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L. HSIAO
P. MARCATI

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Nonhomogeneous Quasilinear Hyperbolic System Arising in Chemical Engineering

L. HSIAO(*) - P. MARCATI(**)

1. - Introduction

We consider the following system

$$(1.1)\qquad \begin{cases} \dfrac{\partial \varepsilon}{\partial t}(x,t) - \dfrac{\partial}{\partial x}[v(x,t)(1-\varepsilon(x,t))] = 0 \\ \dfrac{\partial v}{\partial t}(x,t) + \dfrac{1}{2}\dfrac{\partial}{\partial x}v^2(x,t) - c_0\dfrac{\partial \varepsilon}{\partial x}(x,t) = f(\varepsilon(x,t), v(x,t)) \end{cases}$$

Under the initial conditions

$$(1.2)\qquad v(x,0) = v_0(x), \quad \varepsilon(x,0) = \varepsilon_0(x), \quad x > 0$$

and the boundary conditions

$$(1.3)\qquad v(0,t) = v_1, \quad t > 0, \quad v_1 \text{ is a constant value}$$

and $f(\varepsilon, v)$ is defined on a bounded domain A in the plane (ε, v), such that

$$(1.4)\qquad f_v \leq 0, \text{ for any } (\varepsilon, v) \in A$$

$$(1.5)\qquad - f_v(\varepsilon, v) \leq M, \text{ for any } (\varepsilon, v) \in A,$$

where M depends on the domain A and the parameters on f only. Moreover

$$(1.6)\qquad f(\varepsilon, v) = 0$$

defines a curve $v = v(\varepsilon)$ in the domain and v_1 is chosen so that $v = v_1$ intersects with $v = v(\varepsilon)$, in A, once and only once.

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(*) CNR-Visiting Professor at the University of Rome 2 "Tor Vergata".

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The system (1.1) was developed in chemical engineering [FG] to describe the onset of bubbling in a uniformly fluidized bed, under the assumptions that the idealized bed consists of interlocking horizontal layers of particles arranged into an expanded close packet array, with the voidage essentially constants for all horizontal planes. Here x denotes the distance of the x-layer from the bottom of the bed, $v(x,t)$ denotes the velocity of the x-layer at time t and $\varepsilon(x,t)$ the void fraction in the x-layer, c_0 is constant positive number depending on assigned physical parameters, while $f(\varepsilon, v)$ denotes the external force acting on a single particle. The explicit expression used in [FG] was given by

$$f(\varepsilon, v) = \mu\varepsilon - \gamma|v|^{\alpha-1}v\varepsilon^{-\beta} \tag{1.7}$$

Where μ is a positive constant depending on the same parameters as c_0 and α, β, γ are positive constants depending on Reynold's number and $1 \leq \alpha \leq 2$, $3 \leq \beta \leq 4$. It is easy to see that the function f in (1.7) satisfies the assumption (1.4), (1.5), (1.6) in the bounded domain A, defined by $0 < d \leq \varepsilon \leq 1$, $0 \leq v \leq b < +\infty$, where b and d are arbitrary positive numbers and $d < 1, 0 < v_1 < b$.

The mathematical model presented here is in excellent agreement with the experimental observations reported in the literature ([FG] and [RFHY]), when the initial situation is close to an equilibrium value.

This system, in addition, seems to exhibit some interesting new features worth of an accurate mathematical investigation.

The system (1.1) is a nonhomogeneous quasilinear hyperbolic system for which, as is well known, due to the breaking of waves and formation of shocks, the initial value problem does not generally possess globally defined smooth solutions, even when the initial data are very smooth, and the natural function class in which solutions should be sought is the space BV of functions of bounded variation. The standard method for constructing BV solutions to the initial value problem for a general homogeneous hyperbolic system of conservation laws is the difference scheme of Glimm [G].

For systems with inhomogeneities and/or source terms, there were different efforts ([DH], [LI], [YM]). The algorithm introduced in [DH] is very simple; it combines the fractional steps method with Glimm's sheme and only involves the resolution of discontinuities for homogeneous systems of conservation laws.

Using the same ideas of [DH], we introduce a particular simple algorithm for system (1.1) in section 3 and establish its consistency by showing that, when it converges, the limit is a solution to (1.1), (1.6), (1.7) satisfying the usual entropy admissibility criterion. Then section 4 is devoted to investigating the problem of convergence. We have to deal with new difficulties in this paper (different from the nonhomogeneous system discussed in the literature) caused by the degeneracy of hyperbolicity at $\varepsilon = 1$ and the boundary condition. Moreover, the global existence theorem is not local in the phase space.

2. - Preliminary remarks

Let us consider the corresponding homogeneous system

$$\text{(2.1)}\qquad \begin{cases} \dfrac{\partial \varepsilon}{\partial t}(x,t) - \dfrac{\partial}{\partial x}[v(x,t)(1-\varepsilon(x,t))] \quad = 0 \\ \dfrac{\partial v}{\partial t}(x,t) + \dfrac{1}{2}\dfrac{\partial}{\partial x}v^2(x,t) - c_0 \dfrac{\partial \varepsilon}{\partial x}(x,t) = 0 \end{cases}$$

It is easy to show that (2.1) is a hyperbolic system, since the Jacobian matrix has two real distinct eigenvalues, when $\varepsilon < 1$.

Namely $\lambda_1 = v - \sqrt{c_0(1-\varepsilon)}$, $\lambda_2 = v + \sqrt{c_0(1-\varepsilon)}$. Denote by G the domain $\varepsilon < 1$ in the (v, ε) plane.

The Riemann problem for (2.1) is the initial value problem with data of the form

$$\text{(2.2)}\qquad \{\varepsilon(x,0), v(x,0)\} = \{\varepsilon_0(x), v_0(x)\} = \begin{cases} (\varepsilon_\ell, v_\ell), & x < 0 \\ (\varepsilon_r, v_r), & x > 0 \end{cases}$$

where $(\varepsilon_\ell, v_\ell)$, (ε_r, v_r) are constants.

It has been shown, for more general systems, that (2.1), (2.2) can be solved in the class of functions consisting of constant states, separated by either shock waves or centered rarefaction wave, provided that the system is genuine nonlinear [LA].

There are two distinct types of shock waves for (2.1), 1-shock and 2-shock, which satisfy the Rankine-Hugoniot condition and the entropy criterion. Namely, for a given state $(\varepsilon_\ell, v_\ell)$, the possible states (ε, v) which can be connected to $(\varepsilon_\ell, v_\ell)$ on the right by a 1-shock {2-shock} satisfy

$$\text{(2.3)}\qquad \frac{\varepsilon-\varepsilon_\ell}{v-v_\ell} = \left(\frac{1-\varepsilon+1-\varepsilon_\ell}{2c_0}\right)^{1/2}, \quad \left\{\frac{\varepsilon-\varepsilon_\ell}{v-v_\ell} = -\left(\frac{1-\varepsilon+1-\varepsilon_\ell}{2c_0}\right)^{1/2}\right\}$$

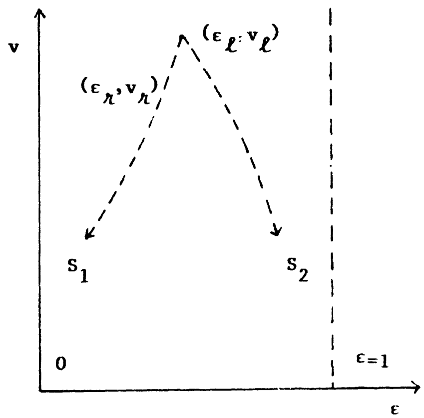
and

$$\text{(2.4)}\qquad \begin{gathered} \lambda_1(\varepsilon, v) < \sigma_1(\varepsilon, v, \varepsilon_\ell, v_\ell) < \lambda_1(\varepsilon_\ell, v_\ell) \\ \{\lambda_2(\epsilon, v) < \sigma_2(\epsilon, v, \epsilon_\ell, v_\ell) < \lambda_2(\epsilon_\ell, v_\ell)\} \end{gathered}$$

where

$$\sigma_1 = \frac{v-v_\ell}{2} - \sqrt{c_0\frac{(1-\varepsilon+1-\varepsilon_\ell)}{2}} \text{ or } \sigma_1 = v - (1-\varepsilon_\ell)\sqrt{\frac{2c_0}{1-\varepsilon+1-\varepsilon_\ell}}$$

$$\left\{\sigma_2 = \frac{v-v_\ell}{2} + \sqrt{c_0\frac{(1-\varepsilon+1-\varepsilon_\ell)}{2}} \text{ or } \sigma_2 = v + (1-\varepsilon_\ell)\sqrt{\frac{2c_0}{1-\varepsilon+1-\varepsilon_\ell}}\right\}$$



(Figure 2.1)

This singles out the half-branch curve, as shown in fig. (2.1), which is called 1-shock curve {2-shock curve}, denoted by $S_1(\varepsilon_\ell, v_\ell)$ $\{S_2(\varepsilon_\ell, v_\ell)\}$.

Now we turn to centered rarefaction waves which are continuous solutions to (2.1) of the form $v = v(x/t)$, $\varepsilon = \varepsilon(x/t)$.

Substituting $v = v(\xi)$, $\varepsilon = \varepsilon(\xi)$, $\xi = x/t$ into (2.1), one obtains

$$\begin{pmatrix} \xi - v & 1 - \varepsilon \\ c_0 & \xi - v \end{pmatrix} \begin{pmatrix} d\varepsilon/d\xi \\ dv/d\xi \end{pmatrix} = 0$$

which possesses three kinds of solutions:

1°) $\varepsilon \equiv$ const., $v \equiv$ const.

2°) $\varepsilon \equiv 1$, $v = \xi$

3°) $\xi = \lambda_i$, $(d\varepsilon/d\xi,\ dv/d\xi)^T // r_i$, where r_i is the right eigenvector corresponding to λ_i, $i = 1, 2$. The solution 1°) is called "constant state", the solution 3°) is an "i-rarefaction wave". We call the solution 2°) a "degenerate simple wave" since in this case the characteristic speeds λ_1 and λ_2 are equal.

Let us look now at the solution 3°). There are two families of rarefaction waves, corresponding to either characteristic family of λ_1 or λ_2. Let us choose

$$r_1 = \left(\sqrt{1-\varepsilon},\ \sqrt{c_0}\right)^T, \qquad r_2 = \left(-\sqrt{1-\varepsilon},\ \sqrt{c_0}\right)^T$$

as the right eigenvectors corresponding to λ_1 and λ_2, respectively. It is known that for any given state $(\varepsilon_\ell, v_\ell)$ the possible states which can be connected to

$(\varepsilon_\ell, v_\ell)$ on the right by a 1-rarefaction wave {2-rarefaction wave} satisfy

$$\frac{d\varepsilon}{dv} = \frac{\sqrt{1-\varepsilon}}{\sqrt{c_0}} \quad \left\{ \frac{d\varepsilon}{dv} = -\frac{\sqrt{1-\varepsilon}}{\sqrt{c_0}} \right\} \tag{2.5}$$

and

$$\lambda_1(\varepsilon, v) > \lambda_1(\varepsilon_\ell, v_\ell) = \lambda_{1\ell} \tag{2.6}$$

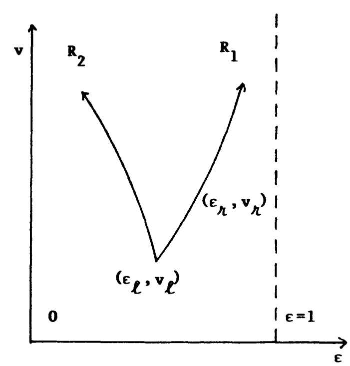
$$\{\lambda_2(\varepsilon, v) > \lambda_2(\varepsilon_\ell, v_\ell) = \lambda_{2\ell}\}$$

Namely,

$$v + 2\sqrt{c_0}\sqrt{1-\varepsilon} = v_\ell + 2\sqrt{c_0}\sqrt{1-\varepsilon_\ell}, \;\; \lambda(\varepsilon, v) > \lambda_{1\ell}$$

$$\left\{ v - 2\sqrt{c_0}\sqrt{1-\varepsilon} = v_\ell - 2\sqrt{c_0}\sqrt{1-\varepsilon_\ell}, \;\; \lambda(\varepsilon, v) > \lambda_{2\ell} \right\}$$

It can be easily seen that λ_i is an increasing function of v along the integral curves of $(2.5)_i$. This shows that (2.1) is genuine nonlinear and singles out the upper half branch curve, as shown in figure (2.2), where R_i denotes the $i-th$ rarefaction curve.



(Figure 2.2)

By means of the Riemann invariants

$$\begin{cases} r = v - 2\sqrt{c_0}\sqrt{1-\varepsilon} \\ s = v + 2\sqrt{c_0}\sqrt{1-\varepsilon} \end{cases} \tag{2.7}$$

the rarefaction wave curve $R_1\{R_2\}$ can be expressed as

$$s = s_\ell, \;\; r > r_\ell \qquad \{r = r_\ell, \;\; s > s_\ell\}. \tag{2.8}$$

If we set $\tilde{G} = \{(r,s) : 0 < s - r\}$, then the mapping (2.7) from G onto $\tilde{G}$ is $1-1$. Therefore we may use either (r,s) or (ε, v) according to our convenience.

It can be shown that the shock curves can be expressed in the (r,s) plane in the following way. Set $\alpha = (1-\varepsilon)/(1-\varepsilon_\ell)$

$$(2.9) \qquad S_1(\varepsilon_\ell, v_\ell) : \begin{cases} r_\ell - r = \sqrt{2c_0(1-\varepsilon_\ell)} \left(\dfrac{\alpha - 1}{\sqrt{\alpha + 1}} + \sqrt{2}\,(\sqrt{\alpha} - 1) \right) \\ s_\ell - s = \sqrt{2c_0(1-\varepsilon_\ell)} \left(\dfrac{\alpha - 1}{\sqrt{\alpha + 1}} - \sqrt{2}\,(\sqrt{\alpha} - 1) \right) \end{cases}$$

$$(2.10) \qquad S_2(\varepsilon_\ell, v_\ell) : \begin{cases} r_\ell - r = \sqrt{2c_0(1-\varepsilon_\ell)} \left(\dfrac{1-\alpha}{\sqrt{\alpha + 1}} - \sqrt{2}\,(1 - \sqrt{\alpha}) \right) \\ s_\ell - s = \sqrt{2c_0(1-\varepsilon_\ell)} \left(\dfrac{1-\alpha}{\sqrt{\alpha + 1}} + \sqrt{2}\,(1 - \sqrt{\alpha}) \right) \end{cases}$$

for $0 < \alpha < 1$

Obviously

$$(2.11) \qquad \begin{cases} r < r_\ell \\ s > s_\ell \end{cases} \text{for } S_1, \qquad \begin{cases} s < s_\ell \\ r > r_\ell \end{cases} \text{for } S_2,$$

which means that (2.1) belongs to the K class [DI], for which the interaction of elementary waves exhibit different patterns than 2×2 systems of gas-dynamics type.

For instance, the result of overtaking of two 1-shock waves is a transmitted 1-shock and a reflected 2-shock wave instead of a transmitted 1-shock and a reflected 2-rarefaction wave as in the case of gas-dynamics [S]. Furthermore, we obtain more precise properties to the shock curves.

PROPOSITION (2.1). *S_1 is monotone and convex in the (r,s) plane (cf. figure 2.3). Moreover*

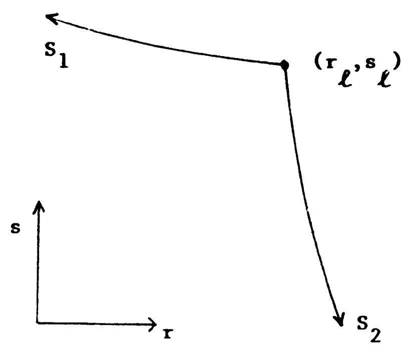
$$(2.12) \qquad S_1 : s_\ell - s = g_1(r_\ell - r, \varepsilon_\ell), \qquad r < r_\ell$$

$$(2.13) \qquad S_2 : r_\ell - r = g_2(s_\ell - s, \varepsilon_\ell), \qquad s < s_\ell$$

where $g_i'(\beta, \varepsilon)$ and $g_i''(\beta, \varepsilon_\ell)$ satisfy

$$(2.14) \qquad -1 < g_i'(\beta, \varepsilon_\ell) \le 0, \qquad i = 1,2$$

$$(2.15) \qquad g_i''(\beta, \varepsilon_\ell) \le 0, \qquad i = 1,2$$



(Figure 2.3)

PROOF. Set

$$\frac{\beta}{\sqrt{2c_0(1-\varepsilon_\ell)^{1/2}}} = \phi_1(\alpha) = \frac{\alpha-1}{\sqrt{\alpha+1}} + \sqrt{2}\ (\sqrt{\alpha}-1)$$

Due to $\phi_1'(\alpha) > 0$, the inverse function can be determined as $\alpha = \alpha_1(\beta/\sqrt{2c_0(1-\varepsilon_\ell)})$. That, combined with the expression $(2.9)_2$, implies (2.12). Let

$$h_1(\alpha) = \frac{\partial(s_\ell - s)/\partial\alpha}{\partial(r_\ell - r)/\partial\alpha} \text{ along } S_1$$

It is easily seen that

$$\frac{\partial(s_\ell - s)/\partial\alpha}{\partial(r_\ell - r)/\partial\alpha} = \frac{y-1}{y+2},$$

where $y = (\alpha+3)\ \sqrt{a}/\sqrt{2}(\alpha+1)^{3/2}$. Since y tends to 1 as α tends to 1, one has that $h_1(\alpha)$ tends to 0 as α tends to 1.

However, as α tends to ∞, y tends to $1/\sqrt{2}$ and $h_1(\alpha)$ tends to $-(1-\frac{1}{\sqrt{2}})/(1+\frac{1}{\sqrt{2}})$. Thus $-1 < h_1(\alpha) < 0$, and $(2.14)_1$ follows with $-1 < h_1(\alpha) < 0$,

$$g_i'(\beta, \varepsilon_\ell) = h_1(\alpha)_{|\alpha=\alpha_1(\beta/\sqrt{2c_0(1-\varepsilon_\ell)})}.$$

Similarly, set

$$\frac{\beta}{\sqrt{2c_0(1-\varepsilon_\ell)^{1/2}}} = \phi_2(\alpha) = \frac{1-\alpha}{(1+\alpha)^{1/2}} + \sqrt{2}\ (1-\sqrt{\alpha}).$$

Due to $\phi_2'(\alpha) < 0$, the inverse function can be determined as $\alpha = \alpha_2(\beta/\sqrt{2c_0(1-\varepsilon_\ell)})$ which, combined with the expression $(2.10)_1$, implies (2.13).

Letting $h_2(\alpha) = \frac{\partial(r_\ell - r)/\partial\alpha}{\partial(s_\ell - s)/\partial\alpha}$ – along S_2, it easily seen that

$$h_2(\alpha) = \frac{y-1}{y+2} \text{ where } y = \frac{\sqrt{\alpha}(3+\alpha)}{\sqrt{2}(1+\alpha)^{3/2}}.$$

One has that, as α tends to 1, y tends to 1 and $h_2(\alpha)$ tends to 0, while as α tends to 0, y tends to 0, and $h_2(\alpha)$ tends to -1. Thus $(2.14)_2$ follows with

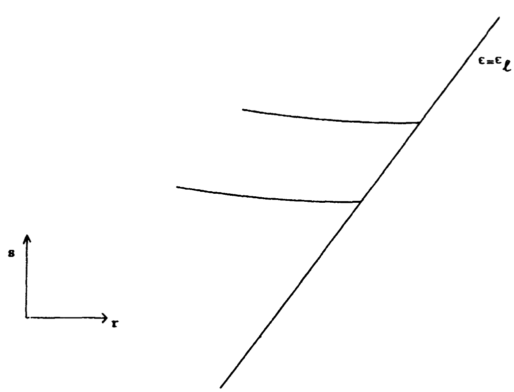
$$g_2'(\beta, \varepsilon_\ell) = h_2(\alpha)_{|\alpha=\alpha_2(\beta/\sqrt{2c_0(1-\varepsilon_\ell)})}.$$

Moreover

$$g''(\beta, \varepsilon) = \frac{dh_i}{d\alpha}\bigg|_{\alpha=\alpha_1(\vartheta)} \frac{d\alpha_i}{d\vartheta}\bigg|_{\vartheta=\beta\sqrt{2c_0(1-\varepsilon_\ell)}} \cdot (2c_0(1-\varepsilon_\ell))^{-1/2}$$

which shows (2.15).

REMARK 1. The shock curves from different points having the same ε_ℓ coincide with each other by shifting the starting point along $\varepsilon = \varepsilon_\ell$. (cf. figure 2.4).



(Figure 2.4)

REMARK 2. For any given state (ε_r, v_r) the possible state (ε, v) which can be connected to (ε_r, v_r) on the left by a 1-shock {2-shock} satisfy

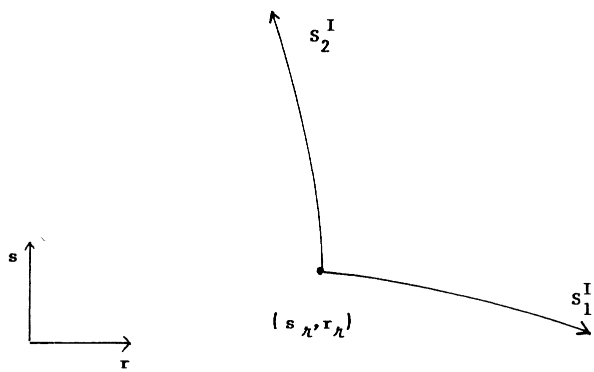
$$\frac{\varepsilon-\varepsilon_r}{v-v_r} = \left(\frac{1-\varepsilon+1-\varepsilon_r}{2c_0}\right)^{1/2} \quad \left\{\frac{\varepsilon-\varepsilon_r}{v-v_r} = -\left(\frac{1-\varepsilon+1-\varepsilon_r}{2c_0}\right)^{1/2}\right\}$$

and

$$\lambda_1(\varepsilon, v) > w_1(\varepsilon, v, \varepsilon_r, v_r) > \lambda_1(\varepsilon_r, v_r) = \lambda_{1r}$$

$$\{\lambda_2(\varepsilon, v) > w_2(\varepsilon, v, \varepsilon_r, v_r) > \lambda_2(\varepsilon_r, v_r) = \lambda_{2r}\}$$

In this way, we single out two half branch curves denoted by S_1^I and S_2^I respectively (cf. figure 2.5)



(Figure 2.5)

Similarly, it can be shown that

$$S_1^I : s - s_r = g_1(r - r_r, \varepsilon_r), \qquad r > r_r$$

$$S_2^I : r - r_r = g_1(s - s_r, \varepsilon_r), \qquad s > s_r$$

where $-1 < g_i'(\beta, \varepsilon_r) < 0$ and $g_i''(\beta, \varepsilon_r) < 0$.

For any given state (ε_r, v_r), the possible states (ε, v) which can be connected to (ε_r, v_r) on the left by a 1-rarefraction wave {2-rarefraction wave} satisfy

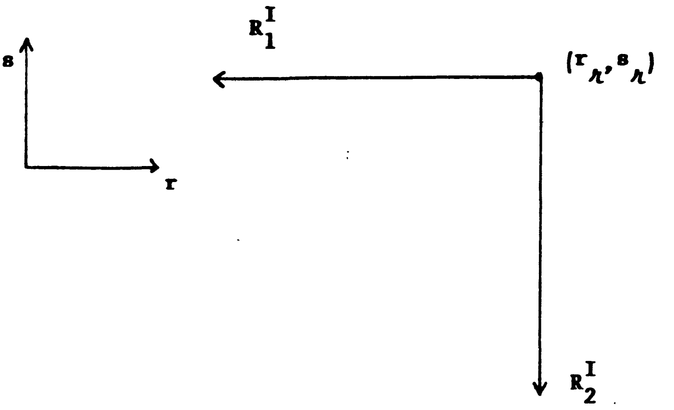
$$\frac{d\varepsilon}{dv} = \frac{\sqrt{1-\varepsilon}}{\sqrt{c_0}} \qquad \left\{\frac{d\varepsilon}{dv} = -\frac{\sqrt{1-\varepsilon}}{\sqrt{c_0}}\right\}$$

and

$$\lambda_1 < \lambda_{1r} \qquad \{\lambda_2 < \lambda_{2r}\}.$$

In this way, we single out two half branch curves denoted by R_1^I and R_2^I respectively. Using the Riemann invariants, one has (cf. figure 2.6)

$$R_1^I : \begin{cases} s = r_r \\ r < r_r \end{cases} \qquad R_2^I : \begin{cases} r = r_r \\ s < s_r \end{cases}$$



(Figure 2.6)

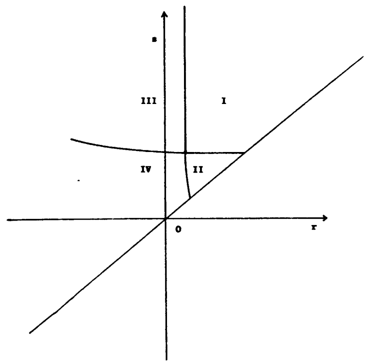
In addition to the Riemann problem, we have to discuss certain special initial-boundary value problems, because of the boundary condition (1.3), namely (2.1) under the conditions

(2.16) $\quad (\varepsilon, v)(x, 0) = (\varepsilon_0, v_0), x > 0, (\varepsilon_0, v_0)$ is a constant state with $\varepsilon_0 < 1$.

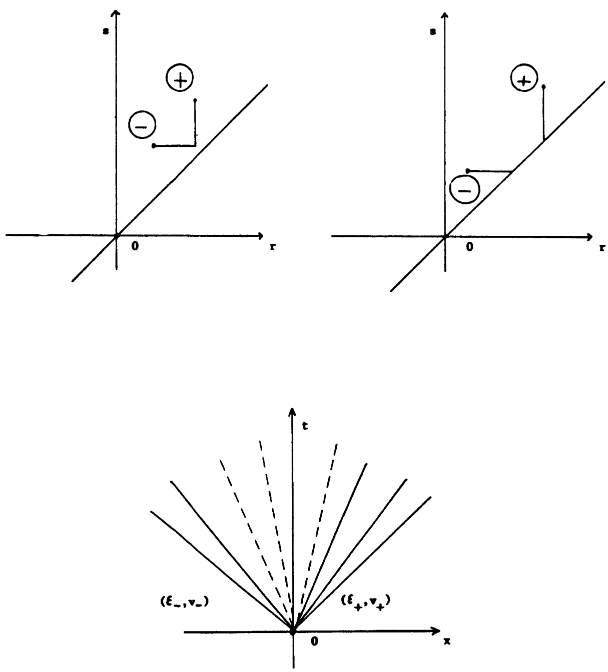
(2.17) $\quad v(0, t) = v_1, \; t > 0, \; 0 \leq v_1.$

PROPOSITION (2.2). *Assume $(v_-, \varepsilon_-), (v_+, \varepsilon_+) \varepsilon G$; then the Riemann Problem (2.1) (2.2) has a unique solution $(v(\xi), \varepsilon(\xi)), \xi = x/t$ such that $(v(\xi), \varepsilon(\xi)) \to (v_\mp, \varepsilon_\mp)$ as ξ tends $\mp\infty$, respectively.*

PROOF. For any given $(r_-, s_-) \in \tilde{G}$, draw the first wave curve $W_1(r_-, s_-)$ which is $S_1(r_-, s_-)$ for $r < r_-$ and $R_1(r_-, s_-)$ for $r > r-$. Draw the second wave curve $W_2(r_-, s_-)$, which is $S_2(r_-, s_-)$ for $s < s_-$ and $R_2(s_-, r_-)$ for $s > s_-$. $W_1(r_-, s_-)$ divide the domain $\tilde{G}$ into four regions $I(r_-, s_-) \cdots IV(r_-, s_-)$ as shown in figure 2.7(1).



(Figure 2.7(1))



(Figure 2.7(2))

By virtue of the basic properties of the shock curves S_i, S_i^I and the rarefaction wave curves R_i, R_i^I given by the proposition (2.1), formula (2.8) and the remarks, it can be proved that for any given state $(r_\mp, s_\mp) \in \tilde{G}$, the problem (2.1), (2.2) is solvable.

When $(r_+, s_+) \in II(r_-, s_-)$, $S_2^I(r_+, s_+)$ intersects with $R_1(r_-, s_-)$, while it does not intersect with $S_2(s_-, r_-)$ for $s_+ \leq s \leq s_-$, unless $(r_+, s_+) \in S_2(r_-, s_-)$, (in this case $S_2^I(r_+, s_+)$ intersects with $S_2(r_-, s_-)$ at (r_-, s_-) only, for $s_+ \leq s \leq s_-$). This shows that the solution of (2.1), (2.2) consists of a 1-rarefaction wave and a 2-shock wave. When $(r_+, s_+) \in III(r_-, s_-)$, $R_2^I(r_+, s_+)$ intersects with $S_1(s_-, r_-)$ while the corresponding solution consists of a 1-shock wave and a 2 rarefaction wave. When $(r_+, s_+) \in IV(r_-, s_-)$, $S_2^I(r_+, s_+)$ intersects with $S_1(s_-, r_-)$ at (r^*, s^*) and the corresponding solution consists of a 1-shock wave and a 2-shock wave. When $(r_+, s_+) \in I(r_-, s_-)$, the curve $R_2^I(r_+, s_+)$ either intersects with $R_1(r_-, s_-)$ for $\varepsilon < 1$ or intersects with $\varepsilon = 1$, first. (cf. figure 2.7(2)). In the former case, the corresponding solution consists of a 1-rarefaction wave a 2-rarefaction wave; in the latter case the corresponding solution consists of a 1-rarefaction wave, a degenerate simple wave $\varepsilon = 1$, $v = \xi$ and a 2-rarefaction wave (figure (2.7)(2)). More precisely

$$(v(\xi), \varepsilon(\xi)) = \begin{cases} (v_-, \varepsilon_-), \quad -\infty < \varsigma < \lambda_1(v_-, \varepsilon_-) = v_- - \sqrt{c_0(1-\varepsilon_-)} \\ \text{1-rarefaction wave, } v_- - \sqrt{c_0(1-\varepsilon_-)} \\ \leq \xi \leq v_- + 2\sqrt{c_0(1-\varepsilon_-)} \\ \text{degenerate simple wave, } v_- + 2\sqrt{c_0(1-\varepsilon_-)} \\ \leq \xi \leq v_+ + \sqrt{c_0(1-\varepsilon_+)} \\ \text{2-rarefaction wave, } v_+ - 2\sqrt{c_0(1-\varepsilon_+)} \\ \leq \xi \leq v_+ + \sqrt{c_0(1-\varepsilon_+)} \\ (v_+, \varepsilon_+), \quad v_+ + \sqrt{c_0(1-\varepsilon_+)} \leq \xi < +\infty \end{cases}$$

Next, let us consider the initial-boundary value problem (2.1), (2.16), (2.17).

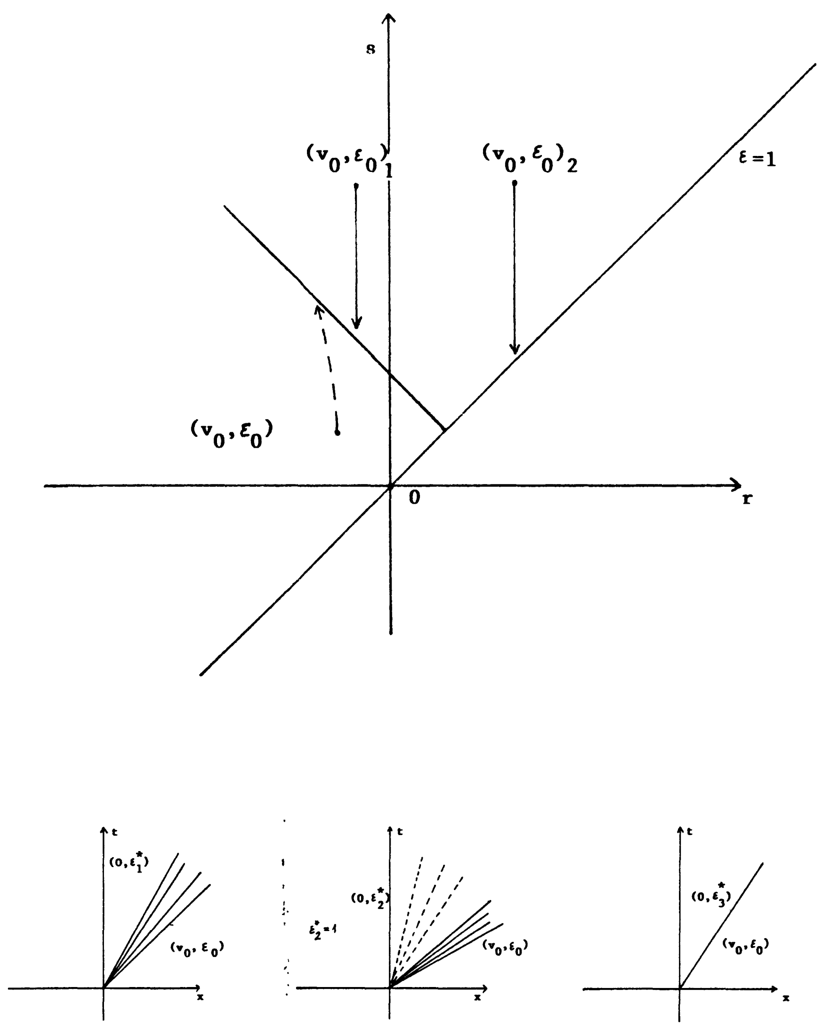
In order to guarantee existence and uniqueness we need to restrict the domain G. Define

$$\begin{aligned} H = &\left\{(\varepsilon, v) : |v| < \sqrt{c_0(1-\varepsilon)}, \varepsilon < 1\right\} \\ &\cap \left\{(\varepsilon, v) : v - 2\sqrt{c_0(1-\varepsilon)} \leq v_1 - 2\sqrt{c_0(1-\varepsilon^*)}\right\} \end{aligned}$$

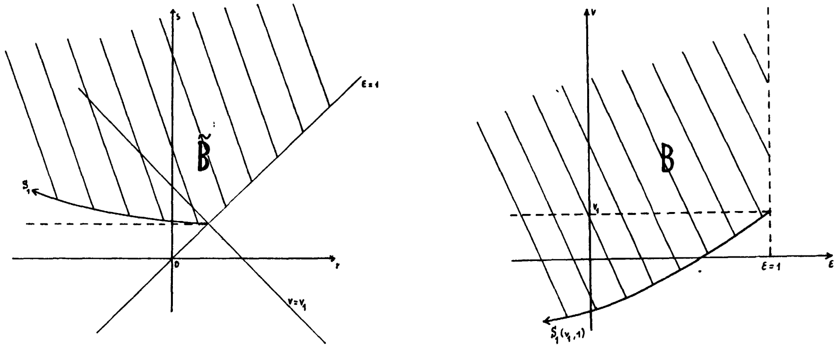
where ε^* is defined by $v_1 = \sqrt{c_0(1-\varepsilon^*)}$. Obviously one has, $H \subset G$.

PROPOSITION (2.3). *Assume that* $(\varepsilon_0, v_0) \varepsilon\, H$, *then the problem* (2.1), (2.16), (2.17) *has a unique solution* $(v(\xi), \varepsilon(\xi))$, $\xi + x/t$, *such that* $(v(\xi), \varepsilon(\xi)) \in G$ *for* $0 < \xi < +\infty$ *and* $(v(\xi), \varepsilon(\xi)) \longrightarrow (v_0, \varepsilon_0)$ *as* $\xi \longrightarrow +\infty$, $v = v_1$ *at* $\xi = 0$.

PROOF. For any given (v_0, ε_0) with $v_0 < v_1$, we draw $S_2^I(v_0, \varepsilon_0)$ which intersects with $v = v_1$ at $(\overline{v}, \overline{\varepsilon})$. Then the corresponding solution consists of a 2-shock, when $v_0 < v_1$. If $v_0 > v_1$, we draw $R_2^I(v_0, \varepsilon_0)$ which intersects with $v = v_1$, then the corresponding solutions consist of a 2-rarefaction wave (figure 2.8). In order to discuss the interaction of waves we also have to restrict the domain G. Denote by B (or $\tilde{B}$ in the (r, s) plane) the domain confined by the curve $S_1(v_1, 1)$ and $\varepsilon \leq 1$. (cf. figure 2.9)



(Figure 2.8)



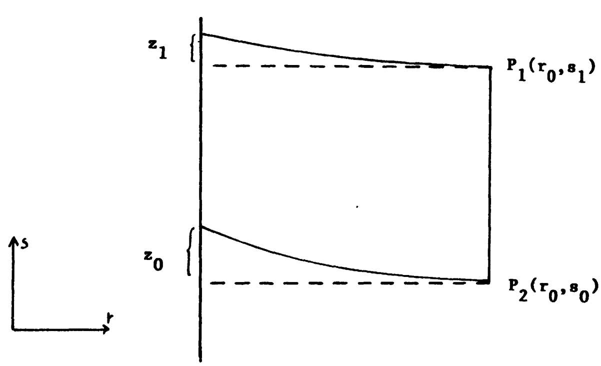
(Figure 2.9)

the control of the wave interaction, which is essential for proving the global existence theorem, requires the following additional results.

PROPOSITION (2.4). *For any two given points* $P_1 : (r_0, s_1)$ *and* $P_0 : (r_0, s_0)$ *with* $s_1 > s_0$ *and* $P_i \in \tilde{B}$ *the* 1*-shock curve* $S_1(P_1)$ *is "flatter" than* $S_1(P_0)$ *namely*

$$0 < z_0 - z_1, \text{ for any } r < r_0,$$

where $z_i (i = 0, 1)$ *is shown in figure* (2.10)(1).

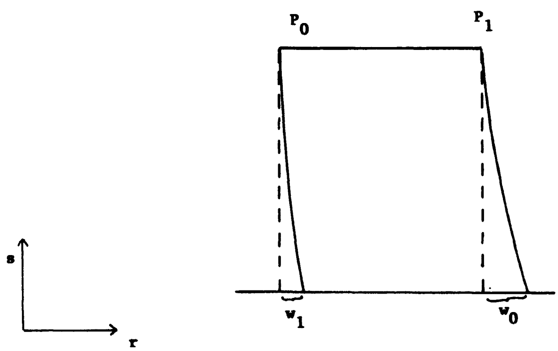


(Figure 2.10(1))

Similarly for any two given points $P_1 : (r_1, s_0)$ *and* $P_0 : (r_0, s_0)$ *with* $r_1 < r_0$ *and* $P_i \in \tilde{B}$ $(i = 0, 1)$, *the 2-shock curve* $S_2(P_1)$ *is "more straight" than* $S_2(P_0)$, *namely*

$$0 < w_0 - w_1, \text{ for any } s < s_0,$$

where $w_i (i = 0, 1)$ *is shown in figure* (2.10)(2).



(Figure 2.10(2))

PROOF.

$$z_1 - z_0 = \int_0^{r_0 - r} \left\{ h_1[\alpha_1(\beta/\sqrt{2c_0(1-\varepsilon_1)})] - h_1[\alpha_1(\beta/\sqrt{2c_0(1-\varepsilon_0)})] \right\} d\beta =$$

$$= \int_0^{r_0 - r} \frac{dh_1}{d\alpha}\Big|_{\alpha = \alpha_1(\vartheta)} \frac{d\alpha_1}{d\vartheta} \left[\frac{\beta}{(2c_0(1-\varepsilon_1))^{1/2}} - \frac{\beta}{(2c_0(1-\varepsilon_0))^{1/2}} \right] d\beta$$

where $\beta/\sqrt{2c_0(1-\varepsilon_1)} < \vartheta < \beta/\sqrt{2c_0(1-\varepsilon_0)}$.

Then $z_1 < z_0$, since the integrand is negative. In a similar way we prove $w_1 < w_0$.

Let us measure the strength of the waves by the difference in the v direction.

Due to the propositions (2.1) and (2.3), we get the following result concerning the interaction of elementary waves.

PROPOSITION (2.5). *Assume that all the elementary waves before interaction are contained in* $\tilde{B}$. *Then the sum of the strength of waves after the interaction is less than or equal to the sum of the strength before the interaction.*

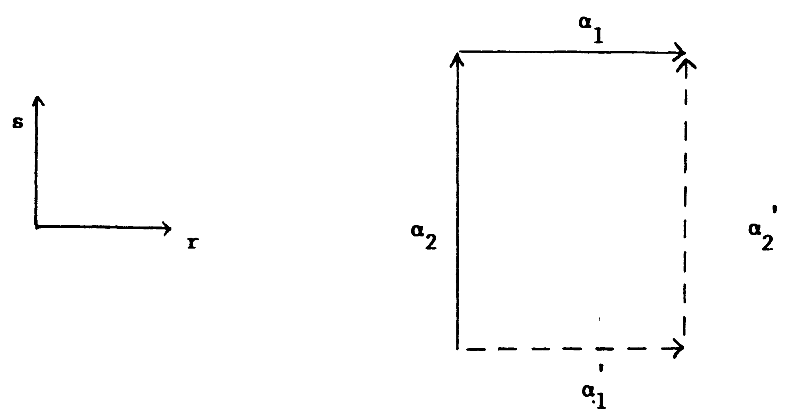
PROOF. The proposition will be proved by discussing each kind of interaction of elementary waves. Let us denote a rarefaction wave by $\alpha \cdots$ and its strength by $|\alpha|$, a shock wave by $\beta \cdots$ and its strength by $|\beta|$. In order to distinguish the waves after the interaction we shall denote them α' and β'. We denote by α_1 and α_2 (respectively β_1 and β_2) the 1-rarefaction wave and the 2-rarefaction wave (respectively 1-shock and 2-shock); by α_d the degenerate simple wave. Now we go on to study all the possible cases.

(i) $$\alpha_2 + \alpha_1 \longrightarrow \alpha_1' + \alpha_2' \text{ (cf. figure 2.11)}$$

or

$$\alpha_2 + \alpha_1 \longrightarrow \alpha_1' + \alpha_d' + \alpha_2'$$

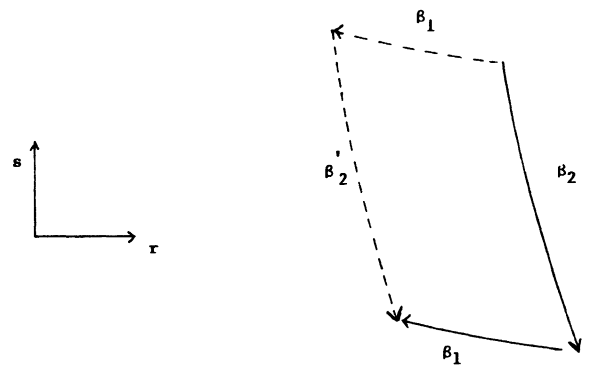
It is obvious that $|\alpha_i'| = |\alpha_i|$, $i = 1, 2$ for the first case and $|\alpha_1'| + |\alpha_d|$ $+|\alpha_2'| = |\alpha_1| + |\alpha_2|$ for the second case



(Figure 2.11)

(ii) $$\beta_2 + \beta_1 \longrightarrow \beta_1' + \beta_2' \qquad \text{(cf. figure 2.12)}$$

It is clear that $|\beta_1'| + |\beta_2'| = |\beta_1| + |\beta_2|$, because of the proposition (2.1), which that v is changing monotonically along the shock wave curves.

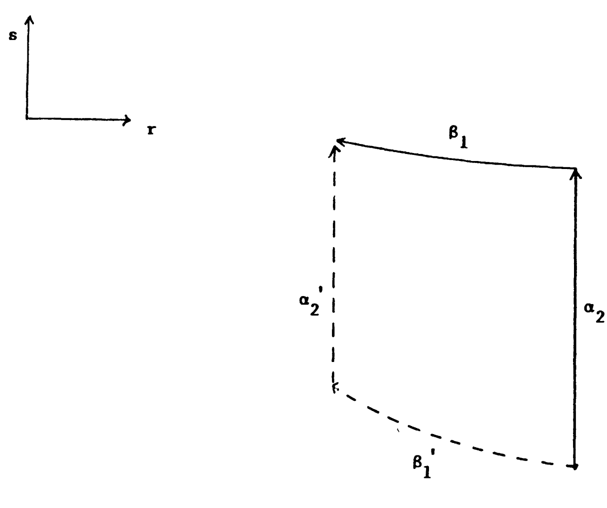


(Figure 2.12)

(iii) $$\alpha_2 + \beta_1 \longrightarrow \beta_1' + \alpha_2' \qquad \text{(cf. figure 2.13)}$$

From proposition (2.4), it follows

$$|\alpha_2'| < |\alpha_2|, \quad |\beta_1'| < |\beta_1|$$



(Figure 2.13)

(iv) $$\beta_2 + \alpha_1 \longrightarrow \alpha_1' + \beta_2'.$$

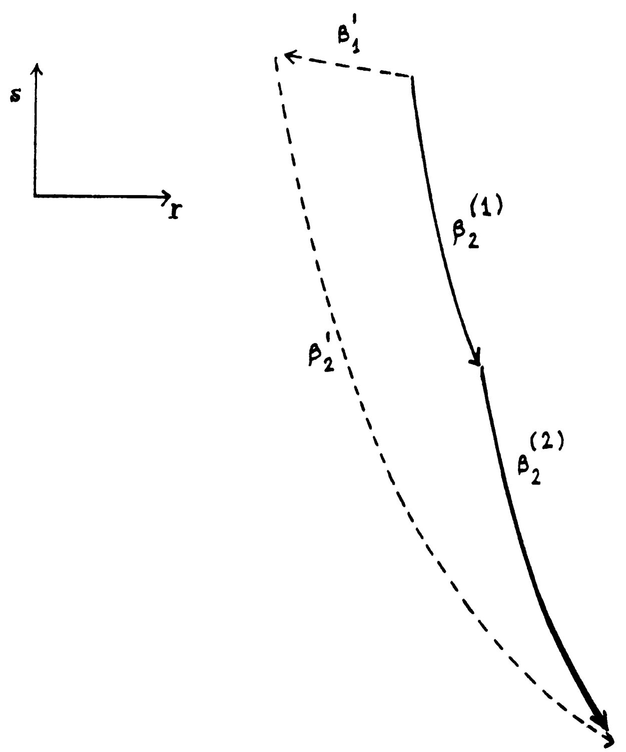
Similarly to (iii) one has $|\alpha_2'| \leq |\alpha_1|$, $|\beta_2'| \leq |\beta_2|$

(v) $$\beta_2^{(1)} + \beta_2^{(2)} \longrightarrow \beta_1' + \beta_2' \text{ (or } \beta_1^{(1)} + \beta_1^{(2)} \longrightarrow \beta_1' + \beta_2')$$

(cf. figure 2.14).

The same argument as in the case (ii) yields

$$|\beta_1'| + |\beta_2'| = |\beta_2^{(1)}| + |\beta_2^{(2)}|$$



(Figure 2.14)

(vi) $$\beta_2 + \alpha_2 \longrightarrow \alpha_1' + \beta_2' \text{ or } \alpha_1' + \alpha_2' \qquad \text{(cf. figure 2.15)}$$

For the second possibility it is clear that

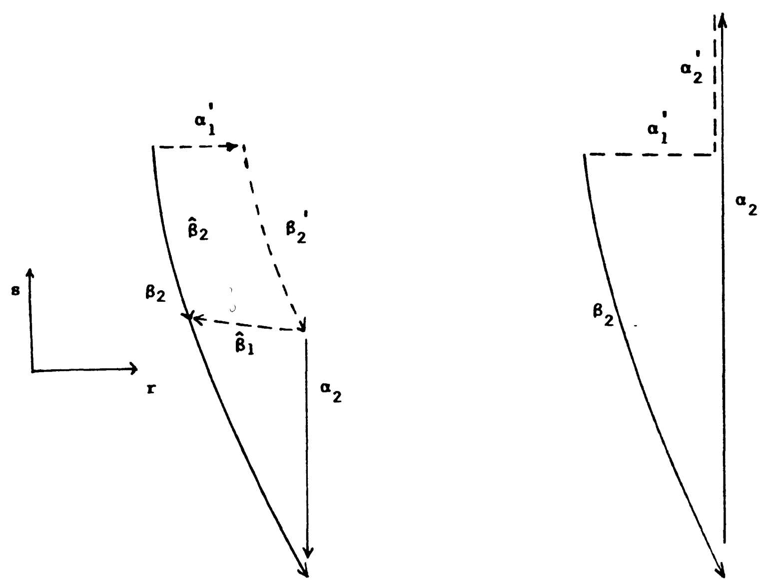
$$|\alpha_1'| + |\alpha_2'| \leq |\alpha_2| + |\beta_2|$$

For the first possibility we may consider an auxiliary interaction $\hat{\beta}_2 + \hat{\beta}_1$, as shown in figure (2.15), and compare $\hat{\beta}_2 + \hat{\beta}_1$ with $\alpha_1' + \beta_2'$ and $\beta_2 + \alpha_2$ respectively. It is obvious that

$$|\hat{\beta}_2| + |\hat{\beta}_2| < |\beta_2| + |\alpha_2|$$

hence

$$|\alpha_1'| + |\beta_2'| \leq |\beta_2| + |\alpha_2|$$



(Figure 2.15)

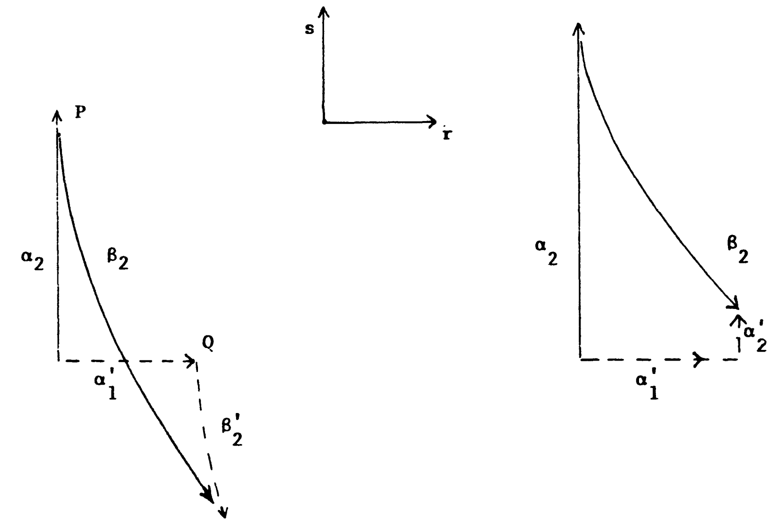
(vii) $\alpha_2 + \beta_2 \longrightarrow \alpha_1' + \beta_2'$ or $\alpha_1' + \alpha_2'$ (cf. figure 2.16)

It is clear for the second case that

$$|\alpha_1'| + |\alpha_2'| \leq |\alpha_2| + |\beta_2|$$

For the first case, noting that $v_Q < v_P$ because of the proposition (2.1) and the remark 2, it follows

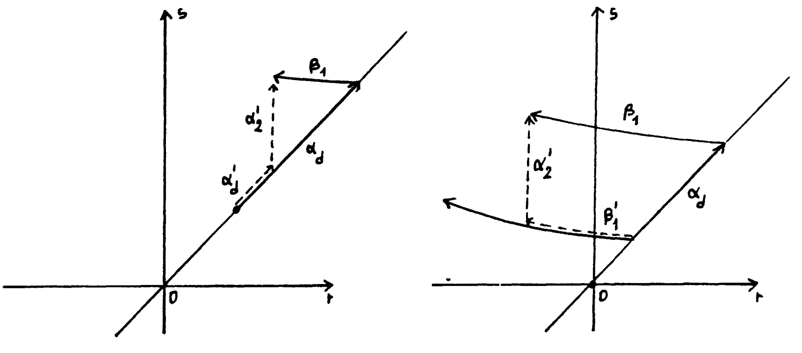
$$|\alpha_1'| + |\beta_2'| \leq |\alpha_2| + |\beta_2|$$



(Figure 2.16)

Similarly it is possible to discuss the interaction $\beta_1 + \alpha_1$ or $\alpha_1 + \beta_1$.

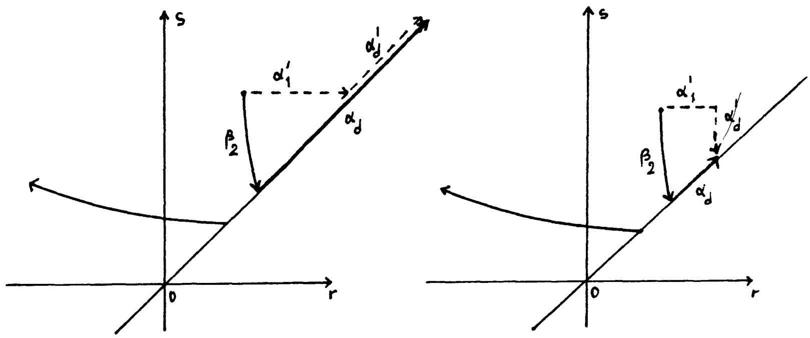
(viii) $$\alpha_d + \beta_1 \longrightarrow \alpha_d' + \alpha_2' \text{ or } \beta_1' + \alpha_2'$$



(Figure 2.17)

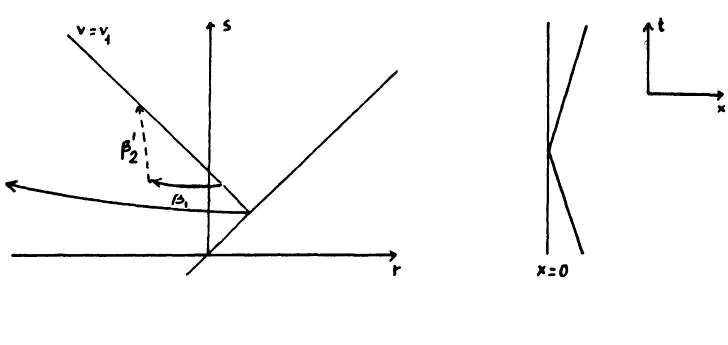
where α_d is defined for $\xi \geq 0$ i.e. $v \leq 0$. It is clear for the first case that $|\alpha'_d| < |\alpha_d|$ and α'_d is still defined for $\xi \geq 0$, $|\alpha'_d| + |\alpha'_2| < |\alpha_d| + |\beta_1|$.

(ix) $$\alpha_d + \alpha_d \longrightarrow \alpha'_1 + \alpha'_d \text{ or } \alpha'_1 + \beta'_2$$



(Figure 2.18)

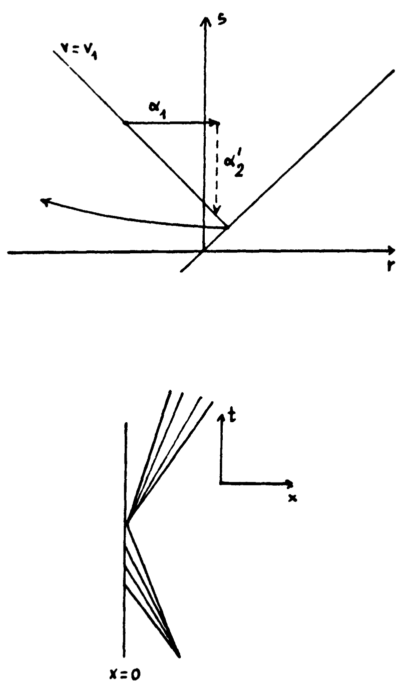
By using a similar argument we obtain the following result for the interaction of waves with the boundary $x = 0$.



(Figure 2.19)

PROPOSITION (2.6). *The reflected wave to a* 1-*shock wave* β_1 *will be a* 2-*shock wave* β'_2 *(cf. figure* (2.19)), *while the reflected wave to a* 1-*rarefaction wave* α_1 *will be a* 2-*rarefaction wave* α'_2 *(cf. figure* (2.20)). *Furthermore*

$$|\alpha_2| = |\beta_1|, \quad |\beta'_2| = |\alpha_1|$$



(Figure 2.20)

3. - The scheme and its consistency

We describe, here, an algorithm to construct a family $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ of approximate solutions to (1.1), (1.6), (1.7).

A pair of bounded measurable functions $\{\varepsilon(x,t), v(x,t)\}$ is called a *weak solution* of (1.1), (1.6), (1.7) if

$$\int\limits_{t>0}\int\limits_{x>0} \{\varepsilon\phi_t - [v(1-\varepsilon)]\phi_x\}dxdt + \int\limits_{t=0} \varepsilon_0\phi dx = 0 \tag{3.1}$$

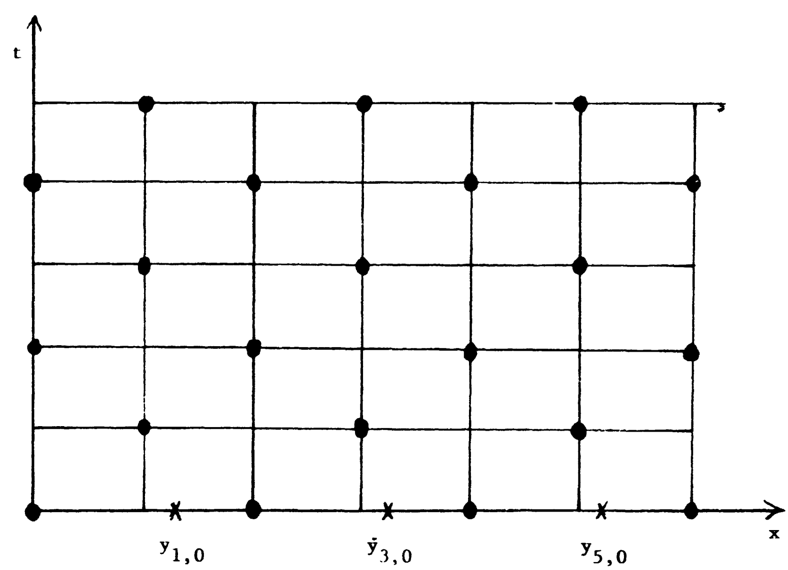
and

$$\int_{t>0}\int_{x>0} \{v\psi_t + [(1/2)v^2 - c_0\varepsilon]\psi_x + f(\varepsilon, v)\psi\}dxdt + \int_{t=0} v_0\psi dx \tag{3.2}$$

for all smooth functions ϕ and ψ with compact support in $t \geq 0$, $x \geq 0$, such that $\phi(0,t) = \psi(0,t) = 0$, for all $t \geq 0$.

Let us introduce the scheme. We begin, as in the Glimm scheme, by generating a sequence $\alpha_0, \alpha_1, \cdots, \alpha_n \cdots$ of random equidistributed numbers in the interval $(-1,1)$. We fix a value $h > 0$ of the space mesh-length and we determine the corresponding time mesh-length as $s = \Lambda^{-1}h$, were Λ is a fixed upper bound of the supremum of the moduli of the characteristic speeds $\lambda_i (i = 1,2)$ on a suitable bounded domain $D \subset B$, in the (ε, v) plane. We need that the ranges of all the approximate solutions be included in D. Such a domain will be determined later. Hence, since the (CFL) condition is fulfilled, the waves emanating simultaneously at a distance $2h$ apart, will not interact on a time interval of length s.

We partition the upper quarterplane of the (x,t) plane into strips $S_n = \{(x,t) : 0 < x,\ ns \leq t < (n+1)s\}$, $n = 0, 1 \cdots\}$ and we identify the mesh points $x = kh$, $t = ns$, with $k + n$ even, $k = 0, 1 \cdots$ (cf. figure 3.1)



(Figure 3.1)

Assuming that $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ has already been determined in $\bigcup_{l=0}^{n-1} S_l$, we describe below its construction in the strip S_n. For every m, with $m+n$ odd, we identify, following Glimm, the random mesh points $x = y_{m,n}$, $t = ns$, where $y_{m,n} = (m+\alpha_n)h$, $m \geq 1$, and we define

$$v_{m,n} = v_h(y_{m,n}, ns^-)$$
$$\varepsilon_{m,n} = \varepsilon_h(y_{m,n}, ns^-)$$

For n odd, we define $v_{0,n} = v_h(0^+, ns^-)$, $\varepsilon_{0,n} = \varepsilon_h(0^+, ns^-)$.
To initiate the algorithm at $n = 0$, we set, for all $x > 0$,

$$v_h(x, 0^-) = v_0(x)$$
$$\varepsilon_h(x, 0^-) = \varepsilon_0(x)$$

Employing $(v_{m,n}, \varepsilon_{m,n})$ we compute

$$\begin{cases} \omega_{m,n} = v_{m,n} + f_{m-1,n}s \\ w_{m,n} = v_{m,n} + f_{m+1,n}s \end{cases}$$

where

$$f_{m+1,n} \overset{\text{def}}{=} f(\varepsilon((m+1)h, ns^-), \frac{1}{2}[v((m+1)h^-, ns^-) + \\ + v((m+1)h^+, ns^-)])$$

and, if $m > 1$

$$f_{m-1,n} \overset{\text{def}}{=} f(\varepsilon((m-1)h, ns^-), \frac{1}{2}[v((m-1)h^-, ns^-) + \\ + v((m-1)h^+, ns^-)]),$$

if $m = 1$

$$f_{0.n} \overset{\text{def}}{=} f(\varepsilon(0^+, ns^-), v(0^+, ns^-)),$$

Assuming that $k+n$ is even, we let $\{\varepsilon_h(x,t),\ v_h(x,t)\}$, on the rectangle $\{(x,t) : (k-1)h \leq x < (k+1)h,\ ns \leq t < (n+1)s,\ k \geq 1\}$, be the restriction of the solution to the Riemann problem (2.1) with

$$(3.3) \qquad (\varepsilon_h, v_h)(x, ns) = \begin{cases} (\varepsilon_{k-1,n}, w_{k-1.n}), & x < kh \\ (\varepsilon_{k+1,n}, \omega_{k+1,n}), & x > kh \end{cases}$$

When n is even, we let $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ on $\{(x,t) : 0 < x < h,\ ns \leq t < (n+1)s\}$ be the restriction on the above rectangle of the solution to the following initial-boundary value problem for (2.1) with the conditions

$$(3.4) \qquad (\varepsilon_h, v_h)(x, ns) = (\varepsilon_{1.n}, v_{1.n} + f_{0.n}s), \quad 0 < x$$

$$(3.5) \qquad \varepsilon_h(0, t) = \varepsilon_1, \quad ns \leq t$$

The solvability of (2.1), (3.3) follows from the proposition (2.2) in section 2 and the solvability of (2.1), (3.4), (3.5) follows from the proposition (2.3) in section 2. The above procedure determines $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ in the strip S_n. We note that $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ experiences jump discontinuities across shocks emanating from the mesh points $x = kh,\ t = ns$, but also across the line segments $\{x = (k-1)h,\ ns \le t < (n+1)s\}$, since $v_h((k-1)h^-,t) = w_{k-1,n}$ while $v_h((k-1)h^+,t) = \omega_{k-1,n}$.

The computation may proceed for as long as the range of $\{\varepsilon_h(x,t),\ v_h(x,t)\}$ remains in D. The consistency of the scheme is demonstrated by the following.

PROPOSITION (3.1). *Assume that each selection of the random sequence* $\{\alpha_i\}$, *yields a family* $\{\varepsilon_h(x,t),\ v_h(x,t)\}$, $0 < h < h_0$, *of approximate solutions which are defined and have uniformly* (*in* h) *locally bounded variation on* $[0,\infty)\times[0,T)$. *Then there is a sequence* $(h_\ell), h_\ell \downarrow 0$ *as* $\ell \uparrow \infty$, *such that, for almost every sequence* $\{\alpha_i\}$

$$\{\varepsilon_{h_\ell}(x,t), v_{h_\ell}(x,t)\} \xrightarrow{\text{a.e.}} \{\varepsilon(x,t), v(x,t)\},\quad \ell \longrightarrow \infty \tag{3.6}$$

where $\{\varepsilon, v\}$ *is a function of locally bounded variation on* $[0,\infty)\times[0,T)$ *which is a* (*weak*) *solution to the initial boundary value problem* (1.1), (1.6), (1.7).

The proof follows from the more general framework in [DH] and so we omit it.

For a general nonhomogeneous hyperbolic system

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F(u) + G(u) = 0 \tag{3.7}$$

where u is an N-vector field and $F(u),\ G(u)$ are smooth functions from a bounded domain $\tilde{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$, a (smooth) function $\eta(u)$ is called an *entropy* for (3.7) with (smooth) *entropy flux* $q(u)$ and (smooth) *entropy production* $p(u)$ if

$$h_{uu} \le 0,\ q_u(u) = \eta_u(u)F_u(u),\ p(u) \ge \eta_u(u)G(u),\quad u \in \tilde{D}.$$

A "*BV* solution" $u(x,t)$ of (3.7) is said to satisfy the *entropy admissibility criterion* if

$$\frac{\partial}{\partial t}\eta(u(x,t)) + \frac{\partial}{\partial x}q(u(x,t)) + p(u(x,t)) \ge 0$$

in the sense of measures.

It can be shown in the same way as in [DH] that the limit function in (3.6) satisfies the above entropy condition, namely

PROPOSITION (3.2). *Under the assumptions of proposition* (3.1), *the limit function* $\{\varepsilon(x,t), v(x,t)\}$ *in* (3.6) *satisfies the entropy admissibility criterion.*

4. - Convergence of the algorithm

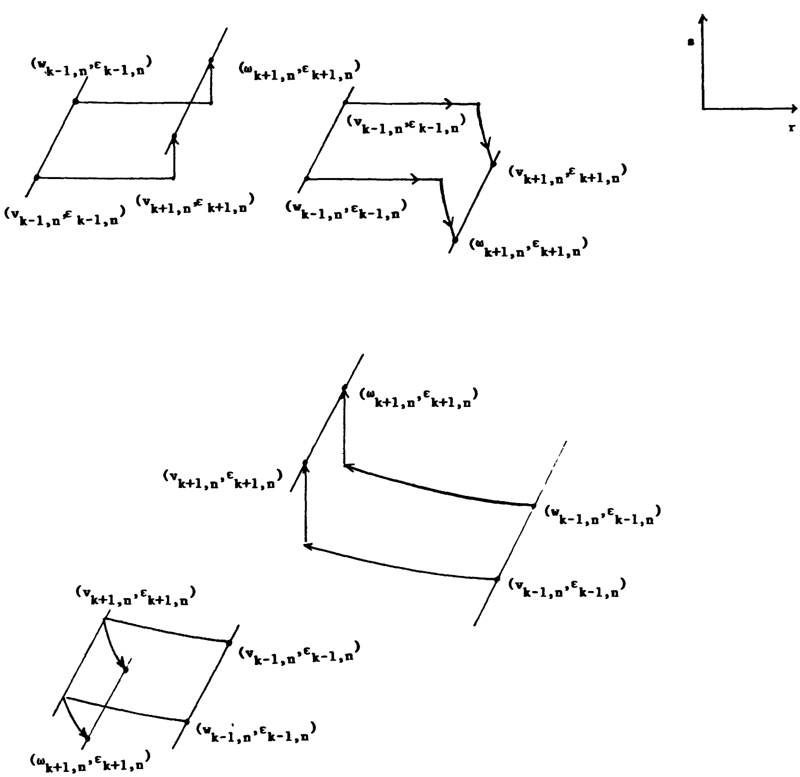
In this section we establish bounds on the total variation of the family $\{\varepsilon_h, v_h\}$ of the approximate solutions, constructed in section 3, which will induce, via the propositions (3.1) and (3.2), the existence of admissible solutions to the initial boundary-value problem (1.1), (1.6), (1.7).

First, let us compare the solution of (2.1), (3.3), with the solution of (1.1) under the following condition

$$(4.1) \qquad (\varepsilon_h, v_h)(x, ns) = \begin{cases} (\varepsilon_{k-1,n}, v_{k-1,n}), & x < kh \\ (\varepsilon_{k+1,n}, v_{k+1,n}), & x > kh \end{cases}$$

Since the initial value of ε is the same in both (3.3) and (4.1) and the difference in v is the same for both $v_{k-1,n} - w_{k-1,n}$ and $v_{k+1,n} - \omega_{k-1,n}$, it follows by proposition (2.1) (Remark 1), that

PROPOSITION (4.1). *The solution of* (1.1), (3.3) *is the same as the solution of* (1.1), (4.1) *in the following sense. The wave pattern is the same and the strength of the corresponding waves is the same* (*cf. figure* 4.1).



(Figure 4.1)

Now, we compare the solution of (1.1), (3.4), (3.5) with the solution of (1.1), (3.5) under the condition

(4.2) $$(\varepsilon_h, v_h)(x, ns) = (\varepsilon_{1,n}, v_{1,n}), \qquad x > 0.$$

As above, we get

PROPOSITION (4.2). *The solution of* (2.1), (3.4), (3.5) *is the same as the solution of* (2.1), (4.2), (3.5), *in the sense that the wave pattern is the same and the strengths of the corresponding waves are the same.*

Consider the curve $f(\varepsilon, v) = 0$, namely

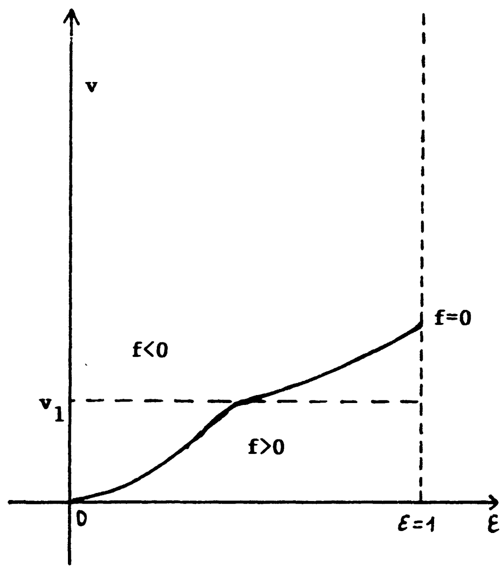
(4.3) $$v = v(\xi)$$

and

$$A = \{(\varepsilon, v) : 0 \leq \varepsilon \leq 1, \ |v| \leq b\}, \quad v_1 < b;$$

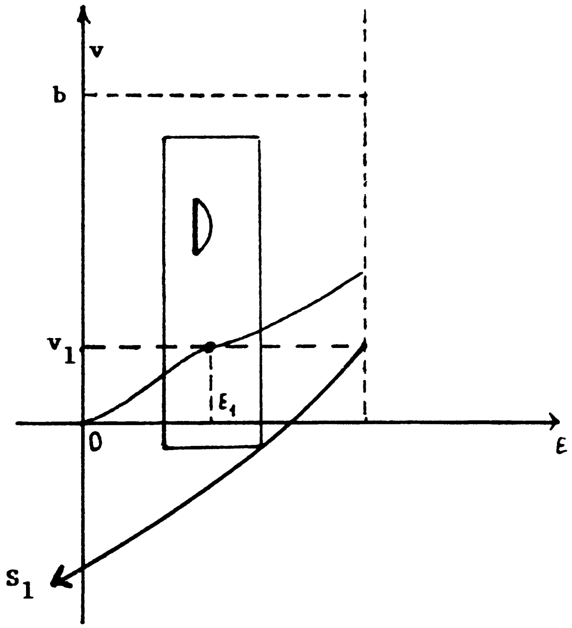
without loss of generality assume $v(0) = 0$.

It is obvious that $f < 0$ for $v > v(\xi)$, $0 \leq \varepsilon \leq 1$, while $f > 0$ for $v < v(\xi)$, $0 \leq \varepsilon \leq 1$ (cf. figure (4.2))



(Figure 4.2)

Define ε_1 such that $v_1 = v(\varepsilon_1)$. For given $(\varepsilon_0(x), v_0(x))$, let us define the domain D as follows (cf. figure (4.3)). We say that $(\varepsilon, v) \in D$ if and only if



(Figure 4.3)

the following inequalities are fulfilled:

$$|v_1 - v| \le \sup |v_0(x) - v_1| + (3 + 2c)(TV_x v_0(x) + TV_x \varepsilon_0(x))$$

$$|\varepsilon - \varepsilon_1| \le \sup |\varepsilon_0(x) - \varepsilon_1| + c(TV_x v_0(x) + TV_x \varepsilon_0(x))$$

where $c = \sqrt{2/c_0}$.

PROPOSITION (4.3). *For any given* $(\tilde{\varepsilon}, \tilde{v}), 0 < \tilde{\varepsilon} < 1$, *the value* $|\frac{d\varepsilon}{dv}| < (2/c_0)^{1/2}$ *along the rarefaction wave curve* $R_i(\tilde{\varepsilon}, \tilde{v}), R_i^I(\tilde{\varepsilon}, \tilde{v})$ *as well as the shock wave curve* $S_i(\tilde{\varepsilon}, \tilde{v}), S_i^I(\tilde{\varepsilon}, \tilde{v}),\ i = 1, 2$.

PROOF. It is obvious, from (2.5), that the statement is true for $R_i(\tilde{\varepsilon}, \tilde{v})$.

For $S_1(\tilde{\varepsilon}, \tilde{v})$ one has

$$\frac{d\varepsilon}{dv} = (2c_0)^{-1/2} w(1-\varepsilon+1-\tilde{\varepsilon})^{3/2}(1-\varepsilon+3(1-\tilde{\varepsilon}))^{-1}$$
$$< (2/c_0)^{1/2}(2-\tilde{\varepsilon})^{3/2}(1+3(1-\tilde{\varepsilon})) < (2/c_0)^{1/2}.$$

The proof is similar in the other cases.

THEOREM. *Assume that and the initial data* $v_0(x), \varepsilon_0(x)$ *are bounded measurable functions with bounded variation such that* $D \subset B \cap \{0 < \varepsilon < 1\} \cap A \cap H$. *Then there exists a solution* $(\varepsilon(x,t), v(x,t))$ *of* (1.1), (1.6), (1.7) *on* $[0,\infty) \times [0,\infty)$ *which satisfies the entropy admissibility criterion.*

PROOF. Following Glimm [G], we consider the I-curves, namely connected polygonal arcs whose edges are also edges of the diamond shaped domains and whose (directed) tangent always points toward increasing x. The set of I-curves is partially ordered according to $J' \geq J$ if J' lies toward larger time. In particular, J' will be called $(m-n)$-consecutive to J if $(y_{m,n}, ns)$ is a node of J, $(y_{m,n+2}, (n+2)s)$ is a node of J' and the remaining nodes of J and J' coincide.

With each I-curve J, we associate the functional

$$L(J) = \Sigma\{|\alpha| : \alpha \text{ crosses } J\} \tag{4.4}$$

where α are the elementary waves emanating from the mesh-points.

Let us fix now a I-curve J_n which originates and terminates on the x-axis and is confined in the strip $0 \leq t \leq ns$. We construct a decreasing sequence $J_n \geq J_{n-1} \geq \cdots \geq J_1$ of I-curves by using the following inductive procedure: J_k is confined in the strip $0 \leq t \leq ks$; J_k and J_{k+1}, share all nodes in the strip $0 < t < ks$; if J_{k+1} contains a note $(y_{m,k+1}, (k+1)s)$, then J_k contains the node $(y_{m,k-1}, (k+1)s)$. It is clear that we can interpolate between J_k and J_{k+1} by an increasing finite sequence of I-curves each one being $(m, k-1)$ consecutive of the previous one.

Denote by "a" the maximum of

$$\sup_D |f(\varepsilon, v)|, \ \sup_D |f_\varepsilon(\varepsilon, v)|, \ \sup_D |f_v(\varepsilon, v)|,$$

and consider the approximate solution in the first strip $0 \leq t \leq s$.

It is clear that $(\varepsilon_0(x), v_0(x) \in D$. By the definition of D and the sign of f, it is easy to see that $(\varepsilon_{m,0}, \omega_{m,0})$ or $(\varepsilon_{m,0}, w_{m,0})$ is contained in D, provided that s is small enough. Indeed, there exists s^* such that for any point (ε, v) ε $\{f < 0\} \cap D$, $-f(\varepsilon, v)s < v - v(\varepsilon)$ for all $s < s^*$, where $v(\varepsilon)$ is defined by $f(\varepsilon, v(\varepsilon)) = 0$; moreover the constant s^* depends only on the physical parameters in $f(\varepsilon, v)$ and on the domain D. The situation is similar for $(\varepsilon, v) \in \{f > 0\} \cap D$. By virtue of the propositions 4.1, 4.2 and 2.1, it can be shown that

$$L(J_1) \leq TV_x v_0(x) + TV_x \varepsilon_0(x) \tag{4.5}$$

The variation of $v_h(x,t)$ along J_1 is induced by two factors, namely (a) the elementary waves that cross J_1 and (b) the jump discontinuities of $v_h(x,t)$ along the line segments $\{x = mh,\ 0 \leq t < s,\ m = 1,3,5\cdots\}$ crossed by J_1. The contribution to $TV_{J_1}V_h$ of factor (a) is majorized by $L(J_1)$. Considering the contribution of factor (b) to $TV_{j_1}v_h$, one has

$$v(mh^-,s^-) - v(mh^+,s^-) = s\{f(\varepsilon((m-1)h,0), z_{m-1,h}/2) \\ - f(\varepsilon((m+1)h,0), z_{m-1,h}/2)\}$$

where we set

$$z_{m,h} = v(mh^-,0) + v(mh^+,0).$$

Since $(\varepsilon_0(x), v_0(x)) \in D$, one has

$$|v(mh^-,s^-) - v(mh^+,s^-)| \leq sa\{|\varepsilon((m-1)h,0) - \varepsilon((m+1)h,0)|+ \\ + |z_{m-1,h} - z_{m+1,h}|1/2\} \leq sa\{|\varepsilon((m-1)h,0) - \varepsilon((m+1)h,0)|+ \\ + |v(m-1)h^+,0) - v((m+1)h^-,0)|+ \\ + |v(m-1)h^-,0) - v((m-1)h^+,0)|1/2+ \\ + |v(m+1)h^-,0) - v((m+1)h^+,0)|1/2\}.$$

Setting

$$\sigma_n = \sum_{\substack{m\geq 1 \\ m+1 \text{ even}}} |v(mh^-,ns^-) - v(mh^+,ns^-)|$$

then

$$\sigma_1 \leq sa\{TV_x\varepsilon_0(x) + TV_x v_0(x)\}.$$

Combining this inequality with (4.5), it follows

$$TV_{J_1}v_h \leq (1+as)[TV_x\varepsilon_0(x) + TV_x v_0(x)] \\ \leq \frac{3}{2}[TV_x\varepsilon_0(x) + TV_x v_0(x)],$$

if $s \leq min\{s^*, 1/2a\}$ and

$$\sup_{J_1}|v_h - v_1| \leq \sup|v_0(x) - v_1| + TV_{J_1}v_h \\ \leq \sup|v_0(x) - v_1| + \frac{3}{2}[TV_x\varepsilon_0(x) + TV_x v_0(x)].$$

The variation of ε_h along J_1, induced only by the above factor (a), is majorized by $cL(J_1)$, by the proposition 4.3.

Therefore

$$TV_{J_1}\varepsilon_h \leq c[TV_x\varepsilon_0(x) + TV_x v_0(x)]$$

and

$$\sup_{J_1} |\varepsilon_h(x) - \varepsilon_1| \leq \sup |\varepsilon_0(x) - \varepsilon_1| + c[TV_x\varepsilon_0(x) + TV_x v_0(x)].$$

Thus, the approximate solution constructed in the first strip $0 \leq t \leq s$ is contained in D. Proceding by induction, we assume

$$L(J_{n-1}) \leq L(J_1) \tag{4.6}$$

$$\sup_{J_{n-1}} |v - v_1| \leq \sup |v_0(x) - v_1| + (3 + 2c)[TV_x\varepsilon_0(x) + TV_x v_0(x)] \tag{4.7}$$

$$\sup_{J_{n-1}} |\varepsilon - \varepsilon_1| \leq \sup |\varepsilon_0(x) - \varepsilon_1| + c[TV_x\varepsilon_0(x) + TV_x v_0(x)]. \tag{4.8}$$

Namely, the approximate solution constructed in the strip $0 \leq t \leq (n-1)s$ is contained in D. Thus for all $s < s^*$

$$(\varepsilon_{m.n-1}, \omega_{m.n-1}) \in D \text{ and } (\varepsilon_{m,n-1}, w_{m,n-1}) \in D.$$

By account of propositions (4.1), (4.2), (2.5) and (2.6) it can be shown that $L(J_n) \leq L(J_{n-1})$, hence using the assumption (4.6) it follows

$$L(J_n) \leq L(J_1). \tag{4.9}$$

Therefore the contribution to $TV_{J_n} v_h$ of factor (a) is majorized by $L(J_1)$ and $TV_{J_n}\varepsilon_h$ is majorized by $cL(J_1)$ by proposition (4.3). Let us consider now the contribution of the factor (b) to $TV_{J_n} v_h$:

$$\begin{aligned}
&|v(mh^-, ns^-) - v(mh^+, ns^-)| \leq sa\{|\varepsilon((m-1)h, (n-1)s^-) \\
&- \varepsilon((m+1)h, (n-1)s^-)| + |v(m-1)h^+, (n-1)s^-) \\
&- v((m+1)h^-, (n-1)s^-)| + |v((m-1)h^-, (n-1)s^-) \\
&- |v(m-1)h^+, (n-1)s^-)|/2 + |v((m+1)h^-, (n-1)s^-) \\
&- v((m+1)h^+, (n-1)s^-)|/2\}
\end{aligned}$$

This implies

$$\sigma_n \leq sa\{cL(J_1) + L(J_1) + c_{n-1}\}$$

because of (4.6), proposition (4.3) and the definition of σ_{n-1}.

Then it follows

$$\sigma_n \leq (1 - sa)^{-1}(1 + c)L(J_1) \leq 2(1 + c)L(J_1),$$

since $s < 1/2a$. Thus

$$TV_{J_n} v_h \leq (3 + 2c)[TV_x\varepsilon_0(x) + TV_x v_0(x)] \tag{4.10}$$

$$\sup_{J_{n-1}} |v_h - v_1| \leq \sup |v_0(x) - v_1| + (3 + 2c)[TV_x\varepsilon_0(x) + TV v_0(x)]. \tag{4.11}$$

(4.12) $TV_{J_n}\varepsilon_h \leq c[TV_x\varepsilon_0(x) + TV_x v_0(x)]$

(4.13) $\sup_{J_n} |\varepsilon_h - \varepsilon_1| \leq \sup |\varepsilon_0(x) - \varepsilon_1| + c[TV_x\varepsilon_0(x) + TV_x v_0(x)]$

The argument by induction can be completed by using (4.9), (4.11) and (4.13). Therefore we proved that (4.10) to (4.13) hold. By means of the propositions (3.1) and (3.2) we get the uniform boundedness of the approximate solutions and of their total variation. This is sufficient to conclude the proof.

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REFERENCES

[DH] C.M. Dafermos - L. Hsiao, *Hyperbolic systems of balance laws with inhomogeneity and dissipation.* Ind. Univ. Math. J. **31** (1982), 471-491.

[DI] R.J. Di Perna, *Global solutions to a class of nonlinear hyperbolic systems of equations.* Comm. Pure and Appl. Math. **26** (1973), 1-28.

[FG] P.V. Foscolo - L.G. Gibilaro, *A fully predictive criterion for the transition between particulate and aggregate fluidization.* Chemical engineering Science **39** (1984), 1667-1675.

[G] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations* Comm. Pure Appl. Math. **18** (1965), 95-105.

[LA] P. Lax, *Hyperbolic systems of conservation laws*, II. Comm. Pure and Appl. Math. **10** (1957), 537-566.

[LI] T.P. Liu, *Quasilinear hyperbolic systems.* Comm. Math. Phys. **68** (1979), 141-172.

[RFHY] P.N. Rowe - P.U. Foscolo - A.C. Hoffmann - I.C. Yates. Proc. 4th International Conference in Fluidization, Japan 1983.

[S] J. Smoller, *Shock Waves and Reaction-Diffusion Equations.* Grundleheren der Mathematischen Wissenschaften 258. Springer-Verlag New York, Heidelberg, Berlin 1983.

[Y] J.C. Yates, *Fundamentals of fluidized-bed. Chemical Processes.* Butterworths. London 1983.

[YW] L.A. YING - C.H. WANG, *Global solutions to the Cauchy problem for a nonhomogeneous quasilinear hyperbolic system.* Comm. Pure and Appl. Math. **33** (1980), 579-597.

Institute of Mathematics
Academia Sinica
Beijing, People's Republic of China

Dipartimento di Matematica
Pura e Applicata
Università dell'Aquila
67100 L'Aquila