ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 15, nº 2 (1988), p. 193-218
<http://www.numdam.org/item?id=ASNSP_1988_4_15_2_193_0>

Varieties whose Hyperplane Section are $\mathbb{P}^k_{\mathbb{C}}$ Bundles

MARIA LUCIA FANIA - ANDREW JOHN SOMMESE

In this article we study the following problem.

PROBLEM. *Let X be a normal projective variety. Let L be an ample line bundle on X that is spanned at all points of X by global sections. Assume that some normal $A \in |L|$ is a $\mathbb{P}^k_{\mathbb{C}}$ bundle $f : A \to Y$ over a projective variety Y. Describe X.*

The second author studied this earlier in [So1] where he showed (as a consequence of an extension theorem of his) that if A is a smooth ample divisor on a smooth projective X and $k \geq 2$ then f extends holomorphically to a $\mathbb{P}^{k+1}$ bundle $\overline{f} : X \to Y$ with L restricted to a general fibre isomorphic to $\mathcal{O}_{\mathbb{P}^{k+1}}(1)$. Some technical improvements were made in this result by Fujita [Fu1, Fu2] and Silva [Si]. We include a quite general extension theorem subsuming all these results in a short appendix. This paper is concerned with the much more subtle case when the fibre of f is $\mathbb{P}^1$.

The key to analyze X is to show that the map $f : A \to Y$ extends to a holomorphic map $\overline{f} : X \to Y$. This is not always true - examples with $Y = \mathbb{P}^n$ for some $n \geq 1$ are easy to construct. We rule this sort of example out by assuming that Y has a nontrivial top degree holomorphic form.

THEOREM. *Let X, A, Y, and f be as in the above problem with $k = 1$. If $h^0(KY) > 0$ and A is smooth then f extends to a meromorphic map $\overline{f} : X \to Y$ holomorphic in a neighborhood of A. If X is also a local complete intersection then $\overline{f}$ is a holomorphic $\mathbb{P}^2$ bundle.*

If $\overline{f}$ is holomorphic, then it is an easy consequence of an earlier result of the second author [So1] that $\dim Y \leq 2$ and in the case that $\dim Y = 2$, $\overline{f} : X \to Y$ is a holomorphic $\mathbb{P}^2$ bundle with L restricted to a fibre of $\overline{f}$ isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$. The classical case when $\dim Y = 1$ has been thoroughly investigated by Bădescu ([B1], [B2], [B3]).

The above theorem is proved as a consequence of a more general meromorphic extension theorem.

One form of it is the following.

Pervenuto alla Redazione il 28 Aprile 1986 ed in forma definitiva il 23 Febbraio 1988.

THEOREM. *Let L be an ample line bundle on a normal projective variety X. Assume that L is spanned at all points by global sections and that there is a normal $A \in |L|$ which fibres holomorphically $f : A \to Y$ over a normal projective variety Y. Assume that:*

a) *X is a local complete intersection.*

b) *The general fibre of f is $\mathbb{P}^1$ and both A and Y have at most rational singularities.*

c) *There is a desingularization $\overline{Y}$ of Y with $h^0(K\overline{Y}) > 0$.*

Then f extends to a meromorphic map $\overline{f} : X \to Y$ which is holomorphic in a neighborhood of the open set $U \subset \mathrm{reg}(A)$ such that $f_U : U \to f(U)$ is a $\mathbb{P}^1$ bundle.

The most natural approach to such extension theorems is to choose a very ample line bundle E on Y, show that f^*E extends to a line bundle $\mathcal{E}$ on X, and show that a "lot" of sections of f^*E extends to $\mathcal{E}$. This was the approach in [So1] (cf. the appendix to this paper) but it works if dim $A = 1+$ dim Y only in very special cases, e.g. [B1] for the case when A is a $\mathbb{P}^1$ bundle over Y.

The second approach is to attempt to construct $\overline{f}$ geometrically. The idea is to take a general fibre λ of f and look at the closure F of all deformations λ' of λ such that $\lambda \cap \lambda'$ is not empty. F should be the general fibre of $\overline{f}$. The main trouble in this approach is showing that dim $F = 2$. A counterexample with $Y = \mathbb{P}^n$ shows that F can equal X. A modified form of the above approach does work. We use a non-trivial holomorphic form on the desingularization of Y to guarantee that dim $F = 2$. To do this we need control over the parameter space of the set of deformations λ' of λ that meet λ. For this reason we restrict to deformations λ' of λ such that λ' meets λ and such that λ' is a fibre of a deformation $f' : A' \to Y'$ of $f : A \to Y$ where $A' \in |L|$. This requires us to show first that for most $A' \in |L|$, an appropriate $f' : A' \to Y'$ exists.

The contents of this paper are as follows.

In §0 we present background material for which there is no good reference (especially material on vanishing theorems and extension of line bundles). We also present the standard counterexamples to extension. We close the section with the following extension theorem for threefolds.

(0.8) THEOREM. *Let L be an ample line bundle on a normal variety X. Assume that there is a normal $A \in |L|$ and a holomorphic map $f : A \to Y$ where Y is a smooth curve of positive genus and the general fibre of f is $\mathbb{P}^1$. Then f extends to a meromorphic map $\overline{f} : X \to Y$, that is holomorphic in a neighborhood of A. If X is Cohen-Macaulay then $\overline{f}$ is holomorphic.*

In an appendix to §0 we give the strongest known version of the extension theorem for holomorphic surjections $f : A \to Y$ with dim $A-$ dim $Y \geq 2$ that was originally given for manifolds in [So1].

In §1 we prove some results on descent of holomorphic forms.

In §2 we prove the general meromorphic extension theorem.

In §3 we use the extension theorems to analyze the global structure of X. We also deduce some results on when a modification of a hyperplane section extends to a modification of a projective variety; these results which are in the same vein as [Fa1, Fa2, Fa+So, So2, So3, So4] were one of our main motivations to study the problem stated at the beginning of this introduction.

In §4 we discuss a conjecture on extension.

We would like to thank J. Noguchi for helpful remarks on the De Franchis theorem. We would like to express our thanks to the Max Planck Institute for Mathematics for making this joint research possible. The second author would like to thank the University of Notre Dame and the National Science Foundation (DMS 82-00629 and DMS 84-20315) for their support.

0. - Background Material

Our notation is the same as in [So2] and [Fa1]. For the convenience of the reader we review it here.

(0.1) We work over the complex numbers. All spaces are complex analytic and all maps are holomorphic. By **variety** we mean an irreducible and reduced complex analytic space. If X is a complex analytic space, we denote its holomorphic structure sheaf by $\mathcal{O}_X$. We do not distinguish notationally between a locally free coherent analytic sheaf and its associated holomorphic vector bundle.

We denote the sections of a sheaf $\mathcal{S}$ over X by $\Gamma(X, \mathcal{S})$, or $\Gamma(\mathcal{S})$ when no confusion will result. We say that a line bundle L on a complex space, X, is **semi-ample** if there is some $t > 0$ such that $\Gamma(L^t)$ spans L^t at all points.

Assume that X is a compact variety and L is a semi-ample line bundle on X. If $\Gamma(L^t)$ spans L^t, where $t > 0$, and dim $\varphi(X) =$ dim X for the map $\varphi : X \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma(L^t)$, then L is said to be **big**.

If X is a connected complex manifold then we have the dualizing sheaf $K_X = \wedge^n T_{X^*}$ where $n =$ dim X and T_{X^*} is the cotangent bundle of X. If X is a normal variety then the dualizing sheaf $K_X = j_* K_{\mathrm{Reg}(X)}$, where

$$j : \mathrm{Reg}(X) \to X$$

is the inclusion of the smooth points of X into X.

Given an effective normal Cartier divisor A on a normal Cohen-Macaulay variety X we have by [A + K], pg. 7:

$$(K_X \otimes [A]) \doteq K_A.$$

Let X be a variety and let $p : \overline{X} \to X$ be a resolution of singularities, i.e. $\overline{X}$ is a complex manifold and p is a surjective holomorphic map which gives a biholomorphism from $\overline{X} - p^{-1}(\mathrm{Sing}(X))$ to $X - \mathrm{Sing}(X)$. The Leray sheaves $p_{(i)}(\mathcal{O}_{\overline{X}})$ for $i \geq 0$ are independent of the resolution; we denote them by $\mathcal{S}_i(X)$. X is normal if and only if $\mathcal{S}_0(X) \cong \mathcal{O}_X$.

Assume that X is normal. If $\mathcal{S}_i(X) = 0$ for $i > 0$ then the singularities of X are said to be rational. It is a theorem of Kempf ([Ke], pg. 50) that X has rational singularities if and only if X is Cohen-Macaulay and $p_*(K_{\overline{X}}) \cong K_X$. We denote by **Irr**$(X)$, the **irrational locus of** X, which is the union of the supports of the sheaves $\mathcal{S}_i(X)$ for $i > 0$.

(0.2) We need the following vanishing theorem ([$(F+L)2$]; theorem (0.2)) due to Picard, Kodaira, Ramanujam, Mumford, Grauert and Riemenschneider [Gr+Ri], Kawamata [Ka] and Viehweg [V], and Kempf [Ke]. See [Sh+So] for a discussion of similar results.

(0.2.1) **Vanishing Theorem.** *Let L be a negative line bundle on a normal projective variety X. If $c_1(L)^{n-t} \cdot H^t > 0$ for some ample divisor H and some $t \geq 0$ then*:

a) $H^i(X, L^{-1}) = 0$ *for* $i <$ min $\{\dim X - t, 2\}$,

b) $H^i(X, K_X \otimes L) = 0$ *for* $i >$ max $\{t, \dim \mathrm{Irr}(X)\}$.

PROOF. a) is a simple variant of Mumford's vanishing theorem that uses [Ka] and [V] in place of the usual Kodaira vanishing theorem (cf. [Sh+So]).

To see b), let $p : \overline{X} \to X$ be a projective desingularization of X. Consider the exact sequence:

$$(0.2.1*) \qquad 0 \to p_*(K_{\overline{X}}) \otimes L \to K_X \otimes L \to \mathcal{S} \otimes L \to 0.$$

By Kempf's theorem support $(\mathcal{S} \otimes L) \subset \mathrm{Irr}(X)$. Therefore by the long exact cohomology sequence associated to $*$), the theorem will follow from $H^i(p_* K_{\overline{X}} \otimes L) = 0$ for $i >$ max $\{t, \dim \mathrm{Irr}(X)\}$. Since

$$p_{(i)}(K_{\overline{X}} \otimes p^* L) \cong p_{(i)}(K_{\overline{X}}) \otimes L = 0 \text{ for } i > 0$$

by the Grauert-Riemenschneider vanishing theorem, theorem (0.2.1b) will follow from the Leray spectral sequence for p and p^*L, and the fact that $H^i(K_{\overline{X}} \otimes p^* L) = 0$ for $i > t$. This last fact follows immediately from the Kawamata-Viehweg vanishing theorem ([Ka], [V]).

□

(0.2.2) THEOREM. *Let $f : X \to Y$ be a holomorphic surjective map from a compact normal projective variety X to a projective variety Y. Assume that* $\dim X - \dim Y \geq 2$. *Assume that L is a semi-ample and big line bundle on X. Then given any locally free sheaf E on Y, $H^1(X, L^{-k} \otimes f^* E) = 0$ for $k \geq 1$.*

PROOF. Let $\pi : \overline{X} \to X$ be a projective resolution of singularities of X. It is clear by the Leray spectral sequence that $H^1(X, L^{-k} \otimes f^* E)$ injects into

$H^1(\overline{X}, (\pi^* L)^{-k} \otimes (f \circ \pi)^* E)$. Therefore, using $\overline{X}$, $f \circ \pi$, and $\pi^* L$ instead of X, f, and L respectively, we have reduced to the case when X is smooth.

Using dim $X-$ dim $Y \geq 2$ the result follows from [Fu1; Corollary A6]. $\square$

(0.3) We need some information about extension of line bundles.

(0.3.1) LEMMA. *Let A be an effective ample divisor on a projective variety X of dimension ≥ 4. Assume that $A \subset \operatorname{Reg}(X)$. Then for any desingularization $\overline{X}$ of X the restriction map* $\operatorname{Pic}(\overline{X}) \to \operatorname{Pic}(A)$ *has finite cokernel.*

PROOF. Since $A \subset \operatorname{Reg}(X)$, X has isolated singularities and we can assume without loss of generality that X is normal.

Let $\pi : \overline{X} \to X$ denote a desingularization of X. Since π is a biholomorphism from $\overline{X} - \pi^{-1}(\operatorname{Sing}(X)) \to X-\operatorname{Sing}(X)$ we identify A and $\pi^{-1}(A)$.

Consider the long exact cohomology sequences associated to the exponential sequences on $\overline{X}$ and A where the vertical maps are restrictions:

$$\begin{array}{ccccccc} H^1(\overline{X}, \mathcal{O}_{\overline{X}}) & \to & \operatorname{Pic}(\overline{X}) & \to & H^2(\overline{X}, \mathbb{Z}) & \to & H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(A, \mathcal{O}_A) & \to & \operatorname{Pic}(A) & \to & H^2(A, \mathbb{Z}) & \to & H^2(A, \mathcal{O}_A) \end{array}$$

From the vanishing theorem (0.2.1), we know that $H^i(\overline{X}, [A]^{-1}) = 0$ for $i \leq \dim A$ and hence that $H^i(\overline{X}, \mathcal{O}_{\overline{X}}) \triangleq H^i(A, \mathcal{O}_A)$ for $i \leq \dim A - 1$. Therefore we will be done by a diagram chase if we show that the restriction $H^2(\overline{X}, \mathbb{Z}) \to H^2(A, \mathbb{Z})$ has finite cokernel. This will follow if we show that $H^2(\overline{X}, \mathbb{Q}) \to H^2(A, \mathbb{Q})$ is onto.

Choose $n > 0$ such that $[A]^n$ is very ample and embeds X in $\mathbb{P}^N_{\mathbb{C}}$ using $\Gamma([A]^n)$. There is a hyperplane $H' = \mathbb{P}^{N-1}_{\mathbb{C}}$ that meets X in nA. The hyperplanes sufficiently near H' meet X in sets contained in a neighborhood $V \subset \operatorname{Reg}(X)$ of A which is a deformation retract of A. The basic result of [So5] shows that for any of these nearby hyperplanes H, the restriction mapping $R_H : H^j(V, \mathbb{Z}) \to H^j(H \cap X, \mathbb{Z})$ is an isomorphism for $j \leq \dim X - 2$. Choosing an H near H' so that $A' = H' \cap X$ is smooth we see that

$$H^2(\overline{X}, \mathbb{Q}) \to H^2(A, \mathbb{Q}) \to 0$$

is equivalent to showing that

$$H^2(\overline{X}, \mathbb{Q}) \to H^2(A', \mathbb{Q}) \to 0.$$

Indeed consider

$$H^2(\overline{X}, \mathbb{Q}) \to H^2(V, \mathbb{Q}) \begin{cases} H^2(A, \mathbb{Q}) \\ H^2(A', \mathbb{Q}) \end{cases}$$

By Kronecker duality we are reduced to showing that:

$$0 \to H_2(A', \mathbb{Q}) \to H_2(\overline{X}, \mathbb{Q}).$$

Since the intersection homology of a manifold is equal to its usual homology $[(G+M)3]$ and since the rational intersection homology of a complex algebraic variety X injects into the rational intersection homology of any desingularization $\overline{X}$ $[(G+M)1]$, we are reduced to showing that:

$$0 \to IH_2(A', \mathbb{Q}) \to IH_2(\overline{X}, \mathbb{Q}),$$

where IH_* denotes the intersection homology. This last injection follows from the beautiful result $[(G+M)3]$ that, for a hyperplane section of a variety by a hyperplane transverse to all strata of a Morse stratification of the variety, (which $A' \subset \text{Reg}(X)$ certainly is) the usual first Lefschetz theorem holds with intersection homology replacing the usual homology.

□

(0.3.2) LEMMA. *Let A be an effective ample divisor on an irreducible projective local complete intersection X. Assume that* cod $\text{Irr}(X) \geq 3$. *Under restriction* $\text{Pic}(X) \hat{=} \text{Pic}(A)$ *if* dim $X \geq 4$ *and*

$$0 \to \text{Pic}(X) \to \text{Pic}(A)$$

has torsion free cokernel if dim $X = 3$.

PROOF. By the usual argument, using the long exact cohomology sequence associated to the exponential sequence of X and A, the above result will follow if we show that $\pi_i(X, A, a) = 0$, with $i \leq \dim A$ and any basepoint $a \in A$, and also that $H^i(X, [A]^{-1}) = 0$ for $i = 1, 2$. The former is the very useful Lefschetz theorem of Hamm [H] (see $[(G+M)2]$ also) and the latter is just (0.2.1).

□

In the same spirit as the above results we need information about holomorphic forms on the desingularization of a variety.

(0.3.3) LEMMA. *Let L be a line bundle on a normal projective variety, X, of dimension n. Assume that L is semi-ample and big. Let $A \in |L|$ be normal with at most rational singularities. Let $\pi_2 : \overline{X} \to X$ be a desingularization of X and let $\pi_1 : \overline{A} \to A'$ be a desingularization of the proper transform, A', in A of $\overline{X}$. Let $R : \Gamma(\wedge^k T_{\overline{X}}^*) \to \Gamma(\wedge^k T_{\overline{A}}^*)$ be the map induced by π_1. Then R is a surjection for $k <$* dim A.

PROOF. By Hodge theory it suffices to show that the map

$$\overline{R} : H^k(\mathcal{O}_{\overline{X}}) \to H^k(\mathcal{O}_{\overline{A}})$$

induced by π_1 is a surjection for $k < \dim A$. By (0.2.1) the restriction

$$H^k(\mathcal{O}_{\overline{X}}) \to H^k(\mathcal{O}_{A''})$$

is an isomorphism for $k < \dim A$ where $A'' = \pi^{-1}(A)$.

Using this and considering the commutative diagram:

$$\begin{tikzcd}
H^k(\mathcal{O}_{\overline{A}}) & \\
H^k(\mathcal{O}_{A'}) \arrow[u] & H^k(\mathcal{O}_{\overline{X}}) \arrow[ul, "\overline{R}"'] \arrow[dl] \\
H^k(\mathcal{O}_{A''}) \arrow[u] & \\
H^k(\mathcal{O}_{A}) \arrow[u] &
\end{tikzcd}$$

it suffices to show that the map

$$\pi^* : H^k(\mathcal{O}_A) \to H^k(\mathcal{O}_{\overline{A}})$$

induced by the composition $\pi : \overline{A} \to A$ of π_1 and $A' \to A$ is an isomorphism. Using the Leray spectral sequence for π and the fact that A, having only rational singularities, is equivalent to $\pi_{(i)}(\mathcal{O}_{\overline{A}}) = 0$ for $i > 0$, this is clear.

□

(0.4) LEMMA. *Let $\varphi : Z \to \mathbb{P}_{\mathbb{C}}$ be a holomorphic map of a projective variety Z with* $\dim \varphi(Z) \geq 2$. *Given a general hyperplane H on $\mathbb{P}_{\mathbb{C}}, \varphi^{-1}(H)$ is irreducible. Given any hyperplane H on $\mathbb{P}_{\mathbb{C}}, \varphi^{-1}(H)$ is connected.*

PROOF. This is a standard fact, e.g. [Sh+So; theorem (3.42)].

□

(0.5) We give here a few standard counterexamples to the extension problem discussed in the introduction. The most obvious is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. This can be generalized slightly. Let $H_d \subset \mathbb{P}^3$ be a smooth degree d hypersurface that contains a line λ, e.g. let H_d be defined by

$$z_0^d + z_1^d + z_2^d + z_3^d = 0.$$

Then $L \doteq \mathcal{O}_{\mathbb{P}^3}(1) \otimes [\lambda]^{-1}$ is spanned by global sections and gives a holomorphic surjection $f : H_d \to \mathbb{P}^1$ with general fibre biholomorphic to a curve of degree $d-1$ in $\mathbb{P}^2$; see [So1] for more on this type of fibration. Clearly f cannot extend holomorphically to $\mathbb{P}^3$.

Many examples of non-extendable maps with $\dim \overline{Y} = 1$ can be given. There is one example of a $\mathbb{P}^1$ bundle A over Y with $\dim Y > 1$, where X is not a $\mathbb{P}^2$ bundle. The following simple argument was given to us by E. Sato.

Let $Y = \mathbb{P}^n$ with $n > 1$. Let γ be a non-trivial element of $H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$ and let $F^* =$ a direct sum of $n+1$ copies of $\mathcal{O}_{\mathbb{P}^1}(-2)$. Let E^* be the unique extension

$$0 \to F^* \to E^* \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

such that $1 \in \Gamma(\mathcal{O}_{\mathbb{P}^1})$ goes to

$$\underbrace{\gamma \oplus \cdots \oplus \gamma}_{n+1 \text{ times}} \in H^1(F^*).$$

Note that $\mathbb{P}(F)$ is a very ample divisor on $\mathbb{P}(E)$. To see this it must just be noted that E is ample. By dualizing the above exact sequence we can easily check that E is spanned by global sections. We must only check that E does not contain a trivial summand.

If it did then E^* would have a nowhere vanishing section. Since F^* has no section, the image of this section would split the above sequence contradicting the non-triviality of γ. Thus $\mathbb{P}(F)$ is a very ample divisor of $\mathbb{P}(E)$.

Note that $\mathbb{P}(F) = \mathbb{P}^1 \times \mathbb{P}^n$. Since there are no non-trivial maps from $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$, the map $\mathbb{P}(F) \to \mathbb{P}^n$ cannot extend to a map from the $\mathbb{P}^{n+1}$ bundle $\mathbb{P}(E)$ to $\mathbb{P}^n$.

(0.6) THEOREM. *Let L be an ample line bundle on a normal projective local complete intersection X. Assume that L is spanned at all points by global sections,* $\dim X \geq 4$, *and that* cod $\mathrm{Irr}(X) \geq 3$. *Assume that there is an $A \in |L|$ and surjective holomorphic map $f : A \to Y$ into a normal projective variety Y. If f extends to a meromorphic map $\overline{f} : X \to Y$ holomorphic in a neighborhood of A and* $\dim A > \dim Y$, *then $\overline{f}$ is holomorphic.*

PROOF. Let E be a very ample line bundle on Y and let $\mathcal{E}$ be the extension of $f^*\mathcal{E}$ to X that exists by lemma (0.3.2). If we knew that pullbacks under f of sections of E extended to sections of $\mathcal{E}$, then we would be done by an argument of [So1] in the smooth case that was nicely generalized to arbitrary X in [Fu1]. Indeed $\dim Y + 1$ sections span E. Thus $\dim Y + 1$ sections span $\mathcal{E}$ off an analytic set $\mathbb{A} \subset X - A$. Therefore $\mathbb{A}$ is empty or $\dim \mathbb{A} \geq \dim X - \dim Y - 1 > 0$. But since $\mathbb{A} \subset X - A$, $\dim \mathbb{A} = 0$. Thus since $\mathcal{E}$ is spanned by $\dim Y + 1$ sections, the map associated to pullbacks of sections has a $\dim Y$ dimensional image. It is easy to see that this must be Y.

If $\overline{f}^* D \in |\mathcal{E}|$, when $D \in |E|$, then we would be done by the above reasoning. If $\overline{f}^* D$ were Cartier this would thus be clear since $0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(A)$. Unfortunately it is not obvious that f^*D is Cartier.

Let $\pi : \overline{X} \to X$ be a desingularization of the graph of $\overline{f}$. Choose an $A' \in |L|$ such that $\overline{A} = \pi^{-1}(A')$ is smooth and $\overline{f}$ is holomorphic in a neighborhood of A'. This is possible since $\overline{f}$ is holomorphic in a neighborhood of A.

Let $f' : \overline{X} \to Y$ be the holomorphic map induced by $\overline{f}$. Let E and $\mathcal{E}$ be as before and let $M = f'^*E$. If we show that $M \cong \pi^*\mathcal{E}$ we will be done.

Consider:

$$0 \to \pi^*(\mathcal{E} \otimes L^{-1}) \otimes M^{-1} \to \pi^*\mathcal{E} \otimes M^{-1} \to (\pi^*\mathcal{E} \otimes M^{-1})_{\overline{A}} \to 0.$$

Since $\pi^*\mathcal{E} \otimes M^{-1} \doteq \mathcal{O}_{\overline{A}}$ it suffices to show that $H^1(\pi^*(\mathcal{E} \otimes L^{-1}) \otimes M^{-1}) = 0$. Since the map associated to $\Gamma((\pi^*L)_{\overline{A}})$ has a dim A dimensional image, it follows that $H^1((\pi^*(\mathcal{E} \otimes L^{-t}) \otimes M^{-1})_{\overline{A}}) = H^1((\pi^*L^{-t})_{\overline{A}}) = 0$ for $t > 0$. Therefore, by tensoring the above exact sequence with π^*L^{-t}, for $t > 0$, and using the associated long exact cohomology sequence, we are reduced to showing that

$$H^1(\overline{X}, \pi^*(\mathcal{E} \otimes L^{-t}) \otimes M^{-1}) = 0 \text{ for some } t > 0.$$

By Serre duality and the Leray spectral sequence we are reduced to showing that:

$$H^i(X, L^t \otimes \mathcal{E}^* \otimes \pi_{(j)}(K_{\overline{X}} \otimes M)) = 0 \text{ for } i + j = \dim A \text{ and } t >> 0.$$

Since M is spanned, it follows from [Gr+Ri] that $\pi_{(j)}(K_{\overline{X}} \otimes M) = 0$ for $j > 0$. Since L is ample $H^{\dim A}(X, L^t \otimes \mathcal{E}^* \otimes \pi_*(K_{\overline{X}} \otimes M)) = 0$ for $t >> 0$. □

(0.7) SLICING LEMMA. *Let $f : X \to Y$ be a holomorphic surjection between projective manifolds. If H is a general hyperplane section of Y then*

a) *H and $H' = f^{-1}(H)$ are smooth,*

b) dim *support* $(f_{(i)}(\mathcal{O}_X))$ = dim *support* $((f_{H'(i)}(\mathcal{O}_{H'})) + 1$ *whenever $f_{(i)}(\mathcal{O}_X)$ is non-trivial (here we adopt the convention that the empty set has dimension* -1).

PROOF. a) is true by Bertini's theorem. We have the exact sequence:

$$0 \to [H']^{-1} \to \mathcal{O}_X \to \mathcal{O}_{H'} \to 0.$$

The long exact sequence of direct image sheaves gives:

$$\to f_{(i)}(\mathcal{O}_X) \otimes [H]^{-1} \to f_{(i)}(\mathcal{O}_X) \to f_{H'(i)}(\mathcal{O}_{H'}) \to .$$

If $\mathcal{S}$ is any coherent sheaf in a manifold Y, then a general hyperplane section will not contain the support of any subsheaf of $\mathcal{S}$. Thus

$$0 \to \mathcal{S} \otimes [H]^{-1} \to \mathcal{S}.$$

From this and the long exact sequence of direct image sheaves above, we get the exact sequence

$$0 \to f_{(i)}(\mathcal{O}_X) \otimes [H]^{-1} \to f_{(i)}(\mathcal{O}_X) \to f_{H'(i)}(\mathcal{O}_{H'}) \to 0$$

and the lemma. □

(0.8) THEOREM. *Let L be an ample line bundle on a normal variety X. Assume that there is a normal $A \in |L|$ and a holomorphic map $f : A \to Y$ where Y is a smooth curve of positive genus and the general fibre of f is $\mathbb{P}^1$. Then f extends to a meromorphic map $\overline{f} : X \to Y$, that is holomorphic in a neighborhood of A. If X is Cohen-Macaulay then $\overline{f}$ is holomorphic.*

PROOF. We need a small lemma.

(0.8.1) LEMMA. *A has at worst rational singularities.*

PROOF. Let $p : \overline{A} \to A$ denote a desingularization of A. Note that

$$*) \qquad (f \circ p)_* \mathcal{O}_{\overline{A}} \doteq \mathcal{O}_Y \text{ and } (f \circ p)_{(i)} \mathcal{O}_{\overline{A}} = 0 \text{ for } i > 0.$$

This follows from the Leray spectral sequence for $f \circ p$ and $\mathcal{O}_{\overline{A}}$ upon noting that $\overline{A}$ is ruled and $f \circ p$ gives an isomorphism of $H^i(\mathcal{O}_Y)$ with $H^i(\mathcal{O}_{\overline{A}})$ for all i.

Since A and Y are normal and f has a generic fibre connected, it follows that all fibres are connected and therefore that

$$**) \qquad f_* \mathcal{O}_A \doteq \mathcal{O}_Y.$$

Consider the Leray spectral sequence for $\mathcal{O}_{\overline{A}}$ and the two maps f and p. From $*)$, $**)$, and $p_* \mathcal{O}_{\overline{A}}$ it follows that $p_{(1)} \mathcal{O}_{\overline{A}} = 0$.

□

Since A is a normal it is Cohen-Macaulay. From this and the fact that it is a Cartier divisor we conclude that X is Cohen-Macaulay in a neighborhood $\mathcal{U}$ of A. Since A is ample $X - \mathcal{U}$ is a finite set.

Let x be an arbitrary point of A and let T be a defining function for A in some neighborhood U of x. The map $T : U \to \mathbb{C}$ is flat for a possibly smaller U. This follows since U is Cohen-Macaulay, $\mathbb{C}$ is smooth, and the fibres of T are equal dimensional. By the above lemma and Elkik's theorem [E] we conclude that X has only rational singularities in a neighborhood of A.

Let $\pi : \overline{X} \to X$ be a desingularization of X. Recall that $S_1(X)$ equals the support of $\pi_{(1)} \mathcal{O}_{\overline{X}}$. Since a neighborhood of A has rational singularities and since A is ample $S_1(X)$ is a finite set in the complement of A. Let $\mathcal{X} = X - S_1(X)$. Let $F : \overline{X} \to \mathrm{Alb}(\overline{X})$ be the Albanese mapping of $\overline{X}$. It is easy to check, e.g. (0.3.3) of [So6], that holomorphic one forms on $\overline{X}$ restrict to 0 on fibres of π over $\mathcal{X}$ and therefore that there is a commutative diagram of holomorphic maps:

$$\begin{tikzcd}
\pi^{-1}(\mathcal{X}) \arrow[d] \arrow[dr, "F"] & \\
\mathcal{X} \arrow[r, "\mathfrak{f}"'] & \mathrm{Alb}(\overline{X})
\end{tikzcd}$$

By (0.3.3) and lemma (0.8.1) it follows that $\mathcal{F}_A$ is the composition of f with the Albanese mapping of Y.

Let λ be a general fibre of f. Since $\lambda \subset \operatorname{reg}(A)$ and since A is Cartier, $\lambda \subset \operatorname{reg}(X)$. Since f is of maximal rank in a neighborhood of λ, we have:

$$0 \to \mathcal{O}_\lambda \to N_\lambda \to L_\lambda \to 0$$

where N_λ is the normal bundle of λ in X. From this we conclude that N_λ is spanned by global sections and also that $H^1(N_\lambda) = 0$. Therefore smooth deformations of λ in $\operatorname{reg}(X)$ cover a neighborhood of λ in X. For any such deformation λ' we know that $\mathcal{F}(\lambda')$ is a point. Since $\lambda' \cdot A = \lambda \cdot A = \lambda \cdot L > 0$ we conclude that $\mathcal{F}(\mathcal{X}) = \mathcal{F}(A) = Y$.

All that it remains to show is that $\mathcal{X} = X$ when X is Cohen-Macaulay. This is obvious since for Cohen-Macaulay X, $\mathcal{S}_1(X)$ has pure codimension 2 support.

□

Appendix - Extension Theorems for Maps of Fibre Dimension at least 2

(A.1) THEOREM. *Let A be an ample divisor on a normal projective variety X. Assume that A is normal and that there is a holomorphic surjection $f : A \to Y$ of A onto a projective variety Y such that* $\dim A - \dim Y \geq 2$. *If there is an ample line bundle L on Y such that f^*L extends to a holomorphic line bundle $\mathcal{L}$ on X, then f extends to a holomorphic map $\overline{f} : X \to Y$. In particular, extension takes place if X is a local complete intersection with the locus of non rational singularities having codimension* ≥ 3.

PROOF. The proof follows that of [So1] very closely; we incorporate the improvements of [Fu1] and [Fu2]. By raising L to a sufficiently high positive power we get a very ample line bundle whose pullback extends to X. Thus we can assume that L is very ample without loss of generality. Consider:

*) $$0 \to \mathcal{L} \otimes [A]^{-1} \to \mathcal{L} \to \mathcal{L}_A \to 0.$$

If the sections of $\mathcal{L}_A$ extend to sections of $\mathcal{L}$, we will get an extension of f to a meromorphic $\overline{f} : X \to \mathbb{P}_{\mathbb{C}}$ by using $\Gamma(\mathcal{L})$ as sketched in (0.6) (cf. [Fu1] also).

To show that the sections of $\mathcal{L}_A$ extend to sections of $\mathcal{L}$ it suffices by *) to show that $H^1(\mathcal{L} \otimes [A]^{-1}) = 0$. Considering the above exact sequence tensored with $[A]^{-r}$ for $r \geq 1$ we see that $H^1(A, \mathcal{L}_A \otimes [A]_A^{-r}) = 0$ for $r \geq 1$ would imply that

$$h^1(\mathcal{L}_A \otimes [A]_A^{-1}) \leq h^1(\mathcal{L}_A \otimes [A]_A^{-2}) \leq \cdots.$$

Since X is normal, $H^1(\mathcal{L} \otimes [A]^{-r}) = 0$ for $r >> 0$ [Ha, Cor. III.7.8]. Therefore we have reduced to showing that

$$h^1(\mathcal{L}_A \otimes [A]_A^{-r}) = 0 \text{ for } r \geq 1.$$

This follows from (0.2.2).

Note that under the local complete intersection condition, extension occurs by (0.3.2).

□

(A.1.1) REMARK. If A and X are as in theorem (A.1) so that the holomorphic extension $\overline{f} : X \to Y$ exists and if $f : A \to Y$ is a $\mathbb{P}^k$ bundle with $k \geq 2$, then it follows from [So1] that $\overline{f}$ is a $\mathbb{P}^{k+1}$ bundle.

(A.2) THEOREM. *Let X be a normal projective variety with isolated singularities. Let A be an ample divisor on X which is normal and such that $A \subset \operatorname{reg}(X)$. If there is a holomorphic surjection $f : A \to Y$ onto a projective variety with* $\dim A - \dim Y \geq 2$, *then f extends to a meromorphic map $\overline{f} : X \to Y$ which is holomorphic on* $\operatorname{reg}(X)$.

PROOF. Let L be an ample line bundle on Y. Let $\pi : \overline{X} \to X$ be a desingularization of X. By lemma (0.3.1), f^*L^m extends to a holomorphic line bundle $\mathcal{L}$ on $\overline{X}$ for some $m > 0$. The proof of the last result and a standard Hartogs' theorem argument would prove this result if we show that for some neighborhood U of A

$$H^1(U, \mathcal{L} \otimes [A]^{-r}) = 0 \text{ for } r >> 0.$$

This is true by a result of Griffiths ([Gri], see also [LP]); we have followed the idea of [Si]. Instead of using Griffiths' theorem we could work on the formal completion of A in X as done by Fujita [Fu2].

□

1. - Some Results on Holomorphic Forms

(1.1) THEOREM. *Let $f : X \to Y$ be a holomorphic surjection with connected fibres between normal projective varieties X and Y. Assume that there is a non-empty Zariski open set $V \subset Y$ such that V and $f^{-1}(V)$ are smooth and $f : f^{-1}(V) \to V$ is of maximal rank. Assume that X and Y have at most rational singularities. If $h^i(\mathcal{O}_F) = 0$ for $0 < i \leq q$, where F is a general fibre of f, then $f_{(i)}(\mathcal{O}_X) = 0$ for $0 < i \leq q$.*

PROOF. It can be assumed without loss of generality that X and Y are smooth. To see this let $g : \overline{Y} \to Y$ be a desingularization of Y and let $\overline{X}$ be a desingularization of the irreducible component of the fibre product $X \times_Y \overline{Y}$

which surjects onto both X and $\overline{Y}$ under the natural projections. We have the commutative square:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{g}} & X \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{Y} & \xrightarrow[g]{} & Y \end{array}$$

The horizontal maps are birational morphisms and since the singularities of X and Y are rational:

$$*) \qquad \overline{g}_{(i)}(\mathcal{O}_{\overline{X}}) = 0 = g_{(i)}(\mathcal{O}_{\overline{Y}}) \qquad \text{for } i > 0$$

and since X and Y are normal $\overline{g}_*(\mathcal{O}_{\overline{X}}) \doteq \mathcal{O}_X$ and $g_*(\mathcal{O}_{\overline{Y}}) \doteq \mathcal{O}_Y$. The condition on the general fibre of f and the fact that $g : g^{-1}(V) \to V$ and $\overline{g} : \overline{g}^{-1}(f^{-1}(V)) \to f^{-1}(V)$ are biholomorphisms imply that $h^i(\mathcal{O}_{\overline{F}}) = 0$, for $0 < i \le q$, where $\overline{F}$ is a general fibre of $\overline{f}$. If the theorem is true for $\overline{f}$, then using $*)$ and $**)$ and the Leray spectral sequence for g and $\overline{f}$ we see that

$$**) \qquad g \circ \overline{f}_{(i)}(\mathcal{O}_{\overline{X}}) = 0 \text{ for } 0 < i \le q \text{ and } g \circ \overline{f}_*(\mathcal{O}_{\overline{X}}) \doteq \mathcal{O}_{\overline{Y}}.$$

Using this, $g \circ \overline{f} = f \circ \overline{g}$, $*)$, $**)$, and the Leray spectral sequence for f and $\overline{g}$, we see that:

$$0 = f_{(i)}(\overline{g}_* \mathcal{O}_{\overline{X}}) = f_{(i)}(\mathcal{O}_X) \text{ for } 0 < i \le q.$$

Therefore we can assume, without loss of generality, that X and Y are smooth.

(1.1.1) LEMMA. *$f^* : H^i(\mathcal{O}_Y) \to H^i(\mathcal{O}_X)$ is an isomorphism for $i \le q$.*

PROOF. By standard Hodge theory the map f^* is an injection for all i, e.g. [W]. We must only show that the map is surjective. By conjugation and the Hodge theory anti-isomorphism of $H^i(\mathcal{O}_X)$ with $\Gamma(\wedge^i T_{X^*})$ and of $H^i(\mathcal{O}_Y)$ with $\Gamma(\wedge^i T_{Y^*})$, this is equivalent to showing that every holomorphic i form η on X with $0 < i \le q$ is of the form $f^*\mu$ for a holomorphic i form μ on Y.

This is certainly true over the dense Zariski open set $V \subset Y$ such that $f : f^{-1}(V) \to V$ is of maximal rank. Indeed let $V' = f^{-1}(V)$ and let $\eta_{V'}$ denote the restriction of η to V'. Consider the exact sequence:

$$0 \to f^* T_{V^*} \to T_{V'^*} \to T_{V'/V^*} \to 0.$$

We get a filtration $F_0 \subset F_1 \subset \cdots \subset F_i$ where

$$F_j = (\wedge^{i-j} f^* T_{V^*}) \wedge (\wedge^j T_{V'^*}).$$

The quotients are $F_j/F_{j-1} \doteq (\wedge^{i-j} f^* T_{V^*}) \otimes (\wedge^j T_{V'^*})$. Since by hypothesis we know that $h^0(\wedge^j T_{F^*}) = 0$ for $0 < j \leq q$ we conclude that $h^0(F_j/F_{j-1}) = 0$ for $0 < j \leq q$ and thus $\eta_{V'} = f^* \mu_V$ for a holomorphic i form μ_V on V. We must only show that μ_V extends to a holomorphic i form on Y.

Assume otherwise. Since by Hartogs' theorem holomorphic sections of vector bundles extend over codimension 2 sets, it follows that μ_V extends to a holomorphic i form μ' on $Y - Z$, where Z is a set of pure codimension 1. Choosing dim $X-$ dim Y general hyperplane sections of X and intersecting we get a submanifold X' of X such that $f_{X'}$ is generically finite to one. Further the pullback of μ' to $X' - (f_{X'})^{-1}(Z)$ extends holomorphically to X', since the pullback of μ' agrees on a dense open set with the restriction of the holomorphic form η. Choose a smooth point x of Z such that $f_{X'}$ is finite to one over a neighborhood of x. An easy calculation shows that μ' has at worst poles on Z and extends holomorphically if it has no poles. Slicing Y with sufficiently many hyperplane sections through X, we can choose an i dimensional submanifold $Y' \subset Y$ such that the restriction μ'' of μ' to $Y' - Z \cap Y'$ has poles along $Z \cap Y'$ if μ' has poles along Z. Further desingularizing an irreducible component of $(f_{X'})^{-1}(Y')$, we get a projective i dimensional manifold X'' and a generically finite to one surjective map $f'' : X'' \to Y$ such that the pullback of μ'' under f'' extends holomorphically to $\eta_{X''}$ on all of X''. But this implies that $\int \mu'' \wedge \overline{\mu}'' < \infty$, since

$$\deg(f'') \cdot \int \mu'' \wedge \overline{\mu}'' = \int (f''^* \mu'') \wedge (f''^* \overline{\mu}'') = \int \eta_{X''} \wedge \overline{\eta}_{X''} < \infty.$$

If $\int \mu'' \wedge \overline{\mu}''$ is finite, then an easy calculation shows that μ' has no poles along $Z \cap Y'$. Therefore μ_V has a holomorphic extension to Y.

□

Now assume that the theorem is false. Let i be the smallest integer satisfying $0 < i \leq q$ such that $f_{(i)}(\mathcal{O}_X) \neq 0$. If $f_{(i)}(\mathcal{O}_X)$ is supported in a finite set then by the Leray spectral sequence and the above lemma we have a contradiction. If $f_{(i)}(\mathcal{O}_X)$ is supported on a $k \geq 1$ dimensional set then by lemma (0.7) we can slice with $k \geq 1$ hyperplane sections on Y and reduce to a situation where we get the same contradiction as in the last sentence.

□

The following lemmas will be convenient.

(1.2) LEMMA. *Let $f : X \to Y$ be a holomorphic surjective map of projective varieties. If there is a non-trivial holomorphic k form on a desingularization of Y, then there is a non-trivial holomorphic k form on a desingularization of X.*

PROOF. Let $\pi_1 : \overline{X} \to X$ and $\pi_2 : \overline{Y} \to Y$ be desingularizations. Since holomorphic forms pullback to holomorphic forms under meromorphic maps, the lemma follows by considering $\pi_2^{-1} \circ f \circ \pi_1$.

□

(1.3) LEMMA. *Let $f : X \to Y$ be a meromorphic surjective map between projective varieties. Assume that there is an open set $V \subset \operatorname{reg}(Y)$ such that*

a) $f^{-1}(V) \subset \operatorname{reg}(X)$,

b) *On $f^{-1}(V)$, f has connected fibres and is of maximal rank,*

c) *Given a generic fibre F of f in $f^{-1}(V)$, $h^i(\mathcal{O}_F) = 0$ for $0 < i \leq q$.*

If a desingularization of X has a non-trivial holomorphic q form, then a desingularization of Y has a non-trivial holomorphic q form.

PROOF. Reduce to lemma (1.1.1).

□

2. - The Meromorphic Extension Theorem

(2.0) Let L be a line bundle on a compact normal variety X. Assume that L is spanned at all points by global sections and that X is Cohen-Macaulay, i.e. that the local rings of X are all Cohen-Macaulay local rings. Let

$$e : X \times \Gamma(L) \to L$$

denote the evaluation map on sections. Since $\Gamma(L)$ spans L at all points, it follows that is onto and the kernel K is a vector bundle on X. We denote $\mathbb{P}(K^*)$ by $\mathbb{A}$ and note that $\mathbb{A} \subset X \times |L|$ is the family of pairs (x, A) with $x \in A \in |L|$. Let $p : \mathbb{A} \to X$ and $q : \mathbb{A} \to |L|$ denote the maps induced by the product projections and note that p is the natural projection of $\mathbb{P}(K^*) \to X$.

Since $\mathbb{A}$ is a fibre bundle with smooth fibre over a Cohen-Macaulay variety, it follows that $\mathbb{A}$ is Cohen-Macaulay. Since q has equal dimensional fibres, $\mathbb{A}$ is Cohen-Macaulay, and $|L|$ is smooth, it follows that:

(2.0.1) $$q \text{ is flat.}$$

(2.1) LEMMA. *Let X, L, $\mathbb{A}$ and q be as above. Assume that there is a normal $A \in |L|$ that fibres holomorphically $f : A \to Y$, where Y is a normal variety and where f has connected fibres. Assume further that there is a smooth Zariski open set $V \subset Y$ such that $U = f^{-1}(V)$ is smooth and such that f is of maximal rank on U. Assume that there is an ample line bundle E on Y such that f^*E extends to a line bundle $\mathcal{E}$ on X. Assume that $f_{(i)}(\mathcal{O}_A) = 0$ for all odd i. Then there is a compact normal variety $\mathcal{Y}$, a holomorphic surjection $g : \mathcal{Y} \to |L|$ and a meromorphic surjection $F : \mathbb{A} \to \mathcal{Y}$ such that:*

a) $$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{Y} \arrow[dl, "g"] \\ & {|L|} & \end{tikzcd}$$ *commutes,*

b) *F is holomorphic on a Zariski open set containing $q^{-1}(A), g^{-1}(A)$ is biholomorphic to Y and $F_{q^{-1}(A)} = f$,*

c) *g has equal dimensional fibres in a neighborhood of $g^{-1}(A)$,*

d) *there is a smooth Zariski open set $\mathcal{C} \subset \mathcal{Y}$ such that F is of maximal rank on the set $\mathcal{U} = F^{-1}(\mathcal{C})$ which is smooth and such that $p^{-1}(U) \cap q^{-1}(A) \subset \mathcal{U}$.*

PROOF. Choose n large enough so that E^n is very ample and by Serre's theorem $H^j(Y, f_{(i)}(f^*E^n)) = H^j(Y, f_{(i)}(\mathcal{O}_A) \otimes E^n)$ is zero for $j > 0$ and all i. By the Leray spectral sequence for f and f^*E^n and the hypothesis that $f_{(i)}(\mathcal{O}_A) = 0$ for odd i, it follows that $H^j(A, f^*E^n) = 0$ for odd $j > 0$. By the flatness (2.0.1) of q, it follows that $\chi(\mathcal{E}_{A'^n})$ is independent of $A' \in |L|$. From this and the upper semi-continuity of dimensions of cohomology groups, it follows that $h^0(A', \mathcal{E}_{A'^n})$ is constant for a Zariski open set of $|L|$ that contains A. This and the flatness of q imply, by a theorem of Grauert, that the coherent sheaf:

$$\mathcal{S} = q_*(p^*\mathcal{E}^n)$$

is locally free of rank $h^0(E^n)$ in a neighborhood of A in $|L|$. Since sections of f^*E^n therefore extend to give sections of $\mathcal{S}$, it follows that $p^*(\mathcal{E}^n)_{q^{-1}(\mathcal{O})}$ is spanned by global sections for a Zariski open set $\mathcal{O} \subset |L|$ containing A. Therefore we have a meromorphic map F' from $\mathbb{A}$ into $\mathrm{Proj}(\mathcal{S})$ which is holomorphic in a neighborhood of $q^{-1}(A)$. Let $\mathcal{Y}$ denote the normalization of the image of F' and let F denote the induced meromorphic map. Note that dim $F(q^{-1}(A'))$ is independent of $A' \in \mathcal{O}$. Indeed, since $\mathcal{E}_{A'^n}$ is spanned, then it equals max $\{k | c_1(\mathcal{E})^k \wedge c_1(L)$ is non-trivial in $H^{2k+2}(X, \mathbb{Q})\}$.

This implies c) where $g : \mathcal{Y} \to |L|$ is the induced map.

The assertion d) is straightforward and left to the reader.

□

(2.2) MEROMORPHIC EXTENSION THEOREM. *Let X be an n dimensional normal compact Cohen-Macaulay variety. Assume that $h^0(\wedge^{n-2}T_{\overline{X}^*}) \neq 0$, where $\overline{X}$ is a desingularization of X. Assume that L is a line bundle which is big and which is spanned at all points of X by global sections. Assume that there is a normal $A \in |L|$ such that there is a holomorphic surjection $f : A \to Y$ with generic fibre $\mathbb{P}^1$ onto a compact normal variety Y. Assume that there is an ample line bundle E on Y such that f^*E extends to a holomorphic line bundle $\mathcal{E}$ on X. If either f is flat or A and Y have at most rational singularities, then f extends to a meromorphic map:*

$$\overline{f} : X \to Y$$

holomorphic in a neighborhood of the open set $U \subset$ reg(A) such that $f_U : U \to f(U)$ is a $\mathbb{P}^1$ bundle.

PROOF. Lemma (2.1) applies. Let

$$\begin{tikzcd} X & \mathbb{A} \arrow[l, "p"'] \arrow[r, "F"] \arrow[dr, "q"'] & \mathcal{Y} \arrow[d, "g"] \\ & & |L| \end{tikzcd}$$

be as in the lemma. For simplicity of exposition we assume that X, Y, and A are smooth, that $f : A \to Y$ is a $\mathbb{P}^1$ bundle, and that L is ample. The mainly notational modifications to show the general case are left to the reader.

The property that f is a $\mathbb{P}^1$ bundle over Y is inherited by the maps $F_{A'} : A' \to F(A')$ given by lemma (2.1) for A' near A in $|L|$. Thus:

(2.2.1) F is a $\mathbb{P}^1$ bundle over a smooth Zariski open set $\mathcal{C} \subset \mathcal{Y}$ which contains Y (here we identify Y with $g^{-1}(A)$). Set $\mathcal{U} = F^{-1}(\mathcal{C})$.

Let $B = p^{-1}(A)$ and let B' be the image of B in $\mathbb{A} \times A$ under the map (i, p) where $i : B \to \mathbb{A}$ is the inclusion. Let $F' : \mathbb{A} \times A \to \mathcal{Y} \times A$ be the map (F, id_A).

Note that B is irreducible since it is a fibre bundle over A. Thus the closures $\overline{\mathbb{B}}$ of $\mathbb{B} = F'(B' \cap (\mathcal{U} \times A))$ and Z of $F'^{-1}(\mathbb{B})$ are irreducible, where $\mathcal{U}$ is as in (2.2.1).

REMARK. Roughly Z is the set of triples (x, A', z) where $x \in A$, $x \in A'$, and z belongs to the rational curve $F^{-1}(F(x))$. Thus roughly Z fibres over Y with a general fibre $\mathcal{F}$ fibring over a fibre λ of $f : A \to Y$, with a general fibre Σ of $\mathcal{F} \to \lambda$ fibring over the projective space $\mathbb{P}$ of $A' \in |L|$ that contain a point x of λ, and the general fibre $\Sigma \to \mathbb{P}$ fibring a rational curve. Thus Z is a family of very rational looking varieties parametrized by Y, and the images of these very rational looking varieties in X are candidates for the fibres of the desired meromorphic map $\overline{f}$. The main idea of this paper is that a holomorphic $n-2$ form on a desingularization of X should force this to be true.

(2.2.2) LEMMA. *The meromorphic map F' from B' to $\overline{\mathbb{B}}$ is one to one on* $B' \cap (\mathcal{U} \times A)$.

PROOF. To see this note that if $(v, x) \in \mathcal{C} \times A$, then

$$(F'_{\mathcal{U} \times A})^{-1}(v, x) = \{(w, x) \in \mathcal{U} \times A | F(w) = v\}.$$

Note that

$$\{w \in \mathcal{U} | F(w) = v\} = \{(z, A') \in X \times |L| : g(v) = A', z \in A', F_{A'}(z) = v\}.$$

Thus $F'^{-1}(v, x) \cap B' = \{(x, A', x) \in X \times |L| \times A | g(v) = A', x \in A', F_{A'}(x) = v\}$. □

Let $h : Z \to X$ denote the map onto X induced by the composition of the product projection $\mathbb{A} \times A \to \mathbb{A}$ and p. Let $k : Z \to Y$ denote the surjection induced by the composition of the product projection $\mathbb{A} \times A \to A$ and $f : A \to Y$. Let $c : Z \to Y \times X$ denote the map (k, h). Let $Z' = c(Z)$ and let $k' : Z' \to Y$ and $h' : Z' \to X$ be the maps induced by the product projections.

Choose a general element $H \in |E^N|$, where N is chosen so that E^N is very ample. By lemma (0.4), $H' = k^{-1}(H)$ is irreducible, since Z is irreducible.

(2.2.3) LEMMA. $h(H') \neq X$.

PROOF. Assume that $h(H') = X$. Since a desingularization of X has a non-trivial holomorphic $n-2$ form on it, it follows from lemma (1.2) that the desingularization $\overline{H}'$ of H' has a non-trivial holomorphic $n-2$ form on it. Since $F'_{H'} : H' \to F'(H')$ is a $\mathbb{P}^1$ bundle over a dense open set of $F'(H')$, it follows from lemma (1.3) that a desingularization of $F'(H')$ has a non-trivial holomorphic $n-2$ form on it. Using lemma (2.2.2) it is clear that $F'(H')$ is birational to $p^{-1}(f^{-1}(H))$. Since this is a projective bundle over $f^{-1}(H)$ it follows from lemma (1.2) that the desingularization of $f^{-1}(H)$ has a non-trivial holomorphic $n-2$ form on it. Since $f^{-1}(H)$ maps onto H' with generic fibre $\mathbb{P}^1$ it follows from lemma (1.3) that the desingularization of H' has a non-trivial holomorphic $n-2$ form on it. But since dim $H' = n-3$, this is absurd.

□

We are now in a positiion to show that Z' is the graph of a meromorphic map from X to Y. First note that the dimension of a generic fibre of $h' : Z' \to X$ is 0 dimensional. Indeed, if it was not, then given a general very ample divisor Δ on Y it would follow that $h'(k'^{-1}(\Delta)) = X$. Since $h'(k'^{-1}(\Delta)) = h(k^{-1}(\Delta))$, this is ruled out by lemma (2.2.3).

Therefore since $h'(Z') = h(Z) = X$, it follows that dim $Y+$ dim (generic fibre of k') $=$ dim X or

(2.2.4) dim (generic fibre of k') $= 2$.

(2.2.5) Choose a $y \in Y$ that is general in the sense that $k^{-1}(y)$ is irreducible and dim $h(k^{-1}(y)) = 2$ and fix a point $x \in \lambda = f^{-1}(y)$.

(2.2.6) Choose an $A' \in |L|$ that is general in the following senses:

a) A' is smooth and meets A transversely ,

b) A' doesn't contain x,

c) $q^{-1}(A') \subset \mathcal{U}$ and $F_{q^{-1}(A')}$ is a $\mathbb{P}^1$ bundle.

Let $P \subset |L|$ denote the pencil joining A and A'. Let Γ denote the graph of the meromorphic map $F_{q^{-1}(P)}$. Let X' denote the irreducible component of Γ such that:

$$p \text{ (projection of } X' \text{ on } q^{-1}(P)) = X.$$

Let $m : X' \to X$ be the map composed of p and the projection of X' to $q^{-1}(P)$. Note that

$$(2.2.7) \qquad m \text{ is a birational map.}$$

Let $\mathcal{Y}_P$ denote the irreducible component of $g^{-1}(P)$ such that $g(\mathcal{Y}_P) = P$ and the map $F_P : X \to \mathcal{Y}_P$ induced by F is onto. We have:

$$\begin{tikzcd} & X' \arrow[r, "m"] \arrow[dl, "F_P"'] \arrow[dd, "r"] & X \\ \mathcal{Y}_P \arrow[dr, "g'"'] & & \\ & P & \end{tikzcd}$$

where g' and r are the maps induced by g and q respectively.

By (2.2.6) $m : X' \to X$ is X with $A \cap A'$ blown up.

It follows from (2.2.4) that given an $a \in \lambda \cap A'$, $m[(F_P)^{-1}(F_P(m^{-1}(a)))]$ contains a unique irreducible 2 dimensional component $\mathcal{J}_a$ that contains λ. To see this note that for most $w \in F_P(m^{-1}(a))$, $m[(F_P)^{-1}(w)]$ is a smooth $\mathbb{P}^1$ on an $A'' \in P$ which is also a fibre of $F_{A''}$. From this we see also that $m(\mathcal{J}_a) \subset h(k^{-1}(y))$. Since $\dim h(k^{-1}(y)) = 2$ by (2.2.6a) we conclude $m(\mathcal{J}_a) = h(k^{-1}(y))$. This set which we call $\mathcal{J}_\lambda$ is therefore independent of $a \in \lambda \cap A'$ and of the general A' chosen subject to (2.2.6).

Let $s_{A'} \in \Gamma(L)$ be a section defining A'. There is a short exact sequence:

$$0 \to N_{\lambda/A} \to N_{A|\lambda} \to L_\lambda \to 0$$

of normal bundles. The infinitesimal deformation of λ corresponding to the family $m(r^{-1}(w))$, for w near $A \in P$, has as image in $\Gamma(L_\lambda)$ the restriction $s_{A'|\lambda}$. Since $s_{A'}(x) \neq 0$ by (2.2.6b) we see that $\mathcal{J}_\lambda$ is smooth near x and $\mathcal{J}_\lambda$ is transverse to A near x. Since $x \in \lambda$ was arbitrary we conclude that $\mathcal{J}_\lambda$ is smooth in a neighborhood of λ and that, along λ, $\mathcal{J}_\lambda$ intersects A transversely.

(2.2.8) LEMMA. *With* $\lambda = f^{-1}(y)$ *as in* (2.2.5), $\mathcal{J}_\lambda \cap A = \lambda$.

PROOF. Let $\varphi : X \to \mathbb{P}_{\mathbb{C}}$ denote the map associated to $\Gamma(L)$. Since φ has an image of dimension equal to $\dim X$, and since y is general, in the sense of (2.2.5), it follows that $\dim \varphi(\mathcal{J}_\lambda) = 2$. By (0.4), $\mathcal{J}_\lambda \cap A$ is connected. Since $\mathcal{J}_\lambda$ meets A transversely along λ, it follows that $\mathcal{J}_\lambda \cap A = \lambda$.

□

The above shows that $\mathcal{J}_\lambda$ determines y by $f(\mathcal{J}_\lambda \cap A)$. From this we claim that it follows that Z' is the graph of a meromorphic map $\overline{f} : X \to Y$. This will follow if we show that the map $h' : Z' \to X$ is birational. This will follow if we show that, given a general point $x \in X$, there are not fibres $\lambda, \overline{\lambda}$ with $f(\lambda), f(\overline{\lambda})$ general in the sense of (2.2.5) and $x \in \mathcal{J}_\lambda \cap \mathcal{J}_{\overline{\lambda}}$, where $\mathcal{J}_\lambda$ (respectively $\mathcal{J}_{\overline{\lambda}}$) is the two dimensional set associated to λ (respectively $\overline{\lambda}$) as above, with $\mathcal{J}_\lambda \cap A = \lambda$ (respectively $\mathcal{J}_{\overline{\lambda}} \cap A = \overline{\lambda}$).

To see this we can work on X' since m is birational. For simplicity we let $\mathcal{J}$ (respectively $\overline{\mathcal{J}}$) denote the unique irreducible 2 dimensional component of

$(F_P)^{-1}(F_P(m^{-1}(\lambda\cap A')))$ [respectively $(F_P)^{-1}(F_P(m^{-1}(\overline{\lambda}\cap A')))$] that contains λ (respectively $\overline{\lambda}$).

Assume that $J\cap\overline{J}$ contains a general point of X' which for simplicity we still call x. Let:

a) $\mathcal{L}$ equal the smooth rational curve $(F_P)^{-1}(F_P(x))$ and $B = m^{-1}(A\cap A')$,

b) $T = \bigcup_{a\in\lambda\cap A'} m^{-1}(a)$ and $\overline{T} = \bigcup_{\overline{a}\in\overline{\lambda}\cap A'} m^{-1}(\overline{a})$.

Let t denote the degree of the map $f_{A\cap A'} : A\cap A' \to Y$. Note that

$$*)\qquad \#(\mathcal{L}\cap B) = t \text{ and } \#(\lambda\cap A') = \#(\overline{\lambda}\cap A') = t,$$

where $\#(\ )$ denotes the number of points in a set. Since J and $\overline{J}$ are F_P saturated and since $F_P(x)\in F_P(J)\cap F_P(\overline{J})$ it follows that $\#(\mathcal{L}\cap T) = t = \#(\mathcal{L}\cap\overline{T})$. Further note that

$$J\supset T,\ \overline{J}\supset\overline{T}, \text{ and } \mathcal{L}\cap B\supset(\mathcal{L}\cap T)\cup(\mathcal{L}\cap\overline{T}).$$

Therefore by $*)$ $\mathcal{L}\cap B = \mathcal{L}\cap T = \mathcal{L}\cap\overline{T}$. Therefore $T = m^{-1}(m(T)) = m^{-1}(m(\mathcal{L}\cap B)) = m^{-1}(m(\overline{T})) = \overline{T}$ and $\lambda\cap\overline{\lambda} = (m(J)\cap A)\cap(m(\overline{J})\cap A)\supset m(T)\cap m(\overline{T})\neq\emptyset$. Thus $\lambda = \overline{\lambda}$ and $J = \overline{J}$.

Finally let $\lambda\subset A$ be a fibre of f. Choose A' subject to (2.2.6). Choose a local holomorphic section $\sigma : \mathbb{N}\to A$, where $\mathbb{N}$ is a neighborhood of $f(\lambda)$ and $\sigma(\mathbb{N})\subset A\cap A'$. For a small enough $\mathbb{N}$ and for y in a small enough neighborhood of x in X, there is a well defined holomorphic map which sends y to $f(a)$, where $a\in\sigma(\mathbb{N})$ and $m^{-1}(y)\in m[(F_P)^{-1}(F_P(m^{-1}(a)))]$. This map agrees with $\overline{f}$ on an open set and gives the desired extension.

□

(2.3) COROLLARY. *Let X be a normal projective variety and let L be an ample line bundle on X spanned at all points of X. Assume that there is a normal $A\in|L|$ such that:*

a) $A\subset \mathrm{reg}(X)$,

b) *There is a holomorphic surjection $f : A\to Y$ onto a normal projective variety and the generic fibre of f is $\mathbb{P}^1$,*

c) *A and Y have only rational singularities,*

d) *There is a desingularization $\overline{Y}$ of Y with $h^0(K_{\overline{Y}}) > 0$.*

Then f extends to a meromorphic map $\overline{f} : X\to Y$, holomorphic on a neighborhood of the open set $U\subset \mathrm{reg}(A)$ such that $f_U : U\to f(U)$ is a $\mathbb{P}^1$ bundle.

PROOF. By (0.8) we can assume without loss of generality that $\dim X\geq 4$. Let $\pi : \overline{X}\to X$ be a desingularization of X. Since $A\subset \mathrm{reg}(X)$ we have π giving a biholomorphism of A and $\pi^{-1}(A)$. Let E be an ample line bundle on

Y. By lemma (0.3.1), f^*E^n extends to $\overline{X}$ for some $n > 0$. By (1.2) and (0.3.3) $h^0(\wedge^{\dim Y} T_{\overline{X}^*}) \neq 0$. Thus $\overline{X}$, π^*L, $\pi^{-1}(A)$ and E^n satisfy the hypotheses on X, L, A and E in theorem (2.2). Therefore there is a meromorphic extension $\overline{f} : \overline{X} \to Y$. The composition $\overline{f} \circ \pi^{-1} : X \to Y$ is the desired extension.

□

(2.4) COROLLARY. *Assume the same hypotheses as* (2.3) *except that* $A \subset \operatorname{reg}(X)$ *is replaced by the assumption that* X *is a local complete intersection. Then the same conclusion as in* (2.3) *holds.*

PROOF. By Elkik's theorem [E], as used in (0.8), we can conclude that $\operatorname{Irr}(X)$ does not meet A. Since A is ample this implies $\operatorname{Irr}(X)$ is finite. Since we can assume without loss of generality that $\dim X \geq 4$ by (0.8), it follows that we can assume without loss of generality that $\operatorname{cod} \operatorname{Irr}(X) \geq 3$. Use (0.3.2) instead of (0.3.1).

□

(2.5) THEOREM. *Let* X *be a projective variety which is a local complete intersection. Assume that* L *is an ample line bundle spanned at all points of* X *by global sections. Assume that there exists a smooth* $A \in |L|$ *which is a* $\mathbb{P}^1$ *bundle* $f : A \to Y$ *over a projective manifold* Y. *Assume that there exists an unramified cover* $\pi : T \to Y$ *with* $h^{\dim T, 0}(T) \neq 0$. *Then* f *extends to a holomorphic map* $\overline{f} : X \to Y$.

PROOF. We can assume without loss of generality that T is a regular covering of Y. Suppressing base points for simplicity we note that

$$f_* : \pi_1(A) \to \pi_1(Y) \text{ and } i_* : \pi_i(A) \to \pi_1(X) \text{ are isomorphisms,}$$

where $i : A \to X$ is the inclusion map. Let $H_1 = \pi_*(\pi_1(T))$, $H_2 = (f_*)^{-1}(H_1)$ and $H_3 = i_*(H_2)$. Denote by $\overline{A}$ and $\overline{X}$ the covering spaces of A and X corresponding to H_2 and H_3, subgroups of $\pi_1(A)$ and $\pi_1(X)$ respectively. Thus we have the following diagram

$$\begin{tikzcd}
 & \overline{A} \arrow[r] \arrow[d, "p"] \arrow[ddl, "\varphi"'] & \overline{X} \arrow[d, "q"] \\
 & A \arrow[r, "i"] \arrow[d, "f"] & X \\
T \arrow[r, "\pi"] & Y &
\end{tikzcd}$$

Note that $f \circ p$ and $i \circ p$ lift to a map from $\overline{A}$ to T and from $\overline{A}$ to $\overline{X}$ respectively, since $(f \circ p)_*(\pi_1(\overline{A})) = H_1 = \pi_*(\pi_1(T))$ and $(i \circ p)_*(\pi_1(\overline{A})) = H_3 = q_*(\pi_1(\overline{X}))$. It is easy to see that $\overline{A}$ is an ample divisor on $\overline{X}$ and that $\varphi : \overline{A} \to T$ is a $\mathbb{P}^1$ bundle over T. Using (0.6) and (2.3) we conclude that the map φ extends to a holomorphic map $\overline{\varphi} : \overline{X} \to T$. The group of deck transformations of $\overline{X}$,

$\overline{A}$, and T are all isomorphic to one another by construction. Denote this group by G. Note that everything descends. Therefore we get a holomorphic map $\overline{f} : X \to Y$, where $\overline{f}$ is obtained from $\overline{\varphi} : \overline{X} \to T$ after we have considered the action of G on $\overline{X}$ and T. Clearly $\overline{f}$ is holomorphic and is an extension of the given f.

□

(2.6) COROLLARY. *Let X, L and A be as in* (2.5). *Assume that $K_Y^t \cong \mathcal{O}_Y$ with $t > 0$ minimal. Then the same conclusion as* (2.5) *holds.*

PROOF. Let $\pi : T \to Y$ be the t cyclic unramified cover of Y determined by the torsion line bundle K_Y. Note that $h^{\dim Y,0}(T) \neq 0$. Therefore (2.5) applies.

□

3. - $\mathbb{P}^1$ Bundles as Hyperplane Sections

(3.0) THEOREM. *Let X be an irreducible projective local complete intersection. Assume that L is an ample line bundle on X spanned at all points by global sections. Assume that there is a smooth $A \in |L|$ which is a $\mathbb{P}^1$ bundle $f : A \to Y$ over a projective manifold Y. Then, if $h^0(K_Y) \neq 0$, f extends to a holomorphic map $\overline{f} : X \to Y$.* Dim $Y \leq 2$ *and if* dim $Y = 2$, *$\overline{f}$ is a $\mathbb{P}^2$ bundle with the restriction of L to a fibre of $\overline{f}$ isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$.*

PROOF. By lemma (0.8) and (2.5) the holomorphic extension $\overline{f} : X \to Y$ exists. By Proposition V of [So1], it follows that if $\overline{f}$ is a $\mathbb{P}^1$ bundle then dim $Y \leq 2$.

Let $\mathbb{A}$ be the union of the singular set of X and the set where $\overline{f}$ is not of maximal rank. Since $\mathbb{A} \subset X - A$, the set $\mathbb{A}$ is finite. Choose a smooth connected curve $C \subset Y$ such that $\overline{f}(\mathbb{A}) \cap C =$ empty. Let $f' = f_{f^{-1}(C)}, \overline{f}' = \overline{f}_{\overline{f}^{-1}(C)}$, and let F denote the general fibre of $\overline{f}'$. Suppressing basepoints for simplicity we have the long exact sequence of homotopy groups of fibre bundles:

$$\begin{tikzcd}
 & & 0 & \\
 & \pi_2(F) \arrow[r] & \pi_2(\overline{f}^{-1}(C)) \arrow[u] \arrow[dr] & \\
\pi_3(C) \arrow[ur] \arrow[dr] & & & \pi_2(C) \\
 & \pi_2(\mathbb{P}^1) \arrow[uu, "a"] \arrow[r] & \pi_2(f^{-1}(C)) \arrow[uu, "b"] \arrow[ur] &
\end{tikzcd}$$

Note that b is surjective by the first Lefschetz theorem on hyperplane sections. A diagram chase shows that a is surjective. Since $\mathbb{P}^1$ is a hyperplane section of F, it is very well known, e.g. (0.6.1) of [So2], that F is either $\mathbb{P}^2$

or a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. Since a is surjective, it follows that F is $\mathbb{P}^2$ and $L_F \cong \mathcal{O}_{\mathbb{P}^2}(1)$.

We are done except for the possibility of a singular fibre F of $\overline{f} : X \to Y$. Dim $F \leq$ dim $f^{-1}(\overline{f}(D)) + 1 = 2$. By the above it is clear that F is irreducible (since $L \cdot L \cdot F = 1$). A straightforward argument using (0.6.1) of [So2] shows that $F \cong \mathbb{P}^2$.

To finish the argument note that, since $\overline{f}^{-1}\overline{f}(F) = \mathbb{P}^2$ and since $\overline{f}$ is flat (fibres are equal dimensional, X is a local complete intersection, and Y is smooth), it follows that $\overline{f}$ is of maximal rank in a neighborhood of F.

□

(3.1) THEOREM. *Let L be an ample line bundle on a local complete intersection X. Assume that:*

a) *L is spanned by global sections and $h^0((K_X \otimes L)^N) \neq 0$ for some $N > 0$,*

b) *The singular set of X has codimension ≥ 4.*

Let $H \in |L|$ and assume that there is a holomorphic surjection $f : H \to H'$ which expresses H as a projective variety with a codimension 2 submanifold $A' \subset \operatorname{reg}(H')$ blown up. Assume that $h^0(K_{A'}) \neq 0$. Then dim $A' = 2$ *and f extends to a holomorphic map $\overline{f} : X \to X'$ such that*

$$\begin{array}{ccc} H & \xrightarrow{i} & X \\ f\downarrow & & \downarrow\overline{f} \\ H' & \xrightarrow[i']{} & X' \end{array}$$

commutes where i and i' are inclusions. The map $\overline{f}$ expresses X as X' with the smooth subvariety $i'(A') \subset \operatorname{reg}(X')$ blown up.

PROOF. By the same argument as in [So2] or [Fa2], it can be shown that there exists a normal Cartier divisor D on X which meets H transversely along E, the exceptional divisor of f over A'. By (3.0) the map $f_E : E \to A'$ extends to a holomorphic map $\overline{f} : D \to A'$ and dim $A' \leq 2$. The case dim $A' = 1$ has been done [Fa2]. Thus the only case remaining is dim $A' = 2$. In this case $\overline{f} : D \to A'$ is a $\mathbb{P}^2$ bundle with the restriction of L_D to the general fibre of $\overline{f}$ isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$. It is then clear that the line bundle [D] restricted to the general fibre of $\overline{f}$ is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Therefore by Nakano's theorem we can smoothly blow down D. Thus there exists a variety X and a holomorphic map $\overline{f} : X \to X'$ such that the diagram in the statement of the theorem commutes. Clearly $i'(A') \subset \operatorname{reg}(X')$ since H' is a Cartier divisor on X'. The map $\overline{f}$ expresses X as X' with $i'(X')$ blown up.

□

4. - Concluding Remarks

(4.0) CONJECTURE. *Let $f : A \to Y$ be a holomorphic $\mathbb{P}^1$ bundle over a smooth connected projective manifold Y. Assume that A is an ample divisor on a projective local complete intersection X. If $A \neq \mathbb{P}^n \times \mathbb{P}^1$, then f extends to a holomorphic $\mathbb{P}^2$ bundle $\overline{f} : X \to Y$, and* $\dim Y = 2$.

For simplicity assume that [A] is spanned by global sections.

The above is true for $\dim Y \leq 2$. The case when $\dim Y = 1$ is summarized in [B1, B2, B3]. The case of $\dim Y = 2$ has recently been completely settled. In [F+S+So] the cases, when Y is of Kodaira dimension < 2, but not biholomorphic to $\mathbb{P}(E)$ for a stable vector bundle E over a curve, are settled. E. Sato and H. Spindler have shown how to rule out the case when Y is biholomorphic to $\mathbb{P}(E)$ for a stable vector bundle E over a curve. Further we can weaken the condition of this paper that $h^0(K_Y) \neq 0$ when $\dim Y = 2$ and Y is of general type. Indeed if Y is a general type surface then most fibres of $\mathcal{Y}_P \to P$ in the proof of (2.2) are general type surfaces. The intersection $A \cap A'$ is also of general type and surjects generically finite to one onto most fibres of $\mathcal{Y}_P \to P$. By the 2 dimensional de Franchis theorem ([D+M], [M]) most of the maps from $A \cap A'$ to a fibre of $\mathcal{Y}_P \to P$ are the same except for some blowing up and blowing down. Assuming that the fibre degree of these maps is t we get a meromorphic map from $\mathcal{Y}_P$ to the symmetric product $(A \cap A')^{(t)}$. The image of $\mathcal{Y}_P$ in $(A \cap A')^{(t)}$ is birational to Y and the composition of

$$X \leftarrow X' \to \mathcal{Y}_P$$

with this map gives the needed meromorphic extension $X \to Y$.

If $\dim Y = 3$ and [A] is very ample, then the second author can show, by using [So6], that either the conjecture is true or $Y \neq \mathbb{P}^n$, but there is a non-trivial two to one map of a smooth quadric onto Y. He can also show that similar but slightly weaker restrictions hold if the dimension of Y is odd.

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