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Classe di Scienze

GIANNI DAL MASO

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Introduction

In some recent papers ([11], [12], [13], [5]) the notion of μ-capacity (Definition 2.8) has been used to study a class of differential equations, called "relaxed Dirichlet problems", that arise in the limit of perturbed Dirichlet problems with homogeneous boundary conditions on varying domains, as well as in the limit od Schrödinger equations with non-negative varying potentials.

These equations can be formally written as

$$(0.1) \qquad -\Delta u + \mu u = f \quad \text{in } \Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^n, n \geq 2, f \in L^2(\Omega)$, and the "coefficient" μ is a non-negative Borel measure on Ω which must vanish on all sets of (harmonic) capacity zero, but may take the value $+\infty$ on non-polar subsets of Ω.

According to [13], we denote by $\mathcal{M}_0(\Omega)$ the class of all these measures, and for every $\mu \in \mathcal{M}_0(\Omega)$ we define the μ-capacity of a Borel set $B \subseteq \Omega$ with respect to Ω by

$$\mathrm{cap}_\mu(B) = \inf_{u \in H_0^1(\Omega)} \{\int_\Omega |Du|^2 dx + \int_B (u-1)^2 d\mu\}.$$

The purpose of this paper is to study the measures $\mu \in \mathcal{M}_0(\Omega)$ by means of the corresponding μ-capacities.

To this aim we determine the properties of cap_μ, considered as an increasing set function defined on the Borel subsets of Ω (Theorem 2.9). As remarked in [11], Example 5.4, the μ-capacities are not, in general, Choquet capacities (Definition 1.1), because the continuity along decreasing sequences of compact sets does not hold for every $\mu \in \mathcal{M}_0(\Omega)$.

Therefore we introduce a class of measures (Definition 3.1), denoted by $\mathcal{M}_0^*(\Omega)$, contained in $\mathcal{M}_0(\Omega)$, such that the μ-capacity of a measure $\mu \in \mathcal{M}_0(\Omega)$

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is a Choquet capacity if and only if $\mu \in \mathcal{M}_0^*(\Omega)$ (Theorems 3.6 and 4.6). We prove that each measure of the class $\mathcal{M}_0^*(\Omega)$ is inner regular (Theorem 4.4) and that for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a measure $\mu^* \in \mathcal{M}_0^*(\Omega)$ which is equivalent to μ according to Definition 4.6 of [11], i.e.

$$\int_\Omega u^2 d\mu^* = \int_\Omega u^2 d\mu$$

for every $u \in H_0^1(\Omega)$. More precisely, we prove that μ^* is maximal in the equivalence class of μ (Theorem 3.10).

Then we consider the problem of the reconstruction of a measure $\mu \in \mathcal{M}_0(\Omega)$ from its μ-capacity. If μ is finite on all compact subsets of Ω and if we know a finite measure $\nu \in \mathcal{M}_0(\Omega)$ with respect to which μ is absolutely continuous, then this problem can be solved by a derivation argument ([5], Theorem 2.3). For an arbitrary $\mu \in \mathcal{M}_0(\Omega)$ this method is not available. We prove, however, that μ is the least superaddittive set function which is greater than or equal to cap_μ on every Borel subset of Ω. This allows to obtain μ from cap_μ by means of a general formula (Theorem 4.3), which can be simplified when $\mu \in \mathcal{M}_0^*(\Omega)$ (Theorem 4.5).

These results are used to prove that two measures are equivalent if and only if their μ-capacities agree on all open sets (Theorem 4.9).

The problem we deal with in the last two sections of this paper is the connection between the γ-convergence of a sequence in $\mathcal{M}_0(\Omega)$ and the convergence of the corresponding μ-capacities.

The γ-convergence (Definition 5.1) is a variational convergence for sequences in $\mathcal{M}_0(\Omega)$, which was introduced in [11] to study the dependence on μ of the solutions u of equation (0.1), subject to appropriate boundary conditions. This convergence, which is defined in terms of the Γ-convergence (see [15], [14], [1]) of the quadratic forms associated with (0.1), is equivalent to the strong convergence in $L^2(\Omega)$ of the resolvent operators of equation (0.1) ([3], Theorem 2.1). Moreover, it is equivalent to the stable convergence of the randomized stopping times associated with (0.1) (see [3], where a probabilistic analysis of equation (0.1) is carried out).

In Section 5 we prove that, if (μ_h) γ-convergence to μ, then

$$\mathrm{cap}_\mu(A) \le \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(A)$$

for every finely open set $A \subseteq \Omega$ (Theorem 5.8), and

$$\mathrm{cap}_{\mu^*}(F) \ge \limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(F)$$

for every finely closed set $F \subseteq \Omega$ with finite harmonic capacity with respect to Ω (Theorem 5.9). These inequalities imply that

$$\mathrm{cap}_\mu(E) = \lim_{h\to\infty} \mathrm{cap}_{\mu_h}(E)$$

for every set $E \subseteq \Omega$, with finite harmonic capacity in Ω, such that the fine interior and the fine closure of E have the same μ^*-capacity (Theorem 5.11).

In Section 6 we prove that the γ-convergence of a sequence in $\mathcal{M}_0(\Omega)$ is equivalent to the convergence of the corresponding μ-capacities on a dense (Definition 4.7) family of subsets of Ω (Theorem 6.3).

By using the compactness of the γ-convergence, we can even prove that, if the sequence (cap_{μ_h}) converges weakly (in the sense of [16]) to an inner regular increasing set function α, then the sequence (μ_h) γ-converges to a measure $\mu \in \mathcal{M}_0^*(\Omega)$, and μ can be obtained as the least superaddittive set function which is greater than or equal to α (Theorem 6.1). This improves a result of [5], where the measure μ is assumed to be finite on all compact sets, and a probabilistic result of [2], where, in addition, μ is assumed to have (locally) a bounded potential.

The equivalence between γ-convergence and convergence of μ-capacities is exploited to give a non-probabilistic proof of the following localization result, obtained in [3], Lemma 5.1, by probabilistic methods: if two γ-convergent sequences of measures of the class $\mathcal{M}_0(\Omega)$ agree on all finely open subsets of a finely open set, then the same property holds for their γ-limits (Theorem 5.12).

Finally, the equivalence between γ-convergence and convergence of μ-capacities is used to tackle the non-trivial problem of the continuity, for the γ-convergence, of the restriction operator (Definition 2.4). Given a measure $\mu \in \mathcal{M}_0(\Omega)$, we determine a family of subsets of Ω, depending on μ, such that, for every element E of this family, the γ-convergence of a sequence (μ_h) to μ implies the γ-convergence of the restrictions μ_h^E of μ_h to the restriction μ^E of μ (Theorem 6.6). This family contains the family $\mathcal{R}_\mu$ introduced in [11], Definition 5.6, as well as the family of sets considered in [3], Lemma 5.2, and can be characterized in terms of the capacity cap_{μ^*} (Theorem 6.6), of the measure μ^* (Proposition 6.7), and of the measure μ itself (Proposition 6.8 and Remark 6.9).

1. - Notation and Preliminaries

Let Ω be an open subset of $\mathbb{R}^n, n \geq 2$, and let $\mathcal{P}(\Omega)$ be the set of all subsets of Ω.

DEFINITION 1.1. A *Choquet capacity* on Ω is a set function $\alpha : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ with the following properties:

(a) α is increasing, i.e. $\alpha(E_1) \leq \alpha(E_2)$ whenever $E_1 \subseteq E_2$;
(b) if (E_h) is an increasing sequence of subsets of Ω and $E = \bigcup_h E_h$, then $\alpha(E) = \sup_h \alpha(E_h)$;
(c) if (K_h) is a decreasing sequence of compact sebsets of Ω and $K = \bigcap_h K_h$, then $\alpha(K) = \inf_h \alpha(K_h)$.

If α is increasing, it is easy to see that (c) is equivalent to the following property:

(c′) for every compact set $K \subseteq \Omega$

$$\alpha(K) = \inf \{\alpha(U) : U \text{ open}, \ K \subseteq U\}.$$

A subset E of Ω is said to be *capacitable* (with respect to α) if

$$\alpha(E) = \sup \{\alpha(K) : K \text{ compact}, \ K \subseteq E\}.$$

The abstract definition of Choquet capacity is motivated by the following result, the celebrated Choquet capacitability theorem (see [6]).

THEOREM 1.2. *If α is a Choquet capacity on Ω, then every analytic subset of Ω (in particular every Borel subset of Ω) is capacitable.*

We now recall the variational definition of the harmonic and metaharmonic capacities on Ω.

DEFINITION 1.3. If Ω is bounded, for every compact set $K \subseteq \Omega$ we define the *harmonic capacity* of K with respect to Ω by

$$\operatorname{cap}(K) = \inf \{\int_\Omega |D\varphi|^2 dx : \varphi \in C_0^\infty(\Omega), \ \varphi \geq 1 \text{ on } K\}.$$

If Ω is unbounded, for every compact set $K \subseteq \Omega$ we denote by the same symbol $\operatorname{cap}(K)$ the *metaharmonic capacity* of K with respect to Ω, defined by

$$\operatorname{cap}(K) = \inf \{\int_\Omega |D\varphi|^2 dx + \int_\Omega \varphi^2 dx : \varphi \in C_0^\infty(\Omega), \ \varphi \geq 1 \text{ on } K\}.$$

Both these definitions are extended to open sets $U \subseteq \Omega$ by

$$\operatorname{cap}(U) = \sup \{\operatorname{cap}(K) : K \text{ compact}, \ K \subseteq U\},$$

and to arbitrary sets $E \subseteq \Omega$ by

$$\operatorname{cap}(E) = \inf \{\operatorname{cap}(U) : U \text{ open}, \ E \subseteq U\}. \tag{1.1}$$

The following proposition collects some well-known properties of the harmonic and metaharmonic capacities (see, for instance, [18]).

PROPOSITION 1.4. *The harmonic and metaharmonic capacities are non-negative countably subadditive Choquet capacities on Ω. Moreover they are strongly subadditive, i.e.*

$$\operatorname{cap}(E_1 \cup E_2) + \operatorname{cap}(E_1 \cap E_2) \leq \operatorname{cap}(E_1) + \operatorname{cap}(E_2)$$

for every $E_1, E_2 \subseteq \Omega$.

Let E be a subset of Ω. If a property $P(x)$ holds for all $x \in E$, except for a set $Z \subseteq E$ with $\operatorname{cap}(Z) = 0$, then we say that $P(x)$ holds *quasi everywhere* on E (q.e. on E).

A set $A \subseteq \Omega$ is said to be *quasi open* (resp. *quasi closed, quasi compact*) in Ω if for every $\varepsilon > 0$ there exists an open (resp. closed, compact) set $U \subseteq \Omega$ such that $\operatorname{cap}(A \triangle U) < \varepsilon$, where $\triangle$ denotes the symmetric difference and the topological notions are given in the relative topology of Ω.

It is well known that A is quasi open if and only if $\Omega - A$ is quasi closed and that any countable union or finite intersection of quasi open sets is quasi open (see, for instance, [19], Lemma 2.3).

A function $f : \Omega \to \overline{\mathbf{R}}$ is said to be *quasi continuous* in Ω if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ with $\operatorname{cap}(\Omega - E) < \varepsilon$ such that the restriction of f to E is continuous on E.

The notions of *quasi upper* and *quasi lower semicontinuity* are defined in a similar way.

For every set $E \subseteq \Omega$ we denote by 1_E the characteristic function of E, defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \in \Omega - E$.

It is easy to check that a set $E \subseteq \Omega$ is quasi open (resp. quasi closed) in Ω if and only if 1_E is quasi lower (resp. quasi upper) semicontinuous in Ω.

It can be proven that a function $f : \Omega \to \overline{\mathbf{R}}$ is quasi lower (resp. quasi upper) semicontinuous if and only if the sets $\{x \in \Omega : f(x) > t\}$ (resp. $\{x \in \Omega : f(x) \geq t\}$) are quasi open (resp. quasi closed) for every $t \in \mathbf{R}$ (see, for instance, [4], Proposition IV.2).

For the definition and properties of the *fine topology* on Ω we refer to [17], Part 1, Chapter XI.

For every set $E \subseteq \Omega$ we denote by $\operatorname{int}_f E$, $\operatorname{cl}_f E$, and $\partial_f E$ respectively the fine interior, the fine closure, and the fine boundary of E in Ω.

The following proposition shows the connection between finely open (resp. finely closed) and quasi open (resp. quasi closed) sets (see, for instance, [4], Chapter IV).

PROPOSITION 1.5. *Every finely open (resp. finely closed) subset of Ω is quasi open (resp. quasi closed). If $A \subseteq \Omega$ is quasi open in Ω, then* $\operatorname{cap}(A - \operatorname{int}_f A) = 0$. *If $F \subseteq \Omega$ is quasi closed in Ω, then* $\operatorname{cap}(\operatorname{cl}_f F - F) = 0$.

The following result can be proven in the same way.

PROPOSITION 1.6. *A function $f : \Omega \to \overline{\mathbf{R}}$ is quasi upper (resp. quasi lower) semicontinuous in Ω if and only if f is finely upper (resp. finely lower) semicontinuous quasi everywhere in Ω.*

We denote by $H^1(\Omega)$ the Sobolev space of all functions in $L^2(\Omega)$ with first order distribution derivatives in $L^2(\Omega)$ and by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

For every $x \in \mathbb{R}^n$ and every $r > 0$ we set

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\},$$

and we denote by $|B_r(x)|$ its Lebesgue measure.

It is well known that for every function $u \in H^1(\Omega)$ the limit

$$\lim_{r \to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy$$

exists and is finite quasi everywhere in Ω. We make the following convention about the pointwise values of a function $u \in H^1(\Omega)$: for every $x \in \Omega$ we always require that

$$\liminf_{r \to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \le u(x) \le \limsup_{r \to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy.$$

With this convention, the pointwise value $u(x)$ is determined quasi everywhere in Ω and the function u is quasi continuous in Ω (see, for instance, [18]).

If Ω is bounded, it can be proven that

$$\text{cap}(E) = \min\ \{\int_\Omega |Du|^2 dx : u \in H_0^1(\Omega),\ u \ge 1 \text{ q.e. on } E\},$$

whereas, if Ω is unbounded,

$$\text{cap}(E) = \min\ \{\int_\Omega |Du|^2 dx + \int_\Omega u^2 dx : u \in H_0^1(\Omega),\ u \ge 1 \text{ q.e. on } E\}$$

for every $E \subseteq \Omega$.

It is easy to prove that for every function $u \in H_0^1(\Omega)$ the level sets $\{x \in \Omega : u(x) \ge t\}$ are quasi compact for every $t > 0$. It follows from the equalities above that a set $E \subseteq \Omega$ is quasi compact if and only if it is quasi closed and $\text{cap}(E) < +\infty$.

We now prove the following analogue of the Urysohn lemma for quasi open and quasi closed sets. Compare this result with the quasi normality property of the fine topology ([20], Section 3.10).

PROPOSITION 1.7. *Let F and A be subsets of Ω with F quasi closed in Ω, A quasi open in Ω, and $F \subseteq A$. Then there exists a quasi continuous function $f : \Omega \to [0,1]$ such that $f(x) = 0$ for every $x \in F$ and $f(x) = 1$ for every $x \in \Omega - A$. If, in addition, F is quasi compact in Ω, then f can be chosen so that the sets $\{x \in \Omega : f(x) \le t\}$ are quasi compact for every $t \in [0,1[$.*

PROOF. Since A is quasi open, the function 1_A is quasi lower semicontinuous, therefore there exists an increasing sequence (u_h) of non-negative functions of $H_0^1(\Omega)$ which converges to 1_A quasi everywhere in Ω (see, for instance, [9], Lemma 1.5). Let $g_A : \Omega \to [0,1]$ be the function defined by

$$g_A(x) = \sum_{h=1}^{\infty} 2^{-h} u_h \ (x)$$

for every $x \in \Omega$. Then g_A is quasi continuous, $g_A = 0$ q.e. on $\Omega - A$ and $g_A > 0$ q.e. on A. Let $f_A : \Omega \to [0,1]$ be defined by

$$f_A(x) = \begin{cases} 0 & \text{if } x \in \Omega - A, \\ g_A(x) & \text{if } x \in A \text{ and } g_A(x) > 0, \\ 1 & \text{if } x \in A \text{ and } g_A(x) = 0. \end{cases}$$

Then $f_A = g_A$ q.e. on Ω, hence f_A is quasi continuous, $f_A = 0$ on $\Omega - A$, and $f_A > 0$ on A.

In the same way we construct a quasi continuous function $f_F : \Omega \to [0,1]$ such that $f_F = 0$ on F and $f_F > 0$ on $\Omega - F$.

Since $F \subseteq A$, the function $f_F + f_A$ never vanishes, therefore the function $f = f_F(f_F + f_A)^{-1}$ is quasi continuous, $f = 0$ on F, and $f = 1$ on $\Omega - A$.

If F is quasi compact, then $\mathrm{cap}(F) < +\infty$, therefore there exists $u \in H_0^1(\Omega)$ such that $0 \le u \le 1$ on Ω and $u \ge 1$ q.e. on F. Define $g : \Omega \to \overline{\mathbb{R}}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in F, \\ u(x) & \text{otherwise.} \end{cases}$$

Then g is quasi continuous and the sets $\{x \in \Omega : g(x) \ge t\}$ are quasi compact for every $t > 0$. Finally define $f' = 1 - \min\{1 - f,\ g\}$. Then the function f' is quasi continuous, $f' = 0$ on F, $f' = 1$ on $\Omega - A$, and the sets $\{x \in \Omega : f'(x) \le t\}$ are quasi compact for every $t \in [0,1[$. □

2. - Some Properties of the μ-Capacity

In this section we introduce the class $\mathcal{M}_0(\Omega)$ of all non-negative Borel measures on Ω which vanish on all subsets of Ω with capacity zero, and study the notion of equivalence in $\mathcal{M}_0(\Omega)$ introduced in [11], Section 4. Then for every measure $\mu \in \mathcal{M}_0(\Omega)$ we define the μ-capacity relative to an elliptic operator and study those properties of the μ-capacity that hold for an arbitrary measure $\mu \in \mathcal{M}_0(\Omega)$.

We denote by $\mathcal{B}(\Omega)$ the σ-field of all Borel subsets of Ω. By a Borel measure on Ω we mean a non-negative countably additive set function $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ such that $\mu(\emptyset) = 0$. If μ is a Borel measure, we still denote by μ its extension to $\mathcal{P}(\Omega)$ defined by

$$\mu(E) = \inf \{\mu(B) : B \in \mathcal{B}(\Omega),\ E \subseteq B\} \tag{2.1}$$

for every $E \subseteq \Omega$. Then μ is countably subadditive on $\mathcal{P}(\Omega)$ and satisfies conditions (a) and (b) of Definition 1.1. If μ is finite on all compact subsets of Ω, then it is well known that μ is a Choquet capacity. Simple examples show that property (c) of Definition 1.1 may not hold if μ is infinite on some compact subset of Ω.

DEFINITION 2.1. We denote by $\mathcal{M}_0(\Omega)$ the class of all Borel measures μ on Ω such that $\mu(Z) = 0$ for every $Z \subseteq \Omega$ with $\mathrm{cap}(Z) = 0$.

PROPOSITION 2.2. *Every quasi open set is the union of a sequence of compact sets and of a set of capacity zero. Therefore*

$$\mu(A) = \sup\{\mu(K) : K \text{ compact}, K \subseteq A\}$$

or every $\mu \in \mathcal{M}_0(\Omega)$ and for every quasi open set $A \subseteq \Omega$.

PROOF. Let A be a quasi open subset of Ω. For every $h \in \mathbf{N}$ there exists an open set $U_h \subseteq \Omega$ such that $\mathrm{cap}(U_h \triangle A) < 1/h$. By (1.1) there exists an open set $V_h \subseteq \Omega$ such that $U_h \triangle A \subseteq V_h$ and $\mathrm{cap}(V_h) < 1/h$. Since the set $A \cup V_h$ is open, for every $h \in \mathbf{N}$ there exists a sequence $(K_h^k)_k$ of campact sets such that $A \cup V_h = \bigcup_k K_h^k$. Let

$$E = \bigcup_{h,k} (K_h^k - V_h), \quad Z = A - E.$$

Since every set $K_h^k - V_h$ is compact and contained in A, it remains to prove that $\mathrm{cap}(Z) = 0$. Since $\bigcup_k (K_h^k - V_h) = A - V_h$, we have $Z \subseteq V_h$, so $\mathrm{cap}(Z) \leq \mathrm{cap}(V_h) < 1/h$ for every $h \in \mathbf{N}$, which proves that $\mathrm{cap}(Z) = 0$.

The assertion concerning $\mu(A)$ follows now from the fact that $\mu(Z) = 0$ whenever $\mathrm{cap}(Z) = 0$. □

We observe that the measures of the class $\mathcal{M}_0(\Omega)$ are not required to be regular nor σ-finite. For instance, the measures introduced by the following definition belong to the class $\mathcal{M}_0(\Omega)$.

DEFINITION 2.3. For every set $E \subseteq \Omega$ we denote by ∞_E the Borel measure defined by

$$\infty_E(B) = \begin{cases} 0 & \text{if } \mathrm{cap}(B \cap E) = 0, \\ +\infty & \text{if } \mathrm{cap}(B \cap E) > 0, \end{cases}$$

for every $B \in \mathcal{B}(\Omega)$.

By (2.1) we have $\infty_E(B) = \infty_B(E)$ for every $E \subseteq \Omega$ and for every Borel set $B \subseteq \Omega$ (more generally, for every set $B \subseteq \Omega$ which differs from a Borel set by a set of capacity zero). In particular, this equality holds if B is quasi open or quasi closed in Ω (Proposition 2.2).

We now introduce the restriction of a measure to an *arbitrary* set $E \subseteq \Omega$.

DEFINITION 2.4. For every $\mu \in \mathcal{M}_0(\Omega)$ and for every $E \subseteq \Omega$ we denote by μ^E the measure of the class $\mathcal{M}_0(\Omega)$ defined by

$$\mu^E(B) = \mu(B \cap E)$$

for every $B \in \mathcal{B}(\Omega)$.

By (2.1) we have $\mu^E(B) = \mu^B(E) = \mu(B \cap E)$ for every $E \subseteq \Omega$ and for every Borel set $B \subseteq \Omega$ (more generally, for every $B \subseteq \Omega$ which differs from a Borel set by a set of capacity zero). In particular these equalities hold if B is quasi open or quasi closed in Ω (Proposition 2.2).

Moreover, (2.1) implies that

$$\int_\Omega f d\mu^E = \inf\{\int_B f d\mu : B \in \mathcal{B}(\Omega),\ E \subseteq B\} \tag{2.2}$$

for every $E \subseteq \Omega$ and for every non-negative Borel function $f : \Omega \to [0, +\infty]$.

Following [11], Definition 4.6, we introduce an equivalence relation on $\mathcal{M}_0(\Omega)$.

DEFINITION 2.5. We say that two measures $\mu, \nu \in \mathcal{M}_0(\Omega)$ are *equivalent* if

$$\int_\Omega u^2 d\mu = \int_\Omega u^2 d\nu$$

for every $u \in H_0^1(\Omega)$.

If μ and ν are equivalent, then we obtain easily that

$$\int_U u^2 d\mu = \int_U u^2 d\nu \tag{2.3}$$

for every open set $U \subseteq \Omega$ and for every $u \in H^1(U)$. In particular $\mu(U) = \nu(U)$ for every open set $U \subseteq \Omega$, but this condition is not sufficient for the equivalence of μ and ν (see [11], Example 4.7).

We now prove that two measures are equivalent if and only if they agree on all finely open subsets of Ω (see also [3], Lemma 4.1). The proof is based on the property that every non-negative quasi lower semicontinuous function is the limit of an increasing sequence of functions of $H_0^1(\Omega)$.

THEOREM 2.6. *Let μ and ν be two measures of the class $\mathcal{M}_0(\Omega)$. The following conditions are equivalent:*

(a) $\mu(A) = \nu(A)$ *for every finely open set* $A \subseteq \Omega$*;*
(b) $\mu(A) = \nu(A)$ *for every quasi open set* $A \subseteq \Omega$*;*
(c) μ *and* ν *are equivalent.*

PROOF. Conditions (a) and (b) are equivalent by Proposition 1.5. Let us prove that (b) is equivalent to (c). Assume (b) and let $u \in H_0^1(\Omega)$. Since u is quasi continuous, for every $t > 0$ the sets $A_t = \{x \in \Omega : [u(x)]^2 > t\}$ are quasi open, hence

$$\int_\Omega u^2 d\mu = \int_0^{+\infty} \mu(A_t)dt = \int_0^{+\infty} \nu(A_t)dt = \int_\Omega u^2 d\nu,$$

which implies (c).

Assume (c) and let A be a quasi open subset of Ω. Since the function 1_A is quasi lower semicontinuous, there exists an increasing sequence (u_h) of non-negative functions of $H_0^1(\Omega)$ which converges to 1_A quasi everywhere in Ω (see, for instance, [9], Lemma 1.5). Therefore (u_h^2) converges to 1_A quasi everywhere in Ω and

$$\mu(A) = \lim_{h\to\infty} \int_\Omega u_h^2 d\mu = \lim_{h\to\infty} \int_\Omega u_h^2 d\nu = \nu(A)$$

by the monotone convergence theorem. □

REMARK 2.7. If F_1 and F_2 are quasi closed in Ω, then the measures ∞_{F_1} and ∞_{F_2} introduced in Definition 2.3 are equivalent if and only if $\text{cap}(F_1 \Delta F_2) = 0$. In fact, if $\text{cap}(F_1 \Delta F_2) = 0$, then clearly $\infty_{F_1} = \infty_{F_2}$. Conversely, if ∞_{F_1} and ∞_{F_2} are equivalent, then $\infty_{F_1}(\Omega - F_2) = \infty_{F_2}(\Omega - F_2)$ by Theorem 2.6, hence $\text{cap}(F_1 - F_2) = 0$. In the same way we prove that $\text{cap}(F_2 - F_1) = 0$, hence $\text{cap}(F_1 \Delta F_2) = 0$.

Throughout the paper L will denote a fixed elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j u)$$

where $a_{ij} = a_{ji} \in L^\infty(\Omega)$ and

$$c_1|z|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)z_j z_i \le c_2|z|^2 \quad \forall x \in \Omega,\ \forall z \in \mathbb{R}^n \tag{2.4}$$

for suitable constants $0 < c_1 \leq c_2$. For every open set $U \subseteq \Omega$ and for every Borel set $B \subseteq U$ we denote by Φ_B the quadratic form on $H^1_{\mathrm{loc}}(U)$ defined by

$$\Phi_B(u) = \int_B [\sum_{i,j=1}^{n} a_{ij} D_j u D_i u] \, dx,$$

if Ω is bounded, and by

$$\Phi_B(u) = \int_B [\sum_{i,j=1}^{n} a_{ij} D_j u D_i u] dx + \int_B u^2 dx,$$

if Ω is unbounded.

We now define the μ-capacity relative to the operator L.

DEFINITION 2.8. Let $\mu \in \mathcal{M}_0(\Omega)$ and let $E \subseteq \Omega$. The μ-capacity of E in Ω, relative to the operator L, is defined by

$$\mathrm{cap}_\mu(E) = \min_{u \in H^1_0(\Omega)} \{\Phi_\Omega(u) + \int_\Omega (u-1)^2 d\mu^E\} \tag{2.5}$$

where μ^E is the measure introduced in Definition 2.4.

The minimum in (2.5) is clearly attained, by the lower semicontinuity of the functional in the weak topology of $H^1_0(\Omega)$. If $\mathrm{cap}_\mu(E) < +\infty$, then the minimum point is unique by strict convexity.

If E is μ-measurable (in the sense of Carathéodory) then

$$\mathrm{cap}_\mu(E) = \min_{u \in H^1_0(\Omega)} \{\Phi_\Omega(u) + \int_E (u-1)^2 d\mu\}$$

If Ω is bounded and $\mu = \infty_\Omega$ (Definition 2.3), then cap_μ coincides with the capacity, associated with the operator L, introduced by G. Stampacchia in [22], Definition 3.1. If, in addition, L is the Laplace operator $-\Delta$, then $\mathrm{cap}_\mu(E) = \mathrm{cap}(E)$ for every $E \subseteq \Omega$.

Returning to an arbitrary $\mu \in \mathcal{M}_0(\Omega)$, the set function cap_μ coincides with the capacity cap^L_μ introduced in [13], Definition 3.1, provided that Ω is bounded (see [13], Remark 3.4). If, in addition, $L = -\Delta$, then cap_μ coincides with the μ-capacity introduced in [11], Definition 5.1.

The following theorem collects the main properties of the μ-capacity for an arbitrary $\mu \in \mathcal{M}_0(\Omega)$

THEOREM 2.9. *For every* $\mu \in \mathcal{M}_0(\Omega)$ *the set function* $\mathrm{cap}_\mu : \mathcal{P}(\Omega) \to$ $\to [0, +\infty]$ *satisfies the following properties:*

(a) $\mathrm{cap}_\mu(\emptyset) = 0$;

(b) *if* $E_1 \subseteq E_2 \subseteq \Omega$, *then* $\mathrm{cap}_\mu(E_1) \leq \mathrm{cap}_\mu(E_2)$;

(c) *if* (E_h) *is an increasing sequence of subsets of* Ω *and* $E = \bigcup_h E_h$, *then* $\mathrm{cap}_\mu(E) = \sup_h \mathrm{cap}_\mu(E_h)$;

(d) *if* (E_h) *is a sequence of subsets of* Ω *and* $E \subseteq \bigcup_h E_h$, *then* $\mathrm{cap}_\mu(E) \leq \sum_h \mathrm{cap}_\mu(E_h)$;

(e) $\mathrm{cap}_\mu(E_1 \cup E_2) + \mathrm{cap}_\mu(E_1 \cap E_2) \leq \mathrm{cap}_\mu(E_1) + \mathrm{cap}_\mu(E_2)$ *for every* $E_1,\ E_2 \subseteq \Omega$;

(f) $\mathrm{cap}_\mu(E) \leq k\ \mathrm{cap}(E)$ *for every* $E \subseteq \Omega$, *where* $k = \max\{1,\ c_2\}$, c_2 *being the constant which occurs in* (2.4);

(g) $\mathrm{cap}_\mu(E) \leq \mu(E)$ *for every* $E \subseteq \Omega$;

(h) $\mathrm{cap}_\mu(E) = \inf\{\mathrm{cap}_\mu(B) : B \in \mathcal{B}(\Omega),\ E \subseteq B\}$ *for every* $E \subseteq \Omega$;

(i) $\mathrm{cap}_\mu(A) = \sup\{\mathrm{cap}_\mu(K) : K$ *compact*, $K \subseteq A\}$ *for every quasi open set* $A \subseteq \Omega$;

(j) $\mathrm{cap}_\mu(A) = \inf\{\mathrm{cap}_\mu(U) : U$ *open*, $A \subseteq U\}$ *for every quasi open set* $A \subseteq \Omega$.

PROOF. Properties (a), (b), and (g) are trivial and (h) follows easily from (2.2).

To prove (f), we may assume that $\mathrm{cap}(E) < +\infty$ and, for instance, that Ω is unbounded. Then there exists $u \in H_0^1(\Omega)$ such that $u = 1$ q.e. on E and

$$\int_\Omega |Du|^2 dx + \int_\Omega u^2 dx = \mathrm{cap}(E).$$

By the definition of μ-capacity we have

$$\mathrm{cap}_\mu(E) \leq \Phi_\Omega(u) + \int_\Omega (u-1)^2 d\mu^E \leq k\ [\int_\Omega |Du|^2 dx + \int_\Omega u^2 dx] = k\ \mathrm{cap}(E),$$

which proves (f).

By (h) it suffices to prove the other properties when each set E_h belongs to $\mathcal{B}(\Omega)$.

Property (e) is proved in [11], Proposition 5.3, when Ω is bounded and $L = -\Delta$. The same proof holds in the general case.

Let us prove (c). Let (E_h) be an increasing sequence in $\mathcal{B}(\Omega)$ and let $E = \bigcup_h E_h$. Since cap_μ is increasing (property (b)), we have only to prove that

$\text{cap}_\mu(E) \le \sup_h \text{cap}_\mu(E_h)$, assuming that

$$\sup_h \text{cap}_\mu(E_h) < +\infty. \tag{2.6}$$

For every $h \in \mathbf{N}$ let u_h be the unique function in $H_0^1(\Omega)$ such that

$$\Phi_\Omega(u_h) + \int_{E_h} (u_h - 1)^2 d\mu = \text{cap}_\mu(E_h).$$

By (2.6) the sequence (u_h) is bounded in $H_0^1(\Omega)$, hence there exists a subsequence, still denoted by (u_h), which converges weakly in $H_0^1(\Omega)$ to a function $u \in H_0^1(\Omega)$. For every $k \in \mathbf{N}$ the functional

$$\Psi_k(v) = \Phi_\Omega(v) + \int_{E_k} (v - 1)^2 d\mu$$

is lower semicontinuous in the strong topology of $H_0^1(\Omega)$ (recall that μ vanishes on all sets of capacity zero). Since Ψ_k is convex, it is also weakly lower semicontinuous in $H_0^1(\Omega)$, therefore

$$\Phi_\Omega(u) + \int_{E_k} (u - 1)^2 d\mu \le \liminf_{h\to\infty} [\Phi_\Omega(u_h) + \int_{E_k} (u_h - 1)^2 d\mu]$$
$$\le \liminf_{h\to\infty} [\Phi_\Omega(u_h) + \int_{E_h} (u_h - 1)^2 d\mu] = \lim_{h\to\infty} \text{cap}_\mu(E_h).$$

As k goes to $+\infty$ we obtain

$$\text{cap}_\mu(E) \le \Phi_\Omega(u) + \int_E (u - 1)^2 d\mu \le \lim_{h\to\infty} \text{cap}_\mu(E_h),$$

which concludes the proof of (c).

Property (d) follows easily from (c) and (e).

Let us prove (i). Let A be a quasi open subset of Ω, and denote by S the right hand side of (i). By monotonicity it is enough to prove that $\text{cap}_\mu(A) \le S$. For every $\varepsilon > 0$ there exists an open set $U \subseteq \Omega$ such that $\text{cap}(U \triangle A) < \varepsilon$. By (1.1) there exists an open set $V \subseteq \Omega$ such that $U \triangle A \subseteq V$ and $\text{cap}(V) < \varepsilon$. Let (K_h) be an increasing sequence of compact sets such that $\bigcup_h K_h = U \cup V$. Then $(K_h - V)$ is an increasing sequence of compact sets and $A - V = \bigcup_h (K_h - V)$. By (c) we have

$$\text{cap}_\mu(A - V) = \sup_h \text{cap}_\mu(K_h - V) \le S,$$

therefore (e) and (f) imply that

$$\mathrm{cap}_\mu(A) \leq \mathrm{cap}_\mu(A - V) + \mathrm{cap}_\mu(V) \leq S + k\ \mathrm{cap}(V) < S + k\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\mathrm{cap}_\mu(A) \leq S$.

It remains to prove (j). Let A be a quasi open subset of Ω and denote by I the right hand side of (j). By monotonicity it is enough to prove that $\mathrm{cap}_\mu(A) \geq I$. For every $\varepsilon > 0$ there exists an open set $U \subseteq \Omega$ such that $\mathrm{cap}(U \triangle A) < \varepsilon$. By (1.1) there exists an open set $V \subseteq \Omega$ such that $U \triangle A \subseteq V$ and $\mathrm{cap}(V) < \varepsilon$. Therefore

$$\begin{aligned} I &\leq \mathrm{cap}_\mu(U \cup V) \leq \mathrm{cap}_\mu(A) + \mathrm{cap}_\mu(V) \\ &\leq \mathrm{cap}_\mu(A) + k\ \mathrm{cap}(V) \leq \mathrm{cap}_\mu(A) + k\varepsilon \end{aligned}$$

by (e) and (f), hence $I \leq \mathrm{cap}_\mu(A)$.

□

We remark that, in general, the set function cap_μ is not a Choquet capacity, because there are measures $\mu \in \mathcal{M}_0(\Omega)$ such that cap_μ does not fulfil condition (c) of Definition 1.1 (see [11], Example 5.4).

3. - A Class of Measures

In this section we introduce a class of measures, denoted by $\mathcal{M}_0^*(\Omega)$, contained in $\mathcal{M}_0(\Omega)$, such that for every $\mu \in \mathcal{M}_0^*(\Omega)$ the set function cap_μ is a Choquet capacity on Ω. Then we prove that every measure $\mu \in \mathcal{M}_0(\Omega)$ is equivalent (according to Definition 2.5) to a measure $\mu^* \in \mathcal{M}_0^*(\Omega)$. More precisely, μ^* is characterized as the largest measure in the equivalence class of μ.

DEFINITION 3.1. We denote by $\mathcal{M}_0^*(\Omega)$ the class of all measures $\mu \in \mathcal{M}_0(\Omega)$ such that

$$\mu(E) = \inf\ \{\mu(A) : A \text{ quasi open},\ E \subseteq A\} \tag{3.1}$$

for every $E \subseteq \Omega$.

By (2.1) it is enough to verify (3.1) for every $E \in \mathcal{B}(\Omega)$.

REMARK 3.2. Every measure $\mu \in \mathcal{M}_0(\Omega)$ which is finite on all compact subsets of Ω belongs to the class $\mathcal{M}_0^*(\Omega)$. In fact, in this case

$$\mu(E) = \inf\ \{\mu(U) : U \text{ open},\ E \subseteq U\}$$

for every $E \subseteq \Omega$.

REMARK 3.3. If F is quasi closed in Ω, then the measure ∞_F introduced in Definition 2.3 belongs to $\mathcal{M}_0^*(\Omega)$. In fact, if B is any Borel subset of Ω such that $\mathrm{cap}(B \cap F) = 0$, then $A = (\Omega - F) \cup (B \cap F)$ is quasi open, contains B, and $\mathrm{cap}(A \cap F) = 0$. Therefore

$$\infty_F(B) = \inf \{\infty_F(A) : A \text{ quasi open}, \ B \subseteq A\} \tag{3.2}$$

because both sides are zero. If $\mathrm{cap}(B \cap F) > 0$, then $\infty_F(B) = +\infty$ and (3.2) is trivial.

Conversely, if B is a Borel subset of Ω (more generally, if B differs from a Borel set by a set of capacity zero), and the measure ∞_B belongs to $\mathcal{M}_0^*(\Omega)$, then B is quasi closed in Ω. In fact, since $\infty_B(\Omega - B) = 0$, by (3.1) there exists a quasi open set $A \subseteq \Omega$ such that $\Omega - B \subseteq A$ and $\mathrm{cap}(A \cap B) = 0$. This implies that $\Omega - B$ is quasi open, hence B is quasi closed in Ω.

REMARK 3.4. Theorem 2.6 implies that two measures of the class $\mathcal{M}_0^*(\Omega)$ are equivalent if and only if they are equal.

THEOREM 3.5. *Let* $\mu \in \mathcal{M}_0^*(\Omega)$. *Then*

$$\mathrm{cap}_\mu(E) = \inf \{\mathrm{cap}_\mu(U) : U \textit{ open}, \ E \subseteq U\}$$

for every $E \subseteq \Omega$.

PROOF. By Theorem 2.9(h) we have

$$\mathrm{cap}_\mu(E) = \inf \{\mathrm{cap}_\mu(B) : B \in \mathcal{B}(\Omega), \ E \subseteq B\}$$

for every $E \subseteq \Omega$, and by Theorem 2.9(j) we have

$$\mathrm{cap}_\mu(A) = \inf \{\mathrm{cap}_\mu(U) : U \text{ open}, \ A \subseteq U\}$$

for every quasi open set $A \subseteq \Omega$. Theorefore it is enough to prove that

$$\mathrm{cap}_\mu(B) = \inf \{\mathrm{cap}_\mu(A) : A \text{ quasi open}, \ B \subseteq A\} \tag{3.3}$$

for every $B \in \mathcal{B}(\Omega)$.

Fix $B \in \mathcal{B}(\Omega)$ and denote by I the right hand side of (3.3). By monotonicity we have $\mathrm{cap}_\mu(B) \leq I$. It remains to prove the opposite inequality, assuming that $\mathrm{cap}_\mu(B) < +\infty$. Then there exists $u \in H_0^1(\Omega)$ such that

$$\mathrm{cap}_\mu(B) = \Phi_\Omega(u) + \int_B (u-1)^2 d\mu < +\infty.$$

By a truncation argument we obtain easily that $0 \leq u \leq 1$ q.e. on Ω (see [13], Remark 3.2).

Fix $\varepsilon \in]0,1[$ and define $B_\varepsilon = \{x \in B : u(x) \leq 1-\varepsilon\}$. Since $(u-1)^2 \geq \varepsilon^2$ on B_ε, we have

$$\mu(B_\varepsilon) \leq \frac{1}{\varepsilon^2} \int_B (u-1)^2 d\mu < +\infty.$$

Since $\mu \in \mathcal{M}_0^*(\Omega)$, there exists a quasi open set $A'_\varepsilon \subseteq \Omega$ such that $B_\varepsilon \subseteq A'_\varepsilon$ and $\mu(A'_\varepsilon - B_\varepsilon) < \varepsilon$. Since u is quasi continuous, the set $A''_\varepsilon = \{x \in \Omega : u(x) > 1-\varepsilon\}$ is quasi open, hence the set $A_\varepsilon = A'_\varepsilon \cup A''_\varepsilon$ is quasi open and contains B. Define $u_\varepsilon \in H_0^1(\Omega)$ by

$$u_\varepsilon(x) = \min\left\{1, \frac{u(x)}{1-\varepsilon}\right\}.$$

Since $0 \leq u \leq u_\varepsilon \leq 1$ q.e. on Ω and $u_\varepsilon = 1$ on A''_ε, we have

$$\begin{aligned}
I \leq \mathrm{cap}_\mu(A_\varepsilon) &\leq \Phi_\Omega(u_\varepsilon) + \int_{A_\varepsilon} (u_\varepsilon - 1)^2 d\mu \\
&\leq (1-\varepsilon)^{-2}\Phi_\Omega(u) + \int_{A_\varepsilon - A''_\varepsilon} (u_\varepsilon - 1)^2 d\mu \\
&\leq (1-\varepsilon)^{-2}\Phi_\Omega(u) + \int_{A'_\varepsilon} (u - 1)^2 d\mu \\
&\leq (1-\varepsilon)^{-2}\Phi_\Omega(u) + \int_{B_\varepsilon} (u - 1)^2 d\mu + \mu(A'_\varepsilon - B_\varepsilon) \\
&\leq (1-\varepsilon)^{-2}\mathrm{cap}_\mu(B) + \varepsilon.
\end{aligned}$$

As $\varepsilon \to 0^+$ we obtain $I \leq \mathrm{cap}_\mu(B)$. □

The following theorem, which collects the main properties of cap_μ for $\mu \in \mathcal{M}_0^*(\Omega)$, is an easy consequence of Theorems 2.9 and 3.5 (compare with Proposition 1.4).

THEOREM 3.6. *For every $\mu \in \mathcal{M}_0^*(\Omega)$ the set function cap_μ is a non-negative countably subadditive Choquet capacity on Ω. Moreover it is strongly subadditive, i.e.*

$$\mathrm{cap}_\mu(E_1 \cup E_2) + \mathrm{cap}_\mu(E_1 \cap E_2) \leq \mathrm{cap}_\mu(E_1) + \mathrm{cap}_\mu(E_2)$$

for every $E_1,\ E_2 \subseteq \Omega$.

From Theorem 3.6 and from the Choquet capacitability theorem (Theorem 1.2) we obtain the following result.

THEOREM 3.7. *Let $\mu \in \mathcal{M}_0^*(\Omega)$. Then*

$$\mathrm{cap}_\mu(E) = \sup\ \{\mathrm{cap}_\mu(K) : K \text{ compact},\ K \subseteq E\}$$

for every analytic subset E of Ω, in particular for every Borel subset E of Ω.

We now prove that for every measure $\mu \in \mathcal{M}_0(\Omega)$ there exists a unique measure $\mu^* \in \mathcal{M}_0^*(\Omega)$ which is equivalent to μ. The measure μ^* is given by the following definition.

DEFINITION 3.8. For every $\mu \in \mathcal{M}_0(\Omega)$ we denote by μ^* the set function defined by

$$\mu^*(E) = \inf\{\mu(A) : A \text{ quasi open, } E \subseteq A\}$$

for every $E \subseteq \Omega$.

THEOREM 3.9. *For every $\mu \in \mathcal{M}_0(\Omega)$ the set function μ^* is a Borel measure and belongs to $\mathcal{M}_0^*(\Omega)$.*

PROOF. Let $\mu \in \mathcal{M}_0(\Omega)$. Since the family of quasi open sets is stable under countable union, the set function μ^* is countably subadditive on $\mathcal{P}(\Omega)$. To prove that μ^* is a Borel measure, we will show that every open set is μ^*-measurable (in the sense of Carathéodory), adapting to our case an argument of [16], Theorem 5.1.

Let U be an open subset of Ω. We have to prove that

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E - U) \tag{3.4}$$

for every $E \subseteq \Omega$. We argue by contradiction. Assume that (3.4) is false for a set $E \subseteq \Omega$. Then, by the definition of μ^*, there exists a quasi open set $A \subseteq \Omega$ such that $E \subseteq A$ and

$$\mu(A) < \mu(A \cap U) + \mu^*(A - U).$$

Since $A \cap U$ is quasi open, by Proposition 2.2 there exists a compact set $K \subseteq A \cap U$ such that

$$\mu(A) < \mu(K) + \mu^*(A - U).$$

Since $A - K$ is quasi open and contains $A - U$, we obtain

$$\mu(A) < \mu(K) + \mu(A - K)$$

which contradicts the fact that K is μ-measurable. Therefore, (3.4) holds for every $E \subseteq \Omega$, which implies that μ^* is a Borel measure.

For every subset Z of Ω with $\text{cap}(Z) = 0$ we have $\mu^*(Z) \leq \mu(Z) = 0$ because Z itself is quasi open, hence $\mu^* \in \mathcal{M}_0(\Omega)$. Since $\mu^*(A) = \mu(A)$ for every quasi open set $A \subseteq \Omega$, the definition of μ^* implies that μ^* satisfies (3.1), hence $\mu^* \in \mathcal{M}_0^*(\Omega)$. □

THEOREM 3.10. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then μ^* is equivalent to μ and $\mu^* \geq \nu$ for every measure $\nu \in \mathcal{M}_0(\Omega)$ which is equivalent to μ.*

PROOF. Since $\mu^*(A) = \mu(A)$ for every quasi open set $A \subseteq \Omega$, the equivalence between μ and μ^* follows from Theorem 2.6.

If $\nu \in \mathcal{M}_0(\Omega)$ is equivalent to μ, then $\mu(A) = \nu(A)$ for every quasi open set $A \subseteq \Omega$ by Theorem 2.6, hence

$$\begin{aligned}\nu(E) &\leq \inf\{\nu(A) : A \text{ quasi open, } E \subseteq A\} \\ &= \inf\{\mu(A) : A \text{ quasi open, } E \subseteq A\} = \mu^*(E)\end{aligned}$$

for every $E \subseteq \Omega$. □

The capacity cap^*_μ is related to cap_μ by the following proposition.

PROPOSITION 3.11. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\mathrm{cap}_{\mu^*}(E) = \inf\{\mathrm{cap}_\mu(U) : U \text{ open, } E \subseteq U\} \tag{3.5}$$

for every $E \subseteq \Omega$ and

$$\mathrm{cap}_{\mu^*}(A) = \mathrm{cap}_\mu(A) \tag{3.6}$$

for every quasi open set $A \subseteq \Omega$.

PROOF. By Theorem 3.10 μ^* is equivalent to μ, thus (2.3) implies that $\mathrm{cap}_{\mu^*}(U) = \mathrm{cap}_\mu(U)$ for every open set $U \subseteq \Omega$. Theorefore (3.5) follows from Theorem 3.5 (recall that $\mu^* \in \mathcal{M}^*_0(\Omega)$) and (3.6) follows from (3.5) and from Theorem 2.9(j). □

We now give a different construction of the measure μ^*, based on the notion of singular set of a mesure introduced by the following definition.

DEFINITION 3.12. For every measure $\mu \in \mathcal{M}_0(\Omega)$ the *set of σ-finiteness* $A(\mu)$ of μ is defined as the union of all finely open subsets A of Ω such that $\mu(A) < +\infty$. The *singular set* $S(\mu)$ of μ is defined as the complement of $A(\mu)$ in Ω.

REMARK 3.13. The set of σ-finiteness $A(\mu)$ is finely open, hence the singular set $S(\mu)$ is finely closed in Ω.

By the quasi-Lindelöf property of the fine topology (see [17], Theorem 1.XI.11) there exists a sequence (A_h) of finely open sets with $\mu(A_h) < +\infty$ and a set Z with $\mathrm{cap}(Z) = 0$ such that

$$A(\mu) = \bigcup_h A_h \cup Z.$$

Therefore μ is σ-finite on $A(\mu)$.

If $A \subseteq \Omega$ is finely open and $A \cap S(\mu) \neq \emptyset$, then $\mu(A) = +\infty$ by the definition of $S(\mu)$. If $A \subseteq \Omega$ is quasi open and $\mathrm{cap}(A \cap S(\mu)) > 0$, then $\mathrm{int}_f A \cap S(\mu) \neq \emptyset$ by Proposition 1.5, hence $\mu(A) = +\infty$.

REMARK 3.14. By Theorem 2.6 we have $A(\mu) = A(\nu)$ and $S(\mu) = S(\nu)$ if μ and ν are equivalent.

REMARK 3.15. If $\mu(K) < +\infty$ for every compact set $K \subseteq \Omega$, then $S(\mu) = \emptyset$. The converse is false. In fact, in dimension $n \geq 2$ there exist an open set $U \subseteq \Omega$ and a point $x_0 \in \overline{U} \cap \Omega$ such that $x_0 \notin \mathrm{cl}_f U$. It is then easy to construct a measure $\mu \in \mathcal{M}_0(\Omega)$ (even absolutely continuous with respect to Lebesgue measure) such that $\mu(\Omega - U) = 0, \mu(U - B_r(x_0)) < +\infty$, and $\mu(U \cap B_r(x_0)) = +\infty$ for every $r > 0$. Then $S(\mu) = \emptyset$, but $\mu(K) = +\infty$ if K is a compact neighbourhood of x_0 in Ω.

PROPOSITION 3.16. *Let μ and ν be two equivalent measures of the class $\mathcal{M}_0(\Omega)$. Then $\mu(E) = \nu(E)$ for every set $E \subseteq \Omega$ with* $\mathrm{cap}(E \cap S(\mu)) = 0$.

PROOF. Let E be a subset of Ω such that $\mathrm{cap}(E \cap S(\mu)) = 0$. By Remark 3.13 there exists an increasing sequence (A_h) of quasi open sets such that $\mu(A_h) < +\infty$ and $E \subseteq \bigcup_h A_h$. By Remark 2.2 we may assume that each A_h is a Borel set. Fix $h \in \mathbf{N}$ and consider the measures $\mu_h = \mu^{A_h}$ and $\nu_h = \nu^{A_h}$ (Definition 2.4). By Theorem 2.6 we obtain

$$\mu_h(U) = \mu(U \cap A_h) = \nu(U \cap A_h) = \nu_h(U)$$

for every open set $U \subseteq \Omega$. Since μ_h and ν_h are finite Borel measures, it follows that $\mu_h = \nu_h$ for every $h \in \mathbf{N}$, hence

$$\begin{aligned}\mu(E) &= \sup_h \ \mu(E \cap A_h) = \sup_h \ \mu_h(E) \\ &= \sup_h \ \nu_h(E) = \sup_h \ \nu(E \cap A_h) = \nu(E)\end{aligned}$$

(recall that each A_h is a Borel set, therefore the above equalities hold even if E is not a Borel set). □

THEOREM 3.17. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\mu^*(E) = \begin{cases} \mu(E) & \textit{if } \mathrm{cap}(E \cap S(\mu)) = 0, \\ +\infty & \textit{if } \mathrm{cap}(E \cap S(\mu)) > 0, \end{cases}$$

for every $E \subseteq \Omega$.

PROOF. Let E be a subset of Ω. Since μ^* is equivalent to μ (Theorem 3.10), by Proposition 3.16 we have $\mu^*(E) = \mu(E)$ whenever $\mathrm{cap}(E \cap S(\mu)) = 0$. If $\mathrm{cap}(E \cap S(\mu)) > 0$, and A is a quasi open subset of Ω containing E, then $\mathrm{cap}(A \cap S(\mu)) > 0$, hence $\mu(A) = +\infty$ by Remark 3.13. By the definition of μ^* this fact implies that $\mu^*(E) = +\infty$. □

REMARK 3.18. If μ is a measure of the class $\mathcal{M}_0^*(\Omega)$ such that $\mu(E) = 0$ or $\mu(E) = +\infty$ for every $E \subseteq \Omega$, then $\mu = \infty_{S(\mu)}$. In fact, by Theorem 3.17 we have $\mu(E) = +\infty$ if $\mathrm{cap}(E \cap S(\mu)) > 0$. If $\mathrm{cap}(E \cap S(\mu)) = 0$, then

$\mu(E) \leq \sum_h \mu(E \cap A_h)$ where (A_h) is the sequence of finely open subsets of Ω considered in Remark 3.13. Since $\mu(E \cap A_h) < +\infty$, by hypothesis we have $\mu(E \cap A_h) = 0$, hence $\mu(E) = 0$.

Since $S(\mu)$ is finely closed in Ω (Remark 3.13), we can conclude that μ is a measure of the class $\mathcal{M}_0^*(\Omega)$ which takes only the values 0 and $+\infty$ if and only if $\mu = \infty_F$ with F finely closed in Ω (see Remark 3.3).

4. - Properties of μ obtained from its μ-Capacity

In this section we prove an explicit formula which enables us to reconstruct a measure $\mu \in \mathcal{M}_0(\Omega)$ from the corresponding μ-capacity. Then we use this formula to prove that two measures of the class $\mathcal{M}_0(\Omega)$ are equivalent if and only if their μ-capacities agree on all open subsets of Ω.

We begin with two lemmas from measure theory.

LEMMA 4.1. *Let $\alpha : \mathcal{B}(\Omega) \to [0, +\infty]$ be a set function such that $\alpha(\emptyset) = 0$ and let λ be the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to α. Then for every $B \in \mathcal{B}(\Omega)$ we have*

$$\lambda(B) = \sup \sum_{i \in I} \alpha(B_i), \tag{4.1}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B.

If, in addition, α is increasing and countably subadditive, then λ is a Borel measure.

PROOF. It is easy to check that the set function λ defined by (4.1) is superadditive and greater than α. To prove that λ is minimal, let β be a superadditive set function on $\mathcal{B}(\Omega)$ such that $\beta \geq \alpha$. Then

$$\beta(B) \geq \sum_{i \in I} \beta(B_i) \geq \sum_{i \in I} \alpha(B_i)$$

for every $B \in \mathcal{B}(\Omega)$ and for every finite Borel partition $(B_i)_{i \in I}$ of B, hence $\beta \geq \lambda$.

Suppose now that α is increasing and countably subadditive. Let us prove that λ is a Borel measure. Since λ is superadditive and $\lambda(\emptyset) = 0$, it is enough to show that λ is countably subadditive. Let (B_h) be a sequence in $\mathcal{B}(\Omega)$ and let $B = \bigcup_h B_h$. By (4.1) for every $t < \lambda(B)$ there exists a finite Borel partition $(B^i)_{i \in I}$ of B such that

$$t < \sum_{i \in I} \alpha(B^i). \tag{4.2}$$

Since α is countably subadditive, we have

$$\sum_{i\in I} \alpha(B^i) \leq \sum_{i\in I} \sum_h \alpha(B^i \cap B_h) = \sum_h \sum_{i\in I} \alpha(B^i \cap B_h). \tag{4.3}$$

Noting that $(B^i \cap B_h)_{i\in I}$ is a finite Borel partition of B_h, we obtain

$$\sum_h \sum_{i\in I} \alpha(B^i \cap B_h) \leq \sum_h \lambda(B_h). \tag{4.4}$$

Since $t < \lambda(B)$ is arbitrary, from (4.2), (4.3), and (4.4) it follows that

$$\lambda(B) \leq \sum_h \lambda(B_h),$$

hence λ is countably subadditive on $\mathcal{B}(\Omega)$. □

LEMMA 4.2. *Let α and λ be as in Lemma 4.1. Suppose that α is increasing and*

$$\alpha(B) = \sup \ \{\alpha(K) : \ K \text{ compact}, \ K \subseteq B\} \tag{4.5}$$

for every $B \in \mathcal{B}(\Omega)$. Then

$$\lambda(B) = \sup \ \{\lambda(K) : \ K \text{ compact}, \ K \subseteq B\} \tag{4.6}$$

for every $B \in \mathcal{B}(\Omega)$. If, in addition, α is countably subadditive, then

$$\lambda(B) = \lim_{h\to\infty} \sum_{i\in\mathbf{Z}^n} \alpha(B \cap Q_h^i) \tag{4.7}$$

for every $B \in \mathcal{B}(\Omega)$, where Q_h^i denotes the cube

$$Q_h^i =]\, i_1 2^{-h}, \ (i_1 + 1)\ 2^{-h}\ ] \times ... \times]\, i_n 2^{-h}, \ (i_n + 1)\ 2^{-h}]$$

for every $h \in \mathbf{N}$ and for every $i = (i_1, ..., i_n) \ \in \ \mathbf{Z}^n$.

PROOF. To prove (4.6), fix $B \in \mathcal{B}(\Omega)$ and denote by S the right hand side of (4.6). By monotonicity it is enough to prove that $\lambda(B) \leq S$. By (4.1) for every $t < \lambda(B)$ there exists a finite Borel partition $(B_i)_{i\in I}$ of B such that

$$t < \sum_{i\in I} \alpha(B_i).$$

By (4.5) there exists a family $(K_i)_{i\in I}$ of compact sets such that $K_i \subseteq B_i$ for every $i \in I$ and

$$t < \sum_{i\in I} \alpha(K_i).$$

Let $K = \bigcup_{i \in I} K_i$. Then K is compact, $K \subseteq B$, and

$$t < \sum_{i \in I} \alpha(K_i) \leq \lambda(K) \leq S$$

by (4.1). Since $t < S$ for every $t < \lambda(B)$, we obtain $\lambda(B) \leq S$, which concludes the proof of (4.6).

Suppose now that α is increasing and countably subadditive. To prove (4.7), for every $B \in \mathcal{B}(\Omega)$ we set

$$\beta(B) = \sup_h \sum_{i \in \mathbf{Z}^n} \alpha(B \cap Q_h^i). \tag{4.8}$$

Note that for every $h \in \mathbf{N}$

$$\sum_{i \in \mathbf{Z}^n} \alpha(B \cap Q_h^i) \leq \sum_{i \in \mathbf{Z}^n} \alpha(B \cap Q_{h+1}^i) \tag{4.9}$$

by the subadditivity of α, thus the supremum in (4.8) is a limit.

The set function β is clearly increasing. Let us prove that

$$\beta(B) = \sup\{\beta(K) : \ K \text{ compact}, \ K \subseteq B\} \tag{4.10}$$

for every $B \in \mathcal{B}(\Omega)$. Fix $B \in \mathcal{B}(\Omega)$ and denote by M the right hand side of (4.10). By monotonicity it is enough to prove that $\beta(B) \leq M$. By (4.8) for every $s < \beta(B)$ there exist $h \in \mathbf{N}$ and a finite set $I \subseteq \mathbf{Z}^n$ such that

$$s < \sum_{i \in I} \alpha(B \cap Q_h^i)$$

By (4.5) there exists a family $(K_i)_{i \in I}$ of compact sets such that $K_i \subseteq B \cap Q_h^i$ and

$$s < \sum_{i \in I} \alpha(K_i)$$

Let $K = \bigcup_{i \in I} K_i$. Then K is compact, $K \subseteq B$, and $K_i = K \cap Q_h^i$, therefore

$$s < \sum_{i \in I} \alpha(K \cap Q_h^i) \leq \beta(K) \leq M.$$

Since $s < M$ for every $s < \beta(B)$, we obtain $\beta(B) \leq M$, which concludes the proof of (4.10).

Let us prove that β is superadditive. Let $B_1, \ B_2 \in \mathcal{B}(\Omega)$ with $B_1 \cap B_2 = \emptyset$, and let $t_1 < \beta(B_1), \ t_2 < \beta(B_2)$. By (4.10) there exist two compact sets $K_1, \ K_2$

such that $K_1 \subseteq B_1$, $K_2 \subseteq B_2$, and $t_1+t_2 < \beta(K_1)+\beta(K_2)$. By (4.8) and (4.9) there exist $h \in \mathbf{N}$ such that

$$t_1 + t_2 < \sum_{i \in \mathbf{Z}^n} \alpha(K_1 \cap Q_h^i) + \sum_{i \in \mathbf{Z}^n} \alpha(K_2 \cap Q_h^1).$$

By (4.9) we may assume that h is large enough, so that $K_1 \cap Q_h^1 \neq \emptyset$ implies $K_2 \cap Q_h^2 = \emptyset$. Since $\alpha(\emptyset) = 0$, we obtain

$$t_1 + t_2 < \sum_{i \in \mathbf{Z}^n} \alpha((K_1 \cup K_2) \cap Q_h^i) \leq \beta(K_1 \cup K_2) \leq \beta(B_1 \cup B_2),$$

hence $\beta(B_1) + \beta(B_2) \leq \beta(B_1 \cup B_2)$, which proves the superadditivity of β.

Since β is superadditive and $\beta \geq \alpha$, by the minimality of λ we have $\beta \geq \lambda$. To conclude the proof of the lemma it remains to show that $\beta \leq \lambda$. Let $B \in \mathcal{B}(\Omega)$. By (4.8) for every $s < \beta(B)$ there exist $h \in \mathbf{N}$ and a finite set $I \subseteq \mathbf{Z}^n$ such that

$$s < \sum_{i \in I} \alpha(B \cap Q_h^i).$$

Therefore (4.1) yields $s < \lambda(B \cap \bigcup_{i \in I} Q_h^i) \leq \lambda(B)$, hence $\beta(B) \leq \lambda(B)$.

□

THEOREM 4.3. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then for every $B \in \mathcal{B}(\Omega)$ we have*

$$\mu(B) = \sup \sum_{i \in I} \mathrm{cap}_\mu(B_i), \tag{4.11}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B.

PROOF. By Theorem 2.9 the set function cap_μ is countably subadditive on $\mathcal{B}(\Omega)$ and satisfies $\mathrm{cap}_\mu(\emptyset) = 0$. Let λ be the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to cap_μ. By Lemma 4.1 λ is a Borel measure and, for every $B \in \mathcal{B}(\Omega)$, $\lambda(B)$ equals the right hand side of (4.11). Therefore we have to prove that $\lambda = \mu$. Since $\mu \geq \mathrm{cap}_\mu$ by Theorem 2.9(g), the minimality of λ implies that $\mu \geq \lambda$, hence $\lambda \in \mathcal{M}_0(\Omega)$.

To prove that $\mu \leq \lambda$ we fix $B \in \mathcal{B}(\Omega)$. If $\lambda(B) = +\infty$, then the inequality $\mu(B) \leq \lambda(B)$ is trivial. If $\lambda(B) < +\infty$, we consider the measures μ^B and λ^B introduced in Definition 2.4. Note that λ^B is finite on Ω and that $\mathrm{cap}_{\mu^B}(E) = \mathrm{cap}_\mu(E \cap B)$ for every $E \in \mathcal{B}(\Omega)$. Therefore, adopting the convention $0/0 = 1$, we have

$$\frac{\mathrm{cap}_{\mu^B}(B_r(x))}{\lambda^B(B_r(x))} = \frac{\mathrm{cap}_\mu(B_r(x) \cap B)}{\lambda(B_r(x) \cap B)} \leq 1$$

for every ball $B_r(x)$ contained in Ω. Since λ^B is finite on Ω, from the derivation theorem for the μ-capacities ([5], Theorem 2.3) we deduce that $\mu^B \leq \lambda^B$, hence $\mu(B) = \mu^B(B) \leq \lambda^B(B) = \lambda(B)$, which implies $\mu \leq \lambda$. □

If $\mu \in \mathcal{M}_0^*(\Omega)$ and we take $\alpha = \mathrm{cap}_\mu$ in Lemma 4.2, then $\lambda = \mu$ by Theorem 4.3 and α satisfies (4.5) by Theorem 3.7. Therefore the next two theorems follow immediately from Lemma 4.2.

The first theorem concerns the inner regularity of every measure μ of the class $\mathcal{M}_0^*(\Omega)$.

THEOREM 4.4. *Let* $\mu \in \mathcal{M}_0^*(\Omega)$. *Then*

$$\mu(B) = \sup \quad \{\mu(K) : \ K \text{ compact}, K \subseteq B\}$$

for every $B \in \mathcal{B}(\Omega)$.

The second theorem provides an easy way to obtain μ from cap_μ when $\mu \in \mathcal{M}_0^*(\Omega)$.

THEOREM 4.5. *Let* $\mu \in \mathcal{M}_0^*(\Omega)$. *Then for every* $B \in \mathcal{B}(\Omega)$ *we have*

$$\mu(B) = \lim_{h \to \infty} \sum_{i \in \mathbf{Z}^n} \mathrm{cap}_\mu(B \cap Q_h^i),$$

where Q_h^i *denotes the cube*

$$Q_h^i =]\ i_1 2^{-h},\ (i_1+1)\ 2^{-h}] \times ... \times]\ i_n 2^{-h},\ (i_n+1)\ 2^{-h}]$$

for every $h \in \mathbf{N}$ *and for every* $i = (i_1, ..., i_n) \ \in \mathbf{Z}^n$.

The following theorem is the converse of Theorem 3.6.

THEOREM 4.6. *If* $\mu \in \mathcal{M}_0(\Omega)$ *and* cap_μ *is a Choquet capacity on* Ω, *then* $\mu \in \mathcal{M}_0^*(\Omega)$.

PROOF. Let $\mu \in \mathcal{M}_0(\Omega)$ and suppose that cap_μ is a Choquet capacity on Ω. By property (c') after Definition 1.1 and by Proposition 3.11 we have

$$\mathrm{cap}_\mu(K) = \inf \quad \{\mathrm{cap}_\mu(U) : \ U \text{ open}, \ K \subseteq U\} = \mathrm{cap}_{\mu^*}(K)$$

for every compact set $K \subseteq \Omega$. By Theorem 3.7 and by the Choquet capacitability theorem (Theorem 1.2) we obtain

$$\mathrm{cap}_\mu(B) = \mathrm{cap}_{\mu^*}(B)$$

for every $B \in \mathcal{B}(\Omega)$, hence $\mu = \mu^*$ by Theorem 4.3. Since $\mu^* \in \mathcal{M}_0^*(\Omega)$ (Theorem 3.9), we conclude that $\mu \in \mathcal{M}_0^*(\Omega)$. □

In the rest of this section we shall prove that two measures of the class $\mathcal{M}_0(\Omega)$ are equivalent if and only if the corresponding μ-capacities agree on a family of subsets of Ω which satisfies one of the conditions considered in the following definition.

DEFINITION 4.7. Let $\mathcal{E}$ be a family of subsets of Ω. We say that $\mathcal{E}$ is *dense* (resp. *finely dense*) in $\mathcal{P}(\Omega)$ if for every pair (K, U), with K compact (resp. quasi compact in Ω), U open (resp. quasi open in Ω), and $K \subseteq U \subseteq \Omega$, there exists $E \in \mathcal{E}$ such that $K \subseteq E \subseteq U$. We say that $\mathcal{E}$ is *rich* (resp. *finely rich*) in $\mathcal{P}(\Omega)$ if, for every chain (resp. fine chain) $(E_t)_{t\in T}$ in $\mathcal{P}(\Omega)$, the set $\{t \in T : \ E_t \notin \mathcal{E}\}$ is at most countable. By a *chain* (resp. *fine chain*) in $\mathcal{P}(\Omega)$ we mean a family $(E_t)_{t\in T}$ of subsets of Ω, such that T is a non-empty open interval of $\mathbb{R}$, $\overline{E}_t$ is compact (resp. $\mathrm{cl}_f E_t$ is quasi compact) in Ω for every $t \in T$, and $\overline{E}_s \subseteq \overset{o}{E}_t$ (resp. $\mathrm{cap}(\mathrm{cf}_f E_s - \mathrm{int}_f E_t) = 0$) for every $s,\ t \in T$ with $s < t$.

It is easy to check that any countable intersection of rich (resp. finely rich) families is rich (resp. finely rich).

PROPOSITION 4.8. *Every rich (resp. finely rich) family is dense (resp. finely dense).*

PROOF. Let $\mathcal{E}$ be a finely rich family in $\mathcal{P}(\Omega)$. Let F and A be two subsets of Ω with F quasi compact in Ω, A quasi open in Ω, and $F \subseteq A$. By Proposition 1.7 there exists a quasi continuous function $f:\ \Omega\ \to [0,1]$ such that $f(x) = 0$ for every $x \in F$, $f(x) = 1$ for every $x \in \Omega - A$, and the sets

$$E_t = \{x \in \Omega : \ f(x) \leq t\}$$

are quasi compact in Ω for every $t \in \,]0,1[$. Let $T = \,]0,1[$. By Proposition 1.5 the family $(E_t)_{t\in T}$ is a fine chain, and $F \subseteq E_t \subseteq A$ for every $t \in T$. Since $\mathcal{E}$ is finely rich, there exists $t \in T$ such that $E_t \in \mathcal{E}$. This proves that there exists $E \in \mathcal{E}$ with $F \subseteq E \subseteq A$, therefore we can conclude that $\mathcal{E}$ is finely dense.

The proof for rich families is similar. □

THEOREM 4.9. *Let μ and ν be two measures of the class $\mathcal{M}_0(\Omega)$. The following conditions are equivalent:*

(a) *μ and ν are equivalent;*

(b) cap_μ *and* cap_ν *agree on all open subsets of Ω;*

(c) cap_μ *and* cap_ν *agree on a dense family in $\mathcal{P}(\Omega)$;*

(d) cap_μ *and* cap_ν *agree on a rich family in $\mathcal{P}(\Omega)$;*

(e) cap_μ *and* cap_ν *agree on all quasi open subsets of Ω;*

(f) cap_μ *and* cap_ν *agree on a finely dense family in $\mathcal{P}(\Omega)$;*

(g) cap_μ *and* cap_ν *agree on a finely rich family in $\mathcal{P}(\Omega)$;*

(h) cap_μ *and* cap_ν *agree on the finely rich family $\mathcal{E}_1(\mu)$ of all subsets E of Ω such that* $\mathrm{cap}_{\mu^*}(\mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(E)$.

PROOF. (a) $\Rightarrow$ (b). It follows from (2.3).

(b) $\Rightarrow$ (e). See Theorem 2.9(j).

(e) $\Rightarrow$ (h). The family $\mathcal{E}_1(\mu)$ is rich because it contains the family $\mathcal{E}(\alpha)$ considered in Lemma 4.10 proved below, for $\alpha = \mathrm{cap}_{\mu^*}$. By (e) and by Proposition 3.11 we have $\mathrm{cap}_{\mu^*} = \mathrm{cap}_{\nu^*}$ on $\mathcal{P}(\Omega)$ and $\mathrm{cap}_{\mu^*} = \mathrm{cap}_\mu = \mathrm{cap}_\nu = \mathrm{cap}_{\nu^*}$ on all finely open subsets of Ω. Therefore, if $E \in \mathcal{E}_1(\mu)$, then $\mathrm{cap}_{\nu^*}(E) = \mathrm{cap}_{\nu^*}(\mathrm{int}_f E)$. Since $\mathrm{cap}_{\mu^*}(\mathrm{inf}_f E) = \mathrm{cap}_\mu(\mathrm{int}_f E) \le \mathrm{cap}_\mu(E) \le \mathrm{cap}_{\mu^*}(E)$, we have $\mathrm{cap}_\mu(E) = \mathrm{cap}_{\mu^*}(E)$ for every $E \in \mathcal{E}_1(\mu)$. In the same way we obtain $\mathrm{cap}_\nu(E) = \mathrm{cap}_{\nu^*}(E)$. Since $\mathrm{cap}_{\mu^*} = \mathrm{cap}_{\nu^*}$, we conclude that $\mathrm{cap}_\mu(E) = \mathrm{cap}_\nu(E)$ for every $E \in \mathcal{E}_1(\mu)$.

(h) $\Rightarrow$ (g). Obvious.
(g) $\Rightarrow$ (f). Every finely rich family is finely dense (Proposition 4.8).
(f) $\Rightarrow$ (c). Every finely dense family is dense.
(g) $\Rightarrow$ (d). Every finely rich family is rich.
(d) $\Rightarrow$ (c). Every rich family is dense (Proposition 4.8).
(c) $\Rightarrow$ (b). It follows easily from Theorem 2.9(i).
(b) $\Rightarrow$ (a). By Proposition 3.11 we have $\mathrm{cap}_{\mu^*} = \mathrm{cap}_{\nu^*}$ on $\mathcal{P}(\Omega)$, therefore $\mu^* = \nu^*$ on $\mathcal{B}(\Omega)$ by Theorem 4.3. Since μ and ν are equivalent to μ^* and ν^* respectively (Theorem 3.10), we conclude that μ is equivalent to ν.

□

LEMMA 4.10. *Let $\alpha : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ be an increasing function such that $\alpha(E_1) = \alpha(E_2)$ whenever $\mathrm{cap}(E_1 \triangle E_2) = 0$. Let $\mathcal{E}(\alpha)$ be the family of all subsets E* of *Ω such that $\mathrm{cl}_f E$ is quasi compact in Ω and $\alpha(\mathrm{int}_f E) = \alpha(\mathrm{cl}_f E)$. Then $\mathcal{E}(\alpha)$ is finely rich in $\mathcal{P}(\Omega)$.*

PROOF. Let $(E_t)_{t \in T}$ be a fine chain in $\mathcal{P}(\Omega)$, and let $f : T \to \overline{\mathbb{R}}$ be the function defined by $f(t) = \alpha(E_t)$. Then f is increasing and

$$\lim_{s \to t-} f(s) \le \alpha(\mathrm{int}_f E_t) \le \alpha(\mathrm{cl}_f E_t) \le \lim_{s \to t+} f(s)$$

for every $t \in T$, therefore $E_t \in \mathcal{E}(\alpha)$ for every $t \in T$ where f is continuous. This implies that the set $\{t \in T : E_t \notin \mathcal{E}(\alpha)\}$ is at most countable, hence $\mathcal{E}(\alpha)$ is finely rich in $\mathcal{P}(\Omega)$. □

REMARK 4.11. If F_1 and F_2 are quasi closed in Ω, and $\mathrm{cap}(U \cap F_1) = \mathrm{cap}(U \cap F_2)$ for every open set $U \subseteq \Omega$, then ∞_{F_1} and ∞_{F_2} are equivalent by Theorem 4.9, hence $\mathrm{cap}(F_1 \triangle F_2) = 0$ by Remark 2.7. This is a well known result with an easy direct proof (see, for intance, [19], Lemma 2.6).

5. - γ-Convergence

In this section we prove that the γ-convergence of a sequence of measures in $\mathcal{M}_0(\Omega)$ implies the convergence of the corresponding μ-capacities in a finely rich subfamily of $\mathcal{P}(\Omega)$.

The γ-convergence is a variational convergence for sequences (μ_h) in $\mathcal{M}_0(\Omega)$ which is defined in terms of the Γ-convergence of the corresponding functionals

$$\Phi_\Omega(u) + \int_\Omega u^2 d\mu_h.$$

We refer to [11] and [3] for the motivation and the main properties of the γ-convergence and for the applications of this notion of convergence to the study of the asymptotic behaviour of Dirichlet problems in domains with many small holes.

For the more general definition of Γ-convergence (also called epi-convergence) and for its applications to the study of perturbation problems in calculus of variations, we refer to [15], [14], [1], and the bibliography therein.

In this paper we need only to recall the definition of γ- convergence and the compactness property proved in [11], Theorem 4.14.

DEFINITION 5.1. Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that (μ_h) γ-converges to μ if the following conditions are satisfied:

(a) for every $u \in H_0^1(\Omega)$ and for every sequence (u_h) in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$ we have

$$\Phi_\Omega(u) + \int_\Omega u^2 d\mu \le \liminf_{h\to\infty} [\Phi_\Omega(u_h) + \int_\Omega u_h^2 d\mu_h];$$

(b) for every $u \in H_0^1(\Omega)$ there exists a sequence (u_h) in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$ such that

$$\Phi_\Omega(u) + \int_\Omega u^2 d\mu \ge \limsup_{h\to\infty} [\Phi_\Omega(u_h) + \int_\Omega u_h^2 d\mu_h].$$

REMARK 5.2. If properties (a) and (b) hold on Ω, then they also hold for every open set $\Omega' \subseteq \Omega$. Conversely, if (a) and (b) hold for every open set $\Omega' \subset\subset \Omega$, then they hold on Ω (the non-trivial proof of this facts can be found in [3], Proposition 2.8). Therefore, if L is the Laplace operator $-\Delta$ and $\Omega = \mathbb{R}^n$, our definition of γ-convergence is equivalent to Definition 4.8 of [11], and for an arbitrary L our notion of γ-convergence coincides with the γ^L-convergence introduced in [5], Definition 5.1.

REMARK 5.3. The definition of γ-convergence depends, of course, on the operator L which enters in the definition of Φ_Ω. However, it is independent of the choice of μ_h and μ in their equivalence classes in $\mathcal{M}_0(\Omega)$. Therefore, (μ_h) γ-converges to μ if and only if (μ_h^*) γ-converges to μ^*.

REMARK 5.4. The γ-convergence on $\mathcal{M}_0(\Omega)$ (more precisely, on the quotient of $\mathcal{M}_0(\Omega)$ under the equivalence relation of Definition 2.5) is metrizable ([11], Proposition 4.9) and $\mathcal{M}_0(\Omega)$ is compact under γ ([11], Theorem 4.14). Since $\mathcal{M}_0^*(\Omega)$ contains one (and only one) representative for each equivalence class in $\mathcal{M}_0(\Omega)$, it follows that $\mathcal{M}_0^*(\Omega)$ is metrizable and compact with respect to γ-convergence.

The γ-convergence of a sequence (μ_h) implies the convergence of the sequence of the corresponding capacities (cap_{μ_h}) on a rich family of subsets of $\mathcal{P}(\Omega)$. When Ω is bounded, this can be obtained as a conseguence of Theorem 5.11 of [11], which relies on more general results about Γ-convergence and obstacle problems proven in [8].

We prefer to give here a direct proof of this fact which relies on the following lemmas.

LEMMA 5.5. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu \in \mathcal{M}_0(\Omega)$. Let U and V be two open sets such that $U \subseteq V \subseteq \Omega$. Then*

$$\Phi_V(u) + \int_U u^2 d\mu \ \leq \ \liminf_{h\to\infty} \, [\Phi_V(u_h) + \int_U u_h^2 d\mu_h] \tag{5.1}$$

for every $u \in H^1(V)$ and for every sequence (u_h) in $H^1(V)$ converging to u weakly in $L^2(V)$.

PROOF. We assume that Ω is bounded, the proof in the unbounded case being analogous. Let $u \in H^1(V)$ and let (u_h) be a sequence in $H^1(V)$ which converges to u weakly in $L^2(V)$. We may assume that the right hand side of (5.1) is finite and that the lower limit is a limit, so that the sequence (u_h) converges to u weakly in $H^1(V)$ by the coerciveness of the quadratic form Φ_V. Let K be a compact subset of U and let $\varphi \in C_0^\infty(U)$ with $0 \leq \varphi \leq 1$ on U and $\varphi = 1$ in a neighbourhood of K. Then the sequence (φu_h) is in $H_0^1(\Omega)$ and converges to φu strongly in $L^2(\Omega)$, so by condition (a) of Definition 5.1 we have

$$\Phi_\Omega(\varphi u) + \int_\Omega (\varphi u)^2 \, d\mu \leq \liminf_{h\to\infty} \, [\Phi_\Omega(\varphi u_h) + \int_\Omega (\varphi u_h)^2 d\mu_h],$$

hence

$$\int_U [\sum_{i,j=1}^n a_{ij} D_j\varphi D_i\varphi] u^2 dx + 2\int_U [\sum_{i,j=1}^n a_{ij} D_j\varphi D_i u]\varphi u dx$$
$$+ \int_U [\sum_{i,j=1}^n a_{ij} D_j u D_i u]\varphi^2 dx + \int_U \varphi^2 u^2 d\mu$$
$$\leq \liminf_{h\to\infty} \{\int_U [\sum_{i,j=1}^n a_{ij} D_j\varphi D_i\varphi] u_h^2 dx + 2\int_U [\sum_{i,j=1}^n a_{ij} D_j\varphi D_i u_h]\varphi u_h dx$$
$$+ \int_U [\sum_{i,j=1}^n a_{ij} D_j u_h D_i u_h]\varphi^2 dx + \int_U \varphi^2 u_h^2 d\mu_h\}.$$

Since (u_h) converges to u weakly in $H^1(U)$, the first and the second term on the left hand side are equal to the limits of the corresponding terms on the right hand side, hence

$$\int_U [\sum_{i,j=1}^n a_{ij} D_j u D_i u]\varphi^2 dx + \int_K u^2 d\mu$$
$$\leq \liminf_{h\to\infty} \{\int_U [\sum_{i,j=1}^n a_{ij} D_j u_h D_i u_h]\varphi^2 dx + \int_U u_h^2 d\mu_h\}.$$

By lower semicontinuity we also have

$$\int_V [\sum_{i,j=1}^n a_{ij} D_j u D_i u](1-\varphi^2) dx$$
$$\leq \liminf_{h\to\infty} \int_V [\sum_{i,j=1}^n a_{ij} D_j u_h D_i u_h](1-\varphi^2) dx.$$

By adding the last two inequalities and by taking the limit as $K \uparrow U$ we obtain (5.1). □

LEMMA 5.6. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu \in \mathcal{M}_0(\Omega)$. Let K be a compact set and let U and V be two open sets such that $K \subseteq U \subseteq V \subseteq \Omega$. Then for every $u \in H^1(V)$ there exists a sequence (u_h) in $H^1(V)$ such that $u_h - u \in H_0^1(V)$ for every h,*

$$\Phi_V(u) + \int_U u^2 d\mu \geq \limsup_{h\to\infty} [\Phi_V(u_h) + \int_K u_h^2 d\mu_h], \tag{5.2}$$

and (u_h) converges to u strongly in $L^2(V)$.

PROOF. We assume that Ω is bounded, the proof in the unbounded case being analogous. Let $u \in H^1(V)$. To prove (5.2) we may assume that $u \in L^2(U,\mu)$. By a diagonal argument it is enough to show that for every $\varepsilon > 0$ there exists a sequence (u_h) in $H^1(V)$, with $u_h - u \in H^1_0(V)$ for every h, such that (u_h) converges to u strongly in $L^2(V)$ and

$$\Phi_V(u) + \int_U u^2 d\mu + \varepsilon \geq \limsup_{h\to\infty} [\Phi_V(u_h) + \int_K u_h^2 d\mu_h].$$

Given $\varepsilon > 0$, let W be an open set such that $K \subseteq W \subseteq \overline{W} \subseteq U$ and $\Phi_{\overline{W}-K}(u) < \varepsilon$. Let $\varphi \in C_0^\infty(U)$ with $0 \leq \varphi \leq 1$ on U and $\varphi = 1$ in a neighbourhood of $\overline{W}$. Define $v = \varphi u$, so that $v \in H^1_0(\Omega) \cap L^2(\Omega,\mu)$. By condition (b) of Definition 5.1 there exists a sequence (v_h) in $H^1_0(\Omega)$ converging to v in $L^2(\Omega)$ such that

$$\Phi_\Omega(v) + \int_\Omega v^2 d\mu \geq \limsup_{h\to\infty} [\Phi_\Omega(v_h) + \int_\Omega v_h^2 d\mu_h].$$

Then, putting $A = \Omega - \overline{W}$, we have

$$\Phi_{\overline{W}}(u) + \int_{\overline{W}} u^2 d\mu + \Phi_A(v) + \int_A v^2 d\mu$$
$$\geq \limsup_{h\to\infty} [\Phi_W(v_h) + \int_W v_h^2 d\mu_h] + \liminf_{h\to\infty} [\Phi_A(v_h) + \int_A v_h^2 d\mu_h].$$

By Lemma 5.5 we have

$$\Phi_A(v) + \int_A v^2 d\mu \leq \liminf_{h\to\infty} [\Phi_A(v_h) + \int_A v_h^2 d\mu_h],$$

hence, recalling that $v \in L^2(A,\mu)$, we obtain

$$\Phi_{\overline{W}}(u) + \int_{\overline{W}} u^2 d\mu \geq \limsup_{h\to\infty} [\Phi_W(v_h) + \int_W v_h^2 d\mu_h]. \tag{5.3}$$

Let $\psi \in C_0^\infty(W)$ with $0 \leq \psi \leq 1$ on W and $\psi = 1$ in a neighbourhood of K. For every $h \in \mathbb{N}$ define $u_h = \psi v_h + (1-\psi)u$, so that $u_h \in H^1(V)$, $u_h - u \in H^1_0(V)$, $u_h = v_h$ in a neighbourhood of K, and (u_h) converges to u in $L^2(V)$. By convexity, for every $\varepsilon \in]0,1[$ we have

$$\sum_{i,j=1}^n a_{ij} D_j u_h D_i u_h \leq \frac{\psi}{1-\varepsilon} \sum_{i,j=1}^n a_{ij} D_j v_h D_i v_h$$
$$+ \frac{1-\psi}{1-\varepsilon} \sum_{i,j=1}^n a_{ij} D_j u D_i u + \frac{(v_h - u)^2}{\varepsilon} \sum_{i,j=1}^n a_{ij} D_j \psi D_i \psi.$$

Since (v_h) converges to u in $L^2(W)$, from (5.3) we obtain

$$\begin{aligned}
&\limsup_{h\to\infty}\,[\Phi_V(u_h)+\int_K u_h^2 d\mu_h] \\
&\leq \frac{1}{1-\varepsilon}\,\limsup_{h\to\infty}\,[\Phi_W(v_h)+\int_W v_h^2 d\mu_h]+\frac{1}{1-\varepsilon}\Phi_{V-K}(u) \\
&\leq \frac{1}{1-\varepsilon}\,[\Phi_{\overline{W}}(u)+\int_{\overline{W}} u^2 d\mu+\Phi_{V-K}(u)] \\
&\leq \frac{1}{1-\varepsilon}\,[\Phi_V(u)+\int_U u^2 d\mu+\varepsilon].
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the proof of the lemma can be concluded by a diagonal argument. □

PROPOSITION 5.7. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\text{cap}_\mu(U) \leq \liminf_{h\to\infty}\,\text{cap}_{\mu_h}(U) \tag{5.4}$$

and

$$\text{cap}_\mu(U) \geq \limsup_{h\to\infty}\,\text{cap}_{\mu_h}(K) \tag{5.5}$$

for every open set U and for every compact set K with $K \subseteq U \subseteq \Omega$.

PROOF. Let U and K be as required in the proposition. By Theorem 2.9(c) we may assume $U \subset\subset \Omega$. To prove (5.4) we may assume that the right hand side of this inequality is finite and that the lower limit is a limit, so that there exists a bounded sequence (v_h) in $H_0^1(\Omega)$ such that

$$\text{cap}_{\mu_h}(U) = \Phi_\Omega(v_h) + \int_U (v_h-1)^2 d\mu_h.$$

By passing to a subsequence, we may assume that (v_h) converges weakly in $H^1(\Omega)$ to a function $v \in H_0^1(\Omega)$. Therefore the inequalities

$$\begin{aligned}
&\text{cap}_\mu(U) \leq \Phi_\Omega(v)+\int_U (v-1)^2 d\mu \\
&\leq \liminf_{h\to\infty}\,[\Phi_\Omega(v_h)+\int_U (v_h-1)^2 d\mu_h] = \liminf_{h\to\infty}\,\text{cap}_{u_h}(U)
\end{aligned}$$

follow from the definition of $\text{cap}_\mu(U)$ and from Lemma 5.5, applied with $V=\Omega$, $u=v-\varphi$ and $u_h=v_h-\varphi$, where $\varphi \in C_0^\infty(\Omega)$ and $\varphi = 1$ on U.

To prove (5.5), let $w \in H_0^1(\Omega)$ such that

$$\operatorname{cap}_\mu(U) = \Phi_\Omega(w) + \int_U (w-1)^2 d\mu. \tag{5.6}$$

By applying Lemma 5.6 with $V = \Omega$ and $u = w - \varphi$ ($\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ on U), we obtain a sequence (w_h) in $H_0^1(\Omega)$ such that

$$\Phi_\Omega(w) + \int_U (w-1)^2 d\mu \geq \limsup_{h\to\infty} \Big[\Phi_\Omega(w_h) + \int_K (w_h - 1)^2 d\mu_h\Big] \tag{5.7}$$

(it is enough to take $w_h = u_h + \varphi$ in Lemma 5.6). Since

$$\Phi_\Omega(w_h) + \int_K (w_h - 1)^2 d\mu_h \geq \operatorname{cap}_{\mu_h}(K),$$

(5.5) follows from (5.6) and (5.7). □

The previous result can be extended to quasi open and quasi compact sets, as shown in the following theorem.

THEOREM 5.8. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\operatorname{cap}_\mu(A) \leq \liminf_{h\to\infty} \operatorname{cap}_{\mu_h}(A) \tag{5.8}$$

and

$$\operatorname{cap}_\mu(A) \geq \limsup_{h\to\infty} \operatorname{cap}_{\mu_h}(F) \tag{5.9}$$

for every pair of sets A and F, with A quasi open in Ω, F quasi compact in Ω, and $F \subseteq A \subseteq \Omega$.

PROOF. We prove only (5.9), the proof of (5.8) being analogous. Let A and F be as required in the theorem. For every $\varepsilon > 0$ there exist an open set $U \subseteq \Omega$ and a compact set $K \subseteq \Omega$ such that $\operatorname{cap}(U \triangle A) < \varepsilon$ and $\operatorname{cap}(K \triangle F) < \varepsilon$. By (1.1) there exist two open sets V and W contained in Ω such that $U \triangle A \subseteq V$, $K \triangle F \subseteq W$, $\operatorname{cap}(V) < \varepsilon$, and $\operatorname{cap}(W) < \varepsilon$. Then $K - W \subseteq U \cup V$, hence

$$\operatorname{cap}_\mu(U \cup V) \geq \limsup_{h\to\infty} \operatorname{cap}_{\mu_h}(K - W)$$

by Proposition 5.7. By properties (e) and (f) of Theorem 2.9 we have

$$\begin{aligned}\operatorname{cap}_\mu(U \cup V) &\leq \operatorname{cap}_\mu(A) + \operatorname{cap}_\mu(V)\\ &\leq \operatorname{cap}_\mu(A) + k \operatorname{cap}(V) \leq \operatorname{cap}_\mu(A) + k\varepsilon\end{aligned}$$

and

$$\begin{aligned}\operatorname{cap}_{\mu_h}(K-W) &\geq \operatorname{cap}_{\mu_h}(F)-\operatorname{cap}_{\mu_h}(W)\\ &\geq \operatorname{cap}_{\mu_h}(F)-k\operatorname{cap}(W)\geq \operatorname{cap}_{\mu_h}(F)-k\varepsilon,\end{aligned}$$

hence

$$\operatorname{cap}_\mu(A)+k\varepsilon \geq \limsup_{h\to\infty}\ \operatorname{cap}_{\mu_h}(F)-k\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we obtain (5.9). □

The next theorem follows easily from Theorem 5.8 and Proposition 3.11.

THEOREM 5.9. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu\in\mathcal{M}_0(\Omega)$. Then*

$$\operatorname{cap}_{\mu^*}(A)\leq \liminf_{h\to\infty}\ \operatorname{cap}_{\mu_h}(A)$$

for every A quasi open in Ω, and

$$\operatorname{cap}_{\mu^*}(F)\geq \limsup_{h\to\infty}\ \operatorname{cap}_{\mu_h}(F)$$

for every F quasi compact in Ω.

The inequalities of Theorem 5.8 and 5.9 are improved in the following theorem.

THEOREM 5.10. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to $\mu\in\mathcal{M}_0(\Omega)$. Then*

$$\begin{aligned}(5.10)\quad \operatorname{cap}_\mu(A) &= \sup\,\{\liminf_{h\to\infty}\ \operatorname{cap}_{\mu_h}(K):\ K \text{ compact},\ K\subseteq A\}\\ &= \sup\,\{\limsup_{h\to\infty}\ \operatorname{cap}_{\mu_h}(F):\ F \text{ quasi compact},\ F\subseteq A\}\end{aligned}$$

for every A quasi open in Ω, and

$$\begin{aligned}(5.11)\quad \operatorname{cap}_{\mu^*}(F) &= \inf\,\{\liminf_{h\to\infty}\ \operatorname{cap}_{\mu_h}(A):\ A \text{ quasi open},\ F\subseteq A\}\\ &= \inf\,\{\limsup_{h\to\infty}\ \operatorname{cap}_{\mu_h}(U):\ U \text{ open}, F\subseteq U\}\end{aligned}$$

for every F quasi compact in Ω.

PROOF. We prove only (5.10), the proof of (5.11) being analogous. Let A be a quasi open subset of Ω and let

$$S=\sup\,\{\liminf_{h\to\infty}\ \operatorname{cap}_{\mu_h}(K):\ K \text{ compact},\ K\subseteq A\}.$$

By (5.9) it is enuogh to prove that $\operatorname{cap}_\mu(A)\leq S$. By Theorem 2.9(i) for every $\varepsilon>0$ there exists a compact set $K\subseteq A$ such that $\operatorname{cap}_\mu(A)-\varepsilon<\operatorname{cap}_\mu(K)$. Since A is quasi open, there exists an open set $U\subseteq\Omega$ such that $\operatorname{cap}(U\triangle A)<\varepsilon$, and by (1.1) there exists an open set $V\subseteq\Omega$ such that $U\triangle A\subseteq V$ and $\operatorname{cap}(V)<\varepsilon$.

Since $K \subseteq U \cup V$, there exist an open set W and a compact set H such that $K \subseteq W \subseteq H \subseteq U \cup V$. By (5.4) we have

$$\mathrm{cap}_\mu(A) - \varepsilon < \mathrm{cap}_\mu(K) \leq \mathrm{cap}_\mu(W) \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(W).$$

The set $H - V$ is compact and contained in A. By properties (e) and (f) of Theorem 2.9 we have

$$\begin{aligned}\mathrm{cap}_{\mu_h}(W) &\leq \mathrm{cap}_{\mu_h}(H) \leq \mathrm{cap}_{\mu_h}(H-V) + \mathrm{cap}_{\mu_h}(V) \\ &\leq \mathrm{cap}_{\mu_h}(H-V) + k\ \mathrm{cap}(V) \leq \mathrm{cap}_{\mu_h}(H-V) + k\varepsilon,\end{aligned}$$

hence

$$\mathrm{cap}_\mu(A) - (k+1)\varepsilon \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(H-V) \leq S.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\mathrm{cap}_\mu(A) \leq S$. □

We now associate with every measure $\mu \in \mathcal{M}_0(\Omega)$ a finely rich family $\mathcal{E}_2(\mu)$ such that, if (μ_h) γ-converges to μ, then $\mathrm{cap}_{\mu_h}(E)$ converges to $\mathrm{cap}_\mu(E)$ for every $E \in \mathcal{E}_2(\mu)$.

THEOREM 5.11. *For every $\mu \in \mathcal{M}_0(\Omega)$ let $\mathcal{E}_2(\mu)$ be the family of all subsets E of Ω such that* $\mathrm{cl}_f E$ *is quasi compact in Ω and* $\mathrm{cap}_{\mu^*}(\mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(\mathrm{cl}_f E)$. *Then $\mathcal{E}_2(\mu)$ is finely rich in $\mathcal{P}(\Omega)$ and*

$$\mathrm{cap}_\mu(E) = \lim_{h\to\infty} \mathrm{cap}_{\mu_h}(E)$$

for every $E \in \mathcal{E}_2(\mu)$ and for every sequence (μ_h) which γ-converges to μ in $\mathcal{M}_0(\Omega)$.

PROOF. For every $\mu \in \mathcal{M}_0(\Omega)$ the family $\mathcal{E}_2(\mu)$ is finely rich by Lemma 4.10. By Proposition 3.11 we have $\mathrm{cap}_{\mu^*}(\mathrm{int}_f E) = \mathrm{cap}_\mu(\mathrm{int}_f E) \leq \mathrm{cap}_\mu(E) \leq \mathrm{cap}_\mu(\mathrm{cl}_f E) \leq \mathrm{cap}_{\mu^*}(\mathrm{cl}_f E)$ for every $E \subseteq \Omega$, hence $\mathrm{cap}_\mu(E) = \mathrm{cap}_{\mu^*}(\mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(\mathrm{cl}_f E)$ if $E \in \mathcal{E}_2(\mu)$. The conclusion follows now from Theorem 5.9. □

We conclude this section by proving the following fine localization theorem, which was obtained by probabilistic methods in [3], Lemma 5.1.

THEOREM 5.12. *Let (μ_h) and (ν_h) be two sequences in $\mathcal{M}_0(\Omega)$ which γ-converge to μ and ν respectively, and let A be a quasi open subset of Ω. If μ_h and ν_h agree on all quasi open subsets of A for every $h \in \mathbf{N}$, then μ and ν agree on all quasi open subsets of A.*

PROOF. The hypothesis of the theorem implies that μ_h^* and ν_h^* agree on all subsets of A (Definition 3.8). Since (μ_h^*) γ-converges to μ^* and (ν_h^*) γ-converges to ν^* (Remark 5.3), Theorem 5.10 implies that cap_{μ^*} and cap_{ν^*} agree on all finely open subsets of A, therefore cap_{μ^*} and cap_{ν^*} agree on all subsets of A by (3.3) (Theorem 3.5). By Theorem 4.3 μ^* and ν^* agree on all

Borel subsets of A, thus the conclusion follows from the fact that μ^* coincides with μ and ν^* with ν on all quasi open subsets of Ω (Definition 3.8). □

6. - γ-Convergence and Convergence of μ-Capacities

In this section we prove that a sequence of measures of the class $\mathcal{M}_0(\Omega)$ γ-converges if and only if the sequence of the corresponding μ-capacities converges on a dense subfamily of $\mathcal{P}(\Omega)$. We use this result to associate with every $\mu \in \mathcal{M}_0(\Omega)$ a finely rich family $\mathcal{E}_3(\mu)$ such that, if (μ_h) γ-converges to μ, then the sequence of the restrictions (μ_h^E) (Definition 2.4) γ-converges to the restriction μ^E for every $E \in \mathcal{E}_3(\mu)$.

THEOREM 6.1. *Let (μ_h) be a sequence of measures of the class $\mathcal{M}_0(\Omega)$. Define for every $E \subseteq \Omega$*

$$\alpha'(E) = \liminf_{h\to\infty} \operatorname{cap}_{\mu_h}(E), \quad \alpha''(E) = \limsup_{h\to\infty} \operatorname{cap}_{\mu_h}(E).$$

Suppose that for every open set $U \subseteq \Omega$

$$\begin{aligned} &\sup\{\alpha'(K):\ K \text{ compact},\ K \subseteq U\} \\ &= \sup\{\alpha''(K):\ K \text{ compact}, K \subseteq U\}. \end{aligned} \tag{6.1}$$

For every open set $U \subseteq \Omega$ define $\alpha(U)$ as the common value of both sides of (6.1), and extend the definition to arbitrary sets $E \subseteq \Omega$ by

$$\alpha(E) = \inf\{\alpha(U):\ U \text{ open},\ E \subseteq U\}. \tag{6.2}$$

Let λ be the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to α, so that for every $B \in \mathcal{B}(\Omega)$

$$\lambda(B) = \sup \sum_{i\in I} \alpha(B_i), \tag{6.3}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

Then λ is a measure of the class $\mathcal{M}_0^*(\Omega)$, the sequence (μ_h) γ-converges to λ, and $\alpha(E) = \operatorname{cap}_\lambda(E)$ for every $E \subseteq \Omega$.

PROOF. First we note that (6.3) follows from Lemma 4.1. Since the γ-convergence on $\mathcal{M}_0(\Omega)$ is metrizable and compact (Remark 5.4), and α does not change if we pass to a subsequence of (μ_h), we may assume that (μ_h) γ-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$, and we have only to prove that $\lambda = \mu^*$ and $\alpha = \operatorname{cap}_{\mu^*}$. By (6.1) and by Theorem 5.10 we have $\alpha(U) = \operatorname{cap}_\mu(U)$ for every open set $U \subseteq \Omega$, hence $\alpha = \operatorname{cap}_{\mu^*}$ on $\mathcal{P}(\Omega)$ by (6.2) and by Proposition 3.11. Therefore we conclude that $\lambda = \mu^*$ by (6.3) and by Theorem 4.3. □

REMARK 6.2. If the measure λ defined by (6.3) is finite on all compact subsets of Ω, then Theorem 6.1 can be obtained also from a derivation argument (see [5], Theorem 5.2, applied with $\nu = \lambda$). If, in addition, λ has (locally) a bounded potential, and each μ_h has the form $\mu_h = \infty_{E_h}$ for a suitable closed set $E_h \subseteq \Omega$, then the same result was obtained in [2] by probabilistic methods.

THEOREM 6.3. *Let* (μ_h) *be a sequence in* $\mathcal{M}_0(\Omega)$ *and let* $\mu \in \mathcal{M}_0(\Omega)$. *Then the following conditions are equivalent:*

(a) (μ_h) γ*-converges to* μ*;*

(b) the inequalities

$$\mathrm{cap}_\mu(K) \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(U)$$
$$\mathrm{cap}_\mu(U) \geq \limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(K)$$

hold for every compact set K *and for every open set* U *with* $K \subseteq U \subseteq \Omega$*;*

(c) for every open set $U \subseteq \Omega$

$$\begin{aligned}\mathrm{cap}_\mu(U) &= \sup\,\{\liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(K) :\ K \text{ compact},\ K \subseteq U\}\\ &= \sup\,\{\limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(K) :\ K \text{ compact},\ K \subseteq U\};\end{aligned}$$

(d) the family of all sets $E \subseteq \Omega$ *such that*

$$\mathrm{cap}_\mu(E) = \lim_{h\to\infty} \mathrm{cap}_{\mu_h}(E)$$

is dense in $\mathcal{P}(\Omega)$*;*

(e) the equality

$$\mathrm{cap}_\mu(E) = \lim_{h\to\infty} \mathrm{cap}_{\mu_h}(E)$$

holds for every set E *in the finely rich family* $\mathcal{E}_2(\mu)$ *of Theorem* 5.11.

PROOF. (a) $\Rightarrow$ (e). It follows from Theorem 5.11.

(e) $\Rightarrow$ (d). Every finely rich set is dense by Proposition 4.8.

(d) $\Rightarrow$ (b). It follows immediately from the definition of dense set (Definition 4.7) and from the monotonicity of the μ-capacity.

(b) $\Rightarrow$ (c). Let U be an open subset of Ω and let

$$S = \sup\,\{\liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(K) :\ K \text{ compact},\ K \subseteq U\}.$$

By (b) it is enough to prove that $\mathrm{cap}_\mu(U) \leq S$. By Theorem 2.9(i) for every $\varepsilon > 0$ there exists a compact set $H \subseteq U$ such that $\mathrm{cap}_\mu(U) - \varepsilon < \mathrm{cap}_\mu(H)$. Let

V be an open set and let K be a compact set such that $H \subseteq V \subseteq K \subseteq U$. By (b) we have

$$\mathrm{cap}_\mu(U) - \varepsilon < \mathrm{cap}_\mu(H) \le \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(V) \le \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(K) \le S.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\mathrm{cap}_\mu(U) \le S$.

(c) $\Rightarrow$ (a). By (c) the sequence (μ_h) satisfies hypothesis (6.1) of Theorem 6.1 with $\alpha(U) = \mathrm{cap}_\mu(U)$ for every open set $U \subseteq \Omega$. By (6.2) and by Proposition 3.11 we have $\alpha = \mathrm{cap}_{\mu^*}$ on $\mathcal{P}(\Omega)$, hence $\lambda = \mu^*$ by (6.3) and by Theorem 4.3. The conclusion follows now Theorem 6.1. □

REMARK 6.4. E. De Giorgi and G. Letta introduced a notion of weak convergence in the space $\Lambda(\Omega)$ of all increasing set functions defined on $\mathcal{P}(\Omega)$ vanishing on the empty set ([16], Definition 7.3). This notion depends on the choice of two families of subsets of Ω, denoted in their paper by $\mathcal{U}$ and $\mathcal{K}$. If we choose $\mathcal{U}$ equal to the family of all open subsets of Ω and $\mathcal{K}$ equal to the family of all compact subsets of Ω, then Theorem 6.3 implies that a sequence (μ_h) γ-converges to μ in $\mathcal{M}_0(\Omega)$ if and only if (cap_{μ_h}) converges weakly to cap_μ in $\Lambda(\Omega)$, and Theorem 6.1 implies that the set of all μ-capacities is compact in $\Lambda(\Omega)$. Compare this result with the density in $\Lambda(\Omega)$ of the class of all Choquet capacities ([7], Theorem 4.5) and with the compactness in $\Lambda(\Omega)$ of the class of all strongly subadditive Choquet capacities ([7], Proposition 4.9).

REMARK 6.5. Let F and F_h $(h \in \mathbf{N})$ be closed subsets of Ω, let ∞_F and ∞_{F_h} be the corresponding measures (see Definition 2.3), and let $K = H_0^1(\Omega - F)$ and $K_h = H_0^1(\Omega - F_h)$ (considered as subspaces of $H_0^1(\Omega)$). Then (K_h) converges to K in $H_0^1(\Omega)$ in the sense of Mosco ([21], Definition 1.1) if and only if (∞_{F_h}) γ- converges to ∞_F with respect to the Laplace operator $L = -\Delta$ ([11], Proposition 4.13). By Theorem 6.3 the last condition is satisfied if and only if there exists a dense family $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ such that

$$\mathrm{cap}(E \cap F) = \lim_{h\to\infty} \mathrm{cap}(E \cap F_h)$$

for every $E \in \mathcal{E}$. Compare this result with Theorem 3.3 of [10].

We now attack the problem of the continuity, with respect to γ-convergence, of the restriction operator introduced in Definition 2.4.

THEOREM 6.6. *For every $\mu \in \mathcal{M}_0(\Omega)$ let $\mathcal{E}_3(\mu)$ be the family of all subsets E of Ω such that $\mathrm{cap}_{\mu^*}(U \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(U \cap \mathrm{cl}_f E)$ for every open set $U \subseteq \Omega$. Then $\mathcal{E}_3(\mu)$ is finely rich in $\mathcal{P}(\Omega)$ and (μ_h^E) γ-converges to μ^E for every $E \in \mathcal{E}_3(\mu)$ and for every sequence (μ_h) which γ-converges to μ in $\mathcal{M}_0(\Omega)$.*

PROOF. Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\mathcal{V}$ be a countable dense family of open subsets of Ω. For every $V \in \mathcal{V}$ we can apply Lemma 4.10 with $\alpha(E) = \mathrm{cap}_{\mu^*}(V \cap E)$, therefore for every $V \in \mathcal{V}$ there exists a finely rich family $\mathcal{E}_V$ such that $\mathrm{cap}_{\mu^*}(V \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(V \cap \mathrm{cl}_f E)$ for every $E \in \mathcal{E}_V$. Let $\mathcal{E} = \bigcap_{V \in \mathcal{V}} \mathcal{E}_V$. Since $\mathcal{V}$ is countable, the family $\mathcal{E}$ is finely rich. Let us prove that $\mathcal{E} \subseteq \mathcal{E}_3(\mu)$. Let $E \in \mathcal{E}$ and let U be an open subset of Ω. Since $\mathcal{V}$ is dense, there exists an increasing sequence (V_h) of elements of $\mathcal{V}$ whose union is U. Since $\mathrm{cap}_{\mu^*}(V_h \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(V_h \cap \mathrm{cl}_f E)$ for every $h \in \mathbf{N}$, by Theorem 2.9(c) we obtain $\mathrm{cap}_{\mu^*}(U \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(U \cap \mathrm{cl}_f E)$. Since U is arbitrary, we have $E \in \mathcal{E}_3(\mu)$, hence $\mathcal{E} \subseteq \mathcal{E}_3(\mu)$. This implies that $\mathcal{E}_3(\mu)$ is finely rich.

Let $E \in \mathcal{E}_3(\mu)$ and let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ which γ-converges to μ. We have to prove that (μ_h^E) γ-converges to μ^E. Since $\mathrm{cap}_{\mu_h^E}(B) = \mathrm{cap}_{\mu_h}(B \cap E)$ and $\mathrm{cap}_{\mu^E}(B) = \mathrm{cap}_{\mu}(B \cap E)$ for every $B \in \mathcal{B}(\Omega)$, by Theorem 6.3 it is enough to prove that

$$\mathrm{cap}_{\mu}(K \cap E) \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(U \cap E) \tag{6.4}$$

and

$$\mathrm{cap}_{\mu}(U \cap E) \geq \limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(K \cap E) \tag{6.5}$$

for every pair (K, U) of subsets of Ω with K compact, U open, and $K \subseteq U$.

Let (K, U) be such a pair. Then $\mathrm{cap}_{\mu}(K \cap E) \leq \mathrm{cap}_{\mu^*}(U \cap \mathrm{cl}_f E) = \mathrm{cap}_{\mu^*}(U \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu}(U \cap \mathrm{int}_f E)$ by the definition of $\mathcal{E}_3(\mu)$ and by Proposition 3.11. Therefore Theorem 5.8 implies that

$$\begin{aligned} &\mathrm{cap}_{\mu}(K \cap E) \leq \mathrm{cap}_{\mu}(U \cap \mathrm{int}_f E) \\ &\leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(U \cap \mathrm{int}_f E) \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}(U \cap E), \end{aligned}$$

which yields (6.4).

To prove (6.5) we observe that $\mathrm{cap}_{\mu}(U \cap E) \geq \mathrm{cap}_{\mu}(U \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(U \cap \mathrm{int}_f E) = \mathrm{cap}_{\mu^*}(U \cap \mathrm{cl}_f E) \geq \mathrm{cap}_{\mu^*}(K \cap \mathrm{cl}_f E)$ by the definition of $\mathcal{E}_3(\mu)$ and by Proposition 3.11. Therefore Theorem 5.9 implies that

$$\begin{aligned} &\mathrm{cap}_{\mu}(U \cap E) \geq \mathrm{cap}_{\mu^*}(K \cap \mathrm{cl}_f E) \\ &\geq \limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(K \cap \mathrm{cl}_f E) \geq \limsup_{h\to\infty} \mathrm{cap}_{\mu_h}(K \cap E), \end{aligned}$$

which yields (6.5). □

The following proposition characterizes the family $\mathcal{E}_3(\mu)$ in terms of the measure μ^*. This shows that the definition of $\mathcal{E}_3(\mu)$ is independent of the operator L occurring in the definition of cap_{μ^*}.

PROPOSITION 6.7. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\mathcal{E}_3(\mu)$ be the family of sets introduced in Theorem 6.6. Then a set $E \subseteq \Omega$ belongs to $\mathcal{E}_3(\mu)$ if and only if $\mu^*(A \cap \mathrm{int}_f E) = \mu^*(A \cap \mathrm{cl}_f E)$ for every finely open set $A \subseteq \Omega$.*

PROOF. Let E be a subset of Ω and let μ_1 and μ_2 be the measures of the class $\mathcal{M}_0(\Omega)$ defined by $\mu_1(B) = \mu^*(B \cap \mathrm{int}_f E)$ and $\mu_2(B) = \mu^*(B \cap \mathrm{cl}_f E)$

for every $B \in \mathcal{B}(\Omega)$. Since $\mathrm{cap}_{\mu_1}(B) = \mathrm{cap}_{\mu^*}(B \cap \mathrm{int}_f E)$ and $\mathrm{cap}_{\mu_2}(B) = \mathrm{cap}_{\mu^*}(B \cap \mathrm{cl}_f E)$ for every $B \in \mathcal{B}(\Omega)$, the set E belongs to $\mathcal{E}_3(\mu)$ if and only if $\mathrm{cap}_{\mu_1}(U) = \mathrm{cap}_{\mu_2}(U)$ for every open set $U \subseteq \Omega$. This condition is satisfied if and only if μ_1 and μ_2 are equivalent (Theorem 4.9), that is only if μ_1 and μ_2 agree on all finely open subsets of Ω (Theorem 2.6). Since the last condition is satisfied if and only if $\mu^*(A \cap \mathrm{int}_f E) = \mu^*(A \cap \mathrm{cl}_f E)$ for every finely open set $A \subseteq \Omega$ (by the definition of μ_1 and μ_2), the proof of the proposition is complete. □

The following proposition characterizes the family $\mathcal{E}_3(\mu)$ in terms of the measure μ and of its singular set $S(\mu)$ introduced in Definition 3.12.

PROPOSITION 6.8. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\mathcal{E}_3(\mu)$ be the finely rich family introduced in Theorem 6.6. Let E be a subset of Ω, let $E_0 = \mathrm{int}_f E$, and let*

$$E^* = \partial_f S(\mu) \;\cap\; \partial_f(A(\mu) \cap E_0) - \partial_f(S(\mu) \cap E_0).$$

Then $E \in \mathcal{E}_3(\mu)$ if and only if all the following conditions are satisfied:

(a) $\mu(A(\mu) \cap \partial_f E) = 0$,

(b) $\mathrm{cap}(S(\mu) \cap (\partial_f E - \partial_f E_0)) = 0$,

(c) $\mu(A \cap A(\mu) \cap E_0) = +\infty$ *for every finely open set $A \subseteq \Omega$ such that* $\mathrm{cap}(A \cap E^*) > 0$.

PROOF. Assume $E \in \mathcal{E}_3(\mu)$. By Proposition 6.7 we have

$$\mu^*(A \cap E_0) = \mu^*(A \cap \mathrm{cl}_f E) \tag{6.6}$$

for every finely open set $A \subseteq \Omega$. Therefore $\mu(A \cap \partial_f E) \leq \mu^*(A \cap \partial_f E) = 0$ for every finely open set $A \subseteq \Omega$ such that $\mu^*(A) < +\infty$. Since $A(\mu)$ is the union of these sets (Proposition 3.11 and Definition 3.12), (a) follows from the quasi-Lindelöf property of the fine topology ([17], Theorem 1.XI.11).

To prove (b) we take $A = \Omega - \mathrm{cl}_f E_0$. Since A is finely open and $A \cap E_0 = \emptyset$, by (6.6) we have $\mu^*(\partial_f E - \partial_f E_0) = \mu^*(A \cap \mathrm{cl}_f E) = 0$, which implies (b) by Theorem 3.17.

To prove (c), let A be a finely open subset of Ω such that $\mathrm{cap}(A \cap E^*) > 0$ and let $A' = A - \mathrm{cl}_f(S(\mu) \cap E_0)$. Since $A' \cap \mathrm{cl}_f E \cap S(\mu) \supseteq A' \cap E^* = A \cap E^*$, we have $\mathrm{cap}(A' \cap \mathrm{cl}_f E \cap S(\mu)) > 0$, hence $\mu^*(A' \cap \mathrm{cl}_f E) = +\infty$ by Theorem 3.17. Since A' is finely open and $A' \cap E_0 = A \cap A(\mu) \cap E_0$, from (6.6) we obtain $\mu^*(A \cap A(\mu) \cap E_0) = \mu^*(A' \cap \mathrm{cl}_f E) = +\infty$, hence $\mu(A \cap A(\mu) \cap E_0) = +\infty$, which completes the proof of (c).

Conversely, assume (a), (b), and (c). By Proposition 6.7, to prove that $E \in \mathcal{E}_3(\mu)$ it is enough to show that (6.6) holds for every finely open set $A \subseteq \Omega$. Let A be such a set.

If $\mathrm{cap}(A \cap E^*) > 0$, then (c) implies $\mu^*(A \cap \mathrm{cl}_f E) \geq \mu^*(A \cap E_0) \geq \mu(A \cap A(\mu) \cap E_0) = +\infty$, which yields (6.6).

If $A \cap S(\mu) \cap E_0 \neq \emptyset$, then Remark 3.13 implies $\mu^*(A \cap \mathrm{cl}_f E) \geq \mu^*(A \cap E_0) \geq \mu(A \cap E_0) = +\infty$, which yields (6.6).

If $\mathrm{cap}(A \cap E^*) = 0$ and $A \cap S(\mu) \cap E_0 = \emptyset$, then $A \cap \mathrm{cl}_f(S(\mu) \cap E_0) = \emptyset$ and the identity

$$S(\mu) \cap \mathrm{cl}_f E = \mathrm{cl}_f(S(\mu) \cap E_0) \cup (S(\mu) \cap (\partial_f E - \partial_f E_0)) \cup E^*,$$

together with condition (b), implies that $\mathrm{cap}(A \cap S(\mu) \cap \mathrm{cl}_f E) = 0$. By (a) and by Theorem 3.17 we obtain

$$\begin{aligned}\mu^*(A \cap \mathrm{cl}_f E) &= \mu^*(A \cap A(\mu) \cap \mathrm{cl}_f E) \\ &= \mu^*(A \cap A(\mu) \cap E_0) + \mu^*(A \cap A(\mu) \cap \partial_f E) \\ &= \mu^*(A \cap E_0) + \mu(A \cap A(\mu) \cap \partial_f E) = \mu^*(A \cap E_0),\end{aligned}$$

which proves (6.6). □

REMARK 6.9. If $\mathrm{cap}(E^*) = 0$, then condition (c) of Proposition 6.8 is trivial, therefore conditions (a) and (b) imply that $E \in \mathcal{E}_3(\mu)$.

Since $E^* \subseteq \partial_f S(\mu) \cap \partial_f E_0$, it follows that the set E belongs to $\mathcal{E}_3(\mu)$ if the following conditions are satisfied:

(a′) $\mu(A(\mu) \cap \partial_f E) = 0$,

(b′) $\mathrm{cap}(S(\mu) \cap (\partial_f E - \partial_f E_0)) = 0$,

(c′) $\mathrm{cap}(\partial_f S(\mu) \cap \partial_f E_0) = 0$.

Compare this result with Lemma 5.2 of [3].

REMARK 6.10. Let F be a quasi closed subset of Ω and let $\mu = \infty_F$. Then $\infty_F = \infty_{S(\mu)}$ by Remark 3.18, hence $\mathrm{cap}(F \Delta S(\mu)) = 0$ by Remark 2.7. Therefore a subset E of Ω belongs to $\mathcal{E}_3(\mu)$ if and only if the following conditions are satisfied:

$$\mathrm{cap}(F \cap (\partial_f E - \partial_f E_0)) = 0, \tag{6.7}$$

$$\mathrm{cap}(\partial_f F \cap \partial_f(E_0 - F) - \partial_f(E_0 \cap F)) = 0. \tag{6.8}$$

In fact condition (c) of Proposition 6.8 is equivalent to (6.8) because $\mu(A(\mu)) = 0$.

REMARK 6.11. The rich family $\mathcal{R}_\mu$ introduced in Definition 5.6 of [11] is contained in the finely rich family $\mathcal{E}_3(\mu)$ of Theorem 6.6. In fact, by applying the inequality (5.6) of [11] to μ^*, for every $E \in \mathcal{R}_\mu = \mathcal{R}_{\mu^*}$ we obtain

$$\int\limits_{\mathrm{int}_f E} u^2 d\mu^* = \int\limits_{\mathrm{cl}_f E} u^2 d\mu^*$$

for every $u \in H_0^1(\Omega)$, hence the measures μ_1 and μ_2 considered in the proof of Proposition 6.7 are equivalent, which implies $E \in \mathcal{E}_3(\mu)$.

In general, the inclusion $\mathcal{R}_\mu \subseteq \mathcal{E}_3(\mu)$ is strict. For instance, if $\mu \in \mathcal{M}_0(\Omega)$ is finite on all compact subsets of Ω, then $\mathcal{R}_\mu$ is the family of all $B \in \mathcal{B}(\Omega)$ such that $B \subset\subset \Omega$ and $\mu(\partial B) = 0$ ([11], Proposition 5.7), whereas $\mathcal{E}_3(\mu)$ is the family of all sets $E \subseteq \Omega$ such that $\mu(\partial_f E) = 0$ (Remark 6.9).

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S.I.S.S.A.
Strada Costiera 11
34014 TRIESTE