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Time-Delay Operators in Semiclassical Limit, Finite Range Potentials

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1. Introduction

In this paper we consider the semi-classical approximation of time-delay operators in quantum scattering theory. Let us recall briefly some related aspects of scattering theory. Let V be a C^∞ real function on $\mathbb{R}^n$ so that for some $\varepsilon > 0$ and all $\alpha \in \mathbb{N}^n$, one has:

$$|\partial_x^\alpha V(x)| \leq C_\alpha (1+|x|)^{-|\alpha|-1-\varepsilon}. \tag{1.1}$$

For $h > 0$ a small parameter proportional to Planck constant, set: $H_0^h = -\frac{h}{2}\Delta, H^h = H_0^h + V_h$ where $V_h(x) = V(h^{1/2}x)$. Define the unitary groups $U_0^h(t)$ and $U^h(t)$ as:

$$U_0^h(t) = \exp(-ih^{-1}tH_0^h), \ \ U^h(t) = \exp(-ih^{-1}tH^h), \quad t \in \mathbb{R}.$$

Then the wave operators $\Omega_\pm(h)$:

$$\Omega_\pm(h) = s - \lim_{t \to \pm\infty} U^h(t)^* \ U_0^h(t) \qquad \text{in } L^2(\mathbb{R}^n)$$

exist and are complete, that is to say, the ranges of $\Omega_+(h)$ and $\Omega_-(h)$ are both identical with the continuous spectral subspace of H^h. The scattering operator $S(h)$ is defined by: $S(h) = \Omega_+(h)^* \ \Omega_-(h)$. $S(h)$ is a unitary operator in $L^2(\mathbb{R}^n)$. We denote by $F_h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+; P)$, $P = L^2(S^{n-1})$, the spectral representation for $H_0^h : F_h H_0^h F_n^{-1} = \lambda$. For $f \in L^2(\mathbb{R}_+, P)$, one has:

$$F_h S(h) F_h^{-1} f = S(\lambda, h) f$$

where $S(\lambda, h)$ is a multiplication operator in λ with values in $B(P)$. Under suitable assumptions on the potential V, one knows that $S(\lambda, h)$ is continuously

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differentiable in $\lambda \in \mathbb{R}_+$. (See Jensen [14]). Then the Eisenbud-Wigner time-delay operator $T(h)$ is defined by:

$$F_h T(h) F_h^{-1} f = -i\ S(\lambda, h)^* \frac{dS(\lambda, h)}{d\lambda} f \tag{1.2}$$

for $f \in C_0(\mathbb{R}_+; P)$. For fixed $h > 0$, the existence of $T(h)$ as self-adjoint operator in $L^2(\mathbb{R}^n)$ has been studied in [14]. Various expressions for $T(h)$ have been obtained (see [8], [15], [16] and [23]). For a more extensive review of time-delay in scattering theory, we refer to Martin [18]. However, we would like to mention the definition of time-delay put forward by Narnhofer [22]: the operator $\tilde{T}(h)$, which we will call modified time-delay operator, is defined by:

$$\begin{aligned} < f, H^h \tilde{T}(h) g > = & \lim_{t \to -\infty} < f, U^h(t)^* U_0^h(t) A(h) U_0^h(t)^* U^h(t) g > \\ & - \lim_{t \to +\infty} < f, U^h(t)^* U_0^h(t) A(h) U_0^h(t)^* U^h(t) g > \end{aligned} \tag{1.3}$$

if the limit exists. Here $A(h) = h(x \cdot \nabla_x + \nabla_x \cdot x)/2i$ is the generator of analytic dilation group with parameter h and $< \cdot, \cdot >$ is the scalar product in $L^2(\mathbb{R}^n)$.

Recall also the definition of classical time-delay. Consider the trajectory $(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))$ for the Hamiltonian $p(x, \xi) = \frac{|\xi|^2}{2} + V(x)$:

$$\begin{cases} \dfrac{dx}{dt} = \xi & , \quad x(0) = x_0 \\ \dfrac{d\xi}{dt} = -\nabla V(x) & , \quad \xi(0) = \xi_0. \end{cases} \tag{1.4}$$

A scattering trajectory $(x(t; x_0, \xi_0), \xi(t, x_0, \xi_0))$ is the solution of (1.4) satisfying: $\lim_{|t| \to \infty} |x(t; x_0, \xi_0)| = +\infty$. For $R > 0$, put: $B_R = \{x; |x| \le R\}$ and $S_R = \{x; |x| = R\}$. For a scattering trajectory $(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))$, denote by $t_-(R)$ $(t_+(R))$ the time at which the particle enters (leaves resp.) S_R. The sojourn time T_R of the trajectory in the region B_R is given by $T_R = t_+(R) - t_-(R)$. We denote by T_R^0 the sojourn time in B_R for the free particle with the same initial data (x_0, ξ_0). Then we can define the classical time-delay τ by:

$$\tau(x_0, \xi_0) = \lim_{R \to +\infty} (T_R - T_R^0). \tag{1.5}$$

For the potential V satisfying (1.1), we can easily check that:

$$\tau(x_0, \xi_0) = \frac{1}{|\xi_-|^2} \left(\lim_{t \to -\infty} (x(t) - t\xi(t)) \cdot \xi(t) - \lim_{t \to +\infty} (x(t) - t\xi(t)) \cdot \xi(t) \right) \tag{1.6}$$

where $|\xi_-|^2 = \lim_{t \to -\infty} |\xi(t)|^2 = 2p(x_0, \xi_0)$, $p(y, \eta) = \frac{|y|^2}{2} + V(y)$.

The main purpose of this paper is to study the asymptotic behaviour of $T(h)$ and $\tilde{T}(h)$ when h tends to 0. The study of time delay operator is closely related to that of the scattering matrix. We notice that there is a large literature on the semi-classical approximation of scattering quantities, such as scattering amplitude ([3], [4], [41]), scattering phase ([30], [31]) and scattering matrix ([40]). For classical wave scattering by an obstacle, there are also papers concerning the relations between time-delay, sojourn time and scattering matrix (see [7], [17] and [24], for instance). Seeing the definition (1.2) for $T(h)$, one might think it easy to get the semiclassical result for $T(h)$ using the known results for scattering matrix $S(\lambda, h)$. But it is not the case, for the known results for $S(\lambda, h)$ are of complicated forms (see Yajima [40]), it is difficult to establish the connection between quantum and classical time-delay. The method we will use is based on the results of [38], roughly speaking, the results on the correspondence between quantum and classical dynamics (see also [2], [11], [33], [34]). To show that a direct study of time-delay operator is preferable, let us indicate the following fact: if V satisfies the assumption (1.1), $U^h(t)$ can be approximated only by Fourier integral operators. But if B is a pseudo-differential operator, we can approximate $U^h(t)^* B U^h(t)$ by pseudo-differential operators when t is fixed. This makes our work easier.

In this paper, our general assumptions on V are:

(A) V is a C^∞ real function with compact support;

(B) (non-trapping condition) there exists an open set $J \subset \mathbb{R}_+$ such that for any interval $I \subset\subset J$ and for any $R > 0$, there exist positive constants C and t_0:

$$|x(t; y, \eta)| \geq C|t| \qquad \text{for } |t| \geq t_0$$

$p(y, \eta) \in I$ and $|y| \leq R$, where $(x(t, y, \eta), \xi(t, y, \eta))$ is the solution of (1.4) with initial conditions $x(0) = y$, $\xi(0) = \eta$.

The main results of this paper may be summed up as follows: let $W_h(x_0, \xi_0)$ be the Weyl operator $\exp(ih^{-1/2}(\xi_0 \cdot x - x_0 \cdot D_x))$. For $(x_0, \xi_0) \in P^{-1}(J)$, take a function $\varphi \in C_0^\infty(J)$ and $\varphi = 1$ in a neighbourhood of $p(x_0, \xi_0)$. For $f \in L^2(\mathbb{R}^n)$ define f_h by

$$f_h = \varphi(H^h) W_h(x_0, \xi_0) f.$$

Then for modified time-delay operator $\tilde{T}(h)$, one has:

$$\lim_{h \to 0_+} < f_h, \tilde{T}(h) f_h >= 2\tau(x_0, \xi_0) \|f\|^2. \tag{1.7}$$

For Eisenbud-Wigner time-delay operator $T(h)$, we set $f_h = \psi(H_0^h) W_h(x_0, \xi_0) f$ where ψ is C_0^∞ function and equal to 1 in a neighbourhood of $|\xi_0|^2/2$. Then one has, for $(x_0, \xi_0) \in \mathbb{R}^{2n}$, $|\xi_0|^2/2 \in J$,

$$\lim_{h \to 0_+} < f_h, T(h) f_h >= \tau \circ \Omega_-^{cl}(x_0, \xi_0) \|f\|^2 \tag{1.8}$$

where Ω_-^{cl} is the classical wave operator (see Reed-Simon [26]). We also obtain a semi-classical expansion for modified time-delay operator in terms of pseudo-differential operators in class T_1^1 (for the definition, see section 2). Notice that $\tau(x_0, \xi_0)$ is the classical time-delay for scattering trajectory $(x(t; x_0, \xi_0), \xi(t, x_0, \xi_0))$ whereas $\tau \circ \Omega_-^{cl}(x_0, \xi_0)$ is that for a scattering trajectory $(\tilde{x}(t, x_0, \xi_0), \tilde{\xi}(t, x_0, \xi_0))$ which behaves like $(x_0 + t\xi_0, \xi_0)$ as $t \to -\infty$. Since $p \circ \Omega_-^{cl}(x_0, \xi_0) = p_0(x_0, \xi_0)$, (1.7) and (1.8) show the difference between $\tilde{T}(h)$ and $T(h)$: the modified time-delay operator is the quantization of two times the classical time-delay function τ while the Eisenbud-Wigner time-delay operator is the quantization of $\tau \circ \Omega_-^{cl}$, or in other word, the quantization of time-delay function applied to incoming data.

The organisation of this paper is as follows. In section 2, we establish some results which are important to this work.

We introduce also a class of pseudo-differential operators with symbols in $T_{\rho.\delta}^{s.r}$. This class of pseudo-differential operators appears naturally in the study of time-delay operators. It seems interesting to study the composition and continuity properties of these operators. But we have not had time to do this. In section 3, some properties of classical time-delay are studied. Since the function τ is only constructed for each fixed (x, ξ) in phase space, global control over the increase of τ seems difficult. We only arrive at showing that if V is of compact support and if we localise τ in some nontrapping energy interval, the truncated classical time-delay is in class T_1^0. In section 4, we prove the semi-classical expansion for modified time-delay operator. We use a semi-classical version of Egorov's theorem (see [38]) and the results on the local energy decay for wave functions in weighted L^2-spaces ([42]). The difficulty we encounter here is that usually we have to work out estimates uniform with respect to the large parameter t and the small parameter h. We believe that the remainder estimates obtained here are not the best possible and can be improved by studying the continuity of pseudo-differential operators with symbol in the class T_1^1. In section 5 we prove (1.7). As a matter of fact, the uniform continuity of time-delay operators localized in non-trapping energy interval enables us to only prove (1.7) for a dense subset of functions in L^2. In section 6, we prove (1.8).

The proof is similar to that for (1.7). But since we are only concerned with the first term in the semi-classical limit, it suffices to give approximations uniform in $h \in]0, 1]$ and $h^{-N} \leq |t| \leq h^{-N-1}$ for some N large enough. Once these approximations are obtained, we use the Weyl operator $W_h(x_0, p_0)$ to get classical limit as in section 5 (see also [11], [33] and [38]).

Finally we should remark that the proof of our results is rather technical and could be considerably simplified if one had a nice result on the continuity of pseudo-differential operators with symbol in the exotic class mentioned above. In this connection, a recent work of Bony-Lerner [44] might be useful.

An abstract of this work has been presented in [39] and [43].

2. Some preliminaires

Assume V to be a C^∞ real valued function on $\mathbb{R}^n$ satifying (1.1) for some $\varepsilon > 0$. Let J be a non-trapping interval defined by (B). Let $U^h(t)$ and $U_0^h(t)$ be the unitary groups associated with H^h and H_0^h respectively (see §-1).

Let $A(h)$ be the generator of analytic dilation group with parameter $h : A(h) = h(x \cdot \nabla_x + \nabla_x \cdot x)/2i$. Then we have the following estimates which are useful in the study of time-delay operators by time-dependent method.

THEOREM 2.1. *Under the above assumptions, let φ_1 and φ_2 be $C_0^\infty(\mathbb{R}_+)$ functions and* supp $\varphi_2 \subset J$. *Then there is $C > 0$ so that for every μ, $0 \le \mu \le 1$, one has*

$$\|(A(h)^2+1)^{-\mu/2}\varphi_1(H_0^h)U_0^h(r)^*U^h(s)\varphi_2(H^h)(A(h)^2+1)^{-\mu/2}\| \le C(1+|r-s|)^{-\mu} \tag{2.1}$$

uniformly in $r, s \in \mathbb{R}$ and $h \in]0,1]$. For every μ, $1 < \mu \le 2$, there is $\rho > 1$ so that

$$\|(A(h)^2+1)^{-\mu/2}\varphi_1(H_0^h)U_0^h(r)^*U^h(s)\varphi_2(H^h)(A(h)^2+1)^{-\mu/2}\| \le C'(1+|r-s|)^{-\rho} \tag{2.2}$$

uniformly in $r, s \in \mathbb{R}$ and $h \in]0,1]$.

The proof of Theorem 2.1 is rather technical and will be given in Appendix.

Introduce the space $B^m(h)$ by: $B^m(h) = \{f \in L^2(\mathbb{R}^n); x^\alpha D_x^\beta f \in L^2(\mathbb{R}^n)$ for all $|\alpha|+|\beta| \le m\}$, equipped with the norm $\|\cdot\|_{m,h}$

$$\|f\|_{m,h} = \{\sum_{|\alpha|+|\beta|\le m} \|(h^{1/2}x)^\alpha(h^{1/2}D_x)^\beta f\|^2)\}^{1/2}, \qquad f \in B^m(h).$$

It is known that $U^h(t)$ is a continuous mapping of $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$ and may be extended to a bounded operator on $B^m(h)$ (see [6] and [38]):

$$\|U^h(t)f\|_{m.h} \le C_m\|f\|_{m.h}\ , \qquad \text{for } f \in B^m(h) \tag{2.3}$$

uniformly in $h \in]0,1]$. Thus as operators from S to S, one has:

$$A(h)U_0^h(t) = U_0^h(t)A(h) + 2tH_0^hU_0^h(t) \tag{2.4}$$

$$A(h)U^h(t) = U^h(t)A(h) + 2tH^hU^h(t) + \int_0^t U^h(t-s)\tilde{V}_hU^h(s)ds \tag{2.5}$$

where $\tilde{V}_h = \tilde{V}(h^{1/2}x)$, $\tilde{V} = x \cdot \nabla V - 2V$.

LEMMA 2.2. *For any $g \in C_0^\infty(\mathbb{R})$, $[g(H^h), A(h)]$ is a bounded operator in $L^2(\mathbb{R}^n)$ $\|[g(H^h), A(h)]\| \le Ch$.*

More precisely, one has:

$$[g(H^h), A(h)] = h\ g_1(H^h) + h\ R_1(h) \tag{2.6}$$

where $g_1(s) = 2sg'(s)$ *and* $R_1(h)$, $0 < h \le 1$, *is uniformly bounded as operators of* $L^{2,s}(h)$ *to* $L^{2,s+\rho}(h)$, *for some* $\rho > 1$. *Here* $L^{2,s}(h)$ *is the completion of* $\mathcal{S}(\mathbb{R}^n)$ *in the norm* $\|\cdot\|_{(s).h}$:

$$\|f\|_{(s).h} = \{\int_{\mathbb{R}^n} (1 + hx^2)^s |f|^2 dx\}^{1/2}.$$

PROOF. Notice that since $g \in C_0^\infty(\mathbb{R})$, we can define $g(H^h) = (2\Pi)^{-1} \int \exp(isH^h)\hat{g}(s)ds$ where $\hat{g}$ is the Fourier transform of g. Using a commutator relation similar to (2.5) and the results on functional calculus (see [10] and [28]), one gets (2.6).

Under the assumptions of theorem 2.1, one knows that for every $\sigma > 1/2$,

$$\|(1 + h|x|^2)^{-\sigma/2} R(\lambda \pm io, h)(1 + h|x|^2)^{-\sigma/2}\| \le Ch^{-1}, \quad h \in]0, 1] \tag{2.7}$$

uniformly in λ in any compact of J (see Robert-Tamura [31]). By (2.7) one can easily prove the following result.

PROPOSITION 2.3. *Let* φ *be a* C^∞ *function with compact support in* J. *Then there is* $C > 0$ *so that:*

$$\int_{-\infty}^{+\infty} | < \varphi(H^h)U^h(t)f, V_h\varphi(H^h)U^h(t)f > |dt \le C\|f\|^2, \qquad f \in L^2(\mathbb{R}^n)$$

uniformly in $h \in]0, 1]$, *here* $< \cdot, \cdot >$ *is the scalar product in* $L^2(\mathbb{R}^n)$.

PROOF. By (2.7) and the results on smooth perturbations (Theorem XIII.25 and Theorem XIII.30 [27]), one gets:

$$\int_{-\infty}^{+\infty} \|(1 + hx^2)^{-\sigma/2} e^{-itH^h} \phi(H^h)f\|^2 dt \le Ch^{-1}\|f\|$$

for $f \in L^2(\mathbb{R}^n)$, uniformly in $h \in]0, 1]$. Since $U^h(t) = e^{-ih^{-1}tH^h}$, the proposition follows from the Cauchy-Schwarz inequality.

The following result is important in this work.

THEOREM 2.4. *Let* V *be a short-range potential satisfying* (1.1) *and* (B). *Let* $\psi \in C_0^\infty(J)$. *Then for every* $s > 0$, *one has:*

$$\|(1 + h|x|^2)^{-s/2} \psi(H^h)U^h(t)(1 + h|x|^2)^{-s/2})\| \le C(1 + |t|)^{-s}$$

uniformly in $h \in]0, 1]$ *and* $t \in \mathbb{R}$.

Theorem 2.4 is proved in [42]. Comparing with the free evolution, we see that the decay rate obtained in Theorem 2.4 is the best possible. By commutator techniques we are able to show that for every μ, $0 < \mu \leq 1$,

$$\|(A(h)^2+1)^{-\mu/2}\varphi(H^h)U^h(t)(A(h)^2+1)^{-\mu/2}\| \leq C(1+|t|)^{-\mu}, \quad t \in \mathbb{R}. \tag{2.8}$$

For every μ, $1 < \mu \leq 2$, there is $\rho > 1$ so that

$$\|(A(h)^2+1)^{-\mu/2}\varphi(H^h)U^h(t)(A(h)^2+1)^{-\mu/2}\| \leq C(1+|t|)^{-\rho}, \quad t \in \mathbb{R}. \tag{2.9}$$

The estimates (2.8) and (2.9) are uniform in $h \in]0,1]$.

From Theorem 2.1, we get the following.

COROLLARY 2.5. *Let $\Omega_\pm(h)$ denote the wave operators. Then for $\varphi \in C_0^\infty(J)$, one has:*

$$\begin{aligned}\|(A(h)^2+1)^{-\mu/2}U^h(t)\Omega_\pm(h)\varphi(H_0^h)(A(h)^2+1)^{-\mu/2}\|\\ \leq C(1+|t|)^{-\mu},\ 0 \leq \mu \leq 1\end{aligned} \tag{2.10}$$

uniformly in $h \in]0,1]$. For every $\mu > 1$, there is $\rho > 1$ so that:

$$\|(A(h)^2+1)^{-\mu/2}U^h(t)\Omega_\pm(h)\varphi(H_0^h)(A(h)^2+1)^{-\mu/2}\| \leq C(1+|t|)^{-\rho}. \tag{2.11}$$

Using (2.9), one gets also:

COROLLARY 2.6. *For every $\mu > 1/2$, one has:*

$$\|(A(h)^2+1)^{-\mu}R(\lambda \pm io, h)(A(h)^2+1)^{-\mu}\| \leq Ch^{-1} \tag{2.12}$$

uniformly in $h \in]0,1]$ and λ in compact subset of J. Here $R(z,h) = (H^h - z)^{-1}$ for $z \notin \sigma(H^h)$.

Notice that (2.12) may be compared with the known results of uniform resolvent bounds (see [30] and [31]). For fixed $h > 0$, (2.12) is proved in [21].

Recall also the following expression for Eisenbud-Wigner time-delay operator $T(h)$. Let F_h be a spectral representation of H_0^h, that is to say, F_h is a unitary mapping of $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}_+, L^2(S^{n-1}))$ so that $F_h H_0^h F_h^{-1} = \lambda$. Then for all $f_j \in L^2(\mathbb{R}^n)$ so that $F_h f_j \in C_0^1(\mathbb{R}_+; L^2(S^{n-1}))$, $j = 1, 2$, one has:

$$< f_1, H_0^h T(h) f_2 >= -1/2 < f_1, S^*(h)[A(h), S(h)] f_2 > \qquad \text{(see [14])}. \tag{2.14}$$

PROPOSITION 2.7. *Let $h > 0$ be fixed. For $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}_+)$, $\varphi_2 = 1$ on* supp φ_1. *Set:*

$$\Omega(t,h) = U^h(t)^* U_0^h(t) A(h) U_0^h(t)^* U^h(t)$$

$$S(t,h) = U_0^h(t)\varphi_2(H^h)U^h(2t)^* U_0^h(t) A(h) U_0^h(t)^* U^h(2t)\varphi_2(H^h)U_0^h(t)^*.$$

Then for $f \in L^2(\mathbb{R}^n)$ *so that* $A(h)f \in L^2(\mathbb{R}^n)$, $\Omega(t,h)\varphi_1(H^h)f$ *and* $S(t,h)\varphi_1(H_0^h)f$ *are in* $L^2(\mathbb{R}^n)$ *and one has*:

$$\lim_{t\to\pm\infty} \varphi_1(H^h)\Omega(t,h)\varphi_1(H^h)f = \varphi_1(H^h)\Omega_\pm(h)A(h)\Omega_\pm(h)^*\varphi_1(H^h)f \tag{2.15}$$

$$\lim_{t\to+\infty} \varphi_1(H_0^h)S(t,h)\varphi_1(H_0^h)f = \varphi_1(H_0^h)S(h)^*A(h)S(h)\varphi_1(H_0^h)f \quad \text{in } L^2(\mathbb{R}^n). \tag{2.16}$$

PROOF. By the definition of wave operators and scattering operator, it is easy to check that (2.15) and (2.16) are true in weak topology of $L^2(\mathbb{R}^n)$.

In order to show (2.15) and (2.16) are true in L^2-norm we use the following estimates:

$$\sup_t \|A(h)\varphi_1(H^h)\Omega(t,h)\varphi_1(H^h)(A(h)^2+1)^{-1}\| < +\infty$$
$$\sup_t \|A(h)\varphi_1(H_0^h)S(t,h)\varphi_1(H_0^h)(A(h)^2+1)^{-1}\| < +\infty$$

These estimates may be proved by the same method as that used for proving theorem 2.1. (cf. Appendix).

For a symbol $a(\cdot,\cdot)$ on $\mathbb{R}^{2n}$, we denote by $a^w(h^{1/2}x, h^{1/2}D_x)$ the associated pseudo-differential operators defined by:

$$(a^w(h^{1/2}x,h^{1/2}D_x)f)(x) = \iint_{\mathbb{R}^{2n}} \exp(i(x-y)\cdot\xi)a(h^{1/2}(x+y)/2,\xi)f(y)\,dy\,d\xi, \quad f\in S(\mathbb{R}^n) \tag{2.17}$$

Sometimes we note also $\mathrm{op}_h^w a$ instead of $a^w(h^{1/2}x, h^{1/2}D)$.

DEFINITION 2.8. *Let* $\rho,\delta\in[0,1]$ *and* $s,r\in\mathbb{R}$. *We denote by* $T^{s,r}_{\rho,\delta}$ *class of* C^∞ *functions on* $\mathbb{R}^{2n}$ *which satisfy*:

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|x|)^{s+\rho|\beta|}(1+|\xi|)^{r+\delta|\alpha|}, \quad a\in T^{s,r}_{\rho,\delta}$$

for all multi-indices $\alpha,\beta\in\mathbb{N}^n$. *We note also*: $T^s_\rho = \bigcap_{r\in\mathbb{R}} T^{s,r}_{\rho,\delta}$.

In comparing with the usual classes of symbols, we see that the class $T^{s,r}_{\rho,\delta}$ is worse in that we lose the control over the increase of symbol each time we derivate it. But the study of such symbols is of interest in itself. For example if $f\in C^\infty(\mathbb{R})$ with bounded derivatives, the principal symbol of $f(A(h))$ may be regarded as in $T^{0,0}_{1,1}$.

For $a\in T^{s,r}_{\rho,\delta}$, we can define an operator $a^w(h^{1/2}x,h^{1/2}D_x) : S(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ by formula (2.17). If $\rho,\delta<1$, we can easily check that $a^w(h^{1/2}x,h^{1/2}D)$ is continuous from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. In general case, we have the following result.

LEMMA 2.9. *There exists a constant k depending only on n such that if $a \in T^s_\rho, 0 \leq \rho \leq 1$, $a(h^{1/2}x, h^{1/2}D)$ is a continuous operator from $L^{2,k+s}(h)$ into $L^{2,-k-s}(h)$ and one has:*

$$\|a^w(h^{1/2}x, h^{1/2}D_x)f\|_{(-k-s),h} \leq C\|f\|_{(k+s),h} \text{ for all } h \in \left]0,1\right]. \tag{2.18}$$

Lemma 2.9 is an easy consequence of L^2-continuity result of h-pseudo-differential operators ([28]). We omit the proof.

3. Properties of classical time-delay

The classical time-delay is closely related to the classical particle scattering. Suppose condition (1.1) to be satisfied. Let ϕ^t and ϕ_0^t denote the interacting and free dynamics respectively. Then we can define the wave operators $\Omega^{cl}_\pm$ by:

$$\Omega^{cl}_\pm(y,\eta) = \lim_{t\to\pm\infty} \phi^{-t} \circ \phi_0^t(y,\eta) \qquad \text{for } (y,\eta) \in \mathbb{R}^n \ x\ \mathbb{R}^n\backslash\{0\};$$

$\Omega^{cl}_\pm$ are C^∞ canonical maps. (See [26], [32]). Let Σ_b be the set of all bounded trajectories in $\mathbb{R}^{2n}$.

We put $G = \text{Ran } \Omega^{cl}_+ \cap \text{ Ran } \Omega^{cl}_- \cap \mathbb{R}^{2n}\backslash\Sigma_b$. Then G is the set of all points $(y,\eta) \in \mathbb{R}^{2n}$ for which there are (y_-, η_-) and $(y_+, \eta_+) \in \mathbb{R}^{2n}$ so that $\phi^t(y,\eta)$ behaves like $(y_- + t\eta_-, \eta_-)$ at $-\infty$ and like $(y_+ + t\eta_+, \eta_+)$ at $+\infty$. If we write $\phi^t(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$, $(y,\eta) \in G$, there exists a constant $C(y,\eta) > 0$ such that:

$$|x(t,y,\eta)| \geq C|t|$$

for $|t|$ large enough. We note: $d(x,\xi) = x \cdot \xi$. The classical time-delay τ defined in (1.6) may be written as:

$$\tau(y,\eta) = \frac{1}{|\xi_-|^2}\{\lim_{t\to-\infty} d \circ \phi_0^{-t} \circ \phi^t(y,\eta) - \lim_{t\to+\infty} d \circ \phi_0^{-t} \circ \phi^t(y,\eta)\}, \qquad (y,\eta) \in G \tag{3.1}$$

where $|\xi_-|^2 = \lim_{t\to-\infty} |\xi(t,y,\eta)|^2 = 2p(y,\eta)$. Notice that τ is invariant by $\phi^t : \tau(y,\eta) = \tau \circ \phi^t(y,\eta)$ for $t \in \mathbb{R}$.

LEMMA 3.1. *Let $G_\pm = (\Omega^{cl}_\pm)^{-1}G$. Let $S^{cl} : G_- \to G_+$ be the S-transformation defined by:*

$$S^{cl}(y,\eta) = (\Omega^{cl}_+)^{-1}\Omega^{cl}_-(y,\eta).$$

Then for $(y,\eta) \in G_-$, one has: $\tau \circ \Omega^{cl}_-(y,\eta) = \frac{1}{|\xi_-|^2}(d(y,\eta) - d \circ S^{cl}(y,\eta))$.

The proof of the lemma is quite easy and we omit it.

LEMMA 3.2. *Let $E \subset \mathbb{R}^{2n}$ be an open set so that* $\inf_{(y,\eta)\in E} p(y,\eta) > 0$ *and for any $R > 0$, there are C and $t_0 > 0$ so that*

$$|x(t, y, \eta)| \geq C|t|, \qquad |t| \geq t_0 \tag{3.2}$$

for $(y, \eta) \in E$ with $|y| \leq R$. Then one has:

$$|\partial_y^\alpha \partial_\eta^\beta x(t, y, \eta)| \leq C_{\alpha\beta,R}(1 + |t|), \quad |\partial_y^\alpha \partial_\eta^\beta \xi(t, y, \eta)| \leq C_{\alpha\beta,R}, \quad t \in \mathbb{R}, \tag{3.3}$$

for $(y, \eta) \in E$ with $|y| \leq R$.

PROOF. For $(y, \eta) \in E$, we can write $(x(t, y, \eta), \xi(t, y, \eta))$ as:

$$\begin{cases} x(t,y,\eta) = y + \int_0^{+\infty} s\nabla V(x(s))ds \\ \qquad + (\eta - \int_0^{+\infty} \nabla V(x(s))ds)t - \int_t^{+\infty} (s-t)\nabla V(x(s))ds \\ \xi(t,y,\eta) = \eta - \int_0^{+\infty} \nabla V(x(s))ds \\ \qquad + \int_t^{+\infty} \nabla V(x(s))ds, \qquad t > 0; \end{cases} \tag{3.4}$$

(3.3) follows from (3.4) by an elementary argument.

Now introduce the functions $\tau_\pm$ by:

$$\tau_\pm(y, \eta) = \lim_{t\to\pm\infty} d \circ \phi_0^{-t} \circ \phi^t(y, \eta), \qquad (y, \eta) \in E.$$

By definition, $\tau(y, \eta) = (\tau_-(y, \eta) - \tau_+(y, \eta))/2p(y, \eta)$. Using lemma 3.2, one gets easily:

LEMMA 3.3. *Under the assumptions of lemma 3.2, $\tau_\pm$ are C^∞ functions of $(y, \eta) \in E$. For $R > 0$, one has:*

$$|\partial_y^\alpha \partial_\eta^\beta \tau_\pm(y, \eta)| \leq C_{\alpha\beta,R}(1 + |\eta|)$$

for $(y, \eta) \in E$ with $|y| \leq R$.

It is not clear how to control the increase of $\tau_\pm(y, \eta)$ for large y, since one has no knowledge on the behaviour of $\phi^t(y, \eta)$ in large t and y. However if the potential V is of compact support, we do have some control over the increase of $\tau_\pm(y, \eta)$ in (y, η). This is sufficient for the present paper, for in the following we will always assume V of compact support.

More precisely, one has:

LEMMA 3.4. *In addition to the assumptions of lemma 3.2, suppose V to be of compact support. Then one has:*

$$|\partial_y^\alpha \partial_\eta^\beta \tau_\pm(y,\eta)| \le C_{\alpha,\beta}(1+|y|)^{1+|\beta|}(1+|\eta|) \tag{3.5}$$

and

$$|\partial_y^\alpha \partial_\eta^\beta \tau(y,\eta)| \le C_{\alpha\beta}(1+|y|)^{|\beta|}(1+|\eta|), \qquad \text{for } (y,\eta)\in E. \tag{3.6}$$

PROOF. Take $R_0 > 0$ so that supp $V \subset \{|x| < R_0\}$. Put $R_1 = R_0 + 1$. By lemma 3.3, (3.5) is true for $(y,\eta) \in E$ with $|y| \le R_1$. Now define $W \subset \mathbb{R}^{2n}$ by $W = \{(y,\eta) \in E;\ |y| > 2\ R_1\}$. Let W_1 and W_2 be two subsets of W defined by:

$$W_1 = \{(y,\eta) \in W;\ |x(t;y,\eta)| > R_0,\ \text{for all } t \in \mathbb{R}\},$$
$$W_2 = \{(y,\eta) \in W;\ \exists\ T \in \mathbb{R} \text{ so that } |x(T;y,\eta)| < R_1\}.$$

For $(y,\eta) \in W_1$, $\phi^t(y,\eta) = (x(t;y,\eta),\xi(t;y,\eta))$ will behave just as the free dynamics $\phi_0^t(y,\eta)$. Thus for $(y,\eta) \in W_1$, $\tau_+(y,\eta) = \tau_-(y,\eta) = y\cdot\eta$. For $(y,\eta) \in W_2$, one sees easily that $y\cdot\eta \ne 0$ and there exists a unique T_0 so that $|x(t;y,\eta)| > R_1$ for $|t| < |T_0|$ and $|x(T_0;y,\eta)| = R_1 \cdot T_0$ is determined by the equation:

$$|\eta|^2T^2 + 2\ y\cdot\eta T - R_1^2 + |y|^2 = 0.$$

Since we must have $T_0 > 0$ if $y\cdot\eta < 0$ and $T_0 < 0$ if $y\cdot\eta > 0$, we get:

$$T_0 = (-y\cdot\eta + \text{sgn}\ y\cdot\eta(|y\cdot\eta|^2 - |\eta|^2(|y|^2 - R_1^2))^{1/2})/|\eta|^2$$

T_0 is a C^∞ functions in W_2 and we have:

$$|\partial_y^\alpha \partial_\eta^\beta T_0(y,\eta)| \le C_{\alpha\beta}(1+|y|)^{(1-|\alpha|)_+} \tag{3.8}$$

$(y,\eta) \in W_2$ and $\alpha,\beta \in \mathbb{N}^n$. Now, one has:

$$\tau_\pm(y,\eta) = \lim_{t\to\pm\infty}(x(t,y,\eta) - t\xi(t,y,\eta))\cdot\xi(t,y,\eta)$$
$$= \tau_\pm(y_0,\eta) + T_0(y,\eta)|\xi_\pm(y_0,\eta)|^2 \qquad \text{for } (y,\eta)\in W_2$$

where $y_0 = y + T_0\eta$ and $\xi_\pm(y_0,\eta) = \lim_{t\to\pm\infty}\xi(t;y_0,\eta)$. Thus $\tau(y,\eta) = \tau(y_0,\eta)$. Since $|y_0|^2 = R_1^2$, from lemmas 3.2, 3.3 and the estimate (3.8), we conclude that:

$$|\partial_y^\alpha \partial_\eta^\beta \tau_\pm(y,\eta)| \le C_{\alpha\beta}(1+|y|)^{1+|\beta|}(1+|\eta|)$$
$$|\partial_y^\alpha \partial_\eta^\beta \tau(y,\eta)| \le C_{\alpha\beta}(1+|y|)^{|\beta|}(1+|\eta|) \text{ for } (y,\eta)\in W_2.$$

The lemma is proved.

From now on, we will always assume that conditions (A) and (B) be satisfied. We remark that condition (B) is equivalent to the usual non-trapping condition used in the literature. (See [30], [35]).

PROPOSITION 3.5. *Suppose conditions* (A) *and* (B) *to be satisfied. Let* φ *be a* C^∞ *function on* $\mathbb{R}$ *with compact support in* J. *Put*:

$$\tau_\varphi(y,\eta) = \varphi \circ p(y,\eta)\ \tau(y,\eta). \tag{3.9}$$

Then τ_φ *is in* T_1^0, *that is to say, we have the estimates on* τ_φ:

$$|\partial_y^\alpha \partial_\eta^\beta \tau_\varphi(y,\eta)| \le C_{\alpha\beta.N}(1+|y|)^{|\beta|}(1+|\eta|)^{-N}, \quad (y,\eta) \in \mathbb{R}^{2n} \tag{3.10}$$

where $N > 0$ *is arbitrary.*

PROOF. By lemma 3.3, τ_φ is a C^∞ function on $\mathbb{R}^{2n}$ and the estimates (3.10) follow from lemma 3.4.

Using the non-trapping condition (B), one sees easily that $p^{-1}(J) \subset$ Ran $\Omega_-^{cl} \cap$ Ran Ω_+^{cl}. Define $G_- \subset \mathbb{R}^{2n}$ by: $G_- = (\Omega_-^{cl})^{-1}(p^{-1}(J))$. Since the support of τ_φ is contained in $p^{-1}(J)$, we can define the function $\tau_\varphi \circ \Omega_-^{cl}$, which is C^∞ and supported in G_-:

$$\tau_\varphi \circ \Omega_-^{cl}(x,\xi) = \varphi_1 \circ p_0(\xi)(x\cdot\xi - \tau_+ \circ S^{cl}(x,\xi)), \text{ for } (x,\xi) \in G_-,$$

where $\varphi_1(t) = t^{-1}\varphi(t)$. By the method used in the proof of Lemma 3.4, one can show that (3.10) is also true for $\tau_\varphi \circ \Omega_-^{cl}$.

4. Semi-classical expansion of modified time-delay operator

Suppose conditions (A) and (B) to be satisfied. We will consider the asymptotic behaviour of modified time-delay operator $\tilde{T}(h)$ (see Narnhofer [22], [23]) defined by:

$$\begin{aligned} < f, H^h\tilde{T}(h)g > = &\lim_{t\to-\infty} < f, U^h(t)^*U_0^h(t)A(h)U_0^h(t)^*U^h(t)g > \\ &- \lim_{t\to+\infty} < f, U^h(t)^*U_0^h(t)A(h)U_0^h(t)^*U^h(t)g > . \end{aligned} \tag{4.1}$$

By proposition 2.7, (4.1) is well defined for $f, g \in \mathcal{D}(A(h))$ so that there is $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\varphi(H^h)f = f$ and $\varphi(H^h)g = g$. One has:

$$< f, H^h\tilde{T}(h)g >=< f, (\Omega_-(h)A(h)\Omega_-(h)^* - \Omega_+(h)A(h)\Omega_+(h)^*)g > .$$

Put as before $\Omega(t,h) = U^h(t)^*U_0^h(t)A(h)U_0^h(t)^*U^h(t)$. In order to obtain a semi-classical expansion for $\tilde{T}(h)$, we will study the asymptotic behavior of $\Omega(t,h)$

with respect to t and h. For this purpose, we have to localise $\Omega(t,h)$ in the non-trapping interval. Let χ be a C^∞ function on $\mathbb{R}$ with compact support in J. Define $\Omega(t,h;\chi)$ as:

$$\Omega(t,h;\chi) = \chi(H^h)\Omega(t,h)\chi(H^h). \tag{4.2}$$

Let $\chi_1 \in C_0^\infty(J)$ and $\chi_1 = 1$ on the support of χ. Put: $\chi_0(x,\xi) = \chi_1 \circ p(x,\xi)$. Since $U_0^h(t)A(h)U_0^h(t)^*$ is an operator of symbol $x\cdot\xi - t\xi^2$,

$$G(t,h) \leq \chi_0^w(h^{1/2}x, h^{1/2}D_x)U_0^h(t)A(h)U_0^h(t)^*$$

is a pseudo-differential operator with Weyl symbol $g(x,\xi;t,h)$:

$$g(x,\xi;t,h) = g_0(x,\xi;t) + hg_1(x,\xi;t) + h^2 g_2(x,\xi;t) \tag{4.3}$$

where

$$\begin{aligned} g_0(x,\xi;t) &= \chi_0(x,\xi)(x\cdot\xi - t\xi^2) \\ g_1(x,\xi;t) &= \frac{1}{2i}\chi_1'(p(x,\xi))(|\xi|^2 - (x-2t\xi)\cdot\nabla V(x)) \\ g_2(x,\xi;t) &= \frac{1}{4}(2t\Delta_x\chi_0(x,\xi) + \nabla_x\cdot\nabla_\xi\chi_0(x,\xi)). \end{aligned}$$

It is clear that the symbol $g(x,\xi;t,h)$ satisfies:

$$|\partial_x^\alpha\partial_\xi^\beta g(x,\xi;t,h)| \leq C_{\alpha\beta.N}(1+|t|)(1+|\xi|)^{-N}(1+|x|)^{1-|\alpha|} \tag{4.4}$$

and the support of $g(\cdot,\cdot,t,h)$ is contained in supp χ_0 for all t and h. Now let $R(t,h)$ be defined by:

$$R(t,h) = \Omega(t,h;\chi) - \chi(H^h)U^h(t)^*G(t,h)U^h(t)\chi(H^h).$$

LEMMA 4.1. *With the above notations, $R(t,h)(A(h)+i)^{-1}$ is a bounded operator in $L^2(\mathbb{R}^n)$ and one has:*

$$\|R(t,h)(A(h)+i)^{-1}\| = O(h^\infty) \tag{4.5}$$

uniformly with respect to $t\in\mathbb{R}$, that is to say, for every $N>0$, one has:

$$\|R(t,h)(A(h)+i)^{-1}\| \leq C_N h^N, \qquad t\in\mathbb{R}.$$

PROOF. Using the commutator relation (2.5), one sees that

$$\|\chi_1(H^h)(A(h) - 2tH_0^h)U^h(t)\chi(H^h)(A(h)+i)^{-1}\| \leq C < +\infty \tag{4.6}$$

uniformly in $t \in \mathbb{R}$ and $h \in \,]0, 1]$. We write also:

$$\begin{aligned}&(1-\chi_1(H^h))(A(h)-2tH_0^h)U^h(t)\\&=(A(h)-2tH_0^h)U^h(t)(1-\chi_1(H^h))\\&+[A(h),\chi_1(H^h)]U^h(t)-2t\chi_1(H^h)V_hU^h(t).\end{aligned}$$

Noticing that χ and $1-\chi_1$ are of disjoint support, we obtain:

$$\begin{aligned}&\|(1-\chi_1(H^h))(A(h)-2tH_0^h)U^h(t)\chi(H^h)(A(h)+i)^{-1}\|\\&\leq\|[A(h),\chi_1(H^h)]U^h(t)\chi(H^h)(A(h)+i)^{-1}\|\\&+2|t|\|\chi_1(H^h)V_hU^h(t)\chi(H^h)(A(h)+i)^{-1}\|\leq C,\\&\qquad\text{for all } t\in\mathbb{R} \text{ and } h\in\,]0,1];\end{aligned}\tag{4.7}$$

here we have used lemmas 2.2 and 2.4. Notice that by results on functional calculus (see [10], [28] and [29]), one has:

$$\|\chi(H^h)(1-\chi_0^w(h^{1/2}x,h^{1/2}D))\|=O(h^\infty);$$

(4.5) follows from (4.6) and (4.7).

By lemma 4.1, we are reduced to study the asymptotic behavior of operator $U^h(t)^*G(t,h)U^h(t)$ with respect to t and h. Applying a semi-classical version of Egorov theorem (see [38]), we get:

PROPOSITION 4.2. *As continuous operators of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, one has:*

$$\begin{aligned}&U^h(t)^*g_j^w(h^{1/2}x,h^{1/2}D;t)U^h(t)\\&=\sum_{k=0}^{N}h^kg_{jk}^w(h^{1/2}x;h^{1/2}D;t)+h^{N+1}R_{jN}(t,h)\end{aligned}\tag{4.8}$$

for $j=0,1,2$ and $N\in\mathbb{N}$, where $g_{jk}(t)$ may be calculated and $R_{jN}(t,h)$ may be extended to a bounded operator on $L^2(R^n)$: $\sup_h\|R_{jN}(t,h)\|<+\infty$, $j=0,1,2$, for every $t\in\mathbb{R}$.

Our main technical result in this section is to get some uniform estimates on the remainders with respect to $t\in\mathbb{R}$.

THEOREM 4.3. *Under conditions* (A) *and* (B) *there is $k\in\mathbb{N}$ depending only on n such that for every $N>0$, one has*

$$\|(1+h^{1/2}|x|)^{-k}\chi(H^h)R_{jN}(t,h)\chi(H^h)(1+h^{1/2}|x|)^{-k}\|\leq C_N<+\infty,\quad j=0,1,2\tag{4.9}$$

uniformly in $t\in\mathbb{R}$ and $h\in\,]0,1]$.

PROOF. We only prove (4.9) for $j = 0$. The other cases can be treated in the same way. Let:

$$g(x, \xi; t) = (x \cdot \xi - t\xi^2)\chi_1 \circ p(x, \xi).$$

Let ϕ^t be the hamiltonian flow associated with $p = |\xi|^2/2 + V(x)$.
Since $U^h(t)$ is continuous from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$, we have:

$$\begin{aligned} U^h(t)^* g^w(h^{1/2}x, h^{1/2}D; t)U^h(t) \\ = \mathrm{op}_h^w g(\phi^t, t) + h^2 \int_0^t U^h(s)^* B(t, s; h) U^h(s) ds \end{aligned} \tag{4.10}$$

where

$$B(t, s; h) = h^{-2}\{\frac{i}{h}[H^h, \mathrm{op}_h^w g(\phi^{t-s}, t)] - \mathrm{op}_h^w \{V, g(\phi^{t-s}, t)\}\}.$$

Here we have denoted by $\mathrm{op}_h^w a$ the operator with Weyl symbol $a, \{\cdot, \cdot\}$ the Poisson bracket. By an easy calculus we get an expansion for the symbol $b(t, s; h)$ of $B(t, s; h)$:

$$b(t, s; h) = \sum_{j=0}^{N} h^{2j} a_j(t, s) + h^{2N+1} r_1(t, s; h)$$

with

$$a_j(t, s) = \sum_{|\alpha|=2j+3} C_\alpha \partial_x^\alpha V \partial_\xi^\alpha g(\phi^{t-s}, t), \qquad j = 0, 1, \cdots, N.$$

Since the symbols in consideration are not in the usual classes, we have to be careful with the remainder estimates.

LEMMA 4.4. *Under the assumptions of Theorem* 4.3, *for every* $M \geq 0$, *one has*:

$$|\partial_x^\alpha \partial_\xi^\beta g(\phi^{t-s}(x, \xi), t)| \leq C_{\alpha\beta}(1 + |s|)(1 + |x|)^{|\beta|+1}(1 + |\xi|)^{-M},$$

for $(x, \xi) \varepsilon R^{2n}$, *uniformly in* $t, s \in R$.

PROOF. Put $\phi^t = (y(t), \eta(t))$. Then we can write

$$g(\phi^{t-s}, t) = g(\phi^{t-s}, t - s) - s\chi_1(p)|\eta(t - s)|^2.$$

By Lemma 3.3 we get for every $R \geq 0$:

$$\begin{cases} |\partial_x^\alpha \partial_\xi^\beta g(\phi^r, r)| \leq C_{\alpha\beta}(1 + |\xi|)^{-M} \\ |\partial_x^\alpha \partial_\xi^\beta \eta(r, x, \xi)| \leq C_{\alpha.\beta} \end{cases} \tag{4.11}$$

for $(x,\xi) \in R^{2n}$ with $|x| \leq R$, uniformly in $r \varepsilon R$. For $|x| > R$, we use the argument of Lemma 3.4. Take $R > 0$ so that supp $V \subset B_{R-1}$. If $y(t;x,\xi)$ does not enter into B_R, $\phi^t(x,\xi) = \phi_0^t(x,\xi)$. It is clear that the result is true for (x,ξ) in such region. If $y(t;x,\xi)$ enters into B_R at some time T, we can use the translation in time

$$\phi^r(x,\xi) = \phi^{r-T}(x+T\xi,\xi)$$

with $T = T(x,\xi)$ determined by $|x+T\xi| = R$. See Lemma 3.4. Then Lemma 4.4 follows from (4.10) as in the proof of Lemma 3.4.

Return now to the proof of Theorem 4.3. Let $R > 0$ so that the support for V is contained in B_{R-1}. Take $\psi \in C_0^\infty(R^n)$ so that $\psi = 1$ on B_R. Put:

$$g_1(x,\xi;t,s) = \psi(x)g(\phi^{t-s}(x,\xi),t)$$
$$g_2(x,\xi;t,s) = (1-\psi(x))g(\phi^{t-s}(x,\xi),t)$$

$g_1(t,s)$ is a compact support in $(x,\xi) \in R^{2n}$. By Lemma 4.4,

$$|\partial_x^\alpha \partial_\xi^\beta g_1(x,\xi;t,s)| \leq C_{\alpha\beta.M}(1+|s|)(1+|x|)^{-M}(1+|\xi|)^{-M}$$

uniformly in $t, s \varepsilon \mathbb{R}$. Since $\psi(x) = 1$ on the support of V the symbol of $h^{-2}\{\frac{i}{h}[V_h, \mathrm{op}_h^w g_1(\phi^{t-s},t)] - \mathrm{op}_h^w\{v, g(\phi^{t-s},t)\}\}$ has the same expansion as $b(t,s;h)$. Hence we get an expression for the remainder $r_1(t,s;h)$:

$$\mathrm{op}_h^w r_1(t,s;h) = \mathrm{op}_h^w r_{1,1}(t,s;h) + h^{-2N-4} i[V_h, \mathrm{op}_h^w g_2(\phi^{t-s},t)]$$

where $r_{1,1}(t,s;h)$ is a symbol satisfying:

$$|\partial_x^\alpha \partial_\xi^\beta r_{1,1}(t,s;h)| \leq C_{\alpha\beta,M}(1+|s|)(1+|x|)^{-M}(1+|\xi|)^{-M} \tag{4.12}$$

for every $M > 0$, uniformly in $t, s \in R$ and $h \in]0,1]$. Take $\rho \in C_0^\infty(R^n)$ which is equal to 0 on the ball B_{R-1} and to 1 outside B_R. Then for every $M > 0$, we have

$$\rho(h^{1/2}x)\mathrm{op}_h^w g_2(\phi^{t-s},t) = \mathrm{op}_h^w g_2(\phi^{t-s},t) + h^{M+1}R_M(t,s;h)$$

where $R_M(t,s;h)$ is an operator defined by

$$R_M(t,s;h)f(x) =$$
$$= (2\pi h)^{-n}\int\int e^{i(x-y)\xi} \sum_{|\alpha|=M+1} \frac{1}{\alpha!}\partial_\xi^\alpha g_2(\phi^{t-s}(h^{1/2}(x+y)/2, h^{1/2}\xi),t)$$
$$\int_0^1 (1-\sigma)^M \partial_x^\alpha \rho(h^{1/2}((x+y)/2+\sigma(x-y)/2))d\sigma f(y)\ dyd\xi.$$

Since V and ρ are of disjoint support, we have:

$$V_h \mathrm{op}_h^w g_2(\phi^{t-s}, t) = -h^{M+1} V_h R_M(t, s; h).$$

Let $r_M(t, s; h)$ be defined by:

$$r_M(t, s; h) = V_h \tilde{r}_M(t, s; h)$$

where $\tilde{r}_M(t, s; h)$ is the symbol of $R_M(t, s; h)$. Then on the support of $r_M(t, s; h)$, $h^{1/2}|x| \leq R - 1$ and $h^{1/2}|y| \geq 1$. Taking notice that for $|\alpha| \geq 1$,

$$\left| \int_0^1 (1-\sigma)^M \partial_x^\alpha \rho(h^{1/2}((x+y) + \sigma(x-y))/2) d\sigma \right| \leq C(h^{1/2}|y|)^{-M-1}$$

for $(x, y) \in \operatorname{supp} r_M(\cdot, \cdot, \xi; t, s; h)$, by Lemma 4.4, we obtain:

$$|\partial_x^\alpha \partial_\xi^\beta r_M(x, y, \xi; t, s; h)| \leq C_{\alpha\beta.M}(1 + h^{1/2}|x| + h^{1/2}|\xi|)^{-M}(1 + h^{1/2}|y|)^{|\beta|}$$

uniformly in $t, s \in R$ and $h \in \,]0, 1]$. By Lemma 2.9, there exists k depending only on n such that for every $N > 0$:

$$\|(1 + h^{1/2}|x|)^N V_h \mathrm{op}_h^w g_2(\phi^{t-s}, t) \; (1 + h^{1/2}|x|)^{-k}\| \leq C_N h^{M+1}(1 + |s|)$$

uniformly in $t, s \in \mathbb{R}$ and $h \in \,]0, 1]$. In the same way, we can prove:

$$\|(1 + h^{1/2}|x|)^{-k} \mathrm{op}_h^w g_2(\phi^{t-s}; t) V_h (1 + h^{1/2}|x|)^N\| \leq C_N h^{M+1}(1 + |s|).$$

On the other hand, we can prove by induction that for every $j \geq 0$:

$$\|U^h(t)\chi(H^h)\|_{\mathcal{L}(L^{2,j}(h); L^{2,j}(h))} \leq C_j(1 + |t|)^j$$

for $t \in \mathbb{R}$, uniformly in $h \in \,]0, 1]$. Applying (4.12) and Theorem 2.4, we get for $j = k + 3$:

$$\|\chi(H^h) \int_0^t U^h(s)^* \mathrm{op}_h^w r_1(t, s; h) U^h(s) \chi(H^h) ds\|_{\mathcal{L}(L^{2,j}(h): L^{2,-j}(h))}$$
$$\leq C| \int_0^t (1 + |s|)^{-j}(1 + |s|)^{k+1} ds| \leq C'$$

uniformly in $t \in R$ and $h \in \,]0, 1]$. By (4.10), we have:

$$\begin{aligned} U^h(t)^* g^w(h^{1/2}x, h^{1/2}D; t) U^h(t) &= \mathrm{op}_h^w g(\phi^t, t) \\ &+ \sum_{j=0}^N h^{2j+2} \int_0^t U^h(s)^* \mathrm{op}_h^w a_j(t, s) U^h(s) ds \\ &+ h^{2N+3} R_1(t, h) \end{aligned} \tag{4.13}$$

with $R_1(t,h)$ satisfying:

$$\sup_{t,h} \|\chi(H^h)R_1(t,h)\chi(H^h)\|_{\mathcal{L}(L^{2,k}(h):L^{2,-k}(h))} < +\infty.$$

The constant k depends only on n.

Recall that $a_j(t,s) = \sum_{|\alpha|=2j+3} C_\alpha \partial_x^\alpha V \partial_\xi^\alpha g(\phi^{t-s}, t)$ satisfies

$$|\partial_x^\alpha \partial_\beta^\beta a_j(t,s)| \leq C_{\alpha.\beta,M}(1+|s|)(1+|x|+|\xi|)^{-M} \tag{4.14}$$

uniformly in $t, s \in \mathbb{R}$. Applying Egorov's Theorem, we get:

$$\int_0^t U^h(s)^* \mathrm{op}_h^w a_j(t,s) U^h(s) ds = \int_0^t \mathrm{op}_h^w a_j(\phi^s; t,s) ds + h^2 \int_0^t U^h(r)^* B_1(r,t;h) U^h(r) dr \tag{4.15}$$

where

$$B_1(r,t;h) = h^{-2} \int_r^t \left(\frac{i}{h}[H^h, \mathrm{op}_h^w a_j(\phi^{s-r}; t,s)] - \mathrm{op}_h^w \{p, a_j(\phi^{s-r}; t,s)\}\right) ds.$$

By the non-trapping condition (B), we deduce from (4.14) that for every $M \geq 1$,

$$|\partial_x^\alpha \partial_\xi^\beta a_j(\phi^{s-r}; t,s)| \leq C_{\alpha\beta}(+|r|)(1+|s-r|+|\xi|)^{-M} \tag{4.26}$$

for $(x,\xi) \in R^{2n}$ with $|x| \leq R$ uniformly in s, r, t in $\mathbb{R}$. Put:

$$f(t,r) = \int_r^t a_j(\phi^{s-r}; t,s) ds.$$

By the argument used in the proof of Lemma 4.4, we can get from (4.16) that

$$|\partial_x^\alpha \partial_\xi^\beta f(t,r)| \leq C_{\alpha\beta}(1+|x|)^{|\beta|+1}(1+|r|)$$

uniformly in t, r in $\mathbb{R}$ and $(x,\xi) \in R^{2n}$. This shows that $f(t,r)$ has the same properties as $g(\phi^{t-s}, t)$. Now repeating the arguments used before, we get:

$$\begin{aligned}\int_0^t U^h(s)^* \mathrm{op}_h^w a_j(t,s) U^h(s) ds &= \int_0^t \mathrm{op}_h^w a_j(\phi^s; t, s) ds \\ &+ \sum_{k=0}^N h^{2k+2} \int_0^t U^h(r)^* \mathrm{op}_h^w c_{jk}(t,r) U^h(r) dr \\ &+ h^{2N+3} R_{jN}(t; h)\end{aligned}$$

where $c_{jk}(t,r)$ is given by:

$$c_{jk}(t,r) = \sum_{|\alpha|=2k+3} c_\alpha \partial_x^\alpha V \partial_\xi^\alpha \int_r^t a_j(\phi^{s-r}; t, s) ds, \;\; k = 0, 1, \cdots, N$$

and $R_{jN}(t,h)$ satisfies:

$$\|\chi(H^h) R_{jN}(t,h) \chi(H^h)\|_{\mathcal{L}(L^{2,k'}(h); L^{2,-k'}(h))} \le C$$

uniformly in $t \in \mathbb{R}$ and $h \in]0,1]$. k' is an integer depending only on n. Consequently we have proved by (4.13) and (4.15) that:

$$\begin{aligned}&U^h(t)^* \mathrm{op}_h^w g(t) U^h(t) \\ &= \mathrm{op}_h^w g(\phi^t, t) + \sum_{j=0}^N h^{2j+2} \int_0^t \mathrm{op}_h^w a_j(\phi^s, t, s) ds \\ &+ \sum_{j=0}^N \sum_{k=0}^N h^{2j+2k+4} \int_0^t U^h(r)^* \mathrm{op}_h^w c_{jk}(t,r) U^h(r) dr \\ &+ h^{2N+3} R_2(t,h)\end{aligned}$$

where $R_2(t,h) = R_1(t,h) + \sum_{j=0}^N h^{2j+2} R_{jN}(t,h)$ satisfies the same estimate as $R_1(t,h)$ (see (4.13)). Notice that $c_{jk}(t,r)$ satisfies also (4.14) with r instead of s. We can use an induction to prove that for every $N \ge 0$,

$$U^h(t)^* \mathrm{op}_h^w g(t) \; U^h(t) = \sum_{k=0}^N h^{2k} \mathrm{op}_h^w b_k(t) + h^{2N+1} R_N(t,h)$$

with $R_N(t,h)$ satisfying the same estimate as $R_1(t,h)$. This finishes the proof for Theorem 4.3.

We can give an expression for $b_k(t)$, $k = 0, 1, \cdots$. Put:

$$c_{1.j}(t, t_1) = \sum_{|\alpha|=2j+1} \frac{1}{\alpha!} \partial_x^\alpha V \; D_\xi^\alpha g(\phi^{t-t_1}, t) \qquad \text{for } j \geq 1$$

$$\begin{aligned} &c_{m.j}(t, t_1, \cdots, t_m) \\ &= \sum_{\ell+k=j} \sum_{|\alpha|=2k+1} \frac{1}{\alpha!} \partial_x^\alpha V \; D_\xi^\alpha c_{m-1.\ell}(\phi^{t_{m-1}-t_m}; t, t_1, \cdots t_{m-1}) \end{aligned}$$

for $j \geq m$. Then we have:

$$\begin{aligned} &b_0(t) = g(\phi^t, t) \\ &b_k(t) = \sum_{m=1}^{k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} c_{m.k}(\phi^{t_m}; t, t_1, \cdots, t_m) dt_m dt_{m-1} \cdots dt_1 \end{aligned} \tag{4.17}$$

for $k \geq 1$.

LEMMA 4.5. *For every $j \geq 0$, the limits $p_{j.\pm} = \lim_{t \to \pm\infty} p_j(t)$ exist in $C^\infty(R^{2n})$ and $p_{j.\pm}$ is in the class T_1^1. In particular,*

$$p_{0.\pm}(x, \xi) = \chi_1 \text{op}(x, \xi) \tau_\pm(x, \xi). \tag{4.18}$$

PROOF. We only prove that $\lim_{t \to \pm\infty} b_j(t)$ exists in $C^\infty(R^{2n})$ and the limiting functions are in the class T_1^1. Notice that this result for $j = 0$ has been proved in Lemma 3.4. Since $p_0(t) = b_0(t)$, (4.18) follows from (4.17) and Lemma 3.4. For $j \geq 1$, V being of compact support, by condition (B), we have:

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta c_{1,j}(t, t_1)| \leq C_{\alpha\beta,M}(1 + |x| + |\xi|)^{-M}(1 + |t_1|) \\ &|\partial_x^\alpha \partial_\xi^\beta c_{m.j}(t, t_1, \cdots, t_m)| \\ &\leq C_{\alpha\beta,M} \; (1 + |x| + |\xi| + |t_1 - t_2| + \cdots + |t_m - t_{m-1}|)^{-1}(1 + |t_m|), \end{aligned}$$

for $2 \leq m \leq j$. These estimates are uniform in $(t, t_1, \cdots, t_m)$ and $(x, \xi) \in R^{2n}$. For fixed $t_1, t_2, \cdots, t_m$, we can prove that $\lim_{t \to \pm\infty} c_{m.j}(t; t_1, \cdots t_m)$ exists in $C^\infty(R^{2n})$, since $g(\phi^t, t)$ is so. From (4.17) we deduce that $b_k(t)$ converges in $C^\infty(R^{2n})$. By the methods used in the proof of Theorem 4.3, we can show that the limits are in the class T_1^1. This proves Lemma 4.5.

Define $p_j \in T_1^1$ by

$$p_j(x, \xi) = p_{j.-}(x, \xi) - p_{j.+}(x, \xi).$$

According to Lemma 2.9, the operator $p_j^w(h^{1/2}x, h^{1/2}D)$ is uniformly continuous from $L^{2,k}(h)$ to $L^{2,-k}(h)$ for some k depending only on n. Now we are able to prove the main result of this section.

THEOREM 4.6. *Under assumptions* (A) *and* (B), *let* $\chi, \chi_1 \in C_0^\infty(J)$ *so that* $\chi_1 = 1$ *on* supp χ. *Then the modified time-delay operator* $\tilde{T}(h)$ *defined in* (4.1) *admits a complete semi-classical expansion in terms of pseudo-differential operators*:

$$H^h \tilde{T}(h) = \sum_{j=0}^{N} h^j \; p_j^w(h^{1/2}x, h^{1/2}D) + h^{N+1} R_N(h) \tag{4.19}$$

with the estimate on the remainder:

$$\|\chi(H^h) R_N(h) \chi(H^h)\|_{\mathcal{L}(L^{2,k}(h):L^{2,-k}(h))} \leq C_N \; h^{N+1}$$

where k *depends only on* n. *For* $j \geq 0$, p_j *belongs to the class* T_1^1. *In particular,* $p_0 = 2p\chi_1 \circ p \; \tau$, τ *being the classical time-delay function defined in* §3.

PROOF. With the notations of Proposition 2.4, we have for $f \in \mathcal{S}(R^n)$:

$$\chi(H^h) H^h \tilde{T}(h) \chi(H^h) f = \lim_{t \to +\infty} (T(-t, h) - T(t, h)) f.$$

Applying Lemma 4.1 and (4.10), one gets an expansion for $T(t, h)$:

$$T(t, h) = \chi(H^h) \sum_{j=0}^{N} h^j \mathrm{op}_h^w p_j(t) \chi(H^h) + h^{N+1} R_N(t, h) \tag{4.20}$$

with the estimate:

$$\sup_{t,h} \|R_N(t, h)\|_{\mathcal{L}(L^{2,k}(h):L^{2,-k}(h))} \leq C_N,$$

(4.19) follows by taking the limits in (4.20).

5. - Classical limit for modified time-delay operator

The semi-classical expansion for modified time-delay operator given in §4 is not so satisfying in that the expansion depends on the truncating function χ, even the principal term. However it gives clearly a correspondence between quantum and classical time-delay. Using this result we can find the classical limit for modified time-delay operator $\tilde{T}(h)$.

Suppose assumptions (A) and (B) to be satisfied. Let $(x_0, \xi_0) \in \mathbb{R}^{2n}$, $p(x_0, \xi_0) \in J$. The function χ used in §4 is chosen to be of compact support in J and to be equal to 1 in some neighbourhood of $p(x_0, \xi_0)$.

Introduce the Weyl operator $W_h(x_0, \xi_0)$ by:

$$W_h(x_0, \xi_0) = \exp(ih^{-1/2}(\xi_0 \cdot x - x_0 \cdot D_x)) \tag{5.1}$$

(see [11], [33] and [38]). Then $W_h(x_0, \xi_0)$ is a pseudo-differential operator with symbol $\exp(-ih^{-1}(\xi_0 \cdot x - x_0 \cdot \xi))$. For every temperate weighted symbol b, one has:

$$\begin{aligned} &W_h(x_0, \xi_0)^* b^w(h^{1/2}x, h^{1/2}D_x) W_h(x_0, \xi_0) f \\ &= b^w(h^{1/2}x + x_0, h^{1/2}D_x + \xi_0) f \qquad \text{for } f \in \mathcal{S}(\mathbb{R}^n). \end{aligned} \tag{5.2}$$

For $f, g \in L^2(\mathbb{R}^n)$, we put:

$$f_h = \chi(H^h) W_h(x_0, \xi_0) f \text{ and } g_h = \chi(H^h) W_h(x_0, \xi_0) g,$$

where (x_0, ξ_0) and χ are chosen as above. In this section, we will prove the following result.

THEOREM 5.1. *Under assumptions* (A) *and* (B), *with the above notations, one has*:

$$\lim_{h \to 0} < f_h, \tilde{T}(h) g_h >= 2\tau(x_0, \xi_0) \quad < f, g > \tag{5.3}$$

for any $f, g \in L^2(\mathbb{R}^n)$.

REMARK 5.2. Notice that $\tau(x_0, \xi_0)$ is the classical time-delay for a scattering trajectory $(x(t), \xi(t))$ passing through (x_0, ξ_0), (5.3) shows that the modified time-delay operator $\tilde{T}(h)$ introduced in [22] is a quantization of 2τ.

Before proving theorem 5.1, we need some preparations.

LEMMA 5.3. *For* $\psi \in C_0^\infty(\mathbb{R})$, $\psi(H^h)$ *is continuous of* $L^{2,s}(h)$ *into* $L^{2,s}(h)$ *for every* $s \in \mathbb{R}$ *and one has*:

$$\sup_{h \in]0,1]} \|\psi(H^h)\|_{B(L^{2,s}(h), L^{2,s}(h))} < +\infty.$$

PROOF. It suffices to use the results on functional calculus for h-pseudo-differential operators (see [10], [29]).

LEMMA 5.4. *For* $s \geq 0$, $0 \leq \rho \leq 1$, *let* b *be a symbol in class* T_ρ^s. *Then there is an integer* k *depending only on* n, *so that for* $f, g \in L^{2,s+k}$, *one has*

$$\lim_{h \to 0_+} < W_h(x_0, \xi_0) f, b^w(h^{1/2}x, h^{1/2}D_x) W_h(x_0, \xi_0) g >= b(x_0, \xi_0) < f, g > .$$

PROOF. Notice that for every $s \in \mathbb{R}$, $W_h(x_0, \xi_0)$ is uniformly continuous in $L^{2,s}(h)$. By lemma 2.9, there is an integer $k = k(n)$ so that $W_h^*(x_0, \xi_0) b^w(h^{1/2}x, h^{1/2}D_x) W_h(x_0, \xi_0)$ is uniformly continuous of $L^{2,s+k}(h)$ into $L^{2,-s-k}(h)$. Thus by an argument of density, it suffices to prove the lemma for $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then it is an easy consequence of (5.2). Lemma 5.4 is proved.

LEMMA 5.5. *Let $\varphi \in C_0^\infty(\mathbb{R})$. For every $s \in \mathbb{R}$, one has:*

$$\|(W_h^*(x_0, \xi_0)\varphi(H^h)W_h(x_0, \xi_0) - \varphi \circ p(x_0, \xi_0) f\|_{(s),h} \leq Ch\|f\|_{(s),h}$$

for $f \in L^{2,s}(h)$.

PROOF. It is sufficient to use results on functional calculus for h-pseudo-differential operators which say that $\varphi(H^h)$ is a pseudo-differential operator with h-principal symbol $\varphi \circ p$. The result follows from (5.2).

The following result enables us to only prove Theorem 5.1 for a dense subset of functions in $L^2(\mathbb{R}^n)$.

LEMMA 5.6. *Let $\chi \varepsilon C_0^\infty(J)$. Then $\chi(H^h)\tilde{T}(h)$ is uniformly bounded on $L^2(\mathbb{R}^n)$:*

$$\sup_{h \in]0,1]} \|\chi(H^h)\tilde{T}(h)\| \leq C < +\infty.$$

PROOF. For $f, g \in \mathcal{S}(R^n)$, define f_1, g_1 by $f_1 = \chi(H^h)f$ and $g_1 = \psi(H^h)g$, where $\psi \varepsilon C_0^\infty(J)$ is a function so that $\psi(s) = s^{-1}$ on the support of χ. By the definition of $\tilde{T}(h)$,

$$< f, \chi(H^h)\tilde{T}(h)g >= \lim_{t \to +\infty} \int_{-t}^{t} \frac{d}{ds} < f_1, T(s,h)g_1 > ds$$

where $T(s,h) = U^h(s)^* U_0^h(s) A(h) U_0^h(s)^* U^h(s)$. By a simple calculus, we get

$$\int_{-t}^{t} \frac{d}{ds} < f_1, T(s,h)g_1 > ds$$
$$= \int_{-t}^{t} < U^h(s)f_1, \tilde{V}_h U^h(s)g > ds$$
$$- 2t(< f_1, U^h(t)^* V_h U^h(t) g_1 > - < f_1, U^h(t) V_h U^h(t)^* g_1 >)$$

where $\tilde{V} = 2V + x \cdot \nabla V$. Applying Theorem 2.4, we have:

$$< f, \chi(H^h)T(h)g >= \int_{-\infty}^{+\infty} < U^h(s)f_1, \tilde{V}_h U^h(s)g_1 > ds. \tag{5.4}$$

This is Lavine's formula for time-delay. By Proposition 2.3, we deduce from (5.4) that $\chi(H^h)\tilde{T}(h)$ is uniformly bounded on L^2.

PROOF OF THEOREM 5.1. Making use of Lemma 5.6, one sees that it suffices to prove (5.3) for $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then it is an easy consequence of Theorem 4.3, Lemmas 5.3, 5.4 and 5.5.

6. - Classical limit for time delay operator $T(h)$

In this section, we study the classical limit of Eisenbud-Wigner time-delay operator $T(h)$. Since the method used here is similar to that used for modified time-delay operator $\tilde{T}(h)$, we shall be brief in the details.

Let χ, χ_1 and χ_2 be in $C_0^\infty(J)$ such that $\chi_j = 1$ on the support of χ_{j-1} for $j = 1, 2$. Here we take $\chi_0 = \chi$. Put:

$$S(t,h) = U_0^h(t)U^h(2t)^*\chi_1(H^h)U_0^h(t)A(h)U_0^h(t)^*U^h(2t)\chi_1(H^h)U_0^h(t)^*.$$

For $g_h = \chi(H_0^h)g$, $f_h = \chi(H_0^h)f$ with $f, g \in \mathcal{D}(A(h))$, we put: $F(t,h) = < g_h, S(t,h)f_h >$. Then by proposition 2.7, one has:

$$< g_h, H_0^h T(h) f_h >= 1/2(< g_h, A(h)f_h > - \lim_{t\to+\infty} F(t,h)). \tag{6.1}$$

As is seen in §4, $\mathrm{op}_h^w \chi_2(p)U_0^h(t)A(h)U_0^h(t)^*$ is a pseudo-differential operator, which we denote by $G(t,h)$. $G(t,h)$ may be written as:

$$G(t,h) = \sum_{j=0}^{2} h^j \; g_j^w(h^{1/2}x, h^{1/2}D; t) \tag{6.2}$$

with $g_j(t)$ given by (4.3). We define $S_0(t,h)$ by:

$$S_0(t,h) = U_0^h(t)U^h(2t)^*\chi_1(H^h)G(t,h)U^h(2t)\chi_1(H^h)U_0^h(t)^*. \tag{6.3}$$

By Theorem 2.1 and the result on functional calculus ([10], [29]), we obtain:

$$|F(t,h) - < g_h, S_0(t,h)f_h > | \leq C \; h\|g\| \cdot \|(A(h)+i)f\| \tag{6.4}$$

for $h \in \,]0,1]$, uniformly in $t \in \mathbb{R}$.

LEMMA 6.1. *Put $F_0(t) = g_0(\phi^{2t}, t)$. Then for all $f, g \in \mathcal{S}(R^n)$, $N > 0$, one has*:

$$| < f_h, S_0(t,h)g_h > - < f_h, U_0^h(t)\chi_1(H^h) \; \mathrm{op}_h^w F_0(t)\chi_1(H^h)U_0^h(t)^* g_h > | \leq C_N \; h \tag{6.5}$$

uniformly in $|t| \leq h^{-N}$, $h \in \,]0,1]$.

PROOF. We first notice that by the expression of $g_j(t)$ (see (4.3)), we can prove:

$$(6.6)\quad \begin{aligned} &| < f_h, U_0^h(t)U^h(2t)^*\chi_1(H^h)\mathrm{op}_h^w g_j(t)\chi_1(H^h)U^h(2t)U_0^h(t)^* g_h > | \\ &\leq C\ \|f\|\ \|(A(h)+i)g\| \qquad j=1,2 \end{aligned}$$

uniformly in $h \in]0,1]$ and $t \in \mathbb{R}$. As a consequence, it suffices to prove (6.5) for $S_0(t,h)$ defined by (6.3) with $G(t,h)$ replaced by $g_0^w(h^{1/2}x, h^{1/2}D; t)$. As in §4, we have the formula:

$$(6.7)\quad \begin{aligned} &U^h(2t)^*\mathrm{op}_h^w g_0(t)U^h(2t) - \mathrm{op}_h^w F_0(t) \\ &= h^2 \int_0^{2t} U^h(t)^*\mathrm{op}_h^w R(t,r,h)U^h(r)\ dr \end{aligned}$$

where $R(t,r,h)$ is a symbol admitting an asymptotic expansion: $R(t,r,h) = \sum_{j=0}^{N} h^{2j}c_j(t,r) + h^{N+1}d_N(t,r,h)$. Here $c_j(t,r)$ can be computed as in (4.13). In particular, by the assumptions, we see that $c_j(t,r)$ satisfies for every $M \geq 0$:

$$|\partial_x^\alpha \partial_\xi^\beta c_j(t,r)| \leq C_{\alpha\beta.M}(1+|t-r|)(1+|x|+|\xi|)^{-M}.$$

This shows that $\mathrm{op}_h^w c_j(t,r)$ is continuous from $L^{2,-s}(h)$ to $L^{2,s}(h)$ for every $s>0$ and

$$\|(1+h^{1/2}|x|)^s \mathrm{op}_h^w c_j(t,r)(1+h^{1/2}|x|)^s\| \leq C(1+|t-r|)$$

uniformly in $t, r \in R$ and $h \in]0,1]$. By Theorem 2.1, for every $\mu > 1/2$, there exists $\rho > 1/2$ such that:

$$(6.8)\quad \begin{aligned} &\left| \int_0^t \|(A(h)^2+1)^{-\mu} W(t,r;h)\mathrm{op}_h^w c_j(t,r)W(t,r;h)^*(A(h)^2+1)^{-\mu}\| dr \right| \\ &\leq C \left| \int_0^t (1+|t-r|)^{-4\rho+1} dr \right| \leq C' \end{aligned}$$

uniformly in t, r in $\mathbb{R}$ and $h \in]0,1]$. Here we have set $W(t,r;h) = \chi(H_0^h)U_0^h(t)U^h(r)\chi_1(H^h)$. For the remainder $d_N(t,r;h)$, we can apply the argument used in the proof of Theorem 4.3 to show that there is an integer k

dependending only on n such that:

$$\left|\int_0^t \|(1+h^{1/2}|x|)^{-k}W(t,r;h)\mathrm{op}_h^w d_N(t,r;h)W(t,r;h)(1+h^{1/2}|x|)^{-k}\|dr\right| \tag{6.9}$$
$$\leq c(1+|t|)^k$$

uniformly in $t \in \mathbb{R}$ and $h \in\,]0,1]$. For fixed N', take $N \geq kN'+1$. Then we get Lemma 6.2 for $|t| \leq h^{-N'}$ from (6.8) and (6.9).

LEMMA 6.2. *Under the assumptions* (A) *and* (B), *let ρ be in $C_0^\infty(J)$. Then $\rho(H_0^h)T(h)$ is uniformly bounded on $L^2(\mathbb{R}^n)$:*

$$\sup_{0<h\leq 1} \|\rho(H_0^h)T(h)\| < +\infty.$$

Lemma 6.2 follows easily from Lemma 5.6 by noticing that $T(h)$ and $\tilde{T}(h)$ are related by: $T(h) = \frac{1}{2}\Omega_-(h)^*\tilde{T}(h)\Omega_-(h)$. Now we can state the main result of this section.

THEOREM 6.3. *Under the assumptions* (A) *and* (B), *let $(x_0,\xi_0) \in \mathbb{R}^{2n}$ so that $p_0(\xi_0) \in J$. Take $\chi \in C_0^\infty(J)$ and $\chi = 1$ in some neighbourhood of $p_0(\xi_0)$. For $f, g \in L^2(\mathbb{R}^n)$ define f_h and g_h by*

$$f_h = \chi(H_0^h)\ W_h(x_0,\xi_0)f,\quad g_h = \chi(H_0^h)\ W_h(x_0,\xi_0)g$$

Then one has:

$$\lim_{h\to 0_+} < g_h, T(h)f_h >= \tau \circ \Omega_-^{cl}(x_0,\xi_0) < f,g > \tag{6.10}$$

where Ω_-^{cl} is the classical incoming wave operator.

According to Lemma 6.2, we need only to prove Theorem 6.3 for $f,g \in S(\mathbb{R}^n)$. We divide the proof into several steps. Put:

$$S_1(t,h) = U_0^h(t)\chi_1(H^h)\mathrm{op}_h^w F_0(t)\chi_1(H^h)U_0^h(t)^*.$$

LEMMA 6.4. *For all $f,g \in S(R^n)$, $N > 0$, there exists $C > 0$ such that:*

$$|< f_h, S_1(t,h)g_h > - < f_h, \mathrm{op}_h^w F_0(\phi_0^{-t},t)g_h >| \leq Ch$$

uniformly in $|t| \leq h^{-N}$, $h \in\,]0,1]$. Here f_h and g_h are defined in Theorem 6.3.

Lemma 6.4 can be proved by the same argument used in Lemma 6.1. We omit the details here. Now we can apply (5.2) to approximate $< f_h, \mathrm{op}_h^w F_0(\phi_0^{-t},t)g_h >$.

LEMMA 6.5. *For $f,g \in S(R^n)$, we have*

$$|< f_h, \mathrm{op}_h^w F_0(\phi_0^{-t},t)g_h > - F_0(\phi_0^{-t},t)(x_0,\xi_0) < f,g >| \leq C\ h^{1/2} \tag{6.11}$$

uniformly in $t \in R$ and $h \in\,]0,1]$.

PROOF. Let $R > 0$ so that supp V is contained in the ball B_R. For $(x,\xi) \in R^{2n}$ with $|x| \le R$ and $|\xi|^2/2 \in J$, there is $T = T(x,\xi) > 0$ such that:

$$F_0(\phi_0^{-t}, t) = \chi_2(|\xi|^2/2) d \circ \phi_0^{-T} \circ \phi^{2T} \circ \phi_0^{-T},$$

for $|t| > T$. Then it is easy to check that for (x,ξ) in such a region:

$$|\partial_x^\alpha \partial_\xi^\beta F_0(\phi_0^{-t}, t)| \le C_{\alpha\beta.R},$$

uniformly in $t \in \mathbb{R}$. By the method used in the proof of Lemma 4.4, we get

$$(6.12) \qquad |\partial_x^\alpha \partial_\xi^\beta F_0(\phi_0^{-t}, t)| \le C_{\alpha\beta}(1+|x|)^{1+|\beta|}(1+|\xi|)^{-N},$$

for $(x,\xi) \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. This shows that $\mathrm{op}_h^w F_0(\phi_0^{-t}, t)$ is uniformly continuous from $L^{2,k}(h)$ to $L^{2,-k}(h)$ for some $k = k(n)$. Now Lemma 6.5 follows from (5.2).

We observe that $\chi_2(|\xi_0|^2/2) = 1$ and

$$(6.13) \qquad \lim_{t\to+\infty} F_0(\phi_0^{-t}(x_0,\xi_0), t) = d \circ S^{cl}(x_0,\xi_0).$$

Now we can finish the proof of Theorem 6.3.

PROOF OF THEOREM 6.3. By Lemma 6.2, it is sufficient to prove (6.10) for $f, g \in S(\mathbb{R}^n)$. Applying (2.22), one gets:

$$< f_h, H_0^h T(h) g_h >= 1/2 < f_h, (A(h) - S(h)^* A(h) S(h)) g_h > .$$

Set $\rho(s) = s^{-1}\chi(s), \psi_h = \rho(H_0^h) W_h(x_0,\xi_0) f$. Let $S(t,h)$ be defined at the beginning of this section. Put: $S'(t,h) = U_0^h(t) U^h(2t)^* \chi_1(H^h) U_0^h(t)$.

Then $S(t,h) = S'(t,h) A(h) S'(t,h)^*$ and we have:

$$(6.14) \qquad < f_h, T(h) g_h >= 1/2 < \psi_h, (A(h) - \lim_{t\to+\infty} S(t,h)) g_h > .$$

By Theorem 2.1, there exists $\nu > 1$ so that:

$$| < \psi_h, (S(h) - S'(t,h)) g_h > | = | < \psi_h, \int_t^{+\infty} \frac{d}{ds} S'(s,h) ds g_h > |$$
$$\le Ch^{-1}(1+t)^{1-\nu}, \ t > 0.$$

where $C > 0$ depends on f and g but is independent of $h \in]0,1]$ and $t > 0$.

Taking $N' \ge 1$ large enough, we deduce from (6.14) that

$$| < f_h, T(h) g_h > -1/2 < \psi_h, (A(h) - S(t,h)) g_h > | \le Ch$$

uniformly in $h \in]0,1]$ and $t \geq h^{-N'}$. By Lemmas 6.1 and 6.4, we get

$$(6.15) \qquad |< f_h, T(h)g_h > -1/2 < \psi_h, (A(h) - \mathrm{op}_h^w F_0(\phi_0^{-t}, t)g_h > | \leq Ch$$

uniformly in $h \in]0,1]$ and $h^{-N'} < t < h^{-N'-1}$. Lemma 6.5 shows that

$$|< \psi_h, \mathrm{op}_h^w F_0(\phi_0^{-t}, t)g_h > -\rho(|\xi_0|^2/2)F_0(\phi_0^{-t}(x_0, \xi_0), t) < f, g > | \leq Ch^{1/2}$$

for $h \in]0,1]$ and $t \in \mathbb{R}$. Consequently from (6.15), we get:

$$(6.16) \qquad \begin{aligned} &|< f_h, T(h)g_h > -1/2(< \psi_h, A(h)g_h > \\ &- \rho(|\xi_0|^2/2)F_0(\phi_0^{-t}(x_0, \xi_0), t) < f, g > | \leq Ch^{1/2} \end{aligned}$$

for $h \in]0,1]$ and $h^{-N'} < t < h^{-N'-1}$. Notice that $\rho(|\xi_0|^2/2) = 2/|\xi_0|^2$.

Applying Lemma 3.1 and (6.13), taking the limit $h \to 0$ in (6.16), which implies $t \to +\infty$, we get

$$\lim_{h \to 0} < f_h, T(h)g_h >= \tau \circ \Omega_-^{cl}(x_0, \xi_0) < f, g >$$

τ being the classical time-delay function. This proves Theorem 6.3.

Notice that the remainder estimates obtained in this paper are closely related to the continuity of h-pseudo-differential operators with symbol in T_1^1. We believe that by improving Theorem 2.1 or Lemma 2.9, one could get better results. The methods developed here could also be applied to general short range potentials. The main difficulty therein is to establish a result similar to Proposition 3.5 for classical wave operators and time-delay. Unfortunately there are few litteratures in this direction (see however [13] and [32]).

REMARK 6.7. We denote: $(y_0, \eta_0) = \Omega_-^{cl}(x_0, \xi_0)$. Then by the definition of classical wave operators (see [27]), $\phi^t(y_0, \eta_0)$ behaves asymptotically as $\phi_0^t(x_0, \xi_0)$ when t tends to $-\infty$. $\tau \circ \Omega_-^{cl}(x_0, \xi)$ is the classical time-delay for the scattering trajectory $(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0))$, $x(t, y_0, \eta_0) = x_0 + t\xi_0 + o(1)$. $\xi(t, y_0, \eta_0) = \xi_0 + o(1)$ as t tends to $-\infty$. Since $p_0(\xi_0) = p(y_0, \eta_0)$, comparing with theorem 5.1, we can say that the Eisenbud-Wigner time-delay operator $T(h)$ is the quantization of classical time-delay function applied to incoming data, while the modified time-delay operator is the direct quantization of two times the classical time-delay function.

REMARK 6.8. It is interesting to notice that in the proof of theorem 6.3, we have in fact obatined that:

$$\begin{aligned} &\lim_{h \to 0_+} < g_h, S(h)^* A(h) S(h) f_h > \\ &= d \circ S^{cl}(x_0, \xi_0) < g, f > \text{ for } f, g \in \mathcal{D}((A^2 + 1)^{\mu}) \end{aligned}$$

where $d(x, \xi) = x \cdot \xi$ and S^{cl} is the classical scattering transform (see [26]).

More generally, we can show that if b is a bounded symbol on $\mathbb{R}^{2n}$, under the conditions of theorem 6.3, we have:

$$\lim_{h\to 0_+} < g_h, S(h)^* b^w(h^{1/2}x, h^{1/2}D)S(h)f_h >= b\circ S^{cl}(x_0,\xi_0) < g, f > \tag{6.17}$$

for $f, g \in L^2(\mathbb{R}^n)$, where f_h and g_h are the same as those in theorem 6.3. (6.17) may be considered as the classical limit for scattering operator $S(h)$, which gives a close relationship between quantum scattering operator and classical scattering transform. (6.17) can also be compared with the known results on semi-classical approximation of scattering operator (see [40]).

Appendix

In this appendix, we will give the details of the proof of theorem 2.1. We will use commutator techniques. Let us prove first (2.8) and (2.9).

PROOF OF (2.8) AND (2.9). Using (2.5) and proposition 2.3, we can easily get (2.8) just as in the case of $h > 0$ fixed (see Jensen [15]). In particular, the uniformity in h in (2.8) follows from that in Proposition 2.3. To prove (2.9), we apply $A(h)$ to (2.5):

$$\begin{aligned} A(h)^2U^h(t) = A(h)U^h(t)A(h) + 4t^2(H^h)^2U^h(t) + 2tU^h(t)A(h)H^h \\ +2t\ R(t,h)H^h + A(h)R(t,h) \end{aligned} \tag{A.1}$$

where $R(t,h) = \int_0^t U^h(t-s)\tilde{V}_h U^h(s)ds$. (A.1) is valid as operators from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. For $\varphi \in C_0^\infty(J)$, take Ψ a C^∞ function so that supp $\Psi \subset J$ and $\Psi = 1$ on supp φ. Put $\Psi_1(\lambda) = \lambda^{-2}\Psi(\lambda)$. We have:

$$\begin{aligned} \varphi(H^h)U^h(t) = \frac{1}{4t^2}\Psi_1(H^h)\ \{A(h)^2U^h(t) - A(h)U^h(t) - A(h)U^h(t)A(h) \\ - 2tU^h(t)A(h)H^h - 2tR(t,h)H^h - A(h)R(t,h)\}\varphi(H^h). \end{aligned}$$

From lemma 2.2 and (2.8), it follows that in the norm of operators on $L^2(\mathbb{R}^n)$ on has:

$$\begin{aligned} &\|(A(h)^2+1)^{-1}\varphi(H^h)U^h(t)(A(h)^2+1)^{-1}\| \\ &\le C|t|^{-2}(C_1 + C_2|t|(1+|t|)^{-1} \\ &+ |t|\|(A(h)^2+1)^{-1}\Psi_1(H^h)R(t,h)H^h\varphi(H^h)(A(h)^2+1)^{-1}\| \\ &+ \|(A(h)^2+1)^{-1}\Psi_1(H^h)A(h)R(t,h)\varphi(H^h)(A(h)^2+1)^{-1}\|) \end{aligned} \tag{A.2}$$

for $t \neq 0$, (A.2) is uniform with respect to $h \in\ ]0,1]$. Using the assumption

(1.1), for $1 < \rho \le 1 + \varepsilon$, one has:

$$\begin{aligned}&\|(A(h)^2+1)^{-1}\Psi_1(H^h)R(t,h)H^h\varphi(H^h)(A(h)^2+1)^{-1}\|\\&\le C\left|\int_0^t (1+|t-s|)^{-\rho/2}(1+|s|)^{-\rho/2}ds\right| \le C'(1+|t|)^{1-\rho}, \text{ if } \rho \le 1+\varepsilon < 2.\end{aligned}$$

Using (2.8), we obtain that:

$$\begin{aligned}&\|(A(h)^2+1)^{-1}\Psi_1(H^h)\dot{A}(h)R(t,h)\varphi(H^h)(A(h)^2+1)^{-1}\|\\&\le C\ \log(1+|t|), \text{ for } t \ne 0.\end{aligned}$$

Thus it follows from (A.2) that:

(A.3) $$\|(A(h)^2+1)^{-1}\varphi(H^h)U^h(t)(A(h)^2+1)^{-1}\| \le C(1+|t|)^{-\rho}, \text{ if } \rho < 2$$

This proves (2.9) by interpolation.

We notice that if $\varepsilon \ge 1$, we can show that (A.3) is also true for $\rho = 2$. Thus we get:

(A.4) $$\|(A(h)^2+1)^{-\mu/2}\varphi(H^h)U^h(t)(A(h)^2+1)^{-\mu/2}\| \le C(1+|t|)^{-\mu},\ 0 \le \mu \le 2$$

uniformly in $h \in]0,1]$.

PROOF OF THEOREM 2.1. We put: $U^h(r,s) = U_0^h(r)^*U^h(s)$. Using (2.5) one has:

(A.5) $$A(h)U^h(r,s) = U^h(r,s)A(h) + 2(s-r)H_0^hU^h(r,s) + R(r,s;h)$$

where

$$R(r,s;h) = 2sU_0^h(r)^*V_hU^h(s) + U_0^h(r)^*R(s,h).$$

From (2.9), it follows that for every $\mu > 1/2$, one has:

(A.6) $$\|(A(h)^2+1)^{1/2}\varphi_1(H_0^h)U^h(r,s)\varphi_2(H^h)(A(h)^2+1)^{-\mu}\| \le C(1+|r-s|)^{-1}.$$

In order to get (2.1), we take $\phi \in C_0^\infty(J)$, $\phi = 1$ on supp φ_2 and we write:

(A.7) $$\begin{aligned}A(h)U^h(r,s)\phi(H^h) &= 2(s-r)H_0^hU^h(r,s)\phi(H^h)\\&- U_0^h(r)^*[\phi(H^h),A(h)]U^h(s) + U^h(r,s)\phi(H^h)A(h)\\&+ 2sU_0^h(r)^*V_hU^h(s)\phi(H^h) + U^h(r,s)\phi(H^h)R(s,h).\end{aligned}$$

From lemma 2.2, (2.9) and (A.6), one gets:

$$\|(A(h)^2+1)^{-1/2}\varphi_1(H_0^h)U^h(r,s)\varphi_2(H^h)(A(h)^2+1)^{-1/2}\| \le C(1+|r-s|)^{-1}$$

This proves (2.1). Applying $A(h)$ to (A.5) and using (A.7), one has:

$$\begin{aligned}
&(2(s-r)H_0^h)^2U^h(r,s)\phi(H^h) = A(h)^2U^h(r,s)\phi(H^h)\\
&- A(h)U^h(r,s)A(h)\phi(H^h) - A(h)R(r,s;h)\phi(H^h)\\
&- 4(s-r)H_0^hU^h(r,s)\phi(H^h) - 2(s-r)H_0^hU^h(r,s)\phi(H^h)A(h)\\
&- 2(s-r)H_0^hU_0^h(r)^*[\phi(H^h),A(h)]U^h(s)\\
&- 4\ s(s-r)H_0^hU_0^h(r)^*V_hU^h(s)\phi(H^h)\\
&- 2(s-r)H_0^hU^h(r,s)\phi(H^h)R(s,h).
\end{aligned} \tag{A.8}$$

We write (A.8) as

$$\begin{aligned}
&(2(s-r)H_0^h)U^h(r,s)\ (H^h) = P_1(r,s;h) + P_2(r,s;h)\\
&+ P_3(r,s;h) + 4(s-r)P_4(r,s;h)\\
&+ 2(s-r)P_5(r,s;h) + 2(s-r)P_6(r,s;h)\\
&+ 4s(s-r)P_7(r,s;h) + 2(s-r)P_8(r,s;h)
\end{aligned}$$

with the obvious definitions for $P_j(r,s;h)$. Using lemma 2.2, lemma 2.4 and (2.1), we can estimate the $P_j(r,s;h)'s$ as follows:

$$\begin{aligned}
&\|(A(h)^2+1)^{-1}f(H_0^h)P_j(r,s;h)\varphi_2(H^h)(A(h)^2+1)^{-1}\|\\
&\qquad \le C < +\infty,\ j=1,2,3\\
&\|(A(h)^2+1)^{-1}f(H_0^h)P_j(r,s;h)\varphi_2(H^h)(A(h)^2+1)^{-1}\|\\
&\qquad \le C(1+|r-s|)^{-1},\ j=4,5
\end{aligned}$$

where f is defined by $f(\lambda) = \lambda^{-2}\varphi_1(\lambda)$. Using (2.1) and (2.6), one has:

$$\begin{aligned}
&\|(A(h)^2+1)^{-1}f(H_0^h)P_6(r,s;h)\varphi_2(H^h)(A(h)^2+1)^{-1}\|\\
&\qquad \le C(1+|r-s|)^{-1} + \min(1+|r|)^{-1},(1+|s|)^{-1})).
\end{aligned}$$

By the assumption (1.1), for $\rho = 1+\varepsilon$, one has:

$$\begin{aligned}
&\|(A(h)^2+1)^{-1}f(H_0^h)P_7(r,s;h)\varphi_2(H^h)(A(h)^2+1)^{-1}\|\\
&\qquad \le C\ \min(1+|r|)^{-\rho},(1+|s|)^{-\rho}.
\end{aligned}$$

Just as in the proof of (2.9), one has:

$$\|(A(h)^2+1)^{-1}f(H_0^h)P_8(r,s;h)\varphi_2(H^h)(A(h)^2+1)^{-1}\| \le C(1+|r-s|)^{-\rho+1}.$$

All these estimates are uniform with respect to $h \in\ ]0,1]$. Since $|s-r|\min((1+|r|)^{-1},(1+|s|)^{-1})$ is bounded for $r,s\in\mathbb{R}$ and $|s(s-r)|\min((1+|r|)^{-\rho},(1+|s|)^{-\rho}) \le 2(1+|r-s|)^{2-\rho}$, it follows from (A.8) that

$$\|(A(h)^2+1)^{-1}\varphi_1(H_0^h)U^h(r,s)\varphi_2(H^h)(A(h)^2+1)^{-1}\| \le C(1+|r-s|)^{-\rho}$$

uniformly in $h \in]0, 1]$. Since $\rho > 1$, this proves (2.2). Hence theorem 2.1 is proved.

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