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An Asymptotic Formula for the Green's Function of an Elliptic Operator

MARTINO BARDI (*)

1. Introduction

Let $G^\varepsilon(x, y)$ be the Green's function with pole at y of the Dirichlet problem for the uniformly elliptic operator L^ε , i.e., the weak solution of

$$\begin{cases} L^\varepsilon u := -\varepsilon a_{ij} u_{x_i x_j} + b_i u_{x_i} = \delta_y & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \varepsilon \leq 1$, δ_y is the Dirac measure at $y \in \Omega$, $\Omega \subseteq \mathbb{R}^N$ is bounded and open, and we adopt, here and in the following, the summation convention. In this paper we show that under certain conditions on the vector field b , as $\varepsilon \searrow 0$ $G^\varepsilon(\cdot, y)$ converges exponentially to 0, uniformly on compact subsets of $\Omega \setminus \{y\}$, with rate of decay $\frac{-I(x, y) + o(1)}{\varepsilon}$, $I(x, y) \geq 0$, and we give a representation formula for $I(x, y)$.

The exponential decay as $\varepsilon \searrow 0$ of the Green's functions of the parabolic operators $\frac{\partial}{\partial t} + L^\varepsilon$ was studied by Varadhan [17, 18] in the case $b \equiv 0$, and by Friedman [7, 8] in the general case. Friedman employed the Ventcel-Freidlin estimates from the theory of large deviations of stochastically perturbed dynamical systems and some rather delicate parabolic estimates due to Aronson. His result is the following. For any $x(\cdot) \in W_{loc}^{1,2}([0, \infty), \bar{\Omega})$ define

$$(1.1) \quad \|\dot{x}(s) + b(x(s))\|^2 := a^{ij}(x(s))(\dot{x}(s) + b(x(s)))_i (\dot{x}(s) + b(x(s)))_j,$$

where $((a^{ij})) = a^{-1}$ is the inverse matrix of a . If $\partial\Omega$, a_{ij} and b_i are smooth,

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then the Green's function with pole in y , $q^\epsilon(t, x, y)$, of $\frac{\partial}{\partial t} + L^\epsilon$, satisfies

$$(1.2) \quad \lim_{\epsilon \searrow 0} -\epsilon \log q^\epsilon(t, x, y) = \inf \left\{ \frac{1}{4} \int_0^t \|\dot{x}(s) + b(x(s))\|^2 ds \mid x(0) = x, \right. \\ \left. x(t) = y, x(s) \in \Omega \forall 0 \leq s \leq t \right\}.$$

The corresponding formula we propose for the elliptic case is the following:

$$(1.3) \quad \lim_{\epsilon \searrow 0} -\epsilon \log G^\epsilon(x, y) = I(x, y) := \inf \left\{ \frac{1}{4} \int_0^t \|\dot{x}(s) + b(x(s))\|^2 ds; \right. \\ \left. x(\cdot) \in W^{1,2}([0, t], \Omega), x(0) = x, x(t) = y, \text{ for some } t \in [0, \infty) \right\}.$$

It is clear, however, that unlike the parabolic case, such a formula can be true only under some strong assumptions on the vector field b , since in the simple case $b \equiv 0$, $G^\epsilon \rightarrow +\infty$ as $\epsilon \searrow 0$ uniformly on compact subsets of $\Omega \setminus \{y\}$. The main result of this paper is the proof of formula (1.3) in the case that b satisfies the following condition:

$$(B1) \quad \begin{cases} \text{if } x(\cdot) \in W_{loc}^{1,2}([0, \infty), \bar{\Omega}), \text{ then} \\ \int_0^\infty \|\dot{x}(s) + b(x(s))\|^2 ds = \infty. \end{cases}$$

Condition (B1) was first used by Fleming [5] in the study of a singular perturbation problem arising in stochastic control theory. Its physical meaning is that it takes an infinite amount of energy to resist the flow determined by $-b$ and stay forever in $\bar{\Omega}$. In particular, b is "regular", i.e., it has no zeroes in $\bar{\Omega}$. We remark that the definition of $I(x, y)$ coincides with that of "quasipotential" of the vector field b with respect to the point y in Freidlin-Wentzell [6, p. 108].

Our proof of (1.3) is completely independent of formula (1.2) and also of the probabilistic methods used by Friedman. Instead we follow the PDE approach to WKB-type results initiated in the recent paper by L.C. Evans and H. Ishii [4], where new, totally analytic and simpler proofs are given of three results due respectively to Varadhan, Fleming, and Ventcel-Freidlin. The idea of Evans-Ishii is basically the following: 1) apply a logarithmic transformation to the unknown function, in our case

$$v^\epsilon(x, y) := -\epsilon \log G^\epsilon(x, y),$$

and find a PDE that v^ϵ solves; 2) prove estimates, independent of ϵ , on v^ϵ and its gradient; 3) show that a subsequence of v^ϵ converges as $\epsilon \searrow 0$, to the viscosity solution of a Hamilton-Jacobi equation (see Crandall-Lions [2], Crandall-Evans-Lions [1] and P.L. Lions [14]); 4) by deterministic control theory methods find a representation formula for such a solution.

The main difficulties in the implementation of this plan in our case are in the treatment of the two boundary layers that our problem exhibits, one at the boundary $\partial\Omega$, where ν^ϵ goes to $+\infty$, and the other around the singularity y , where ν^ϵ goes to $-\infty$: notice that the limit $I(x, y)$ is positive and bounded. To deal with these problems we shall establish in §2 suitable estimates of ν^ϵ around y , and we shall introduce in §3 various approximating problems.

The pioneering work about singular perturbation of elliptic operators is due to Levinson [13]. We refer to Schuss [15] for an introduction to the physical motivations and an extensive bibliography. The theory of viscosity solutions has been utilized for problems of this type also by P.L. Lions [14, Ch. 6] and Kamin [12]. For results in the nonregular case, i.e., b having one or more zeroes in Ω , we refer to Freidlin-Wentzell [6], Friedman [8, Ch. 14], Kamin [11], Day [3], and the papers quoted therein. Kamin [19] has also treated recently a nonregular problem where the relevant Hamilton-Jacobi equation has more than one viscosity solution.

The paper is organized as follows: in §2 we list the hypotheses, recall a few definitions and basic facts about the Green's function, prove the estimates for ν^ϵ and deduce from them a convergence result; in §3 we prove the representation formula for the limit.

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2. Estimates and Convergence

Throughout the paper we assume the summation convention and $0 < \epsilon \leq 1$. We will write for brevity $H := W_{\text{loc}}^{1,2}([0, \infty), \mathbb{R}^N)$.

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, satisfy

(A1) Ω is open, bounded and connected, with smooth boundary $\partial\Omega$.

Let $S^{N \times N}$ be the space on $N \times N$ symmetric matrices and let $a : \Omega \rightarrow S^{N \times N}$ satisfy

$$(A2) \quad \begin{cases} a \in C^{1,\alpha}(\Omega) & \text{for some } \alpha > 0 \text{ and} \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 & \text{for every } \xi \in \mathbb{R}^N, x \in \Omega. \end{cases}$$

Let $b : \bar{\Omega} \rightarrow \mathbb{R}^N$ satisfy

$$(B0) \quad b \in C^{1,\alpha}(\Omega) \text{ for some } \alpha > 0.$$

and define

$$d_i^\varepsilon := b_i + \varepsilon a_{ij}x_j, \quad D := \sup_{\substack{0 < \varepsilon \leq 1 \\ i=1,\dots,N}} \|d_i^\varepsilon\|_{L^\infty(\Omega)}.$$

It is well known that in the above hypotheses, for every $f \in C^0(\bar{\Omega})$, the unique weak solution of

$$(2.1) \quad \begin{cases} -(\varepsilon a_{ij}u_{x_i})_{x_j} + d_i^\varepsilon u_{x_i} = f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

belongs to $W_{loc}^{2,N}(\Omega) \cap C^0(\bar{\Omega})$ and it solves

$$(2.2) \quad \begin{cases} -\varepsilon a_{ij}u_{x_i x_j} + b_i u_{x_i} = f \text{ a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

see for instance [9, Ch. 9]. The problem adjoint to (2.1) is

$$(2.3) \quad \begin{cases} -(\varepsilon a_{ij}v_{x_i} + d_j^\varepsilon v)_{x_j} = \psi & \text{in } \Omega, \\ v \in W_0^{1,2}(\Omega), \end{cases}$$

which is also uniquely solvable in the weak sense.

DEFINITION [16]. A function $G^\varepsilon(x, y)$ defined for $x, y \in \Omega, x \neq y$ is a Green's function for the problem (2.1) if $G^\varepsilon(\cdot, y) \in L^1(\Omega), \forall y \in \Omega$, and

$$\int_{\Omega} G^\varepsilon(x, y)\psi(x)dx = v(y) \text{ for all } y \in \Omega,$$

for every $\psi \in C^0(\bar{\Omega})$ and v the corresponding weak solution of (2.3).

From the theory of Stampacchia [16] it follows that there exists a unique Green's function for the problem (2.1), and it satisfies the following properties:

$$(2.4) \quad \left\{ \begin{array}{l} G^\varepsilon(x, y) = \tilde{G}^\varepsilon(y, x) \\ \text{where } \tilde{G}^\varepsilon \text{ is the Green's function for problem (2.3), i.e.} \\ \int_{\Omega} \tilde{G}^\varepsilon(x, y)f(x)dx = u(y) \quad \forall y \in \Omega, \\ \text{for all } f \in C^0(\bar{\Omega}) \text{ and } u \text{ the corresponding solution of (2.1);} \end{array} \right.$$

$$(2.5) \quad \tilde{G}^\varepsilon(x, y) \geq 0;$$

$$(2.6) \quad \begin{cases} \text{for each } y \in \Omega \ G^\varepsilon(\cdot, y), \tilde{G}^\varepsilon(\cdot, y) \in W_0^{1,q}(\Omega) \\ \text{for every } q < \frac{N}{N-1}; \end{cases}$$

$$(2.7) \quad \begin{cases} \tilde{G}^\varepsilon(\cdot, y) \in W_{\text{loc}}^{1,2}(\Omega \setminus \{y\}) \\ \text{and it is a weak solution in } \Omega \setminus \{y\} \text{ of} \\ -(\varepsilon a_{ij} v_{x_i} + d_j^\varepsilon v)_{x_j} = 0. \end{cases}$$

PROPOSITION 2.8. *Under assumptions (A1), (A2), (B0) we have $G^\varepsilon(\cdot, y) \in C^{3,\alpha}(\Omega \setminus \{y\})$ and it satisfies*

$$-\varepsilon a_{ij} G_{x_i x_j}^\varepsilon + b_i G_{x_i}^\varepsilon = 0 \text{ for all } x \in \Omega \setminus \{y\}.$$

PROOF. Fix ε and y and define $u(x) = G^\varepsilon(x, y)$. Let f_n be an approximation of the identity and let u^n be the weak solution of

$$\begin{cases} -(\varepsilon a_{ij} u_{x_i}^n)_{x_j} + d_i^\varepsilon u_{x_i}^n = f_n & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega. \end{cases}$$

By [16, Thm. 9.1] we have

$$\|u^n\|_{W_0^{1,q}(\Omega)} \leq K \quad \text{for } q < \frac{N}{N-1}.$$

Then a subsequence of u^n converges in $L^q(\Omega)$ and weakly in $W_0^{1,q}(\Omega)$ to u . Now fix $\Omega' \subset\subset \Omega \setminus \{y\}$ with smooth boundary. For n big enough u^n solves

$$-\varepsilon(a_{ij} u_{x_i}^n)_{x_j} + d_i^\varepsilon u_{x_i}^n = 0 \quad \text{in } \Omega'.$$

Thus, by standard methods we have

$$\int_{\Omega'} |Du^n|^2 \, dx \leq C,$$

so that a subsequence of u_n converges in $L^2(\Omega')$ and weakly in $W^{1,2}(\Omega')$, necessarily to u . Thus u is a weak solution in Ω' of

$$-(\varepsilon a_{ij} u_{x_i})_{x_j} + d_i^\varepsilon u_{x_i} = 0.$$

Thus u is continuous and the proposition follows from the Schauder theory. □

By the above proposition and the strong maximum principle we have $G^\epsilon(x, y) > 0$ for all $x \in \Omega, x \neq y$, so that we can define

$$v^\epsilon(x, y) := -\epsilon \log G^\epsilon(x, y).$$

It is easy to check that $v^\epsilon(\cdot, y)$ satisfies

$$(2.9) \quad \begin{cases} -\epsilon a_{ij} v_{x_i x_j}^\epsilon + a_{ij} v_{x_i}^\epsilon v_{x_j}^\epsilon + b_i v_{x_i}^\epsilon = 0 & \text{in } \Omega \setminus \{y\}, \\ v^\epsilon(x, y) \rightarrow -\infty & \text{as } x \rightarrow y, \\ v^\epsilon(x, y) \rightarrow +\infty & \text{as } x \rightarrow \partial\Omega. \end{cases}$$

For the solution of the above PDE it is possible to obtain interior estimates for the gradient independent of ϵ , as shown by Evans-Ishii [4]:

LEMMA 2.10. *For each $\Omega' \subset\subset \Omega \setminus \{y\}$ there exists a constant $C(\Omega')$, independent of ϵ , such that every C^3 solution of the PDE in (2.9) satisfies*

$$\sup_{\Omega'} |D_x v^\epsilon| \leq C(\Omega').$$

PROOF. See [4, Lemma 2.2]. □

We are now going to prove interior estimates, independent of ϵ , for $|v^\epsilon|$. To do this we will estimate the Green's function of the adjoint problem \tilde{G}^ϵ and exhibit the dependence of the constants on ϵ . The crucial exponential dependence on ϵ^{-1} of the bounds for \tilde{G}^ϵ comes from the constant in the Harnack inequality:

LEMMA 2.11. *Let $\Omega' \subseteq \Omega$ be open and $u \in W^{1,2}(\Omega')$, $u \geq 0$, be a solution of*

$$-(\epsilon a_{ij} u_{x_i} + d_j^\epsilon u)_{x_j} = 0 \quad \text{in } \Omega'.$$

Then, for any ball $B(z, 4r) \subset \Omega', r > \frac{4\epsilon}{3}$, we have

$$(2.12) \quad \sup_{B(z,r)} u \leq C^{r/\epsilon} \inf_{B(z,r)} u,$$

where $C = C(N, D, \|a_{ij}\|_{L^\infty(\Omega)})$.

PROOF. Fix $x_0 \in B(z, r)$ and define

$$\tilde{a}_{ij}(x) := a_{ij}(x_0 + \epsilon x), \quad \tilde{d}_j(x) := d_j^\epsilon(x_0 + \epsilon x), \quad \tilde{u}(x) := u(x_0 + \epsilon x).$$

Then \tilde{u} solves

$$-(\tilde{a}_{ij} \tilde{u}_{x_i} + \tilde{d}_j \tilde{u})_{x_j} = 0 \quad \text{in } \tilde{\Omega} := \{x \mid x_0 + \epsilon x \in \Omega'\},$$

\tilde{a} satisfies (A2) and $\|\tilde{d}_j\|_{L^\infty(\tilde{\Omega})} \leq D$. Since $B(0,4) \subseteq \tilde{\Omega}$, by the Harnack inequality there exists C such that

$$\sup_{B(x_0,\varepsilon)} u = \sup_{B(0,1)} \tilde{u} \leq C \inf_{B(0,1)} \tilde{u} = C \inf_{B(x_0,\varepsilon)} u .$$

Since any two points in $B(z,r)$ can be connected by a chain of $\lceil \frac{2r}{\varepsilon} \rceil$ appropriately overlapping balls of radius ε , we obtain (2.12). □

REMARK 2.13. The dependence on ε of the constant in the Harnack inequality displayed in (2.12) is sharp, as the following simple example shows:

$$-\varepsilon \Delta u + u_{x_i} = 0 \quad \text{in } \mathbb{R}^N$$

has the positive solution $u(x) = e^{x_i/\varepsilon}$ that assumes the values $e^{r/\varepsilon}$ and $e^{-r/\varepsilon}$ on the boundary of $B(0,r)$. □

PROPOSITION 2.14. Assume (A1), (A2), (B0). The function $\tilde{G}^\varepsilon(x,y)$ defined in (2.4) satisfies the inequality

$$\tilde{G}^\varepsilon(x,y) \geq \frac{C_1 e^{-C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^{N-2}}, \quad \text{for } \frac{4\varepsilon}{3} \leq |x-y| \leq 1 \wedge \text{dist}(y, \partial\Omega)/2$$

where C_1 and C_2 are constants independent of ε .

PROOF. Fix $x \neq y$ and define $r = |x-y|$, $u(z) := \tilde{G}^\varepsilon(z,y)$. Define $S_1 := \{z \in \Omega \mid \frac{r}{2} \leq |z-y| \leq r\}$, $S_2 := \{z \in \Omega \mid \frac{r}{4} \leq |z-y| \leq \frac{3r}{2}\}$ and let $\zeta \in C_0^\infty(\Omega)$ be such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in S_1 , $\zeta \equiv 0$ in $\Omega \setminus S_2$, $|D\zeta| \leq \frac{C}{r}$. By (2.7), using $\phi = u\zeta^2$ as a test function, we get

$$\varepsilon \int_{\Omega} |Du|^2 \zeta^2 dz \leq \left(\frac{\varepsilon C}{r} + D\right) \int_{\Omega} \sum_i |u_{x_i}| u \zeta dz + \frac{C}{r} \int_{\Omega} u^2 \zeta dz$$

where we indicate by C any constant depending only on N, D and $\|a_{ij}\|_{L^\infty(\Omega)}$. Thus

$$(2.15) \quad \int_{S_1} |Du|^2 dz \leq \frac{C}{\varepsilon} \left(\frac{C\varepsilon}{r^2} + \frac{C}{r} + \frac{C}{\varepsilon}\right) \int_{S_2} u^2 dz \leq \left(\frac{C}{r^2} + \frac{C}{\varepsilon^2}\right) r^N \sup_{S_2} u^2 .$$

Now let $\phi \in C_0^\infty(\Omega)$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B(y, \frac{r}{2})$, $\phi \equiv 0$ in $\Omega \setminus B(y, r)$, $|D\phi| \leq \frac{C}{r}$. As a consequence of (2.4) and the regularity of the coefficients we have

$$\int_{\Omega} (\varepsilon a_{ij} u_{x_i} \phi_{x_j} + d_i^e u \phi_{x_i}) dx = \phi(y).$$

Then

$$\begin{aligned} 1 &\leq \varepsilon \frac{C}{r} \int_{S_1} |Du| dz + \frac{C}{r} \int_{S_1} u dz \\ &\leq \frac{\varepsilon C}{r} r^{N/2} (r^N (\frac{C}{r^2} + \frac{C}{\varepsilon^2}) \sup_{S_2} u^2)^{1/2} + Cr^{N-1} \sup_{S_2} u \\ &\leq Cr^{N-2} \sup_{S_2} u \end{aligned}$$

for $r \leq 1$, where we have got the second inequality from Schwarz inequality and (2.15). Now, in order to apply the Harnack inequality (Lemma 2.11), we observe that any ball $B(z, R)$ with $z \in S_2$ and $R = \frac{r}{20}$ is such that $B(z, 4R) \subseteq \Omega \setminus \{y\}$, and that any two points in S_2 can be connected by a chain of appropriately overlapping such balls whose number depends only on N . Then we obtain

$$1 \leq Cr^{N-2} C^{r/\varepsilon} u(x),$$

which yields the conclusion. □

REMARK 2.16. For this proof we borrowed some ideas from Grüter-Widman [10]. □

In order to get the estimate from above of G^e around the pole y we shall use hypothesis (B1). Define

$$\Omega_\gamma := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \gamma\}.$$

The main consequence of (B1) is the following Lemma, which is a slight extension of Lemma 4.2 in [4]:

LEMMA 2.17. *Assume (B0) and (B1) and let \tilde{b} be a Lipschitz extension of b to a neighbourhood of Ω . Then there exist $\alpha > 0$, $T > 0$, $\gamma > 0$ such that*

$$\int_0^S |\dot{x}(s) - \tilde{b}(x(s))|^2 ds \geq \alpha S$$

for all $S \geq T$ and for all $x(\cdot) \in W^{1,2}([0, S], \bar{\Omega}_\gamma)$.

PROOF. First observe that (B1) is equivalent to

$$\int_0^\infty |\dot{x}(s) - b(x(s))|^2 ds = \infty$$

for all $x(\cdot) \in W_{loc}^{1,2}([0, \infty), \bar{\Omega})$. Then the proof (by contradiction) is essentially the same as that of Lemma 4.2 in [4]. □

LEMMA 2.18. *Under the assumptions of Lemma 2.17 there exist $\gamma > 0$ and $w \in W^{1,\infty}(\Omega_\gamma)$ such that*

$$(2.19) \quad \tilde{b}_i w_{x_i} \geq 3, \quad \text{a.e. in } \Omega_\gamma.$$

PROOF. For a given $x(\cdot) \in H$, $x(0) = x$, let t_x be the first exit time of $x(\cdot)$ from Ω_γ , i.e.

$$t_x := \inf \{t > 0 : x(t) \notin \Omega_\gamma\}.$$

Let α, T, γ be the constants provided by Lemma 2.17 and define for $x \in \bar{\Omega}_\gamma$

$$w(x) := \inf \left\{ \int_0^{t_x} \left(\frac{3}{\alpha} |\dot{x}(s) - \tilde{b}(x(s))|^2 - 3 \right) ds : x(\cdot) \in H, x(0) = x \right\}.$$

By standard arguments w is Lipschitz continuous in $\bar{\Omega}_\gamma$.

Now, observe that w is the value function of the control problem of minimizing

$$\int_0^{t_x} \left(\frac{3}{\alpha} |\beta(s) - \tilde{b}(x(s))|^2 - 3 \right) ds$$

where $x(\cdot)$ satisfies $\dot{x}(s) = \beta(s)$, $x(0) = x$, and the control $\beta(\cdot) \in L_{loc}^2([0, \infty), \mathbb{R}^N)$. The Hamilton-Jacobi-Bellman equation associated to this problem is

$$\frac{\alpha}{12} |Dw|^2 - \tilde{b}_i w_{x_i} + 3 = 0.$$

Since w is continuous it is a viscosity solution of this equation (see [14, Thm. 1.10]), then by Rademacher's theorem it also satisfies the equation a.e., which implies (2.19). □

PROPOSITION 2.20. *Assume (A1), (A2), (B0), (B1). Then the function $\tilde{G}^\varepsilon(x, y)$ defined in (2.4) satisfies*

$$\tilde{G}^\varepsilon(x, y) \leq \frac{C_1 e^{C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^N}, \quad \text{for } 20\varepsilon/3 < |x-y| < \text{dist}(y, \partial\Omega)/2 \text{ and } \varepsilon < C_3,$$

where the constants C_1, C_2, C_3 are independent of ε .

PROOF. We extend b to a Lipschitz vector field \tilde{b} defined in a neighbourhood of Ω and consider the function w constructed in Lemma 2.18. We call

$$w^\eta(x) := (w * \rho_\eta)(x), \quad \text{for } 0 < \eta < \gamma, \quad x \in \Omega,$$

the convolution of w with a mollifier ρ_η . Then w^η is smooth and satisfies

$$\begin{aligned} \sup_\Omega |w^\eta| &\leq C, \quad \text{for } 0 < \eta < \gamma, \\ b_i w^\eta_{x_i} &\geq 3 - C\eta \quad \text{in } \Omega, \end{aligned}$$

and

$$|w^\eta_{x_i x_j}| \leq C/\eta^2 \quad \text{in } \Omega.$$

Now we choose η_0 small enough so that $v := w^{\eta_0}$ satisfies

$$b_i v_{x_i} \geq 2,$$

and

$$-\varepsilon a_{ij} v_{x_i x_j} \geq -\varepsilon C/\eta_0^2.$$

Therefore v satisfies

$$-\varepsilon a_{ij} v_{x_i x_j} + b_i v_{x_i} \geq 1, \quad \text{for all } \varepsilon \leq \eta_0^2/C =: C_3, \quad x \in \Omega.$$

Now let u^ε be the solution of

$$\begin{aligned} -\varepsilon a_{ij} u^\varepsilon_{x_i x_j} + b_i u^\varepsilon_{x_i} &= 1, \quad \text{a.e. in } \Omega, \\ u^\varepsilon &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

By the Alexandrov-Bakelman-Pucci maximum principle (see e.g. [9, Thm. 9.1]) we then have

$$\sup_\Omega u^\varepsilon \leq \sup_\Omega v + \sup_{\partial\Omega} v^- \leq C, \quad \text{for } \varepsilon \leq C_3.$$

Using the definition of \tilde{G}^ε we get

$$\int_\Omega \tilde{G}^\varepsilon(x, y) dx = u^\varepsilon(y) \leq C, \quad \text{for } \varepsilon \leq C_3, \quad y \in \Omega.$$

Now taking $\rho = |x - y|/5$, $4\varepsilon/3 < \rho < \text{dist}(x, \partial\Omega)/4$, by the Harnack inequality (Lemma 2.11) we have

$$\tilde{G}^\varepsilon(x, y) \leq C^{\rho/\varepsilon} \inf_{z \in B(x, \rho)} \tilde{G}^\varepsilon(z, y) \leq \frac{C^{\rho/\varepsilon}}{\rho^N} \int_\Omega \tilde{G}^\varepsilon(x, y) dx.$$

□

THEOREM 2.21. *Assume (A1), (A2), (B0), (B1). For each $y \in \Omega$ there exists a sequence $\varepsilon_k \searrow 0$ and a function $v(\cdot, y) \in C^{0,1}(\bar{\Omega})$ such that $\lim_k v^{\varepsilon_k}(\cdot, y) = v(\cdot, y)$ uniformly on compact subsets of $\Omega \setminus \{y\}$ and $v(\cdot, y)$ is a viscosity solution of the Hamilton-Jacobi equation*

$$(2.22) \quad a_{ij}v_{x_i}v_{x_j} + b_iv_{x_i} = 0 \quad \text{in } \Omega \setminus \{y\}.$$

Moreover there exists a positive constant C such that

$$(2.23) \quad |v(x, y)| \leq C|x - y|, \text{ for } |x - y| \leq 1 \wedge \text{dist}(y, \partial\Omega)/2.$$

PROOF. Lemma 2.10 and Propositions 2.14 and 2.20 imply that $\{v^\varepsilon(\cdot, y), 0 < \varepsilon \leq 1\}$ is bounded in $W_{loc}^{1,\infty}(\Omega)$. Therefore a subsequence converges uniformly to $v(\cdot, y) \in W_{loc}^{1,\infty}(\Omega)$, which is a viscosity solution of (2.22) because $v^\varepsilon(\cdot, y)$ solves (2.9), see Crandall-Lions [2, §IV.1]. By the results of Crandall-Lions [2, §I.4], (2.22) implies $|Dv(\cdot, y)| \leq C$ in $\Omega \setminus \{y\}$ and then $v(\cdot, y)$ has a unique Lipschitz extension to $\bar{\Omega}$. □

3. The Representation Formula for the Limit

We recall the definition:

$$I(x, y) := \inf \left\{ \frac{1}{4} \int_0^t \|\dot{x}(s) + b(x(s))\|^2 ds \mid 0 \leq t < \infty, \right. \\ \left. x(\cdot) \in W^{1,2}([0, t], \Omega), x(0) = x, x(t) = y \right\}$$

where $\|\dot{x}(s) + b(x(s))\|^2$ is defined by (1.1). Our main result is the following:

THEOREM 3.1. *Assume (A1), (A2), (B0), (B1). Then*

$$\lim_{\varepsilon \searrow 0} v^\varepsilon(\cdot, y) = I(\cdot, y)$$

uniformly on compact subsets of $\Omega \setminus \{y\}$.

PROOF. Let $v(\cdot, y) = \lim_k v^{\varepsilon_k}(\cdot, y)$ uniformly on compact subsets of $\Omega \setminus \{y\}$ for some $\varepsilon_k \searrow 0$. Our goal is to prove that $v(x, y) = I(x, y)$ for all $x, y \in \Omega$. For $x \in \Omega$ let τ_x be the first exit time of $x(\cdot) \in H$, $x(0) = x$, from $\Omega \setminus \{y\}$. Since by Theorem 2.21 $v(\cdot, y)$ is a viscosity solution of (2.22) and we are assuming

(B1), the following representation formula of Evans-Ishii [4, Thm. 4.1] holds:

$$v(x, y) = \inf\left\{\frac{1}{4} \int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds + v(x(\tau_x), y) \mid x(\cdot) \in H, x(0) = x\right\},$$

for all $x \in \Omega \setminus \{y\}$.

Since either $x(\tau_x) = y$ or $x(\tau_x) \in \partial\Omega$, we have

$$\begin{aligned} v(x, y) &\leq \inf\left\{\frac{1}{4} \int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds + v(x(\tau_x), y) \mid x(\cdot) \in H, x(0) = x, \right. \\ &\qquad\qquad\qquad \left. x(\tau_x) = y\right\} \\ &= \inf\left\{\frac{1}{4} \int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds \mid x(\cdot) \in H, x(0) = x, x(\tau_x) = y\right\} \\ &= I(x, y) \end{aligned}$$

because (2.23) implies $v(y, y) = 0$.

We are now going to prove that $v(x, y) \geq I(x, y)$ for all $x, y \in \Omega$. We fix $y \in \Omega$ and in order to simplify the notation we drop the second variable y in v^ϵ , v , I and in all the functions defined in the following. Define $\Omega' = \Omega \cup \{x \notin \Omega \mid \text{dist}(x, \partial\Omega) < \beta\}$, for $\beta > 0$ small, let τ'_x be the exit time of $x(\cdot) \in H$ $x(0) = x \in \Omega'$ from $\Omega' \setminus \{y\}$, and extend a and b to be Lipschitz and bounded in all \mathbb{R}^N . For $\lambda \geq 0$, $x \in \Omega'$, define

$$I'_\lambda(x) := \inf\left\{\frac{1}{4} \int_0^{\tau'_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \mid x(\cdot) \in H, x(0) = x, \right. \\ \left. x(\tau'_x) = y \text{ if } \tau'_x < \infty\right\}.$$

For all $\lambda \geq 0$ I'_λ is locally Lipschitz with $|DI'_\lambda| \leq \frac{1}{4}(1 + \|b\|_{L^\infty(\Omega)})^2$ so that

$$(3.2) \qquad |I'_\lambda| \leq C, \qquad C \text{ independent of } \lambda,$$

and it is not difficult to show, using the argument in [4, Lemma 2.4], that I'_λ is a viscosity solution of

$$\begin{cases} \lambda I'_\lambda + a_{ij} I'_{\lambda x_i} I'_{\lambda x_j} + b_i I'_{\lambda x_i} = 0 & \text{in } \Omega' \setminus \{y\} \\ I'_\lambda(y) = 0. \end{cases}$$

Now define for $0 < \gamma < \beta$ the mollification

$$I'_\lambda{}^\gamma(x) := (I'_\lambda * \rho_\gamma)(x), \qquad x \in \bar{\Omega},$$

where ρ_γ is an approximation of the identity. It is easy to deduce from Jensen's inequality, (A2), (B0) and (3.2) that

$$\lambda I'_\lambda{}^\gamma + a_{ij} I'_{\lambda x_i}{}^\gamma I'_{\lambda x_j}{}^\gamma + b_i I'_{\lambda x_i}{}^\gamma \leq C\gamma, \qquad x \in \Omega,$$

where C is independent of γ and λ . (3.2) implies also

$$-\varepsilon a_{ij} I_{\lambda x_i x_j}^\gamma \leq \frac{C\varepsilon}{\gamma^2}, \quad \text{for all } x \in \Omega,$$

so that I_λ^γ satisfies

$$(3.3) \quad L_\lambda^\varepsilon I_\lambda^\gamma \leq C_0 \left(\gamma + \frac{\varepsilon}{\gamma^2} \right) \quad \text{in } \Omega$$

for a suitable constant C_0 independent of ε , γ and λ , where L_λ^ε is the quasilinear elliptic operator

$$L_\lambda^\varepsilon w := -\varepsilon a_{ij} w_{x_i x_j} + a_{ij} w_{x_i} w_{x_j} + b_i w_{x_i} + \lambda w.$$

Furthermore, since $I_\lambda^\gamma(y) = 0$, we have $I_\lambda^\gamma(y) \leq C_1 \gamma$ and thus

$$(3.4) \quad I_\lambda^\gamma(x) \leq C_1 \gamma + C_2 R \quad \text{for all } x \in \partial B(y, R),$$

where the constants are independent of λ , γ and R , $0 < R < \text{dist}(y, \partial\Omega)$.

Now fix a constant M such that

$$(3.5) \quad I_\lambda^\gamma \leq M, \quad \text{on } \partial\Omega, \quad \text{for all } \lambda, \gamma,$$

and define $v_{\lambda,R}^\varepsilon$ to be the solution of

$$(3.6) \quad \begin{cases} L_\lambda^\varepsilon v_{\lambda,R}^\varepsilon = 0 & \text{in } \Omega \setminus B(y, R), \\ v_{\lambda,R}^\varepsilon = C_1 \gamma + C_2 R & \text{on } \partial B(y, R), \\ v_{\lambda,R}^\varepsilon = M & \text{on } \partial\Omega, \end{cases}$$

(for the existence and regularity of $v_{\lambda,R}^\varepsilon$ see e.g. [9, Thm. 15.10]). By the comparison principle [9, Thm. 10.1] and (3.3-4-5) we have

$$(3.7) \quad I_\lambda^\gamma \leq v_{\lambda,R}^\varepsilon + \frac{C_0}{\lambda} \left(\gamma + \frac{\varepsilon}{\gamma^2} \right) \quad \text{in } \Omega \setminus B(y, R),$$

and

$$(3.8) \quad v_{\lambda,R}^\varepsilon \geq 0 \quad \text{in } \Omega \setminus B(y, R).$$

Again by the comparison principle, (3.6) (3.8) and (2.9) (2.20) we get

$$v_{\lambda,R}^\varepsilon \leq v^\varepsilon + C_1 \gamma + C_2 R + CR + \varepsilon(C - N \log R) \quad \text{in } \Omega \setminus B(y, R).$$

Combining this last inequality with (3.7) and letting $\varepsilon \rightarrow 0$, $\gamma \rightarrow 0$ and $R \rightarrow 0$ in this order we get

$$(3.9) \quad I_\lambda^\gamma(x) \leq v(x) \quad \text{for all } x \in \Omega.$$

We are now going to show that

$$I_\lambda(x) := \inf \left\{ \frac{1}{4} \int_0^{\tau_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \mid x(\cdot) \in H, x(0) = x, x(\tau_x) = y \right\}$$

satisfies

$$(3.10) \quad I_\lambda(x) \leq \liminf_{\beta \searrow 0} I'_\lambda(x) \leq v(x) \text{ for all } x \in \Omega.$$

Fix $\varepsilon > 0$. Let $\beta_n \searrow 0$ as $n \rightarrow \infty$, $\Omega_n := \Omega \cup \{x \notin \Omega \mid \text{dist}(x, \partial\Omega) < \beta_n\}$, let τ_x^n be the exit time from $\Omega_n \setminus \{y\}$ of $x(\cdot) \in H, x(0) = x$ and let $I'_{\lambda,n}$ be I'_λ for $\beta = \beta_n$. Now take $x_n(\cdot) \in H$ such that $x_n(0) = x, x_n(\tau_x^n) = y$ if $\tau_x^n < \infty$ and

$$\frac{1}{4} \int_0^{\tau_x^n} e^{-\lambda s} \|\dot{x}_n(s) + b(x_n(s))\|^2 ds \leq I'_{\lambda,n}(x) + \varepsilon.$$

Define

$$z_n(s) := \begin{cases} x_n(s) & \text{for } s \leq \tau_x^n \\ \text{the solution of } \begin{cases} \dot{z} = -b(z) \\ z(\tau_x^n) = y \end{cases} & \text{for } s > \tau_x^n, \text{ if } \tau_x^n < \infty. \end{cases}$$

It is easy to see, using (3.9), that for all $T > 0$

$$\frac{1}{4} \int_0^T e^{-\lambda s} |\dot{z}_n(s)|^2 ds \leq Cv(x) + C\varepsilon + CT,$$

so that the sequence $\{z_n(\cdot)\}$ is bounded in $W^{1,2}([0, T], \Omega_0)$. Hence there exists $z(\cdot) \in H$ and a subsequence of $z_n(\cdot)$, still denoted by $z_n(\cdot)$, which converges to $z(\cdot)$ weakly in $W^{1,2}([0, T], \Omega_0)$ and uniformly on $[0, T]$.

Now for $x(\cdot) \in H, x(0) = x$ define

$$s_x := \begin{cases} \inf\{s : x(s) = y\} & \text{if } \{s : x(s) = y\} \neq \emptyset \\ +\infty & \text{if } \{s : x(s) = y\} = \emptyset. \end{cases}$$

We claim that $z(s) \in \bar{\Omega}$ for all $0 \leq s \leq s_x$. To prove this assume $z(\tilde{s}) \notin \bar{\Omega}$. Let $\alpha := \text{dist}(z(\tilde{s}), \partial\Omega)$ and fix \bar{n} such that $|z_n(s) - z(s)| < \frac{\alpha}{2}$ for $0 \leq s \leq \tilde{s}, n > \bar{n}$. Let \tilde{n} be such that $\beta_{\tilde{n}} < \frac{\alpha}{2}$ and define $\bar{n} := \max\{\bar{n}, \tilde{n}\}$. Then for all $n > \bar{n}$ we have $z_n(\tilde{s}) \notin \Omega_n$ and thus $x_n(\tau_x^n) = y$ with $\tau_n := \tau_x^n[x_n(\cdot)] < \tilde{s}$. Hence τ_n has a subsequence converging to $\bar{s} \leq \tilde{s}$ and it is easy to see that $z(\bar{s}) = y$, which implies $\tilde{s} > s_x$ and proves the claim.

Now define

$$J_\lambda(x) := \inf \left\{ \frac{1}{4} \int_0^{s_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \mid x \in H, x(0) = x, x(s) \in \bar{\Omega} \right. \\ \left. \text{for } 0 \leq s \leq s_x \right\}.$$

We claim that

$$(3.11) \quad J_\lambda(x) \leq \liminf_n I'_{\lambda,n}(x).$$

To prove this we recall that for each $T > 0$ the functional

$$x(\cdot) \mapsto \int_0^T e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds$$

is sequentially weakly lower semicontinuous by a classical theorem of Tonelli. Then

$$\int_0^T e^{-\lambda s} \|\dot{z}(s) + b(z(s))\|^2 ds \leq \liminf_n \int_0^T e^{-\lambda s} \|\dot{z}_n(s) + b(z_n(s))\|^2 ds \\ \leq \liminf_n \int_0^{r_n} e^{-\lambda s} \|\dot{x}_n(s) + b(x_n(s))\|^2 ds \\ \leq \liminf_n 4I'_{\lambda,n}(x) + 4\varepsilon.$$

Thus

$$J_\lambda(x) \leq \frac{1}{4} \int_0^{s_x} e^{-\lambda s} \|\dot{z}(s) + b(z(s))\|^2 ds \\ \leq \liminf_n I'_{\lambda,n}(x) + \varepsilon,$$

and the claim is proved by the arbitrariness of ε .

Next we claim that

$$(3.12) \quad I_\lambda(x) \leq J_\lambda(x).$$

We first observe that by a Lemma of Evans-Ishii (see [4, Remark 4.3]) hypothesis (B1) implies that there exist T_0, λ_0 such that

$$\frac{1}{4} \int_0^T e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \geq v(x) + 1 \geq J_\lambda(x) + 1,$$

for all $T \geq T_0$, $0 \leq \lambda \leq \lambda_0$, $x(\cdot) \in H$ satisfying $x(s) \in \bar{\Omega}$ for all $0 \leq s \leq T_0$. Then, if we assume $\lambda \leq \lambda_0$ and fix $0 < \varepsilon < 1$, we can find $x(\cdot) \in H$ such that

$$(3.13) \quad \begin{cases} x(0) = x, \quad x(s_x) = y, \quad s_x \leq T_0, \quad x(s) \in \bar{\Omega} \text{ for } 0 < s < s_x, \\ \frac{1}{4} \int_0^{s_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \leq J_\lambda(x) + \varepsilon. \end{cases}$$

Since $\partial\Omega$ is smooth there exists a smooth function $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Omega = \{x \in \mathbb{R}^N \mid \phi(x) > 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^N : \phi(x) = 0\}, \\ |D\phi| = 1 \text{ on } \partial\Omega. \end{cases}$$

Define

$$(3.14) \quad \bar{x}(s) := \begin{cases} x + sD\phi(x) & \text{for } 0 \leq s \leq \varepsilon \\ x(s - \varepsilon) + \varepsilon D\phi(x(s - \varepsilon)) & \text{for } \varepsilon \leq s < s_x + \varepsilon \\ y + (2\varepsilon + s_x - s)D\phi(y) & \text{for } s_x + \varepsilon \leq s \leq s_x + 2\varepsilon. \end{cases}$$

Clearly

$$I_\lambda(x) \leq \frac{1}{4} \int_0^{s_x + 2\varepsilon} e^{-\lambda s} \|\dot{\bar{x}}(s) + b(\bar{x}(s))\|^2 ds,$$

and using the definitions (1.1) (3.13) (3.14) and the smoothness of a , b , and ϕ , it is not hard to show that

$$\int_0^{s_x} |\dot{x}(s)|^2 ds \leq C,$$

and to deduce from it that

$$I_\lambda(x) \leq \frac{1}{4} \int_0^{s_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds + 0(\varepsilon) \leq J_\lambda(x) + 0(\varepsilon).$$

This proves the claim (3.12), and then, by (3.11) and the arbitrariness of $\beta_n \searrow 0$, the proof of (3.10) is complete. It remains to show that

$$(3.15) \quad I_\lambda(x) \rightarrow I(x) \text{ for all } x \in \Omega.$$

Using [4, Remark 4.3] as above, we find λ_0, T_0 such that for all $\lambda \leq \lambda_0$ and fixed $\varepsilon > 0$ there exists $x_\lambda(\cdot) \in H$ such that

$$\begin{cases} x_\lambda(0) = x, \quad x_\lambda(\tau_x) = y, \quad \tau_x < T_0, \\ \frac{1}{4} \int_0^{\tau_x} e^{-\lambda s} \|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 ds \leq I_\lambda(x) + \varepsilon. \end{cases}$$

Then

$$\begin{aligned} I_\lambda(x) + \varepsilon &\geq \frac{1}{4} e^{-\lambda T_0} \int_0^{T_x} \|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 ds \\ &\geq e^{-\lambda T_0} I(x) \geq I(x) - \varepsilon \end{aligned}$$

for λ small enough. This gives (3.15) and completes the proof. □

REMARK 3.16. Several ideas in this proof are taken from Evans-Ishii [4, §2]. □

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