfying (2.5.1)-(2.5.4) and

$$f : B \to \mathbb{C}^\ell\backslash\{0\}, \qquad f = (f_1, -, f_\ell)$$

satisfying

$$f_j(Tu) = e^{i\frac{2\pi}{p}m_j}f_j(u) \quad \forall u \in B \text{ for some integers } m_j \neq 0$$

relatively prime to p, $j = 1, \cdots, \ell$.

Since $B \cap F^- = \phi$ (otherwise $i_p(B) = +\infty$), there exists a continuous map $\overline{f} : E \to \mathbb{C}^\ell$ such that $\overline{f}|_B = f$ and $\overline{f}|_{F^-} = 0$.

Moreover, the map $\hat{f} = (\hat{f}_1, \cdots, \hat{f}_\ell)$ given by

$$\hat{f}_j(u) = \frac{1}{p} \sum_{s=0}^{p-1} e^{-i\frac{2\pi}{p}m_j s}\overline{f}_j(T^s u)$$

satisfies $\hat{f}_j(Tu) = e^{i\frac{2\pi}{p}m_j}\hat{f}_j(u)$ and $\hat{f}|_B = f$, $\hat{f}|_{F^-} = 0$. On the other hand, $h_1 : A \to E^-$ is given by

$$h_1(u) = e^{\alpha(u)L}u + K(u), \text{ with } \alpha \text{ and } K \text{ as in (2.5.2).}$$

Call $\overline{\alpha}$ a bounded extension of α on E, and set

$$\hat{\alpha}(u) = \frac{1}{p} \sum_{j=0}^{p-1} \overline{\alpha}(T^j u).$$

So $\hat{\alpha}$ is a bounded T-invariant functional on E. Similarly construct

$$\hat{K} : E \to E^-$$

a continuous compact, T-equivariant extension of K and

$$\hat{h}_2 : E \to \mathbb{C}^k$$

a continuous extension of h_2, satisfying the same equivariant property of h_2. Set $\hat{h}_1(u) = e^{\hat{\alpha}(u)L}u + \hat{K}(u)$ and define:

$$g : A \cup B \to E^- \times \mathbb{C}^{k+\ell}$$

by

$$g(u) = \left[\hat{h}_1(u), \hat{h}_2(u), \hat{f}(u)\right], \qquad u \in A \cup B.$$

Easily one checks that g satisfies (2.5.1)-(2.5.3). Finally (2.5.4) follows since for every $u \in A \cup B \cap F^-$ necessarily $u \in A$, so $\hat{\alpha}(u) = \alpha(u) = 0$ and $\hat{K}(u) = K(u) = 0$.

(c) Suppose $A \cap (E^-)^\perp = \phi$, that is $0 \notin p_E\text{-}A$. The continuous map:

$$h : A \to E^- \times \{0\}$$
$$u \to (p_E\text{-}u, 0)$$

satisfies (2.5.1)-(2.5.4) and therefore $\tau_p(A) = 0$, a contradiction! $\square$

We shall use the following notation:

$$S_\rho = \{u \in E : \|u\| = \rho\}, \quad B_\rho = \{u \in E : \|u\| < \rho\}$$

$\rho > 0$.

If E is a *complex* Hilbert space, we have:

$$E = \bigoplus_{j=1}^{\infty} E_j \oplus F, \quad E_j \subset E_{j+1}$$

where F is the fixed point set and the subspace E_j satisfies:

$(2.7)_\mathbb{C}^j$
$$\begin{cases} \text{(a) } \dim_\mathbb{C} E_j = j, \ E_j \ T\text{-invariant;} \\[2mm] \text{(b) there exists an isomorphism } \varphi_j : E_j \to \mathbb{C}^j \text{ such} \\ \text{that } \hat{T}_j : \mathbb{C}^j \to \mathbb{C}^j \text{ given by } \hat{T}_j = \varphi_j \circ T \circ \varphi_j^{-1} \\ \text{is a unitary representation of a } \mathbb{Z}_p \text{ group} \\ \text{action on } \mathbb{C}^j \text{ and:} \\[2mm] \hat{T}_j(\xi_1, \cdots, \xi_j) = \left[e^{i\frac{2\pi}{p}m_{j,1}}\xi_1, \cdots, e^{i\frac{2\pi}{p}m_{j,j}}\xi_j\right] \\[2mm] \text{for some integer } m_{j,s} \neq 0 \text{ relatively prime to } p, \\ s = 1, \cdots, j. \end{cases}$$

In case E is a *real* Hilbert space with a $\mathbb{Z}_p$ group acting on it, then applying the above decomposition to its complex extension $\tilde{E} = E + iE$ one concludes the following for E:

$$E = \bigoplus_{j=1}^{\infty} E_j \oplus F, \quad E_j \subset E_{j+1}$$

and the subspaces E_j now satisfies

$(2.7)_{\mathbb{R}}^j$
$$\begin{cases} \text{(a) } \dim_{\mathbb{R}} E_j = 2j, \ E_j \ T\text{-invariant;} \\[2mm] \text{(b) there exists an homeomorphism } \varphi_j : E_j \to \mathbb{C}^j \\ \text{such that } \hat{T}_j : \mathbb{C}^j \to \mathbb{C}^j \text{ with } \hat{T}_j = \varphi_j \circ T \circ \varphi_j^{-1} \\ \text{is a unitary representation of a } \mathbb{Z}_p \text{ group} \\ \text{action on } \mathbb{C}^j \text{ given by:} \\[2mm] \hat{T}_j(\xi_1, \cdots, \xi_j) = \left[e^{i\frac{2\pi}{p} m_{j,1}} \xi_1, \cdots, e^{i\frac{2\pi}{p} m_{j,j}} \xi_j \right] \\[2mm] \text{for some integer } m_{j,s} \neq 0 \text{ relatively prime to } p, \\ s = 1, \cdots, j. \end{cases}$$

REMARK. We have $\varphi_j \circ T = \hat{T}_j \circ \varphi_j$.

DEFINITION. *A subspace $D \subset E$ is called k-nice if D satisfies $(2.7)_{\mathbb{C}}^k$ in case E is a complex Hilbert space or $(2.7)_{\mathbb{R}}^k$ in case E is a real Hilbert space.*

We state here the $\mathbb{Z}_p$ version of the Borsuk-Ulam Theorem needed in the sequel.

$\mathbb{Z}_p$*-Borsuk-Ulam Theorem* (see [12]): Let Ω be an open bounded neighborhood of the origin in $\mathbb{R}^n = \mathbb{R}^b \times \mathbb{C}^a$, $n = b + 2a$, with coordinates $z = (x, z')$, $x = (x_1, \cdots, x_b) \in \mathbb{R}^b$, $z' = (z_1, \cdots, z_a) \in \mathbb{C}^a$. Let T be the unitary representation of the $\mathbb{Z}_p$ group action on $\mathbb{R}^n$ given by

$$Tz = \left[x, e^{i\frac{2\pi}{p} m_1} z_1, \cdots, e^{i\frac{2\pi}{p} m_a} z_a \right]$$

for some integers $m_j \neq 0$ relatively prime to p, $j = 1, \cdots, a$. Assume $\overline{\Omega}$ to be invariant with respect to such action (i.e. $\forall z \in \overline{\Omega}$, $Tz \in \overline{\Omega}$).

Let $f : \partial\Omega \to \mathbb{R}^b \times \mathbb{C}^\ell$ be a continuous map $f = (f_1, \cdots, f_b, f_{b+1}, \cdots, f_{b+\ell})$ with $f_j \in \mathbb{R}$, $j = 1, \cdots, b$, and $f_{b+s} \in \mathbb{C}$, $s = 1, \cdots, \ell$, satisfying the following equivariant properties:

(1) $f_{b+j}(Tz) = e^{i\frac{2\pi}{p} k_j} f_{b+j}(z)$, $\forall z \in \partial\Omega$, for some integer $k_j \neq 0$ relatively prime to p, $j = 1, \cdots, \ell$

(2) $f_j(Tz) = f_j(z)$, $\forall z \in \partial\Omega$, $\forall j = 1, \cdots, b$

(3) $\forall (x, 0) \in \partial\Omega$, $f_j(x, 0) = x_j$, $j = 1, \cdots, b$.

If $\ell < a$ then $0 \in f(\partial\Omega)$.   □

We are now ready to prove the following:

PROPOSITION 2.3.: *Set* $E^+ = (E^-)^\perp$ *and let* $E_k^+ \subset E^+$ *be a k-nice T-invariant subspace.*

For any $\delta > 0$ *and for any* $y \in \mathcal{F}$ *we have*:

$$\tau_\rho(E^- \oplus E_k^+ \cap y(S_\rho)) = k.$$

PROOF. Let $\varphi_k : E_k^+ \to \mathbb{C}^k$ and $m_j \neq 0$ integers relatively prime to p for $j = 1, \cdots, k$ such that $\forall(\xi_1, \cdots, \xi_k) \in \mathbb{C}^k$

$$\hat{T}_k(\xi_1, \cdots, \xi_k) \overset{\text{def.}}{=} \varphi_k \circ T \circ \varphi_k^{-1}(\xi_1, \cdots, \xi_k)$$
$$= \left[e^{i\frac{2\pi}{p}m_1}\xi_1, \cdots, e^{i\frac{2\pi}{p}m_k}\xi_k \right].$$

Define the continuous map:

$$h : E^- \oplus E_k^+ \cap y(S_\rho) \to E^- \times \mathbb{C}^k$$
$$z^- + z_k^+ \longrightarrow (z^-, \varphi_k(z_k^+))$$

$\forall z^- \in E^-$ and $z_k^+ \in E_k^+$. Since $\varphi_k(z_k^+) = 0 \iff z_k^+ = 0$ and $y(z) = 0 \iff z = 0$, we have $(0,0) \notin h(E^- \oplus E_k^+ \cap y(S_\rho))$. Moreover, h obviously satisfies (2.5.2)-(2.5.4), thus $\tau_\rho(E^- \oplus E_k^+ \cap y(S_\rho)) = \ell \leq k$. Arguing by contradiction, assume that
$$\ell < k.$$

Hence there exists a continuous map:

$$h : E^- \oplus E_k^+ \cap y(S_\rho) \to E^- \times \mathbb{C}^\ell$$
$$h(u) = \left[p_{E^-}e^{\alpha(u)L}u + Ku, h_2(u) \right] \neq (0,0)$$

where α is a bounded T-invariant functional, and K is a T-equivariant compact map into E^-. Moreover

$$h_2(Tu) = \left[e^{i\frac{2\pi}{p}k_1}h_{2,1}(u), \cdots, e^{i\frac{2\pi}{p}k_\ell}h_{2,\ell}(u) \right]$$

for some integer $k_j \neq 0$ relatively prime to p, $j = 1, \cdots, \ell$ and

$$\alpha(u) = 0, \quad K(u) = 0 \qquad \forall u \in F^- \cap y(S_\rho).$$

We suppose dim $E^- = +\infty$ and dim $F^- = +\infty$, since the finite dimensional case follows by even easier arguments.

Decompose: $E^- = \overset{\infty}{\underset{j=1}{\oplus}} E_j^- \oplus F^-$ and $F^- = \overset{\infty}{\underset{j=1}{\oplus}} F_j^-$ where $E_j^- \subset E_{j+1}^-$ is j-nice and $F_j^- \subset F_{j+1}^-$ with $\dim_\mathbb{R} F_j^- = j$.

Denote by $p_{E_j^-} : E \to E_j^-$ and $p_{F_j^-} : E \to F_j^-$ the canonical orthogonal projections on E_j^- and F_j^- respectively, and by $\varphi_s : E_s \to \mathbb{C}^s$ the homeomorphism such that

$$\hat{T}_s(\xi_1, \cdots, \xi_s) \overset{\text{def}}{=} \varphi_s \circ T \circ \varphi_s^{-1}(\xi_1, \cdots, \xi_s)$$

$$= \left[e^{i\frac{2\pi}{p}m_{s,1}}\xi_1, \cdots, e^{i\frac{2\pi}{p}m_{s,s}}\xi_s \right]$$

for every $(\xi_1, \cdots, \xi_s) \in \mathbb{C}^s$ with $m_{s,j} \neq 0$ integers relatively prime to p, $j = 1, \cdots, s$.

Define

$$h_s : F_s^- \oplus E_s^- \oplus E_k^+ \cap y(S_\rho) \to F_s^- \oplus E_s^- \times \mathbb{C}^\ell$$

as follows:

$$h_s = (h_{s.1} + h_{s.2}, h_2) \text{ where}$$

$$h_{s,1}(u) = p_{F_s^-}\left[u + e^{-\alpha(u)L}K(u) \right]$$

$$h_{s,2}(u) = p_{E_s^-}\left[u + e^{-\alpha(u)L}K(u) \right].$$

We claim that $h_s(u_s) = 0$ for some $u_s \in F_s^- \oplus E_s^- \oplus E_k^+ \cap y(S_\rho)$. To this end identify F_s^- with $\mathbb{R}^s$ and define

$$\Omega_s = \left\{ \begin{array}{c} (u_s^-, z_1, z_2) \in \mathbb{R}^s \times \mathbb{C}^{s+k} : u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \in y(B_\rho) \\ z_1 \in \mathbb{C}^s, \ z_2 \in \mathbb{C}^k \end{array} \right\}$$

Hence Ω_s defines a bounded neighborhood of the origin in $\mathbb{R}^s \times \mathbb{C}^{s+k}$ and

$$\partial\Omega_s = \{(u_s^-, z_1, z_2) : u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \in y(S_\rho)\}.$$

Furthermore $\overline{\Omega}_s$ is symmetric with respect to the $\mathbb{Z}_p$ group action on $\mathbb{R}^s \times \mathbb{C}^{s+k}$ generated by the unitary operator:

$$\hat{T}(u_s^-, z_1, z_2) = (u_s^-, \hat{T}_s z_1, \hat{T}_k z_2) =$$

$$\left[u_s^-, e^{i\frac{2\pi}{p}m_{s,1}}z_{1,1}, \cdots, e^{i\frac{2\pi}{p}m_{s,s}}z_{1,s}, \right.$$

$$\left. e^{i\frac{2\pi}{p}m_1}z_{2,1}, \cdots, e^{i\frac{2\pi}{p}m_k}z_{2,k} \right]$$

where $z_1 = (z_{1,1}, \cdots, z_{1,s}) \in \mathbb{C}^s$ and $z_2 = (z_{2,1}, \cdots, z_{2,k}) \in \mathbb{C}^k$.

Define the continuous map:

$$f : \partial\Omega_s \to \mathbb{R}^s \times \mathbb{C}^{s+\ell} \quad \text{as follows}$$

$$f(u_s^-, z_1, z_2) = [h_{s,1}(u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2)), \varphi_s \circ h_{s,2}(\cdots), h_2(\cdots)]$$

where still we identify F_s^- with $\mathbb{R}^s$.

Obviously h_s vanishes in $F_s^- \oplus E_s^- \oplus E_k^+ \cap y(S_\rho)$ if and only if f vanishes on $\partial\Omega_s$.

In addition f enjoys the following equivariant properties:

$$f_2[\hat{T}(u_s^-, z_1, z_2)] \overset{\text{def.}}{=} \varphi_s \circ h_{s.2} \left[u_s^- + \varphi_s^{-1}(\hat{T}_s z_1) + \varphi_k^{-1}(\hat{T}_k z_2) \right]$$

$$= \varphi_s \circ h_{s.2} \left[u_s^- + T\varphi_s^{-1}(z_1) + T\varphi_k^{-1}(z_2) \right]$$

$$= \varphi_s \circ h_{s.2} \left[T \left[u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \right] \right]$$

$$= \varphi_s \circ T \circ h_{s.2} \left[u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \right]$$

$$= \hat{T}_s \circ \varphi_s \circ h_{s.2}(\cdots) = \hat{T}_s f_2(u_s^-, z_1, z_2);$$

$$f_3[\hat{T}(u_s^-, z_1, z_2)] \overset{\text{def.}}{=} h_2 \left[u_s^- + \varphi_s^{-1}(\hat{T}_s z_1) + \varphi_k^{-1}(\hat{T}_k z_2) \right]$$

$$= h_2 \left[T \left[u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \right] \right]$$

$$= \left[e^{i\frac{2\pi}{p}k_1} h_{2,1}(\cdots), \cdots, e^{i\frac{2\pi}{p}k_\ell} h_{2,\ell}(\cdots) \right]$$

$$= \left[e^{i\frac{2\pi}{p}k_1} f_{3,1}(u_s^-, z_1, z_2), \cdots, e^{i\frac{2\pi}{p}k_\ell} f_{3,\ell}(u_s^-, z_1, z_2) \right].$$

Furthermore

$$f_1[\hat{T}(u_s^-, z_1, z_2)] \overset{\text{def}}{=} h_{s.1} \left[u_s^- + \varphi_s^{-1}(\hat{T}_s z_1) + \varphi_k^{-1}(\hat{T}_k z_2) \right]$$

$$= h_{s.1} \left[T \left[u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \right] \right]$$

$$= Th_{s.1} \left[u_s^- + \varphi_s^{-1}(z_1) + \varphi_k^{-1}(z_2) \right] = h_{s.1}(\cdots)$$

$$= f_1(u_s^-, z_1, z_2).$$

Finally, for every $u_s^- \in F_s^- \cap y(S_\rho)$ we have

$$f(u_s^-) = \left[h_{s.1}(u_s^-), h_{s.2}(u_s^-), h_2(u_s^-) \right] = (u_s^-, 0).$$

Thus applying the $\mathbb{Z}_p$-Borsuk-Ulam Theorem, we conclude that $0 \in f(\partial\Omega_s)$ and therefore that there exists $u_s \in F_s^- \oplus E_s^- \oplus E_k^+ \cap y(S_\rho)$ such that $h_s(u_s) = 0$. That is

$$(2.8) \qquad h_2(u_s) = 0 \text{ and } p_{F_s^- \oplus E_s^-} u_s + p_{F_s^- \oplus E_s^-} e^{-\alpha(u_s)L} K(u_s) = 0.$$

Set $u_s = u_s^- + u_s^+$ with $u_s^- \in F_s^- \oplus E_s^-$ and $u_s^+ \in E_k^+$.

Since $\|u_s\|$ is uniformly bounded, for some subsequence, that we still call u_s we have:

$$\lim_{s \to +\infty} K(u_s) = v^- \in E^-;$$

and

$$\alpha_s = \alpha(u_s) \to \alpha \text{ as } s \to \infty.$$

Hence by (2.8) we get:

$$u_s^- = -p_{F_s^- \oplus E_s^-} e^{-\alpha_s L} K(u_s) \to -e^{-\alpha L} v^-.$$

On the other hand, since E_k^+ is finite dimensional, we can certainly assume that $u_s^+ \to u_k^+ \in E_k^+$. In conclusion:

$$u_s \to u = -e^{-\alpha L}v^- + u_k^+ \in E^- \oplus E_k^+ \cap y(S_\rho);$$

and consequently $h(u) = (0,0)$. This is clearly impossible since, by definition, $(0,0) \notin h(E^- \oplus E_k^+ \cap y(S_\rho))$. $\qquad\qquad\square$

REMARK 2.1. In the preceding proof the property (2.5.2) of τ_p enters only for the final convergence argument. Therefore for a finite dimensional subspace E^-, the relative index can be defined independent of the functional. In particular in this case one does not need to require ($*$).

REMARK 2.2. In the previous proposition we can replace S_ρ, $\rho > 0$, with the boundary of any bounded T-invariant neighborhood of the origin.

To any relative index map, one can associate a *pseudoindex map*:

$$\tau_p^* : \Sigma \to \mathbb{N} \cup \{\infty\} \qquad \tau_p^* = \tau_p^*(\cdot, E^-)$$

as follows.

Fix $\rho > 0$ and define

$$\tau_p^*(A) = \min_{y \in \mathcal{F}} \tau_p(A \cap y(S_\rho))$$

for any given $A \in \Sigma$.

We summarize in the following proposition the main properties of τ_p^*.

PROPOSITION 2.4.

(a) *If E_k^+ is a k-nice subspace of E^+, then*

$$\tau_p^*(E^- \oplus E_k^+) = k.$$

(b) *(Monotonicity)* $\forall y \in \mathcal{F} \;\; \tau_p^*(y(A)) = \tau_p^*(A)$.

(c) *(Subadditivity)* $\tau_p^*(A \cup B) \le \tau_p^*(A) + i_p(B)$; $\;\; A, B \in \Sigma$.

(d) *If $\tau_p^*(A) \ge 1$ then $A \cap S_\rho \cap E^+ \ne \phi$.*

PROOF. (a) follows immediately from proposition 2.3; and (d) from proposition 2.2.(c).

To prove (b) just notice that for any $\psi \in \mathcal{F}$

$$\tau_p(y(A) \cap \psi(S_\rho)) = \tau_p[y(A \cap y^{-1} \circ \psi(S_\rho))] = \tau_p(A \cap y^{-1} \circ \psi(S_\rho))$$

and

$$y^{-1} \circ \psi \in \mathcal{F}.$$

By similar considerations and the subadditivity of τ_p one obtains (c). $\qquad\square$

REMARK. Propositions 2.3 and 2.4.(a) show that the relative index theory as well as the pseudoindex theory introduced above are indeed consistent. Namely that there exist invariant subsets with arbitrarily large relative (psuedo) indices.

Our next goal is to show that for functionals as in $(*)$ one can construct a deformation map within the family $\mathcal{F}$.

We learned the following version of the deformation theorem from Zhengfang Zhou. It is however a slight modification of Benci's (cf. [2]). We give the proof here for completeness.

DEFORMATION THEOREM. *Let $f \in C^1(E, \mathbb{R})$ be a T-invariant functional satisfying $(*)$ and the (P.S.) condition at the value $c \in \mathbb{R}$ such that $f^{-1}[c - \epsilon_0, c + \epsilon_0] \cap F^- = \phi$ for some $\epsilon_0 > 0$. Then for any neighborhood N of K_c there exist $0 < \bar{\epsilon} < \epsilon_0$, and $y \in \mathcal{F}$ such that for every $0 < \epsilon < \bar{\epsilon}$ we have:*

$$y(A_{c+\epsilon} \backslash N) \subset A_{c-\epsilon},$$

and if $K_c = \phi => y(A_{c+\epsilon}) \subset A_{c-\epsilon}$.

PROOF. Since f satisfies the (P.S.) condition at the value c, we have K_c compact. Let $\delta > 0$ be such that

$$K_c \subset N_\delta(K_c) \subset N$$

where, we recall $N_\delta(K_c) = \{u \in E : \text{dist}(u, K_c) \leq \delta\}$. There exist $\bar{\epsilon} > 0$ and $\beta > 0$ such that

$$\|f'(u)\| \geq \beta \quad \forall u \in (A_{c+\bar{\epsilon}} \backslash A_{c-\bar{\epsilon}}) \backslash N_{\delta/8}(K_c),$$

and we can always assume that

$$(2.9) \qquad\qquad 0 < \bar{\epsilon} < \min \left\{ \frac{\delta \beta}{12}, \ \epsilon_0 \right\}.$$

Recall the following

LEMMA (Benci [2]): *Let $K : E \to E$ be a T-equivariant compact operator. For any $\gamma > 0$, there exists $\tilde{K} : E \to E$ a T-equivariant compact operator such that:*

(a) *$\tilde{K}$ is locally Lipschitz continuous,*

(b) *$\|\tilde{K}(u) - K(u)\| \leq \gamma \quad \forall u \in E$.*

$\square$

Hence take

$$0 < \gamma < \min \left\{ \frac{\bar{\epsilon}}{4}, \ \beta/4 \right\}$$

and let $b : E \to E$ be T-equivariant compact, locally Lipschitz continuous operator such that

$$\|\Psi'(u) - b(u)\| \leq \gamma \qquad \forall u \in E.$$

Set

$$S = [A_{c+\bar{\epsilon}} \backslash A_{c-\bar{\epsilon}}] \backslash N_{\delta/8}(K_c).$$

We have:

$$\|Lu - b(u)\| = \|f'(u) + (\Psi'(u) - b(u))\| \geq \|f'(u)\| - \gamma \geq \frac{3}{4}\|f'(u)\|$$

for every $u \in S$.

Set

$$V(u) = 2\frac{Lu - b(u)}{\|Lu - b(u)\|^2} \qquad \forall u \in S;$$

thus:

(2.10) $$\|V(u)\| \leq \frac{8}{3}\frac{1}{\|f'(u)\|} \leq \frac{8}{3\beta} \qquad \forall u \in S.$$

Moreover $\forall u \in S$

$$\|\Psi'(u) - b(u)\| \leq \frac{1}{4}\|f'(u)\| \leq \frac{1}{4}\|Lu - b(u)\| + \frac{1}{4}\|\Psi'(u) - b(u)\|$$

that is

$$\|\Psi'(u) - b(u)\| \leq \frac{1}{3}\|Lu - b(u)\|.$$

So we obtain:

$$< V(u), f'(u) > \; = 2 < \frac{Lu - b(u)}{\|Lu - b(u)\|^2}, \; Lu - b(u) + b(u) - \Psi'(u) >$$

$$= 2\left[1 + \frac{1}{\|Lu - b(u)\|^2} < Lu - b(u), \; b(u) - \Psi'(u) >\right]$$

$$\geq 2\left[1 - \frac{\|b(u) - \Psi(u)\|}{\|Lu - b(u)\|}\right],$$

i.e.

(2.11) $$< V(u), f'(u) > \; \geq \; \frac{4}{3} \qquad \forall u \in S.$$

Let $\chi : E \to [0, 1]$ be a T-invariant, Lipschitz continuous functional such that

$$\chi(u) = \begin{cases} 0 \text{ if } u \notin f^{-1}[c - \bar{\epsilon}, c + \bar{\epsilon}] \text{ or } u \in N_{\delta/8} \\ 1 \text{ if } u \in f^{-1}[c - \dfrac{\bar{\epsilon}}{2}, \; c + \dfrac{\bar{\epsilon}}{2}]\backslash N_{\delta/4}. \end{cases}$$

Set

$$\tilde{V}(u) = \begin{cases} -\chi(u)V(u) & \text{if } u \in S \\ 0 & \text{if } u \notin S. \end{cases}$$

The initial value problem:

$$\begin{cases} \dfrac{dy}{dt}(t) = \tilde{V}(y(t)) \\ y(0) = u \qquad u \in E \end{cases}$$

admits a unique solution $y(t, u)$ defined for all $t \in \mathbb{R}$. In addition for any fixed $t \in \mathbb{R}$

$$y_t : E \to E \qquad\qquad y_t(u) = y(t, u)$$

is a T-equivariant diffeomorphism, and both y_t, y_t^{-1} map bounded sets in bounded sets. Indeed by (2.10) we have:

$$\|y_t(u) - u\| \le \frac{8t}{3\beta}.$$

Set $y(u) = y(\bar{\epsilon}, u)$. By standard arguments using (2.11) and the fact that $\bar{\epsilon}$ satisfies (2.9) one shows that necessarily:

$$y(A_{c+\epsilon} \backslash N_\delta(K_c)) \subset A_{c-\epsilon} \qquad \forall\ 0 < \epsilon < \frac{\bar{\epsilon}}{2}.$$

Furthermore, to prove that y has the required form, set

$$\hat{\chi}(u) = \begin{cases} \dfrac{-2\chi(u)}{\|Lu - b(u)\|^2} & \text{if } u \in S \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\alpha(t, s, u) = \int_0^{t-s} \hat{\chi}(y(\tau + s, u)) d\tau.$$

Therefore $y(t, u)$ satisfies the integral equation:

$$y(t, u) = e^{\alpha(t, o, u)L} u + \int_0^t e^{\alpha(t, s, u)L}[\hat{\chi}(y(s, u))b(y(s, u))] ds.$$

Thus if we set

$$\alpha : E \to \mathbb{R}, \qquad \alpha(u) = \alpha(\bar{\epsilon}, 0, u)$$

and $K : E \to E$

$$K(u) \to \int_0^{\bar{\epsilon}} e^{\alpha(\bar{\epsilon}, s, u)L}[\hat{\chi}(y(s, u))b(y(s, u))] ds,$$

we have that α is a continuous T-invariant functional such that $|\alpha(u)| \leq \frac{16}{9\beta^2} \bar{\epsilon}$; K is a T-equivariant compact operator and

$$y(u) = e^{\alpha(u)L} u + K(u).$$

Finally $\forall u \in F^-$ we have that $u \notin f^{-1}[c - \bar{\epsilon}, c + \bar{\epsilon}]$, this implies $\tilde{V}(u) = 0$, therefore $y(t, u) = u$, $\forall t$, which gives $\alpha(u) = 0$ and $K(u) = 0$. $\qquad\square$

We can now apply the theory developed so far to find (multiple) critical points for T-invariant functionals.

Notice that, since f' is T-equivariant, if $u(t)$ is a critical point for f, so it is $T^j u$ for $j = 1, \cdots, p - 1$.

DEFINITION. We say that $u_1, u_2 \in E$ are $\mathbb{Z}_p$-distinct if

$$T^j u_1 \neq u_2 \qquad \forall j = 0, 1, \cdots, p - 1.$$

Set

$$F_0 = \{u \in F : f'(u) = 0\}.$$

THEOREM 2.1. *Let $f \in C^1(E, \mathbb{R})$ be a T-invariant functional satisfying* (∗). *Assume that there exist invariant subspaces E^- and E_k^+ satisfying*

$$(2.12) \qquad E_k^+ \subset E^+ = (E^-)^\perp \text{ is } k\text{-nice and } LE^- = E^-$$

and such that for some constants $c_0 < c_\infty$ and $\rho > 0$ we have

(f_1) $f(u) < c_\infty$ $\qquad \forall u \in E^- \oplus E_k^+$;

(f_2) $f(u) \geq c_0$ $\qquad \forall u \in E^+ \cap S_\rho$;

(f_3) $f^{-1}([c_0 - \epsilon_0, c_\infty)) \cap F^- = \phi$, *for some $\epsilon_0 > 0$, with $F^- = E^- \cap F$;*

(f_4) $f^{-1}([c_0, c_\infty)) \cap F_0 = \phi$.

If f satisfies the (P.S.) condition in $[c_0, c_\infty)$, then there exist at least k $\mathbb{Z}_p$-distinct critical points $u_1, \cdots, u_k \in E$ of f such that

$$c_0 \leq f(u_j) < c_\infty \qquad \forall j = 1, \cdots, k.$$

$\qquad\square$

PROOF. Define the numbers

$$c_j = \inf_{\substack{A \in \Sigma \\ \tau_p^*(A) \geq j}} \sup_A f, \qquad j = 1, \cdots, k,$$

so $c_1 \leq c_2 \leq \cdots \leq c_k$. Moreover every $A \in \Sigma$ with $\tau_p^*(A) \geq 1$ satisfies $A \cap E^+ \cap S_\rho \neq \phi$. Hence by (f_2) we obtain $c_1 \geq c_0$.

Since $\tau_p^*(E^- \oplus E_k^-) = k$, by (f_1) follows that $c_k < c_\infty$. Therefore

$$c_0 \le c_1 \le c_2 \le \cdots \le c_k < c_\infty.$$

We claim that c_j is a critical value for f, $\forall j = 1, \cdots, k$. More generally we shall prove that if $c_j = c_{j+1} = \cdots = c_{j+r} = c$ for some $r \ge 0$ then $i_p(K_c) \ge r + 1$. Arguing by contradiction, assume that $i_p(K_c) \le r$. Hence for $\delta > 0$ small enough we have $i_p(N_\delta(K_c)) = i_p(K_c) \le r$.

By the deformation theorem, we can then find $\bar{\epsilon} > 0$ and $y \in \mathcal{F}$ such that

$$y\big(A_{c+\bar{\epsilon}/2} \backslash N_\delta(K_c)\big) \subset A_{c-\bar{\epsilon}/2}.$$

Moreover, by definition, there exists a subset $A \in \Sigma$ such that $A \subset A_{c+\bar{\epsilon}/2}$ and $\tau_p^*(A) \ge j + r$. On the other hand the subset $B = y(\overline{A \backslash N_\delta(K_c)}) \in \Sigma$ satisfies $B \subset A_{c-\bar{\epsilon}/2}$ and

$$\tau_p^*(B) = \tau_p^*(\overline{A \backslash N_\delta(K_\delta)}) \ge \tau_p^*(A) - i_p(N_\delta(K_c)) \ge j + r - r = j$$

which is clearly impossible since $c = c_j$. In addition, from (f_4), it follows that $K_{c_j} \cap F_0 = \phi$, $\forall j = 1, \cdots, k$. Therefore if $c_j = c_{j+1} = \cdots = c_{j+r} = c$ with $r \ge 1$, then necessarily K_c must contain infinitely many $\mathbb{Z}_p$-distinct critical points of f (see [13] for details). The proof is therefore concluded. $\qquad\square$

REMARK 2.3. By Remark 2.2. one can replace S_ρ with the boundary of any bounded T-invariant neighborhood of the origin.

By the proof of Theorem 2.1 and Remark 2.1 follows:

COROLLARY 2.1. *If the subspace E^- in Theorem 2.1 is finite dimensional, then the conclusion of Theorem 2.1 follows without the assumption that f satisfies $(*)$.* $\qquad\square$

Since the hypothesis (f_4) of Theorem 2.1 is used to guarantee multiplicity we have:

COROLLARY 2.2. *Let all the assumptions of Theorem 2.1 hold but (f_4). Then f admits, at least, a critical value in $[c_0, c_\infty)$.* $\qquad\square$

Similarly one obtains:

THEOREM 2.2. *Let $f \in C^1(E, \mathbb{R})$ be a T-invariant functional satisfying $(*)$. Assume that there exist invariant subspaces E^- and E_k^+ satisfying (2.12) and constants $c_0 < c_\infty$, $\rho > 0$ such that*

$(f_1)^*$ $\qquad\qquad f(u) < c_\infty \qquad\qquad \forall u \in E^- \oplus E_k^+ \cap S_\rho,$

$(f_2)^*$ $f(u) \geq c_0 \qquad \forall u \in E^+.$

If f satisfies (f_3) and (f_4) and the (P.S.) condition in $[c_0, c_\infty)$, then there exist at least k $\mathbb{Z}_p$-distinct critical points, $u_1, ..., u_k$, of f such that $c_0 \leq f(u_j) < c_\infty$, $\forall j = 1, ..., k$.

PROOF: It is completely analogous to the previous one, only now define:

$$c_j^* = \inf_{\substack{A \in \Sigma \\ \tau_p(A) \geq j}} \sup_A f, \quad c_1^* \leq c_2^* \leq ... \quad .$$

Since $\tau_p(A) \geq 1$ implies $A \cap E^+ \neq \phi$, by $(f_2)^*$ we have $c_1^* \geq c_0$. Moreover $\tau_p(E^- \oplus E_k^+ \cap S_\rho) = k$, so by $(f_1)^*$ we get $c_k^* < c_\infty$.   □

REMARK 2.4. The analogues of Corollary 2.1 and 2.2 and Remark 2.3 hold with respect to Theorem 2.2.

To understand what motivated Theorems 2.1 and 2.2 observe that

$$c_j = \inf_{\substack{A \in \Sigma \\ \tau_p^*(A) \geq j}} \sup_A f = \inf\{c : \tau_p^*(A_c) \geq j\}$$

and similarly

$$c_j^* = \inf_{\substack{A \in \Sigma \\ \tau_p(A) \geq j}} \sup_A f = \inf\{c : \tau_p(A_c) \geq j\}.$$

Hence all the effort was to generalize (2.4) to unbounded functionals.

It is clear that one could extend the given arguments to other groups. We only mention its possible exstension to the group S^1 where a index theory is already available (see [1]).

Let

$$T_\theta : E \to E, \qquad \theta \in [0, 2\pi]$$

be a unitary representation of S^1 in E, with fixed point set

$$F = \{u \in E : T_\theta u = u, \ \forall \theta \in [0, 2\pi]\}.$$

In this case we say that the subspace $E_k \subset E$ is *k-nice* if

(i) $\dim_{\mathbb{C}} E_k = k$ (or $\dim_{\mathbb{R}} E_k = 2k$ in case E is a real Hilbert space);

(ii) E_k is S^1 invariant, i.e. $T_\theta E_k = E_k$, $\forall \theta \in [0, 2\pi]$;

(iii) there exists a homeomorphism $\varphi : E_k \to \mathbb{C}^k$ such that $\hat{T}_\theta = \varphi \circ T_\theta \circ \varphi^{-1}$, $\theta \in [0, 2\pi]$, is a unitary representation of S^1 in $\mathbb{C}_k$ and $\forall (\xi_1, ..., \xi_k) \in \mathbb{C}^k$

$$\hat{T}_\theta(\xi_1, ..., \xi_k) = (e^{im_1\theta}\xi_1, ..., e^{im_k\theta}\xi_k)$$

for some integers $m_j \neq 0, j = 1, ...k.$

Therefore by the Peter-Weyl theorem (cf. [18]) we have

$$E = \bigoplus_{j=1}^{\infty} E_j \oplus F$$

with E_j a j-nice invariant subspace of E and $E_j \subset E_{j+1}$.

Furthermore, one has the Borsuk-Ulam Theorem for the group S^1 (see [15]).

As above it is then possible to define a S^1 relative index and consequently a S^1 pseudoindex; and so to prove the following improvement of Benci's results (cf. [2]):

THEOREM 2.3: *Let* $f \in C^1(E, \mathbb{R})$ *be a* S^1-*invariant functional satisfying* (*). *Assume that there exist* S^1-*invariant subspaces* E^- *and* $E_k^+ \subset E^+ = (E^-)^\perp$ *and constants* $c_0 < c_\infty \in \mathbb{R}$ *such that either* (2.12), $(f_1) - (f_4)$ *or* (2.12) $(f_1)^*, (f_2)^*, (f_3), (f_4)$ *hold.*

If f *satisfies the* (P.S.) *condition on* $[c_0, c_\infty)$, *then there exist at least* k S^1-*distinct*([1]) *critical points* $u_1, ..., u_k$ *of* f *such that*

$$c_0 \leq f(u_j) < c_\infty \qquad \forall j = 1, ..., k. \qquad \qquad \square$$

We leave the details to the reader.

3. - The superquadratic case

Solutions of $(1)_p$ are the critical points for the functional:

$$\Phi(z) = \int_0^{pT} \frac{1}{2}[J\dot{z}(t) - Qz(t)] \cdot z(t) - H(z(t), t)dt$$

with $z \in H^{1/2}[\mathbb{R}/_{pT\mathbb{Z}}, \mathbb{R}^{2N}]$. Here $E \equiv H^{1/2}[\mathbb{R}/_{pT\mathbb{Z}}, \mathbb{R}^{2N}]$ denotes the real Hilbert space given by the functions

$$z(t) = \sum_{j=-\infty}^{+\infty} a_j e^{i\frac{2\pi}{pT}jt}, a_j \in \mathbb{C}^{2N}$$

with $a_{-j} = \overline{a}_j$ such that

$$\sum_{j=-\infty}^{+\infty} (1 + |j|)|a_j|^2 < +\infty.$$

([1])   Recall that in this case $u_1, u_2 \in E$ are called S^1- distinct if $T_\theta u_1 \neq u_2, \forall \theta \in [0, 2\pi]$.

Naturally one defines

$$\|z\|_E^2 = pT \sum_{j=-\infty}^{+\infty} (1 + |j|)|a_j|^2,$$

however we shall consider an equivalent norm on E more suitable for our purposes.

Following [16] we set

$$\varphi_{j,m}(t) = \xi_j e^{i\frac{2\pi}{pT}mt}, \qquad \substack{j = 1, ..., 2N \\ m \in \mathbb{Z}}$$

where $\xi_j, j = 1, ..., 2N$ is the basis of eingenvectors of (1.1) satisfying (1.2) and (1.3), as defined in section 1.

For any $z \in E$, we can write

$$z(t) = \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} \text{Re}(a_{j.m}\varphi_{j.m}(t))$$

$$= \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} \frac{1}{2}\left[a_{j.m}\varphi_{j,m}(t) + \overline{a}_{j,m}\overline{\varphi}_{j,m}(t)\right], \qquad a_{j.m} \in \mathbb{C}.$$

Easy computation shows that:

$$A(z) \stackrel{\text{def}}{=} \int_0^{pT} [J\dot{z}(t) - Qz(t)] \cdot z(t)dt$$

$$= pT \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} \left[\frac{2\pi}{T}\frac{m}{p} - \omega_j\right]|a_{j.m}|^2.$$

Set

$$E^+ = \left\{ z \in E : z(t) = \sum_{j=1}^{N} \sum_{m > \frac{pT\omega_j}{2\pi}} \text{Re}(a_{j.m}\varphi_{j.m}(t)) \right\}$$

$$E^- = \left\{ z \in E : z(t) = \sum_{j=1}^{N} \sum_{m < \frac{pT\omega_j}{2\pi}} \text{Re}(a_{j,m}\varphi_{j.m}(t)) \right\}$$

and

$$E_0 = \left\{ z \in E : z(t) = \sum_{j=1}^{N} \sum_{m = \frac{pT\omega_j}{2\pi}} \text{Re}(a_{j.m}\varphi_{j.m}(t)) \right\}.$$

Notice that E_0 might be empty, and in any case it has (real) dimension at most $2N$. Moreover, the subspaces E^+, E^- and E_0 are mutually orthogonal and

$$E = E^+ \oplus E^- \oplus E_0.$$

For $z \in E$, write $z = z^+ + z^- + z_0$ with $z^\pm \in E^\pm$ and $z_0 \in E_0$, we shall consider the following equivalent norm on E:

$$\|z\|^2 = A(z^+) - A(z^-) + \|z_0\|_{L^2}^2 ;$$

and denote by $< , >$ the corresponding scalar product.

LEMMA 3.1. *Let $z = z(t)$ be a critical point for Φ. We have:*

(a) $\Phi(z) \geq 0$ *and*

(b) *if $z \in E_0 \oplus E^-$, then $\Phi(z) = 0$.*

PROOF: (a) Since $J\dot{z} - Qz = H_z(z,t)$ we get:

$$(3.1) \qquad \Phi(z) = \int_0^{pT} \frac{1}{2} H_z(z,t) \cdot z - H(z,t) \geq \left[\frac{\alpha}{2} - 1\right] \int_0^{pT} H(z,t)$$

with $\alpha > 2$ and $H(z,t) \geq 0$, $\forall z \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$.

(b) We have

$$0 \geq \int_0^{pT} (J\dot{z} - Qz) \cdot z = \int_0^{pT} H_z(z,t) \cdot z \geq 0.$$

Therefore $H_z(z(t),t) \cdot z(t) = 0$, $\forall t \in \mathbb{R}$, and consequently by (H_2) we get $H(z(t),t) = 0$ $\forall t$.

REMARK: Since $\omega_j \geq 0$, $\forall j = 1, ..., N$, $z(t) = $ const. is necessarily contained in $E_0 \oplus E^-$.

PROPOSITION 3.1: *Under the assumptions of Theorem 1, if $z \notin E_0 \oplus E^-$ is a T-periodic critical point for Φ then:*

$$\Phi(z) \geq \left[\frac{1}{2} - \frac{1}{2}\right] pT \left[\frac{\alpha \rho_T}{C_1^2}\right]^{\frac{1}{2\gamma-1}} \frac{\alpha \sigma_T}{\sigma_T(1 + TC_2) + T\alpha\|Q\|}.$$

PROOF. Write $z(t) = z_0 + z^+(t) + z^-(t)$ where

$$z^+(t) = \sum_{j=1}^N \sum_{m > \frac{T\omega_j}{2\pi}} \text{Re}(a_{j,m}\varphi_{j,m}(t)) \in E^+$$

$$z^-(t) = \sum_{j=1}^N \sum_{m < \frac{t\omega_j}{2\pi}} \text{Re}(a_{j,m}\varphi_{j,mp}(t)) \in E^-$$

and by (Q_2):

$$z_0 = \sum_{j:\omega_j=0} \text{Re}\left[a_j\varphi_{j,0}\right] \in E_0.$$

Since $z \notin E_0 + E^-$ we have $z^+ \neq 0$ which implies $\int_0^T H(z(s), s) ds > 0$. Easy computations show that:

$$\int_0^T |J\dot{z}(t) - Qz(t)|^2 dt = T \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} \left[\frac{2\pi m}{T} - \omega_j\right]^2 \mu_j |a_{j.m}|^2$$

this gives:

(3.2)
$$\int_0^T |z_1|^2 \leq \frac{1}{\rho_T^2} \int_0^T |J\dot{z}(t) - Qz(t)|^2 dt$$

and

(3.3)
$$\int_0^T |\dot{z}|^2 = \int_0^T |\dot{z}_1|^2 \leq \frac{1}{\sigma_T^2} \int_0^t |J\dot{z}(t) - Qz(t)|^2 dt$$

where we set $z_1 = z^+ + z^-$.

Furthermore we have:

$$\int_0^T H(z(t), t) dt \leq \frac{1}{\alpha} \int_0^T H_z(z(t), t) \cdot z(t) dt = \frac{1}{\alpha} \int_0^T (J\dot{z} - Qz) \cdot z$$

$$\leq \frac{1}{\alpha} \int_0^T (J\dot{z} - Qz) \cdot z^+ \leq \frac{1}{\alpha} \left[\int_0^T |z^+|^2\right]^{1/2} \left[\int_0^T |J\dot{z} - Qz|^2\right]^{1/2}$$

$$\leq \frac{1}{\alpha \rho_T} \int_0^T |J\dot{z} - Qz|^2 = \frac{1}{\alpha \rho_T} \int_0^T |H_z(z, t)|^2,$$

that is:

(3.4)
$$\int_0^T H(z, t) \leq \frac{C_1^2}{\alpha \rho_T} \int_0^T (H(z, t))^{2\gamma}$$

$$\leq \frac{C_1^2}{\alpha \rho_t} \|H(z(t), t)\|_{L^\infty}^{2\gamma - 1} \int_0^T H(z(t), t) dt$$

and therefore:

(3.5)
$$\|H(z(t), t)\|_{L^\infty} \geq \left[\frac{\alpha \rho_T}{C_1^2}\right]^{\frac{1}{2\gamma - 1}}.$$

We now estimate from below the L^1-norm of $H(z(t), t)$ in terms of its L^∞ norm.

Indeed for any $t \in [0, T]$, we have

$$H(z(t), t) \leq \frac{1}{T} \int_0^T H(z(s), s)ds + \int_0^T |Qz \cdot \dot{z}| + \int_0^T |H_t(z(s), s)|ds$$

$$\leq \left[\frac{1}{T} + C_2\right] \int_0^T H(z(s), s)ds + \int_0^T |Qz_1 \cdot \dot{z}| \leq \left[\frac{1}{T} + C_2\right] \int_0^T H(z(s), s)ds$$

$$+ \|Q\| \left[\int_0^T |z_1|^2\right]^{\frac{1}{2}} \left[\int_0^T |\dot{z}|^2\right]^{\frac{1}{2}}.$$

Hence using (3.2) and (3.3), we obtain:

$$\|H(z(t), t)\|_{L^\infty} \leq \left[\frac{1}{T} + C_2\right] \int_0^T H(z(s), s)ds + \frac{\|Q\|}{\sigma_T \rho_T} \int_0^T |J\dot{z} - Qz|^2$$

$$\leq \left[\frac{1}{T} + C_2\right] \int_0^T H(z(s), s)ds + \frac{\|Q\|C_1^2}{\sigma_T \rho_T} \int_0^T (H(z(s), s))^{2\gamma} ds \leq$$

$$\leq \left[\frac{1}{T} + C_2 + \frac{\|Q\|C_1^2}{\sigma_T \rho_T} \|H(z(t), t)\|_{L^\infty}^{2\gamma - 1}\right] \int_0^T H(z(s), s)ds.$$

Thus

$$\int_0^T H(z(s), s)ds \geq \frac{T\rho_T \sigma_T \|H(z, t)\|_{L^\infty}^{2(1-\gamma)}}{T\|Q\|C_1^2 + \frac{\sigma_T \rho_T (1 + TC_2)}{\|H(z, t)\|_{L^\infty}^{2\gamma - 1}}}$$

and by (3.5)

(3.6) $$\int_0^T H(z(s), s)ds \geq T \left[\frac{\alpha \rho_t}{C_1^2}\right]^{\frac{1}{2\gamma - 1}} \frac{\sigma_T}{\sigma_T(1 + TC_2) + T\alpha\|Q\|}.$$

Finally, by (3.1) we conclude:

$$\Phi(z) \geq \left[\frac{1}{2} - \frac{1}{\alpha}\right] pT \left[\frac{\alpha \rho_T}{C_1^2}\right]^{\frac{1}{2\gamma - 1}} \frac{\alpha \sigma_T}{\sigma_T(1 + TC_2) + T\alpha\|Q\|}. \qquad \square$$

Set

(3.7) $$c_\infty = \left[\frac{1}{2} - \frac{1}{\alpha}\right] pT \left[\frac{\alpha \rho_T}{C_1^2}\right]^{\frac{1}{2\gamma - 1}} \frac{\alpha \rho_T}{\sigma_T(1 + TC_2) + T\alpha\|Q\|}.$$

COROLLARY 3.1. *Let $p > 1$ be a prime integer. If $z(t)$ is a critical point for Φ and*

$$(3.8) \qquad\qquad 0 < \Phi(z) < c_\infty,$$

then $z(t)$ has minimal period pT.

PROOF. Since $\Phi(z) > 0$, then $z(t) \neq$ const.

Assume that z has minimal period $\frac{pT}{k}, k \geq 1$. By the assumption (H_0), necessarily $\frac{p}{k} \in \mathbb{N}$. Since p is prime, either $k = 1$ or $k = p$. But proposition 3.1 rules out the second possibility.   □

This reduces the problem to finding critical points for Φ satisfying (3.8).

The functional Φ inherits the $\mathbb{Z}_p$ symmetry of problem $(1)_p$. In fact define the unitary operator:

$$\hat{T} : E \to E$$

$$u \to u(t + T);$$

we have that $\hat{T}$ generates a $\mathbb{Z}_p$ group action on E and obviously:

$$\Phi(\hat{T}z) = \Phi(z) \qquad \forall z \in E.$$

Hence, to apply Theorem 2.1 we provide the required estimate on Φ.

LEMMA 3.2. *There exist positive constants ρ and c_0 such that for every $z \in E^+ \cap S_\rho$*

$$\Phi(z) \geq c_0.$$

PROOF. Let $z \in E^+$. We have:

$$\Phi(z) = \frac{A(z)}{2} = \int_0^{pT} H(z,t) \geq \frac{\|z\|^2}{2} - \left[\frac{C_1}{\alpha}\right]^{\frac{1}{1-\gamma}} \int_0^{pT} |z|^{\frac{1}{1-\gamma}}.$$

Since $E \subset\to L^{\frac{1}{1-\gamma}}[0,pT]$, there exists a constant $C > 0$ such that

$$\int_0^{pT} |z|^{\frac{1}{1-\gamma}} \leq C\|z\|^{\frac{1}{1-\gamma}}.$$

Therefore

$$\Phi(z) \geq \|z\|^2 \left[\frac{1}{2} - \left[\frac{C_1}{\alpha}\right]^{\frac{1}{1-\gamma}} C\|z\|^{\frac{2\gamma-1}{1-\gamma}}\right].$$

Now take $\rho > 0$ small enough so that $\left[\frac{C_1}{\alpha}\right]^{\frac{1}{1-\gamma}} C\rho^{\frac{2\rho-1}{1-\rho}} < \frac{1}{4}$. Thus $\forall z \in E^+ \cap S_\rho$ we get:

$$\Phi(z) \geq \rho^2/4 = c_0 > 0. \qquad\qquad □$$

Define the integers:

$$m_j^p = \left[\frac{pT}{2\pi}\omega_j\right], \quad \text{i.e. } m_j^p = \text{ largest integer } \leq \frac{pT}{2\pi}\omega_j.$$

For any $k \in \mathbb{N}$ consider the subspace:

$$E_k = E^- \oplus E_0 \oplus E_k^+$$

where

(3.9) $\quad E_k^+ = \left\{ z(t) = \sum_{j=1}^{N} \sum_{m=m_j^p+1}^{m_j^p+k} \frac{1}{2}\left[a_{j,m}\varphi_{j,m}(t) + \overline{a}_{j,m}\overline{\varphi}_{j,m}(t)\right] \right\}.$

Obviously $E^- \oplus E_0$ and $E_k^+ \subset E^+$ are $\hat{T}$ invariant. Furthermore we have:

LEMMA 3.3. *For every $z \in E_k$*

$$\Phi(z) \leq \left[\frac{1}{2} - \frac{1}{\alpha}\right] pT \left[\left[\frac{2\pi}{T}\frac{1}{\mu}\frac{k}{p}\right]^{\frac{\alpha}{\alpha-2}} \left[\frac{1}{a_0}\right]^{\frac{2}{\alpha-2}} + \tau_0 \right]$$

with μ given in (1.6).

PROOF. Let

$$z(t) = \sum_{j=1}^{N} \sum_{m=-\infty}^{m_j^p+k} \text{Re } \left[\alpha_{j,m}\varphi_{j.m}(t)\right]. \text{ We have}$$

$$\Phi(z) = \frac{1}{2}pT \sum_{j=1}^{N} \sum_{m=-\infty}^{m_j^p+k} |\alpha_{j,m}|^2 \left[\frac{2\pi}{T}\frac{m}{p} - \omega_j\right] - \int_0^{pT} H(z,t)$$

$$\leq \frac{1}{2}pT \sum_{j=1}^{N} \sum_{m=-\infty}^{m_j^p+k} \frac{2\pi}{pT}\left[m - \frac{pT\omega_j}{2\pi}\right]|\alpha_{j,m}|^2 - \frac{a_0}{\alpha}\int_0^{pT} |z|^\alpha + \left[\frac{1}{2} - \frac{1}{\alpha}\right]pT\tau_0$$

$$\leq \frac{1}{2}\frac{2\pi}{T}\frac{k}{p}\left[pT \sum_{j=1}^{N} \sum_{m=-\infty}^{m_j^p+k} |\alpha_{j,m}|^2\right] - \frac{a_0}{\alpha}\|z\|_{L^\alpha}^\alpha + \left[\frac{1}{2} - \frac{1}{\alpha}\right]pT\tau_0$$

$$\leq \frac{1}{2}\frac{2\pi}{T}\frac{k}{p}\frac{1}{\mu}\|z\|_{L^2}^2 - \frac{a_0}{\alpha}\|z\|_{L^\alpha}^\alpha + \left[\frac{1}{2} - \frac{1}{\alpha}\right]pT\tau_0.$$

Thus,

(3.10) $\quad \Phi(z) \leq \frac{2\pi}{T}\frac{k}{p}\frac{1}{\mu}\frac{1}{2}\|z\|_{L^2}^2 - \frac{a_0}{\alpha}\left[\frac{1}{pT}\right]^{\frac{\alpha-2}{\alpha}}\|z\|_{L^2}^\alpha + \left[\frac{1}{2} - \frac{1}{\alpha}\right]pT\tau_0.$

Since $\alpha > 2$, the right hand side of (3.10) admits an absolute maximum b_k^{max} that one easily computes to be:

$$b_k^{max} = \left[\frac{1}{2} - \frac{1}{\alpha}\right] pT \left[\left[\frac{2\pi}{T}\frac{1}{\mu}\frac{k}{p}\right]^{\frac{\alpha}{\alpha-2}} \left[\frac{1}{a_0}\right]^{\frac{2}{\alpha-2}} + \tau_0\right]. \qquad \square$$

However, for the inequality

$$b_k^{max} < c_\infty \qquad\qquad (c_\infty \text{ given in (3.7)})$$

to hold, it is necessary and sufficient that

$$(3.11) \qquad\qquad \frac{k}{p} < \tilde{c}_T \qquad\qquad (\tilde{c}_T \text{ given in (1.9)}).$$

Thus we conclude:

PROPOSITION 3.2: *Let $p > 1$ be a prime integer such that (3.11) holds for some integer $k \geq 1$.*
Then for any $z \in E^- \oplus E_0 \oplus E_k^k$, we have

$$\Phi(z) < c_\infty. \qquad\qquad \square$$

Furthermore

LEMMA 3.4: *Let p be a prime integer. The subspace E_k^+ is kN-nice provided $\frac{k}{p} < r_T$.*

PROOF: Clearly $\dim_{\mathbb{R}} E_k^+ = 2kN$. Moreover given the homeomorphism:

$$\varphi : E_k^+ \to \mathbb{C}^{kN}$$

with

$$\varphi\left[\sum_{j=1}^N \sum_{m=m_j^p+1}^{m_j^p+k} \text{Re}\,(a_{j,m}\varphi_{j,m}(t))\right] = (a_{1,m_1^p+1}, ..., a_{1,m_1^p+k}, ..., a_{N,m_N^p+k}),$$

define

$$\tilde{T} = \varphi \circ \hat{T} \circ \varphi^{-1} : \mathbb{C}^{kN} \to \mathbb{C}^{kN}.$$

We have:

$$\tilde{T}(\xi_1, ..., \xi_{kN}) = \left[e^{\frac{i2\pi}{p}(m_1^p+1)}\xi_1, ..., e^{\frac{i2\pi}{p}(m_1^p+k)}\xi_k, ..., e^{\frac{i2\pi}{p}(m_N^p+k)}\xi_{kN}\right].$$

So we are done if we show that $m_j^p + \ell$ is not a multiple of p, $\forall j = 1, ..., N$, and $\forall \ell = 1, ..., k$. Arguing by contradiction, assume that $m_j^p + \ell = sp$ for

some $1 \leq j \leq N, 1 \leq \ell \leq k$ and $s \in \mathbb{N}$. Since $k < p$, necessarily $\omega_k \neq 0$. Write $\dfrac{T\omega_j}{2\pi} = m_j + r_j$ with $m_j = \left[\dfrac{T\omega_j}{2\pi}\right] \in \mathbb{N}, 0 < r_j < 1$ and $\dfrac{pT\omega_j}{2\pi} = m_j^p + r_j^p$ with $0 \leq r_j^p < 1$. We have:

$$s > m_j \quad \text{and} \quad m_j + r_j = s - \frac{\ell}{p} + \frac{r_j^p}{p}.$$

But $\dfrac{k}{p} < 1 - r_j$, therefore $1 \leq s - m_j = r_j + \dfrac{\ell}{p} - \dfrac{r_j^p}{p} < 1 - \dfrac{r_j^p}{p}$ which is clearly absurd. □

We are finally ready to give:

THE PROOF OF THEOREM 1. In virtue of Corollary 3.1 we have only to show that Theorem 2.1 applies to Φ. Certainly Φ satisfies (*) since $\Phi(z) = \dfrac{1}{2} < Lz, z > -\varphi(z)$ with L given by:

$$< Lz, w >= \int_0^{pT} (J\dot{z} - Az) \cdot w \qquad z, w \in E$$

and

$$\varphi(z) = \int_0^{pT} H(z, t).$$

In addition Propositions 3.1 and 3.2 guarantee respectively (f_4) and (f_1); Lemma 3.2 and 3.3 imply (f_2) and (2.12) and finally (f_3) holds since $\Phi|_{E^- \oplus E_0} \leq 0$. So to conclude we have to show that Φ satisfies the (P.S.) condition in $[c_0, c_\infty)$. More generally we prove that any sequence $(z_n) \in E$ such that $\Phi(z_n) \to c > 0$ and

$$\|\Phi'(z_n)\| \to 0$$

admits a convergent subsequence.

Write $z_n = z_n^- + z_n^+ + z_n^0$ with $z_n^\pm \in E^\pm$ and $z_n^0 \in E_0$. For n large we have

$$\frac{1}{2}\|z_n^+\|^2 - \frac{1}{2}\|z_n^-\|^2 - \int_0^{pT} H(z_n, t) > 0$$

which gives $\|z_n^-\|^2 \leq \|z_n^+\|^2$ and by (1.4)

$$\|z_n^0\|_{L^2} \leq d_1 \|z_n^+\|^{2/\alpha} + d_2$$

for some positive constants d_1, and d_2.

By (H_2) we have:

$$\left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\|z_n^+\|^2 - \|z_n^-\|^2\right] \leq \Phi(z_n) - \frac{1}{\alpha} < \Phi'(z_n), z_n >$$

that is:
$$\|z_n^+\|^2 - \|z_n^-\|^2 \leq d_3\|z_n\| + d_4 \text{ with } d_3, d_4 > 0.$$

It follows that
$$\int_0^{pT} H(z_n, t) \leq \frac{d_3}{2}\|z_n\| + \frac{d_4}{2}.$$

In addition,

(3.12) $$\|z_n^+\|^2 = \;< \Phi'(z_n),\; z_n^+ > + \int_0^{pT} H_z(z_n, t) \cdot z_n^+$$

and
$$\int_0^{pT} H_z(z_n, t) \cdot z_n^+ \leq C_1 \int_0^{pT} (H(z_n, t))^\gamma |z_n^+| \leq$$
$$\leq C_1 \|z_n^+\|_{L^{\frac{1}{1-\gamma}}} \left[\int_0^{pT} H(z_n, t)\right]^\gamma \leq$$
$$\leq d_5\|z_n^+\|^{1+\gamma} + d_6$$

with $d_5, d_6 > 0$ constants and $1 + \gamma < 2$. Thus, from (3.12) it follows that $\|z_n^+\|$ is uniformly bounded, that is, $\|z_n\|$ is uniformly bounded. Now by standard compactness arguments it is easy to show that z_n admits a convergent subsequence, using (H_2). The proof is therefore concluded. □

4. - The subquadratic case.

We shall prove Theorem 2 under the stronger assumption that $H(\cdot, t)$ is *strictly* convex. By the same trick used in [13] one then obtains the proof for $H(\cdot, t)$ convex. In this case, to find solutions of $(1)_p$ we shall use the dual approach as introduced in [6] and subsequently extended in [4] and [11].

Hence let H^* be the Legendre transform of H with respect to the z-variable, namely

$$H^*(u, t) = \max_{z \in \mathbb{R}^{2N}} (u \cdot z - H(z, t)), \quad u \in \mathbb{R}^{2N}.$$

Therefore $H^* \in C^1(\mathbb{R}^{2N} \times [0, T])$ and the following reciprocity formulae hold:

$$z = H_u^*(u, t) \leftrightarrow u = H_z(z, t) \leftrightarrow H^*(u, t) + H(z, t) = z \cdot u.$$

In addition, since H satisfies $(H_1)^*$ and $(H_2)^*$ we have:

$$\frac{1}{\gamma^*}a_2^*|u|^{\gamma^*} - b_2 \leq H^*(u, t) \leq \frac{a_1^*}{\alpha^*}|u|^{\alpha^*} + b_1,$$

$H^*(0,t) = 0 \ \forall t$ and $H^*(u,t) \geq 0 \ \forall u \in \mathbb{R}^{2N}, \ \forall t \in [0,T]$, where

$$(4.0) \quad 2 < \gamma^* = \frac{\gamma}{\gamma-1} \leq \alpha^* = \frac{\alpha}{\alpha-1} \text{ and } a_1^* = \left[\frac{1}{a_1}\right]^{\frac{1}{\alpha-1}}, a_2^* = \left[\frac{1}{a_2}\right]^{\frac{1}{\gamma-1}}.$$

In the Banach space

$$E^* = \left\{ u(t) = \sum_{j=1}^{N} \sum_{m \neq \frac{pT\omega_j}{2\pi}} \text{Re} \ (a_{j,m}\varphi_{j,m}(t)) : u \in L^{\alpha^*}[0,T] \right\}$$

define $K : E^* \to E^*$ to be the inverse of the linear operator $M = J\dfrac{d}{dt} - Q$ (i.e. its unique extension in $H^{1/2}$), namely:

$$J\frac{d}{dt}Ku - QKu = u \qquad \forall u \in E^*.$$

The linear operator K is self-adjoint and compact, moreover for any

$$u(t) = \sum_{j=1}^{N} \sum_{m \neq \frac{pT\omega_j}{2\pi}} \text{Re} \ (a_{j,m}\varphi_{j,m}(t)) \in E^*$$

we have:

$$\int_0^{pT} Ku \cdot u = \left[\sum_{j=1}^{N} \sum_{2\pi m \neq pT\omega_j} \frac{1}{\frac{2\pi m}{pT} - \omega_j}|a_{j,m}|^2 \right] pT.$$

Therefore in E^* is well defined the functional:

$$\Phi^*(u) = \int_0^{pT} H^*(u,t) - \frac{Ku \cdot u}{2}$$

and $\Phi^* \in C^1(E^*, \mathbb{R})$.

Moreover if u is a critical point for Φ^*, there exists

$$u_0 \in E_0 = \left\{ u(t) = \sum_{j=1}^{N} \sum_{m = \frac{pT\omega_j}{2\pi}} \text{Re} \ (a_{j,m}\varphi_{j,m}(t) \right\}$$

such that $Ku + u_0 = H_u^*(u,t)$. Hence

$$(4.1) \qquad\qquad\qquad z = H_u^*(u,t)$$

is a solution of $(1)_p$.

Thus we are reduced to finding critical points for Φ^*. Under the given assumptions, the functional Φ^* enjoys the following properties.

PROPOSITION 4.1.

(a) *There exists a constants $a_p > 0$ (depending on p) such that*

$$\Phi^*(u) \geq -a_p \qquad \forall u \in E^*.$$

(b) *If $u \in E^*$ has period T, then*

$$\Phi^*(u) \geq \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^*-2}} \left[\frac{T}{2\pi r_T \mu} \right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT.$$

(c) *If $u = const.$ is a critical point for Φ^*, then*

$$\Phi^*(u) = 0.$$

PROOF. (a) Given $u(t) = \sum_{j=1}^{N} \sum_{m \neq \frac{pT\omega_j}{2\pi}} \mathrm{Re} \ (a_{j,m} \varphi_{jm}(t))$ we have:

$$\Phi^*(u) = \int_0^{pT} H^*(u, t) - \int_0^{pT} \frac{Ku \cdot u}{2} \geq \frac{1}{\gamma^*} a_2^* \int_0^{pT} |u|^{\gamma^*} - \int_0^{pT} \frac{Ku \cdot u}{2} - b_2 pT$$

$$\geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT} \right]^{\frac{\gamma^*-2}{2}} \|u\|_{L^2}^{\gamma^*} - \frac{pT}{2} \sum_{j=1}^{N} \sum_{m \neq \frac{pT\omega_j}{2\pi}} \frac{1}{\frac{2\pi m}{pT} - \omega_j} |a_{j.m}|^2 - b_2 pT$$

$$\geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT} \right]^{\frac{\gamma^*-2}{2}} \|u\|_{L^2}^{\gamma^*} - \frac{pT}{2\pi} \left[\sum_{j=1}^{N} \sum_{m \neq \frac{pT\omega_j}{2\pi}} \frac{1}{m - \frac{pT\omega_j}{2\pi}} |a_{j.m}|^2 \right] \frac{pT}{2} - b_2 pT.$$

Set $r_p = \min_{\substack{m > \frac{pT\omega_j}{2\pi} \\ j=1.-.N}} \left[m - \frac{pT\omega_j}{2\pi} \right] > 0$. We get

$$\Phi^*(u) \geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT} \right]^{\frac{\gamma^*-2}{2}} \|u\|_{L^2}^{\gamma^*} - \frac{pT}{2\pi} \frac{1}{r_p} \frac{1}{\mu} \frac{1}{2} \|u\|_{L^2}^2 - b_2 pT$$

$$\geq - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{1}{\gamma^*-2}} \left[\frac{1}{2\pi r_p \mu} \right]^{\frac{\gamma^*}{\gamma^*-2}} [pT]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT = -a_p.$$

(b) Let $u \in E^*$ have period T, that is:

$$u(t) = \sum_{j=1}^{N} \sum_{m \neq \frac{T\omega_j}{2\pi}} \mathrm{Re} \ (a_{j.m} \varphi_{j,mp}(t)).$$

As above, we have:

$$\Phi^*(u) \geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT}\right]^{\frac{\gamma^*-2}{2}} \|u\|_{L^2}^{\gamma^*} - \frac{T}{2\pi} \left[\sum_{j=1}^{N} \sum_{m \neq \frac{T\omega_j}{2\pi}} \frac{1}{m - \frac{T\omega_j}{2\pi}} |a_{j,m}|^2\right] \frac{pT}{2}$$

$$-b_2 pT \geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT}\right]^{\frac{\gamma^*-2}{2}} \|u\|_{L^2}^{\gamma^*} - \frac{T}{2\pi} \frac{1}{r_T \mu} \frac{1}{2} \|u\|_{L^2}^2 - b_2 pT$$

and therefore

$$\Phi^*(u) \geq - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*}\right] \left[\frac{1}{a_2^*}\right]^{\frac{1}{\gamma^*-2}} \left[\frac{1}{2\pi r_T \mu}\right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2\right] pT.$$

(c) If $u \in E^*$ is constant, we have:

$$u = \sum_{j:\omega_j \neq 0} \text{Re } a_j \varphi_{j,0}$$

and hence

$$0 \leq \int_0^{pT} H_u^*(u,t) \cdot u = \int_0^{pT} Ku \cdot u = -pT \sum_{\omega_j \neq 0} \frac{1}{\omega_j} |a_j|^2 \leq 0.$$

Thus

$$0 = \int_0^{pT} Ku \cdot u = \int_0^{pT} H_u^*(u,t) \cdot u \geq \int_0^{pT} H^*(u,t) \geq 0$$

that is, $\Phi^*(u) = 0$.   □

COROLLARY 4.1: *Let $p > 1$ be a prime integer. If $u \in E^*$ is a critical point for Φ^* and*

$$(4.2) \qquad \Phi^*(u) < - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*}\right] \left[\frac{T}{2\pi \mu r_T}\right]^{\frac{\gamma^*}{\gamma^*-2}} \left[\frac{1}{a_2^*}\right]^{\frac{2}{\gamma^*-2}} + b_2\right] pT,$$

Then $z(t) = H_u^*(u,t)$ is a solution for $(1)_p$ with minimal period pT.

PROOF. Since u is a critical point for Φ^*, there exists $u_0 \in E^*$ such that

$$z(t) = H_u^*(u(t),t) = Ku(t) + u_0(t).$$

In addition, $z(t) \neq$ constant. In fact, $z(t) = \text{const} \Longrightarrow u(t) = \text{const} \Longrightarrow \Phi^*(u) = 0$, contradicting (4.2).

Hence $z(t)$ has either minimal period T or pT . Arguing by contradiction, assume $z(t)$ to have minimal period T and hence $u(t) = H_z(z(t), t)$ has period T and satisfies (4.2).

This is impossible by proposition 4.1. $\qquad\square$

In order to see that the inequality (4.2) holds somewhere in E^*, define the subspace:

$$E_k^* = \left\{ u \in E^* : u(t) = \sum_{j=1}^{N} \sum_{m=m_j^p+1}^{m_j^p+k} \operatorname{Re} \, a_{j.m} \varphi_{j.m}(t) \right\}$$

for some integer $k \geq 1$.

For any $u \in E_k^*$ we have:

$$\Phi^*(u) \leq \frac{a_1^*}{\alpha^*} \int_0^{pT} |u|^{\alpha^*} - \int_0^{pT} \frac{Ku \cdot u}{2} + b_1 pT$$

$$\leq \frac{a_1^*}{\alpha^*} pT \|u\|_{L^\infty}^{\alpha^*} - \frac{1}{2} \int_0^{pT} \frac{Ku \cdot u}{2} + b_1 pT$$

and

$$\|u\|_{L^\infty} \leq \sum_{j=1}^{N} \sum_{m=m_j^p+1}^{m_j^p+k} |a_{j,m}| \mu_j^{1/2}$$

$$\leq \left[\frac{1}{pT} \sum_{j=1}^{N} \mu_j \sum_{m=m_j^p+1}^{m_j^p+k} \left[\frac{2\pi m}{pT} - \omega_j \right] \right]^{\frac{1}{2}} \left[\int_0^{pT} Ku \cdot u \right]^{\frac{1}{2}}$$

$$\leq \left[\frac{1}{pT} \frac{\pi}{T} \frac{k(k+1)}{p} \overline{\mu} \right]^{\frac{1}{2}} \left[\int_0^{pT} Ku \cdot u \right]^{\frac{1}{2}} .$$

Therefore

$$(4.4) \quad \Phi^*(u) \leq \frac{a_1^*}{\alpha^*} pT \left[\frac{\overline{\mu}\pi}{pT^2} \frac{k(k+1)}{p} \right]^{\frac{\alpha^*}{2}} \left[\int_0^{pT} Ku \cdot u \right]^{\frac{\alpha^*}{2}} - \frac{1}{2} \int_0^{pT} Ku \cdot u + b_1 pT.$$

Since $\alpha^* > 2$, the right hand side of (4.4), as a function of $\rho = (\int_0^{pT} Ku \cdot u)^{1/2}$, achieves its absolute minimum at

$$\rho_{\min} = \left[\frac{1}{a_1^* pT} \right]^{\frac{1}{\alpha^*-2}} \left[\frac{pT^2}{\overline{\mu}\pi} \frac{p}{k(k+1)} \right]^{\frac{\alpha^*}{2(\alpha^*-2)}}$$

with a value of:

$$\Phi^*_{\min} = - \left[\left[\frac{1}{2} - \frac{1}{\alpha^*} \right] \left[\frac{1}{a_1^*} \right]^{\frac{2}{\alpha^* - 2}} \left[\frac{T}{\bar{\mu}\pi} \frac{p}{k(k+1)} \right]^{\frac{\alpha^*}{\alpha^* - 2}} - b_1 \right] pT.$$

On the other hand, for the inequality

$$\Phi^*_{\min} < - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^* - 2}} \left[\frac{T}{2\pi r_T \mu} \right]^{\frac{\gamma^*}{\gamma^* - 2}} + b_2 \right] pT$$

to hold, it is necessary and sufficient that

$$\frac{p}{k(k+1)} > \frac{\bar{\mu}\pi}{T} (a_1^*)^{\frac{2}{\alpha^*}} \left\{ \frac{2\alpha^*}{\alpha^* - 2} \left[\frac{\gamma^* - 2}{2\gamma^*} \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma - 2}} \left[\frac{T}{2\pi r_T \mu} \right]^{\frac{\gamma^*}{\gamma^* - 2}} \right.\right.$$
$$\left.\left. + b_1 + b_2 \right] \right\}^{\frac{\alpha^* - 2}{\alpha^*}}$$

i.e.

$$\frac{p}{k(k+1)} > d_T$$

with d_T given in (1.11).

Setting

$$\Gamma_k = \left\{ u \in E_k^* : pT \sum_{j=1}^{N} \sum_{m=m_j^p+1}^{m_j^p+k} \frac{1}{\frac{2\pi m}{pT} - \omega_j} |a_{j,m}|^2 = \rho^2_{\min} \right\},$$

we conclude:

PROPOSITION 4.2. *Let* $p > 1$ *satisfy* $\frac{p}{k(k+1)} > d_T$ *for some* $k \in \mathbb{N}^+$. *Then*

$$\Phi^*(u) < - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^* - 2}} \left[\frac{T}{2\pi r_T \mu} \right]^{\frac{\gamma^*}{\gamma^* - 2}} + b_2 \right] pT$$

for every $u \in \Gamma_k$.   □

As in the previous section, we have that the norm preserving operator

$$\hat{T} : E \to E$$
$$u \to u(t + T)$$

generates a $\mathbb{Z}_p$ group action on E^* and $\Phi^*(\hat{T}u) = \Phi^*(u)$, $\forall u \in E^*$.

Moreover E_k^* as defined in (4.3) is $\hat{T}$-invariant and it is kN-nice provided $\frac{k}{p} < r_T$ and $p > 1$ is prime.

PROOF OF THEOREM 2. First of all, notice that Φ^* satisfies the (P.S.) condition everywhere. This follows as in [13] with obvious modifications.

Now take $E^- = \{0\}$, $E_k^+ = E_k^*$, $c_0 = -a_p$ and

$$c_\infty = - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^*-2}} \left[\frac{T}{2\pi r_T \mu} \right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT.$$

In virtue of Remark 2.4, one easily checks that, with this choice, Theorem 2.2 applies to Φ^*. The conclusion then follows by Corollary 4.1.   □

5. - Second order systems of O.D.E.

Since the arguments here are rather similar to the Hamiltonian ones, we shall only sketch the proofs, emphasizing simply the possible modifications.

The Superquadratic case

Solutions of $(2)_p$ are the critical points for the functional:

$$I(x) = \int_0^{pT} \left[\frac{1}{2} \left(|\dot{x}|^2 - Qx \cdot x \right) - V(x, t) \right]$$

with $x \in H^1 \left[\mathbb{R}_{/pT\mathbb{Z}}, \mathbb{R}^n \right] = X$. Let $(\xi_1, \cdots, \xi_n)$ be a basis in $\mathbb{R}^n$ such that

$$Q\xi_j = \omega_1^2 \xi_j \qquad \text{and } \xi_j \cdot \xi_k = \delta_{j,k}$$

with $0 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2$. Set

$$\psi_{j,m}(t) = \xi_j e^{i\frac{2\pi m}{pT}t} \qquad m \in \mathbb{Z}, \ j = 1, \cdots, n.$$

Hence, for any, $x \in X$ we can write:

$$x(t) = \sum_{j=1}^n \sum_{m=-\infty}^{+\infty} a_{j,m} \psi_{j,m}(t)$$

with $a_{j,m} \in \mathbb{C}$ and $a_{j,-m} = \bar{a}_{j,m}$.

Define the subspaces:

$$X^+ = \left\{ x \in X : x(t) = \sum_{j=1}^{n} \sum_{m^2 > (\frac{pT\omega_1}{2\pi})^2} a_{j.m} \psi_{j.m}(t) \right\}$$

$$X^- = \left\{ x \in X : x(t) = \sum_{j=1}^{n} \sum_{m^2 < (\frac{pT\omega_1}{2\pi})^2} a_{j.m} \psi_{j.m}(t) \right\}$$

$$X_0 = \left\{ x \in X : x(t) = \sum_{j=1}^{n} \sum_{m^2 = (\frac{pT\omega_1}{2\pi})^2} a_{j.m} \psi_{j.m}(t) \right\}$$

so that $X = X^+ \oplus X^- \oplus X_0$. Notice that now X^- is finite dimensional. Furthermore for $x(t) = \sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} a_{j.m} \psi_{j.m}(t) \in X$ we have:

$$A(x) \overset{\text{def}}{=} \int_0^{pT} |\dot{x}|^2 - Qx \cdot x = pT \sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} \left[\left[\frac{2\pi m}{pT}\right]^2 - \omega_j^2 \right] |a_{j,m}|^2.$$

Thus it is natural to consider the following equivalent norm on X: $\|x\|^2 = A(x^+) - A(x^-) + \|x_0\|_{L^2}^2$ where

$$x = x^+ + x^- + x_0 \in X \text{ with } x^{\pm} \in X^{\pm} \text{ and } x_0 \in X_0.$$

Exactly as for Lemma 3.1 we have:

LEMMA 5.1. *Let $x = x(t)$ be the critical point for I. We have*:

(a) $I(x) \geq 0$ *and*

(b) *if $x \in X_0 + X^-$ then $I(x) = 0$.*   □

As the analogue of Proposition 3.1 we get:

PROPOSITION 5.1. *Under the assumptions of Theorem 3, if $x = x(t) \notin X_0 \oplus X^-$ is a T-periodic critical point for I, then*

$$(5.1) \qquad I(x) \geq \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\frac{\alpha^{2\gamma}}{\beta_T C_1^2 T}\right]^{\frac{1}{2\gamma - 1}} pT$$

with β_T defined in (1.13).

PROOF. Let us write $x(t) = x^+(t) + x_1(t)$ with $x^+ \in X^+$, $x_1 \in X_0 \oplus X^-$, so $x^+ \neq 0$ which implies $\int_0^T V(x, t) > 0$.

Easy computations show that:

$$\|x^+\|^2_{L^\infty} \le \beta_T \int_0^T \left[|\dot{x}^+|^2 - Qx^+ \cdot x^+ \right].$$

Furthermore:

$$\int_0^T |\dot{x}^+|^2 - Qx^+ \cdot x^+ = \int_0^T (-\ddot{x} - Qx) \cdot x^+$$

$$\le \|x^+\|_{L^\infty} \int_0^T |-\ddot{x} - Qx| \le \left[\int_0^T |\ddot{x} - Qx| \right] \left[\beta_T \int_0^T \left[|\dot{x}|^2 - Qx^+ \cdot x^+ \right] \right]^{\frac{1}{2}},$$

that is,

(5.2) $$\int_0^T |\dot{x}^+|^2 - Qx^+ \cdot x^+ \le \beta_T \left[\int_0^T |-\ddot{x} - Qx| \right]^2.$$

Thus we derive a lower bound for $\int_0^T V(x,t)$ as follows:

$$\int_0^T V(x,t) \le \frac{1}{\alpha} \int_0^T V_x(x,t) \cdot x = \frac{1}{\alpha} \int_0^T (-\ddot{x} - Qx) \cdot x$$

$$\le \frac{1}{\alpha} \int_0^T (-\ddot{x} - Qx) \cdot x^+ \le \frac{\beta_T}{\alpha} \left[\int_0^T |-\ddot{x} - Qx| \right]^2$$

$$= \frac{\beta_T}{\alpha} \left[\int_0^T |V_x(x,t)| \right]^2 \le \frac{\beta_T}{\alpha} C_1^2 \left[\int_0^T (V(x,t))^\gamma \right]^2$$

$$\le \frac{\beta_T}{\alpha} C_1^2 T^{2(1-\gamma)} \left[\int_0^T V(x,t) \right]^{2\gamma},$$

that is

$$\int_0^T V(x,t) \ge \left[\frac{\alpha}{\beta_T C_1^2} \frac{1}{T^{2(1-\gamma)}} \right]^{\frac{1}{2\gamma-1}}.$$

This readily implies (5.1).   □

COROLLARY 5.1. *Let $p > 1$ prime. If $x = x(t)$ is a critical point for I such that*

$$(5.3) \qquad 0 < I(x) < \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\frac{\alpha^{2\gamma}}{\beta_T C_1^2 T}\right]^{\frac{1}{2\gamma-1}} pT,$$

then $x(t)$ has minimal *period pT.*   □

Once more Corollary 5.1 together with the fact that I is $\hat{T}$-invariant ($\hat{T} : x(t) \to x(t + T)$) reduces the proof of Theorem 3 to a suitable application of Theorem 2.1.

In fact, in analogy with Lemma 3.2 we have:

LEMMA 5.1. *There exist constants $\rho > 0$ and $\delta > 0$ such that for every $x \in X^+ \cap S_\rho$*

$$I(x) \geq \delta.$$

□

Furthermore, following the footsteps of section 3, define the integers

$$m_j^p = \left[\frac{pT\omega_j}{2\pi}\right]$$

and for any $k \in \mathbb{N}^+$ consider the subspace:

$$X_k = X^- \oplus X_0 \oplus X_k^+$$

where

$$(5.4) \qquad X_k^+ = \left\{ x(t) = \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} a_{j,m} \psi_{j,m}(t); a_{j,-m} = \bar{a}_{j,m} \right\}.$$

LEMMA 5.2. *Let $k \in \mathbb{N}^+$ be such that $\frac{k}{p} < \tilde{r}_T$. For any $x \in X_k$ we have:*

$$(5.5) \qquad I(x) \leq \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\left[\left(\frac{2\pi}{T}\right)^2 s_T \frac{k}{p}\right]^{\frac{\alpha}{\alpha-2}} \left[\frac{1}{a_0}\right]^{\frac{2}{\alpha-2}} + \tau_0\right] pT$$

with s_T as defined in (1.15).

PROOF. Let $x(t) = x_1(t) + \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} a_{j,m} \psi_{j,m}(t)$, and $x_1 \in X^- \oplus X_0$. We have:

$$I(x) \leq \frac{pT}{2} \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \left[\left[\frac{2\pi m}{pT}\right]^2 - \omega_j^2\right] |a_{j,m}|^2 - \int_0^{pT} V(x,t)$$

(by 1.14)

$$\leq \frac{1}{2} \left[\frac{2\pi}{T}\right]^2 \frac{k}{p} \left[\sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \left[\frac{|m|}{p} + \frac{T\omega_j}{2\pi}\right] |a_{j.m}|^2\right] pT - \frac{a_0}{\alpha} \int_0^{pT} |x|^\alpha$$

$$+ \left[\frac{1}{2} - \frac{1}{\alpha}\right] \tau_0 pT,$$

that is,

$$I(x) \leq \frac{1}{2} \left[\frac{2\pi}{T}\right]^2 \frac{k}{p} \, s_T \|x\|_{L^2}^2 - \frac{a_0}{\alpha} \left[\frac{1}{pT}\right]^{\frac{\alpha-2}{2}} \|x\|_{L^2}^\alpha + \left[\frac{1}{2} - \frac{1}{\alpha}\right] \tau_0 pT$$

$$\leq \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\left[\left[\frac{2\pi}{T}\right]^2 s_T \frac{k}{p}\right]^{\frac{\alpha}{\alpha-2}} \left[\frac{1}{a_0}\right]^{\frac{2}{\alpha-2}} + \tau_0\right] pT.$$

$\square$

Clearly $X^- \oplus X_0$ and X_k^+ are $\hat{T}$-invariant. In addition, comparing the right hand side of (5.5) with the right hand side of (5.3) and using the same arguments of Lemma 3.4, we conclude:

LEMMA 5.3. *Let* $p > 1$ *be a prime integer such that* $\frac{k}{p} < e_T$ *for some* $k \in \mathbb{N}^+$. *Then for any* $x \in X_0 \oplus X^- \oplus X_k^+$, *we have*

$$I(x) < \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\frac{(\alpha)^{2\gamma}}{\beta C_1^2 T}\right]^{\frac{1}{2\gamma-1}} pT$$

and X_k^+ *is kn-nice*. $\square$

Finally, since the (P.S.) condition for I in the interval $(0, +\infty)$ follows exactly as for the functional Φ of section 3, we conclude the proof of Theorem 3 by applying Theorem 2.1 (or better, Corollary 2.1) with $E_k^+ = X_k^+$, $E^- = X^- \oplus X_0$, $c_0 = \delta > 0$ (defined in Lemma 5.1) and

$$c_\infty = \left[\frac{1}{2} - \frac{1}{\alpha}\right] \left[\frac{(\alpha)^{2\gamma}}{\beta C_1^2 T}\right]^{\frac{1}{2\gamma-1}} pT.$$

The subquadratic case

In analogy with the subquadratic Hamiltonian case, we shall approach problem $(2)_p$ by means of the duality method. Here too we will be only

concerned with the case $V(\cdot, t)$ *strictly* convex, since the convex case then follows as in [13].

Hence define $V^*(\cdot, t)$ to be the Legendre transform of $V(\cdot, t)$, i.e. $V^*(y, t) = \max\limits_{x \in \mathbb{R}^n} (x \cdot y - V(x, t))$.

We have

$$\frac{1}{\gamma^*} \, a_2^* |y|^{\gamma^*} - b_2 \leq V^*(y, t) \leq \frac{a_1^*}{\alpha^*} \, |y|^{\alpha^*} + b_1$$

with γ^*, α^*, a_i^*, $i = 1, 2$, given in (4.0) and

$$V^*(0, t) = 0, \; \forall t; \quad V^*(y, t) \geq 0.$$

Furthermore, in the Banach space:

$$X^* = \left\{ y(t) = \sum_{j=1}^{n} \sum_{m^2 \neq \left[\frac{pT\omega_1}{2\pi} \right]^2} a_{j,m} \psi_{j.m}(t); a_{j,-m} = \bar{a}_{j,m}; \; y \in L^{\alpha^*} [0, pT] \right\}$$

define $-K$ to be the inverse of the operator $\dfrac{d^2}{dt^2} + Q$ (i.e. its unique extension on $H^1(\mathbb{R}_{/pT\mathbb{Z}}, \mathbb{R}^n)$). The linear operator K is therefore compact and self-adjoint.

Moreover, for every

$$y(t) = \sum_{j=1}^{n} \sum_{m^2 \neq \left[\frac{pT\omega_1}{2\pi} \right]^2} a_{j.m} \psi_{j.m}(t)$$

we have:

$$\int_0^{pT} Ky \cdot y = \left[\sum_{j=1}^{n} \sum_{m^2 \neq \left[\frac{pT\omega_1}{2\pi} \right]^2} \frac{1}{\left[\frac{2\pi m}{pT} \right]^2 - \omega_j^2} \, |a_{j,m}|^2 \right] pT.$$

Thus on X^* is well defined the functional:

$$I^*(y) = \int_0^{pT} V^*(y, t) - \frac{Ky \cdot y}{2}$$

and $I^* \in C^1(X^*, \mathbb{R})$.

In addition, if $y(t)$ is a critical point for I^*, then $x(t) = V_y^*(y(t), t)$ is a solution for $(2)_p$.

We have:

PROPOSITION 5.2. (a) *There exists a constant $a_p > 0$ such that*

$$I^*(y) \geq -a_p \qquad \forall y \in X^*.$$

(b) *If $y = y(t) \in X^*$ is T-periodic, then*

$$I^*(y) \geq -\left[\left[\frac{1}{2} - \frac{1}{\gamma^*}\right] \left[\frac{1}{a_2^*}\right]^{\frac{2}{\gamma^*-2}} \left[\frac{1}{\lambda_T} \left[\frac{T}{2\pi}\right]^2\right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT$$

with λ_T given in (1.17).

(c) *If $y(t) = $ const. is a critical point for I^*, then $I^*(y) = 0$.*

PROOF. (a) and (c) follow exactly as for proposition 4.1.

(b) Let $y(t) \in X^*$ be T-periodic, i.e.

$$y(t) = \sum_{j=1}^{n} \sum_{m^2 \neq \left[\frac{T\omega_j}{2\pi}\right]^2} a_{j,m} \psi_{j,pm}(t), \qquad a_{j,-m} = \overline{a}_{j,m}.$$

We have

$$I^*(y) = \int_0^{pT} V^*(y,t) - \frac{Ky \cdot y}{2} \geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT}\right]^{\frac{\gamma^*-2}{2}} \|y\|_{L^2}^{\gamma^*} - b_2 Tp$$

$$- \left[\frac{T}{2\pi}\right]^2 \left[\sum_{j=1}^{n} \sum_{m^2 \neq \left[\frac{pT\omega_j}{2\pi}\right]^2} \frac{1}{m^2 - \left[\frac{T\omega_j}{2\pi}\right]^2} |a_{j,m}|^2 \right] \frac{pT}{2}$$

$$\geq \frac{1}{\gamma^*} a_2^* \left[\frac{1}{pT}\right]^{\frac{\gamma^*-2}{2}} \|y\|_{L^2}^{\gamma^*} - \left[\frac{T}{2\pi}\right]^2 \frac{1}{\lambda_T} \frac{1}{2} \|y\|_{L^2}^2 - b_2 Tp$$

$$\geq -\left[\left[\frac{1}{2} - \frac{1}{\gamma^*}\right] \left[\frac{1}{a_2^*}\right]^{\frac{2}{\gamma^*-2}} \left[\left[\frac{T}{2\pi}\right]^2 \frac{1}{\lambda_T}\right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT.$$

$\square$

Hence, as for Corollary 4.1, we deduce:

COROLLARY 5.2. *If $y(t)$ is a critical point for I^* and*

$$(5.6) \qquad I^*(y) < -\left[\left[\frac{1}{2} - \frac{1}{\gamma^*}\right] \left[\frac{1}{a_2^*}\right]^{\frac{2}{\gamma^*-2}} \left[\frac{1}{\lambda_T} \left[\frac{T}{2\pi}\right]^2\right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT,$$

with $p > 1$ prime integer, then $x(t) = V_y^*(y(t), t)$ is a solution of $(2)_p$ with minimal period pT. □

The estimates from above for I^* are obtained as follows. For any integer $k \geq 1$, consider

$$y_k(t) = \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} a_{j,m} \psi_{j,m}(t).$$

We have:

$$I^*(y_k) \leq \frac{a_1^*}{\alpha^*} \int_0^{pT} |y_k|^{\alpha^*} - \frac{1}{2} \int_0^{pT} K y_k \cdot y_k + b_1 pT$$

$$\leq \frac{a_1^*}{\alpha} pT \|y_k\|_{L^\infty}^{\alpha^*} - \frac{1}{2} \int_0^{pT} K y_k \cdot y_k + b_1 pT.$$

On the other hand:

$$\|y_k\|_{L^\infty} \leq \left[\frac{1}{pT} \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \left[\left[\frac{2\pi m}{pT} \right]^2 - \omega_j^2 \right] \right]^{\frac{1}{2}} \left[\int_0^{pT} K y_k \cdot y_k \right]^{\frac{1}{2}}$$

and straightforward calculations show that:

$$\sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \left[\left[\frac{2\pi m}{pT} \right]^2 - \omega_j^2 \right] =$$

$$= \left[\frac{2\pi}{pT} \right]^2 k \sum_{j=1}^{n} \left[(m_j^p)^2 + (1+k)m_j^p + \frac{(1+k)(2k+1)}{6} - \left[\frac{pT\omega_j}{2\pi} \right]^2 \right].$$

Hence if we let the integers $p > 1$ and $k \geq 1$ satisfy:

$$(5.7) \qquad \frac{k}{p} < \tilde{r}_T$$

we obtain:

$$\sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \left[\left[\frac{2\pi m}{pT} \right]^2 - \omega_j^2 \right] \leq \left[\frac{2\pi}{T} \right]^2 \frac{k(k+1)}{p} \eta_T$$

where we recall $\eta_T = \sum_{j=1}^{n} \left[\frac{T\omega_j}{2\pi} + \frac{1}{2}(1 - \tilde{r}_j) \right]$. Therefore:

$$I^*(y_k) \le \frac{a_1^*}{\alpha} pT \left[\left[\frac{2\pi}{T} \right]^2 \frac{1}{pT} \frac{k(k+1)}{p} \eta_T \right]^{\frac{\alpha^*}{2}} \left[\int_0^{pT} K y_k \cdot y_k \right]^{\frac{\alpha^*}{2}}$$

$$-\frac{1}{2} \int_0^{pT} K y_k \cdot y_k + b_1 pT,$$

and the right hand side of the above inequality, as a function of $\rho = \left[\int_0^{pT} K y_k(t) \cdot y_k(t) dt \right]^{\frac{1}{2}}$, achieves its absolute minimum for

$$\rho_{\min} = \left[\frac{1}{a_1^* pT} \right]^{\frac{1}{\alpha^*-2}} \left[\left[\frac{T}{2\pi} \right]^2 pT \frac{p}{k(k+1)} \frac{1}{\eta_T} \right]^{\frac{\alpha^*}{2(\alpha^*-2)}}$$

given by

$$I_{\min}^* = -pT \left[-b_1 + \left[\frac{1}{2} - \frac{1}{\alpha^*} \right] \left[\frac{1}{a_1^*} \right]^{\frac{2}{\alpha^*-2}} \left[\left[\frac{T}{2\pi} \right]^2 \frac{p}{k(k+1)} \frac{1}{\eta_T} \right]^{\frac{\alpha^*}{\alpha^*-2}} \right].$$

Furthermore for the inequality

$$I_{\min}^* < - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^*-2}} \left[\frac{1}{\lambda_T} \left[\frac{T}{2\pi} \right]^2 \right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT$$

to hold, it is necessary and sufficient that

(5.8) $$\frac{p}{k(k+1)} > \tilde{d}_T \qquad (\tilde{d}_T \text{ as given in (1.19))}.$$

Setting

$$\Gamma_k = \left\{ y(t) = \sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} a_{j.m} \psi_{j.m}(t) \right.$$

$$: \left. \left[\sum_{j=1}^{n} \sum_{|m|=m_j^p+1}^{m_j^p+k} \frac{1}{\left[\frac{2\pi m}{pT} \right]^2 - \omega_j^2} |a_{j.m}|^2 \right] pT = \rho_{\min}^2 \right\},$$

we conclude:

PROPOSITION 5.3. *Let* $p > 1$ *be a prime integer such that* (5.7) *and* (5.8) *hold for some* $k \geq 1$.

Then for any $y \in \Gamma_k$ *we have*:

$$I^*(y) < - \left[\left[\frac{1}{2} - \frac{1}{\gamma^*} \right] \left[\frac{1}{a_2^*} \right]^{\frac{2}{\gamma^*-2}} \left[\frac{1}{\lambda_T} \left[\frac{T}{2\pi} \right]^2 \right]^{\frac{\gamma^*}{\gamma^*-2}} + b_2 \right] pT.$$

□

At this point, the same arguments as for Theorem 2, allow us to conclude Theorem 4. The details are left to the reader.

REFERENCES

[1]   V. BENCI: *A geometrical index for the group* S^1 *and some applications to the study of periodic solutions of ordinary differential equations*, Comm. Pure Appl. Math. **34** (1981), pp. 393-432.

[2]   V. BENCI: *On the critical point theory for indefinite functionals in the presence of symmetries*, Trans. Am. Math. Soc. **274** (1982), pp. 533-572.

[3]   V. BENCI - D. FORTUNATO: *Subharmonic solutions of prescribed minimal period for nonautonomous differential equations*, preprint.

[4]   H. BERESTYCKI - J.M. LASRY - G. MANCINI - B. RUF: *Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces*, Comm. Pure Appl. Math. **38** (1985), pp. 253-289.

[5]   G.D. BIRKHOFF - D.C. LEWIS: *On the periodic motion near a given periodic motion of a dynamical system*, Ann. Mat. Pure Appl. **12** (1933), pp. 117-133.

[6]   F. CLARKE - I. EKELAND: *Nonlinear oscillations and boundary value problems for Hamiltonian systems*, Archive Rat. Mech. Anal. **78** (1982), pp. 315-333.

[7]   I. EKELAND - H. HOFER: *Subharmonics for convex nonautonomous Hamiltonian systems*, Comm. Pure Appl. Math. **40** (1987), pp. 1-36.

[8]   I. EKELAND - H. HOFER: *Periodic solutions with prescribed minimal period for convex autonomous Hamiltonian systems*, Inventions Math. **81** (1985), pp. 155-188.

[9]   E. FADELL - S. HUSSEINI - P.H. RABINOWIZ: *Borsuk-Ulam theorem for arbitrary* S^1 *actions and applications*, M.R.C. Tech. Summary report 2301 Madison, WI 53706.

[10] M. GIRARDI - M. MATZEU: *Solutions of minimal period for a class of nonconvex Hamiltonian systems and application to the fixed Energy problem*, Nonlinear Analysis T.M.A. **Vol. 10** n. 34 (1986), pp. 371-382.

[11] M. GIRARDI - M. MATZEU: *Periodic solutions of convex autonomous Hamiltonian systems with quadratic growth at the origin and superquadratic at infinity*, preprint.

[12] R. MICHALEK: *A* $\mathbb{Z}^p$ *Borsuk-Ulam theorem and Index theory with a multiplicity result*, preprint.

[13] R. MICHALEK - G. TARANTELLO: *Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems*, J. Diff. Eq. in press.

[14] J. MOSER: *Proof of a generalized fixed point theorem due to G.D. Birkhoff*, Lecture notes in Mathematics, **597** Springer Verlag, New York 1977.

[15] L. NIRENBERG: *Comments on nonlinear problems*, Conference Proceedings, Catania Sept. 1981.

[16] P.H. RABINOWITZ: *On subharmonic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **33** (1980), pp. 609-633.

[17] P.H. RABINOWITZ: *Variational methods for nonlinear eigenvalue problem, in Eigenvalue of nonlinear problems*, ed. G. Prodi, edizioni Cremonese Rome 1974.

[18] H. WEYL: *The theory of groups and quantum mechanics*, Dover, New York, 1950.

[19] V. BENCI - D. FORTUNATO: *A "Birkhoff-Lewis" type result for a class of Hamiltonian systems*, preprint.

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