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Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems

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Concavity Properties of Solutions to Some Degenerate Quasilinear Elliptic Dirichlet Problems

SHIGERU SAKAGUCHI

1. - Introduction

The purpose of this paper is to show that solutions to some degenerate quasilinear elliptic Dirichlet problems have certain concavity properties. Precisely, we show the following two theorems.

THEOREM 1. *Let Ω be a bounded convex domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary $\partial\Omega$. Fix a number $p > 1$. Let $u \in W_0^{1,p}(\Omega)$ be a positive weak solution to the nonlinear eigenvalue problem:*

$$(1.1) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda = \inf \{\int_\Omega |\nabla v|^p dx / \int_\Omega |v|^p dx;\ v \in W_0^{1,p}(\Omega)\}$, the Poincaré constant. Then $v = \log u$ is a concave function.

THEOREM 2. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Fix a number $p > 1$. Let $u \in W_0^{1,p}(\Omega)$ be the unique weak solution to the Dirichlet problem:*

$$(1.2) \qquad \begin{cases} -\operatorname{div}\ (|\nabla u|^{p-2}\nabla u) = 1 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Then $v = u^{\frac{p-1}{p}}$ is a concave function.

REMARKS.
(1.3) The existence of a non-trivial non-negative solution to (1.1) follows from direct methods of the calculus of variations (such a solution is the solution to (2.1) in this paper).

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Positivity of this solution follows from Harnack's inequality of Trudinger [23]. Along the similar line to that in Aubin [1, pp. 102-103] we can show that the solutions to (1.1) are proportional. We give its proof in Appendix (see Theorem A.1). When Ω is a ball, Thelin [20] showed the uniqueness of the radially symmetric eigenfunction u of norm 1 to (1.1).

(1.4) The existence of the solution to (1.2) is due to the direct method in the calculus of variations. The uniqueness of the solution to (1.2) follows from the weak comparison principle due to Tolksdorf [22] (see [22, Lemma 3.1, pp. 800-801] or Lemma A.2 in Appendix).

Recently many results concerning concavity properties of solutions to elliptic boundary value problems were obtained by various authors with the help of *concavity maximum principles*.

Korevaar [14] first established a *concavity maximum principle* and using this he obtained the concavity property of capillary surfaces in $\mathbb{R}^{n+1}$. Furthermore, Korevaar [15] and Caffarelli and Spruck [5] showed the result of Branscamp and Lieb [3], that is, "The first eigenfunction of the Laplacian on a convex domain in $\mathbb{R}^n$ is log concave" by using a *concavity maximum principle*. Kennington [13] and Kawohl [10], [11] improved Korevaar's *concavity maximum principle* and obtained concavity properties of solutions to various nonlinear elliptic boundary value problems. Especially Kennington [13] and Kawohl [11] showed that $\sqrt{u}$ is concave in the case $p = 2$ in our Theorem 2. Strict concavity properties of solutions to semilinear elliptic equations in $\mathbb{R}^2$ were proved by Caffarelli and Friedman [4] (see also Kawohl [12] and Korevaar and Lewis [16]).

On the other hand, the pseudo-Laplacian $-\mathrm{div}\ (|\nabla \cdot|^{p-2}\nabla\cdot)$ has been studied by many authors (see Diaz [6]). Thus we consider the pseudo-Laplacian instead of the Laplacian and obtain results similar to those obtained in the case of the Laplacian. Theorem 1 corresponds to the result of [15] and [5], and Theorem 2 corresponds to that of [11] and [13]. In the case of the Laplacian the solutions to (1.1) and (1.2) are classical (that is, smooth), but in the case of the pseudo-Laplacian they are generally only weak solutions, since the pseudo-Laplacian is degenerate elliptic. Precisely, it was shown in [21] and [22] that the bounded solutions to (1.1) and (1.2) belong to $C^{1+\alpha}(\overline{\Omega})$ for some α $(0 < \alpha < 1)$ and not always belong to $C^2(\Omega)$. For example, when the domain Ω is a ball centered at the origin 0, the function $u(x)$ defined by

$$u(x) = a|x|^{\frac{p}{p-1}} + b,$$

with constants a and b $(a < 0,\ b > 0)$, is a solution to (1.2). Korevaar's *concavity maximum principle* and its improved versions due to Kennington and Kawohl work only for a classical solution. Therefore, we can not directly apply *concavity maximum principle* to our problems, (1.1) and (1.2).

In this paper we introduce certain regularized problems and prove our theorems. In the following sections we first prove Theorem 1, and along the similar line to this we prove Theorem 2. In §7 Appendix we show that the eingenvalue λ of (1.1) is simple if the boundary $\partial\Omega$ is connected, where Ω is not necessarily convex.

2. - A regularized problem

The equation of (1.1) is obtained by the variational problem:
(2.1) Find $u \in K$ satisfying

$$\int_\Omega |\nabla u|^p dx = \min_{v \in K} \int_\Omega |\nabla v|^p dx \ (= \lambda),$$

where $K = \{v \in W_0^{1,p}(\Omega) : ||v||_{L^p(\Omega)} = 1\}$ (see [2, Theorem (6.3.2), p. 325]). It follows from Theorem A.1 that (2.1) has a unique positive solution u, which is a positive solution to (1.1).

Our idea of the proof of Theorem 1 is to introduce the following variational problem:
(2.2.ε) Find $u \in K$ satisfying

$$\int_\Omega (\varepsilon\ u^2 + |\nabla u|^2)^{\frac{p}{2}} dx = \min_{v \in K} \int_\Omega (\varepsilon v^2 + |\nabla v|^2)^{\frac{p}{2}} dx \ (= \lambda_\varepsilon)$$

for sufficiently small number $\varepsilon > 0$.

Concerning this problem we obtain

PROPOSITION 2.1. *There exists at least one non-negative solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ to (2.2. ε) satisfying*

$$\text{(2.3)} \qquad -\operatorname{div}\ [(\varepsilon u_\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon] = \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon - \varepsilon\ (\varepsilon u_\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} u_\varepsilon$$

in Ω, and

$$\text{(2.4)} \qquad u_\varepsilon \to u \text{ in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0,$$

where u is the unique positive solution to (2.1).

PROOF. It follows from Theorem (6.3.2) in [2, p. 325] that there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ to (2.2.ε). Since $|u_\varepsilon| \in W_0^{1,p}(\Omega)$ is also a solution to (2.2.ε), we may assume that u_ε is non-negative in Ω. Of course u_ε satisfies (2.3) in the weak sense. Furthermore, we see that the set $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$, since $\|u_\varepsilon\|_{L^p(\Omega)} = 1$. Hence there exist a subsequence $\{u_{\varepsilon'}\}$ and $\tilde{u} \in W_0^{1,p}(\Omega)$ satisfying

$$u_{\varepsilon'} \to \tilde{u} \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } \varepsilon' \to 0.$$

It follows from the lower semicontinuity of the norm that

$$\int_\Omega |\nabla \tilde{u}|^p dx \le \liminf_{\varepsilon' \to 0} \int_\Omega |\nabla u_{\varepsilon'}|^p dx. \tag{2.5}$$

Since the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we have $u_{\varepsilon'} \to \tilde{u}$ in $L^p(\Omega)$ and almost everywhere in Ω as $\varepsilon' \to 0$, by taking a subsequence, if necessary. Then we see that $\tilde{u} \in K$ and $\tilde{u}$ is non-negative in Ω.

On the other hand, by using the minimizing properties of u and u_ε, we see that

$$\begin{aligned}\int_\Omega |\nabla u|^p dx \le \int_\Omega |\nabla u_\varepsilon|^p dx \le \int_\Omega (\varepsilon u_\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{p}{2}} dx \\ \le \int_\Omega (\varepsilon u^2 + |\nabla u|^2)^{\frac{p}{2}} dx.\end{aligned} \tag{2.6}$$

Hence, with the help of Lebesgue's dominated convergence theorem, we have

$$\int_\Omega |\nabla u_\varepsilon|^p dx \to \int_\Omega |\nabla u|^p dx \text{ as } \varepsilon \to 0. \tag{2.7}$$

Then it follows from (2.5) and (2.7) that

$$\int_\Omega |\nabla \tilde{u}|^p dx \le \int_\Omega |\nabla u|^p dx.$$

Therefore, in view of the uniqueness of the non-negative solution to (2.1), we get $\tilde{u} = u$. Consequently, we have

$$u_\varepsilon \to u \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain (2.4) from the mean convergence theorem of Riesz and Nagy [19, Theorem, §37, p. 78]. This completes the proof.

Concerning the regularity of u_ε we have

PROPOSITION 2.2. *The solution u_ε to* (2.2.ε) *obtained in* Proposition 2.1. *belongs to $C^\beta(\overline{\Omega})$ for some β $(0 < \beta < 1)$ and satisfies*

$$\|u_\varepsilon\|_{C^\beta(\overline{\Omega})} \le M, \tag{2.9}$$

where M and β are constants independent of ε.

PROOF. Since $||u_\varepsilon||_{L^p(\Omega)} = 1$ and $u_\varepsilon \geq 0$, applying Lemma 9.6 in [9, pp. 213-214] to (2.3), we obtain the estimate

$$\sup \{|u_\varepsilon(x)|;\ x \in \Omega\} \leq C_1, \tag{2.10}$$

where C_1 is a constant independent of ε (see also [17, Theorem 7.1, pp. 286-287]). Therefore, using Theorem 1.1 in [17, p. 251], we see that u_ε belongs to $C^\beta(\overline{\Omega})$ for some β $(0 < \beta < 1)$ and satisfies the inequality

$$\sup_{x,y \in \overline{\Omega}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{|x-y|^\beta} \leq C_2, \tag{2.11}$$

where C_2 and β are constants independent of ε. Combining (2.10) and (2.11), we get (2.9). This completes the proof.

Using Arzelà and Ascoli's theorem, we obtain from Proposition 2.1 and Proposition 2.2

COROLLARY 2.3. *$u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.*

3. - Some properties of the unique positive solution u to (2.1)

From Theorem A.1 and Lemma A.3 we get

PROPOSITION 3.1. *There exists a neighborhood N of $\partial\Omega$ in Ω satisfying the following conditions:*

$$|\nabla u| \geq \eta > 0 \text{ in } \overline{N} \text{ for some constant } \eta > 0, \tag{3.1}$$

and

$$u \in C^2(\overline{N}) \cap C^{1+\alpha}(\overline{\Omega}) \text{ for some } \alpha\ (0 < \alpha < 1). \tag{3.2}$$

Furthermore, from Lemma 2.4 in [15, pp. 610-611] and its proof we have

PROPOSITION 3.2. *Let Ω be strongly convex (that is, all the principal curvatures of $\partial\Omega$ are positive.). For $\delta > 0$ let*

$$\Omega_\delta = \{x \in \Omega;\ \text{dist } (x, \partial\Omega) > \delta\}. \tag{3.3}$$

Put $v = \log u$. Then there exists a number δ_0 which satisfies the following:

(3.4) *The matrix $[-D_{ij}v]$ is positive on $\Omega - \Omega_{\delta_0}$ where* $D_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j}$, *and*

(3.5) *For any* δ $(0 < \delta < \delta_0)$ *every tangent plane to the graph of* v *on* $\partial\Omega_\delta$ *lies above the graph on* $\overline{\Omega_\delta}$ *and contacts it only at the tangent point in* $\overline{\Omega_\delta}$.

PROOF. We choose $\delta_0 > 0$ small to get $\Omega - \Omega_{\delta_0} \subset N$, where N is the neighborhood of $\partial\Omega$ obtained in Proposition 3.1. Then, since u belongs to $C^2(\overline{N}) \cap C^{1+\alpha}(\overline{\Omega})$, we can use Lemma 2.4 in [15, pp. 610-611] and its proof to prove this proposition.

4. - An application of Korevaar's concavity maximum principle

In this section we apply Korevaar's concavity maximum principle [15, Theorem 1.3, p. 604] to our problem. Before its application we get more regularity of the solution u_ε in compact subsets of Ω for small $\varepsilon > 0$.

PROPOSITION 4.1. *For any* $\delta > 0$, *if we choose numbers* $\varepsilon_0 > 0$ *and* $\tau > 0$ *sufficiently small, we have the following: for any* ε $(0 < \varepsilon < \varepsilon_0)$,

$$M \geq u_\varepsilon \geq \tau > 0 \text{ in } \Omega_\delta, \tag{4.1}$$

where M *is the constant in* Proposition 2.2 *and* Ω_δ *is the domain defined in* Proposition 3.2 (*see* (3.3)).

PROOF. Combining Corollary 2.3 and the positivity of u (see Theorem A.1), we get (4.1).

With the help of Proposition 4.1, using Tolksdorf's regularity theorem [21, Theorem 1, p. 127], we get

PROPOSITION 4.2. *For any* $\delta > 0$ *and any* ε $(0 < \varepsilon < \varepsilon_0)$, *the solution* u_ε *belongs to* $C^\infty(\Omega_\delta)$ *and satisfies*

$$\|u_\varepsilon\|_{C^{1+\beta}(\overline{\Omega_{2\delta}})} \leq C, \tag{4.2}$$

where β $(0 < \beta < 1)$ *and* C *are constants independent of* $\varepsilon > 0$, *and* ε_0 *is the number in* Proposition 4.1.

PROOF. We apply Theorem 1 in [21, p. 127] to (2.3). To this purpose, choose a non-decreasing function $\psi \in C^\infty(\mathbf{R})$ satisfying

$$\psi(t) = \begin{cases} 2M & \text{if } t \geq 2M, \\ t & \text{if } \tau \leq t \leq M, \\ \tau/2 & \text{if } t \leq \tau/2, \end{cases} \tag{4.3}$$

where M and τ are the constants in Proposition 4.1. We put

$$a_j(x, w, \nabla w) = (\varepsilon\ \psi(w)^2 + \ |\nabla w|^2)^{\frac{p-2}{2}}\ D_j w$$

and

$$a(x, w, \nabla w) = \lambda_\varepsilon \psi(w)^{p-1} - \varepsilon\ (\varepsilon\psi(w)^2 + |\nabla w|^2)^{\frac{p-2}{2}} \psi(w)$$

where $D_j\ =\ \frac{\partial}{\partial x_j}$. Then it follows from Proposition 4.1 that u_ε is also a weak solution to the equation

$$\Sigma\ D_j\ a_j(x, u_\varepsilon, \nabla u_\varepsilon) + a(x, u_\varepsilon, \nabla u_\varepsilon) = 0 \text{ in } \Omega_\delta. \tag{4.4}$$

We observe that

$$C_1\ (\sqrt{\varepsilon} + |\nabla w|) \leq (\varepsilon\ \psi(w)^2 + |\nabla w|^2)^{\frac{1}{2}} \leq C_2(\sqrt{\varepsilon} + |\nabla w|), \tag{4.5}$$

where $C_1 = (1/\sqrt{2})\ \min\ \{(\tau/2), 1\}$ and $C_2 = \max\ \{2M, 1\}$.

Therefore we easily verify the assumptions of Theorem 1 in [21], and applying this theorem to (4.4), we see that u_ε belongs to $C^1(\Omega_\delta)$ and ∇u_ε is Hölder continuous in Ω_δ. Then, since $\varepsilon\ \psi(u_\varepsilon)^2$ is positive (that is, the equation is elliptic), from the regularity theory for the elliptic partial differential equation (see [9]) we see that $u_\varepsilon \in C^\infty(\Omega_\delta)$. Furthermore, using Tolksdorf's interior estimate, we get (4.2). The proof is completed.

Now, we apply Korevaar's concavity maximum principle to our problem. In view of Proposition 4.1 and Proposition 4.2, for $0 < \varepsilon < \varepsilon_0$ we define the function $v_\varepsilon \in C^\infty(\Omega_\delta)$ by

$$v_\varepsilon\ =\ \log\ u_\varepsilon. \tag{4.6}$$

Then v_ε satisfies the equation

$$\Sigma\ a^{ij}(\nabla v_\varepsilon)\ D_{ij} v_\varepsilon\ - b(\nabla v_\varepsilon) = 0 \text{ in } \Omega_\delta, \tag{4.7}$$

where

$$\Sigma a^{ij}(\nabla v_\varepsilon)\ D_{ij} v_\varepsilon = \operatorname{div}\ [(\varepsilon + |\nabla v_\varepsilon|^2)^{\frac{p-2}{2}}\ \nabla v_\varepsilon]$$

and

$$b(\nabla v_\varepsilon) = -\lambda_\varepsilon - \{(p-1)|\nabla v_\varepsilon|^2 - \varepsilon\}(\varepsilon + |\nabla v_\varepsilon|^2)^{\frac{p-2}{2}}.$$

Let $c_\varepsilon\ (y, z, t)$ be the concavity function corresponding to $-\ v_\varepsilon$ in the convex domain Ω_δ, that is,

$$c_\varepsilon(y, z, t) = (1-t)\ v_\varepsilon(y) + t v_\varepsilon(z) - v_\varepsilon((1-t)\ y + tz) \tag{4.8}$$

for $(y, z, t) \in \Omega_\delta \times \Omega_\delta \times [0, 1]$. Note that v_ε is concave if and only if $c_\varepsilon \leq 0$.

Here, in view of (4.7), applying Korevaar's concavity maximum principle [15, Theorem 1.3, p. 604] to $-\ v_\varepsilon$ in the convex domain Ω_δ, we obtain

PROPOSITION 4.3. *For any $\delta > 0$ and any ε $(0 < \varepsilon < \varepsilon_0)$, the function c_ε attains its positive maximum on the boundary $\partial\{\Omega_\delta \times \Omega_\delta\} \times [0, 1]$, provided it is anywhere positive.*

5. - Proof of Theorem 1

First of all we show that it suffices to prove Theorem 1 when Ω is strongly convex (see Proposition 3.2). We can choose a sequence of strongly convex smooth domains $\{\Omega_k\}$ satisfying

$$\overline{\Omega}_k \subset \Omega_{k+1} \text{ for all } k \geq 1 \text{ and } \bigcup_{k=1}^{\infty} \Omega_k = \Omega.$$

Let $u_k \in W_0^{1,p}(\Omega_k)$ be the unique positive solution to (2.1) corresponding to Ω_k. We may extend the function u_k to a function in Ω by putting $u_k = 0$ in $\Omega - \Omega_k$. Thus u_k belongs to $W_0^{1,p}(\Omega)$. Since $||u_k||_{L^p(\Omega)} = 1$, by using the minimizing property of u_k, we see that the set $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$.

Hence there exist a subsequence $\{u_{k'}\}$ and a function $\tilde{u} \in W_0^{1,p}(\Omega)$ satisfying

$$u_{k'} \rightarrow \tilde{u} \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } k' \rightarrow \infty.$$

Of course $\tilde{u} \geq 0$ and $||\tilde{u}||_{L^p(\Omega)} = 1$. By using the minimizing property of u_k and the lower semicontinuity of the norm of $W_0^{1,p}(\Omega)$, we see that $\tilde{u}$ is a solution to (2.1). Then it follows from the uniqueness of the non-negative solution to (2.1) that $\tilde{u} = u$. Therefore we obtain

$$u_k \rightarrow u \text{ weakly in } W_0^{1,p}(\Omega),$$

where u is the unique positive solution to (2.1).

On the other hand, since the $C^\beta(\overline{\Omega})$-estimate of the solution to (2.1) is independent of small smooth perturbations of the boundary $\partial\Omega$, we obtain the estimate

$$||u_k||_{C^\beta(\overline{\Omega}_k)} \leq C,$$

where β and C are positive constants independent of k.

According to Arzelà and Ascoli's theorem we obtain

$$u_k \rightarrow u \text{ uniformly on any compact subset of } \Omega.$$

This shows that it suffices to prove Theorem 1 when Ω is strongly convex.

Now we suppose that Ω is strongly convex. In order to prove Theorem 1 it suffices to show

PROPOSITION 5.1. *For any small $\delta > 0$, the function $v = \log u$ is concave in Ω_δ, where Ω_δ is the (strongly) convex domain in* Proposition 3.2.

Furthermore, in view of Corollary 2.3, it suffices to show

LEMMA 5.2. *For any small $\nu > 0$, if we choose $\varepsilon_1 > 0$ sufficiently small, for any ε with $0 < \varepsilon < \varepsilon_1$ the function $v_\varepsilon = \log u_\varepsilon$ is concave in Ω_ν.*

PROOF of LEMMA 5.2. First of all we choose $\nu > 0$ sufficiently small to get

$$\Omega - \Omega_{2\nu} \subset N, \tag{5.1}$$

where N is the neighborhood of $\partial\Omega$ obtained in Proposition 3.1.

On the other hand, it follows from Proposition 4.2 that for small $\varepsilon > 0$

$$\|u_\varepsilon\|_{C^{1+\beta}(\overline{\Omega}_{\nu/2})} \le C, \tag{5.2}$$

where β and C are constants independent of ε. Then, since the imbedding $C^{1+\beta}(\overline{\Omega}_{\nu/2}) \hookrightarrow C^1(\overline{\Omega}_{\nu/2})$ is compact, we have

$$u_\varepsilon \to u \text{ in } C^1(\overline{\Omega}_{\nu/2}) \text{ as } \varepsilon \to 0. \tag{5.3}$$

Hence, from (3.1) in Proposition 3.1 we obtain

$$|\nabla u_\varepsilon| \ge \frac{1}{2}\eta > 0 \text{ in } \overline{\Omega}_{\nu/2} - \Omega_{2\nu}, \tag{5.4}$$

for small $\varepsilon > 0$. In view of (5.2) and (5.4) we can choose the ellipticity constants of (2.3) independently of ε for small $\varepsilon > 0$ (see [9]). Therefore it follows from Schauder estimates for elliptic partial differential equations (see [9]) that for small $\varepsilon > 0$

$$\|u_\varepsilon\|_{C^{2+\beta}(\overline{\Omega}_{\nu/2} - \Omega_{2\nu})} \le C,$$

where C is a constant independent of $\varepsilon > 0$.

Furthermore, using the compactness of the imbedding

$$C^{2+\beta}(\overline{\Omega}_{\nu/2} - \Omega_{2\nu}) \hookrightarrow C^2(\overline{\Omega}_{\nu/2} - \Omega_{2\nu}),$$

from Proposition 3.2 and (5.3) we obtain the following: for small $\varepsilon > 0$

(5.5) *The matrix $[-D_{ij}\, v_\varepsilon]$ is positive on $\overline{\Omega}_{\nu/2} - \Omega_{2\nu}$,* and

(5.6) *Every tangent plane to the graph of v_ε on $\partial\Omega_\nu$ lies above the graph on $\overline{\Omega}_\nu$ and contacts it only at the tangent point on $\overline{\Omega}_\nu$.*

(Here, of course we choose $\nu > 0$ small to get $2\nu < \delta_0$.)

Now, combining Lemma 2.1 in [15, p. 609] and (5.6), we see that the concavity function c_ε does not attain its positive maximum on the boundary $\partial\{\Omega_\nu \times \Omega_\nu\} \times [0, 1]$ for small $\varepsilon > 0$. Hence it follows from Proposition 4.3

that c_ε is non-positive for small $\varepsilon > 0$. This completes the proof of Lemma 5.2.

6. - Proof of Theorem 2

We can prove Theorem 2 along the similar line to the proof of Theorem 1.

The Dirichlet problem (1.2) is equivalent to the variational problem:

(6.1) Find $u \in W_0^{1,p}(\Omega)$ minimizing the functional

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega u\, dx,$$

in $W_0^{1,p}(\Omega)$. Our idea of the proof of Theorem 2 is to introduce the following variational problem:

(6.2.ε) Find $u \in W_0^{1,p}(\Omega)$ minimizing the functional

$$F_\varepsilon(u) = \frac{1}{p}\int_\Omega (\varepsilon|u|^{\frac{2}{p}} + |\nabla u|^2)^{\frac{p}{2}} dx - \int_\Omega u\, dx$$

in $W_0^{1,p}(\Omega)$ for sufficiently small number $\varepsilon > 0$.

Concerning this problem we obtain

PROPOSITION 6.1. *There exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ to* (6.2.ε) *satisfying*

$$u_\varepsilon \geq 0 \text{ in } \Omega, \tag{6.3}$$

and

$$u_\varepsilon \to u \textit{ in } W_0^{1,p}(\Omega) \textit{ as } \varepsilon \to 0, \tag{6.4}$$

where u is the unique solution to (6.1).

PROOF. Using Young's inequality and Poincaré inequality, we see that the functional F_ε is bounded from below in $W_0^{1,p}(\Omega)$. Furthermore, with the help of a semicontinuity theorem in [7, Theorem 2.3, p. 18] we see that F_ε is sequentially lower semicontinuous with respect to the weak topology of $W_0^{1,p}(\Omega)$. Hence there exists at least one solution to (6.2. ε), say $u_\varepsilon \in W_0^{1,p}(\Omega)$.

Put $u_\varepsilon^+ = \max\,(u_\varepsilon, 0)$. We have $F_\varepsilon(u_\varepsilon^+) \leq F_\varepsilon(u_\varepsilon)$. Then it follows from the minimizing property of u_ε that $\min\,(u_\varepsilon, 0) = 0$. This implies (6.3).

By using Young's inequality and Poincaré inequality, we see that the set $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$. Accordingly, it follows from the lower semicontinuity of F and the unique solvability of (6.1) that

$$u_\varepsilon \to u \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0, \tag{6.5}$$

where u is the unique solution to (6.1). Furthermore, since the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, we have

$$u_\varepsilon \to u \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{6.6}$$

On the other hand, since $F_\varepsilon \geq F$, by using the minimizing properties of u and u_ε, we see that $F(u) \leq F(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(u)$. Hence, with the help of Lebesgue's dominated convergence theorem, we have

$$F(u_\varepsilon) \to F(u) \text{ as } \varepsilon \to 0. \tag{6.7}$$

Therefore we obtain from this and (6.6)

$$\int_\Omega |\nabla u_\varepsilon|^p dx \to \int_\Omega |\nabla u|^p dx \text{ as } \varepsilon \to 0. \tag{6.8}$$

Combining (6.5) and (6.8), we get (6.4) from the mean convergence theorem of F. Riesz and B. Sz-Nagy [19, Theorem, §37, p. 78]). This completes the proof.

Concerning the regularity of u_ε we have

PROPOSITION 6.2. *The solution u_ε to* (6.2.ε) *obtained in* Proposition 6.1 *belongs to $C^\beta(\overline{\Omega})$ for some β $(0 < \beta < 1)$ and satisfies*

$$||u_\varepsilon||_{C^\beta(\overline{\Omega})} \leq M, \tag{6.9}$$

where M and β are constants independent of ε.

PROOF. First of all we put

$$F_\varepsilon(u) = \int_\Omega f(x, u, \nabla u)\ dx.$$

Then, with the help of Young's inequality, we see that the function $f(x, u, \nabla u)$ satisfies the growth condition

$$\frac{1}{p}|\nabla u|^p - b\ (|u|^p + 1) \leq f(x, u, \nabla u) \leq a|\nabla u|^p + b\ (|u|^p + 1),$$

where a and b are positive constants depending only on p. Hence, since the set $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$, using Theorem 3.2 in [17, p. 328], we obtain the inequality

$$\sup_{\Omega} |u_\varepsilon| \leq C_1, \tag{6.10}$$

where C_1 is a constant independent of ε. Therefore, using Theorem 3.1 in [8, p. 36] and the remark following it, we see that u_ε belongs to $C^\beta\ (\overline{\Omega})$ for some $\beta\ (0 < \beta < 1)$ and satisfies the inequality

$$\sup_{x,y\in\overline{\Omega}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{|x-y|^\beta} \leq C_2, \tag{6.11}$$

where C_2 and β are constants independent of ε. Combining (6.10) and (6.11), we get (6.9). This completes the proof.

Using Arzelà and Ascoli's theorem, we get Corollary 2.3 also in this section.

Next, concerning the properties of the unique solution u to (6.1), first from Lemma A.2 we see that u is non-negative. Hence, since u is not identically zero, by the weak Harnack inequality [23, Theorem 1.2, p. 724] we see that u is positive in Ω. Then with the help of Lemma A.3 we get Proposition 3.1 also in this section. Also we get Proposition 3.2 replacing $v = \log u$ by $v = u^{\frac{p-1}{p}}$.

Concerning the Euler equation for the variational problem (6.2. ε) we get

PROPOSITION 6.3. *For any $\delta > 0$, if we choose numbers $\varepsilon_0 > 0$ and $\tau > 0$ sufficiently small, we have the following for any ε $(0 < \varepsilon < \varepsilon_0)$*

$$M \geq u_\varepsilon \geq \tau > 0 \ \text{in}\ \Omega_\delta, \tag{6.12}$$

where M is the constant in Proposition 6.2, *and*

(6.13) *The solution u_ε is a weak solution to the Euler equation*

$$\begin{aligned} &- \operatorname{div}\ [(\varepsilon u_\varepsilon^{\frac{2}{p}} + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon] \\ &= 1 - \frac{\varepsilon}{p}\ (\varepsilon u_\varepsilon^{\frac{2}{p}} + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\ u_\varepsilon^{\frac{2-p}{p}} \ \text{in}\ \Omega_\delta, \end{aligned}$$

where Ω_δ is the domain defined in Proposition 3.2.

PROOF. Combining Corollary 2.3 and the positivity of u, we get (6.12). (6.13) follows from the minimizing property of u_ε.

Therefore, using Tolksdorf's estimate, we have Proposition 4.2 also in this section by the similar argument to that used in the proof of Proposition 4.2.

In this section we use Kennington's concavity maximum principle [13, Theorem 3.1, p. 691] instead of that of Korevaar [15]. In view of Proposition

6.3 and Proposition 4.2, for $0 < \varepsilon < \varepsilon_0$ we define the function $v_\varepsilon \in C^\infty(\Omega_\delta)$ by

$$v_\varepsilon = u_\varepsilon^{\frac{p-1}{p}}. \tag{6.14}$$

Then v_ε satisfies the equation

$$\Sigma\, a^{ij}(\nabla v_\varepsilon)\, D_{ij} v_\varepsilon + b(v_\varepsilon, \nabla v_\varepsilon) = 0 \text{ in } \Omega_\delta, \tag{6.15}$$

where

$$\Sigma\, a^{ij}(\nabla v_\varepsilon)\, D_{ij} v_\varepsilon = \operatorname{div}\left[\{\varepsilon + (\frac{p}{p-1})^2 |\nabla v_\varepsilon|^2\}^{\frac{p-2}{2}} \nabla v_\varepsilon\right]$$

and

$$b(v_\varepsilon, \nabla v_\varepsilon) = \frac{1}{v_\varepsilon}\left[\{\varepsilon + (\frac{p}{p-1})^2 |\nabla v_\varepsilon|^2\}^{\frac{p-2}{2}} \{|\nabla v_\varepsilon|^2 - \frac{\varepsilon\,(p-1)}{p^2}\} + \frac{p-1}{p}\right].$$

Let c_ε be the concavity function corresponding to $-v_\varepsilon$ in Ω_δ as in (4.8).

Here we obtain

PROPOSITION 6.4. *If we choose a number ε_1 sufficiently small corresponding to p, then for any ε with $0 < \varepsilon <$ min $(\varepsilon_0, \varepsilon_1)$ the function c_ε does not attain its positive maximum in $\Omega_\delta \times \Omega_\delta \times [0, 1]$, where ε_0 is the number in* Proposition 6.3.

PROOF. It suffices to verify the assumptions of Theorem 3.1 in [13, p. 691]. Put $b(v_\varepsilon, \nabla v_\varepsilon) = \frac{1}{v_\varepsilon} d(\nabla v_\varepsilon)$. Then we see that $d(\nabla v_\varepsilon)$ is positive for small $\varepsilon > 0$. Indeed, if $|\nabla v_\varepsilon|^2 \geq \frac{\varepsilon\,(p-1)}{p^2}$, we get $d(\nabla v_\varepsilon) \geq \frac{p-1}{p} > 0$. And if $|\nabla v_\varepsilon|^2 < \frac{\varepsilon\,(p-1)}{p^2}$, we get $d(\nabla v_\varepsilon) \geq -C_p\, \varepsilon^{\frac{p}{2}} + \frac{p-1}{p}$ for some positive constant C_p depending only on p, since $\varepsilon \leq \varepsilon + (\frac{p}{p-1})^2 |\nabla v_\varepsilon|^2 \leq \frac{p}{p-1}\, \varepsilon$.

Thus we can choose a positive number ε_1 depending only on p, and we see that $b(v_\varepsilon, \nabla v_\varepsilon)$ is positive for $0 < \varepsilon <$ min $(\varepsilon_0, \varepsilon_1)$, since v_ε is positive. We observe that

$$\frac{\partial b}{\partial v_\varepsilon} = -\frac{1}{v_\varepsilon^2}\, d < 0, \text{ and } \frac{\partial^2}{\partial v_\varepsilon^2}(\frac{1}{b}) = 0. \tag{6.16}$$

Therefore, applying Kennington's concavity maximum principle [13, Theorem 3.1, p. 691], we get Proposition 6.4.

Consequently, by the similar argument to that used in the proof of Theorem 1 (see §5) we can prove Theorem 2.

REMARK.
(6.17) Theorem 2 remains valid without hypothesis "$\partial\Omega$ is smooth". Indeed, since the uniqueness of the solution to (1.2) holds for any bounded domain Ω with the help of the weak comparison principle (see Lemma A.2 in this paper), by the same argument as in the beginning of §5 we get Theorem 2 without hypothesis "$\partial\Omega$ is smooth".

7. - Appendix

The purpose of this section is to show

THEOREM A.1. *Let Ω be a bounded domain (not necessarily convex) in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Fix a number $p > 1$. Then there exists a non-trivial non-negative weak solution $u \in W_0^{1,p}(\Omega)$ to the nonlinear eigenvalue problem:*

$$\text{(A.1)} \qquad \begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2u} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where λ is the Poincaré constant (see Theorem 1), *and any non-trivial solution to* (A.1) *is positive in Ω or negative in Ω and belongs to $C^{1+\alpha}$ $(\overline{\Omega})$ for some α $(0 < \alpha < 1)$. Furthermore, if the boundary $\partial\Omega$ is connected, the solutions are proportional (that is, the eigenvalue λ is simple.).*

In order to prove that the eigenvalue is simple we need the following:

LEMMA A.2. (Weak comparison principle) *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\text{(A.2)} \qquad \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla \psi dx \leq \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \cdot \nabla \psi dx$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$, that is,

$$-\operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \leq -\operatorname{div}\left(|\nabla u_2|^{p-2}\nabla u_2\right) \text{ in } \Omega,$$

in the weak sense.

Then the inequality

$$u_1 \leq u_2 \text{ on } \partial\Omega$$

implies that

$$u_1 \leq u_2 \text{ in } \Omega.$$

PROOF. Let $\psi = \max\{u_1 - u_2, 0\}$. Since $u_1 \leq u_2$ on $\partial\Omega$, so ψ belongs to $W_0^{1,p}(\Omega)$. Inserting this function ψ into (A.2), we have

$$\int_{\{u_1 > u_2\}} \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2)\, dx \leq 0.$$

Therefore, using Lemma 1 in [21, p. 129], we obtain

$$u_1 \leq u_2 \text{ in } \Omega.$$

LEMMA A.3. (Hopf's lemma) *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Let $u \in C^1(\overline{\Omega})$ satisfy*

$$\text{(A.3)} \qquad \begin{cases} -\operatorname{div}\,(|\nabla u|^{p-2}\nabla u) \geq 0 \textit{ in } \Omega \textit{ (in the weak sense)}, \\ u > 0 \textit{ in } \Omega, \textit{ and } u = 0 \textit{ on } \partial\Omega. \end{cases}$$

Then $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the unit exterior normal vector to $\partial\Omega$.

PROOF. Let $x_0 \in \partial\Omega$. There exists an open ball $B_{2r}(y) \subset \Omega$ with $x_0 \in \partial B_{2r}(y) \cap \partial\Omega$, where $B_\delta(z)$ denotes an open ball in $\mathbb{R}^n$ centered at z with radius δ. We can find a smooth function v satisfying

$$\text{(A.4)} \qquad \begin{cases} -\operatorname{div}\,(|\nabla v|^{p-2}\nabla v) = 0 \text{ in } B_{2r}(y) - \overline{B_r(y)}, \\ v = 1 \text{ on } \partial B_r\ (y), v = 0 \text{ on } \partial B_{2r}(y), \\ 0 < v < 1 \text{ in } B_{2r}(y) - B_r(y), \text{ and } |\nabla v| \geq c > 0 \\ \text{for some positive constant } c, \end{cases}$$

in [18, Lemma 2, p. 207]. Since u is positive in Ω, we have

$$\tau = \inf\,\{u\ (x); x \in \partial B_r(y)\} > 0.$$

Put $w = \tau v$. Then w satisfies

$$\text{(A.5)} \qquad \begin{cases} -\operatorname{div}\,(|\nabla w|^{p-2}\nabla w) = 0 \text{ in } B_{2r}(y) - \overline{B_r(y)}, \\ w = \tau \text{ on } \partial B_r(y) \text{ and } w = 0 \text{ on } \partial B_{2r}(y). \end{cases}$$

Since $w \leq u$ on $\partial(B_{2r}(y) - \overline{B_r(y)})$, applying Lemma A.2 to w and u ($u_1 = w$ and $u_2 = u$), we obtain

$$w \leq u \text{ in } B_{2r}(y) \ - \ \overline{B_r(y)}.$$

Of course $w\ (x_0) = u\ (x_0)$. Therefore we get

$$\frac{\partial u}{\partial \nu} \leq \frac{\partial w}{\partial \nu} = \tau\ \frac{\partial v}{\partial \nu} < 0 \text{ at } x_0 \in \partial\Omega.$$

This completes the proof.

PROOF of THEOREM A.1. Consider the variational problem (2.1).

Then it follows from Theorem (6.3.2) in [2, p. 325] that there exists at least one solution u_0 to this variational problem. Since $|u_0| \in W_0^{1,p}(\Omega)$ is also a solution, we may assume that $u_0 \geq 0$.

Of course u_0 satisfies the equation of (A.1) in the weak sense. Let u be any non-trivial solution to (A.1). Then we see that $|u|$ is also a non-trivial solution to (A.1) and is non- negative. Hence, concerning the regularity, first by Lemma 9.6 in [9, pp. 213-214] we get $u \in L^{\infty}(\Omega)$. Next, using Theorem 1.1 in [17, p. 251], we get $u \in C^{\beta}(\overline{\Omega})$ for some β $(0 < \beta < 1)$. Therefore, using Proposition 3.7 in [22, p. 806], we see that $u \in C^{1+\alpha}(\overline{\Omega})$ for some α $(0 < \alpha < 1)$. Positivity of $|u|$ follows from Harnack's inequality due to Trudinger [23, Theorem 1.1, p. 724]. This shows that u is positive in Ω or negative in Ω.

Here it remains to show that the eigenvalue λ is simple. Let u_1 and $u_2 \in W_0^{1,p}(\Omega) \cap C^{1+\alpha}(\overline{\Omega})$ be two positive solutions to (A.1). As in Aubin [1, p. 103], we define the number b by

(A.6) $$b = \sup \{\mu \in \mathbb{R}; u_1 - \mu u_2 > 0 \text{ in } \Omega\}.$$

Applying Lemma A.3 to u_1 and u_2, we get

(A.7) $$\frac{\partial u_1}{\partial \nu} < 0 \text{ and } \frac{\partial u_2}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

Therefore we see that b is positive. Obviously, $u_1 - bu_2 \geq 0$ in Ω. Furthermore we can show that there exists a point $z \in \Omega$ where $u_1 - bu_2$ vanishes. Indeed, suppose $u_1 - bu_2 > 0$ in Ω. Since bu_2 is also a positive solution to (A.1) and $\lambda u_1^{p-1} \geq \lambda(bu_2)^{p-1}$ in Ω, we see that

(A.8) $$-\operatorname{div}\,(|\nabla u_1|^{p-2}\nabla u_1) \geq -\operatorname{div}\,(|\nabla\ (bu_2)|^{p-2}\ \nabla(bu_2)) \text{ in } \Omega,$$

in the weak sense.

On the other hand, since $\partial\Omega$ is smooth, there exists a smooth vectorfield ν (x) on some neighborhood of $\partial\Omega$ in $\mathbb{R}^n$ which is equal to the unit exterior normal vector to $\partial\Omega$ for all $x \in \partial\Omega$.

Then in view of Lemma A.3 and the continuity of the derivatives $\frac{\partial u_1}{\partial \nu}$ and $\frac{\partial u_2}{\partial \nu}$ we get from (A.7)

(A.9) $$\frac{\partial u_1}{\partial \nu} < -\delta \text{ and } \frac{\partial u_2}{\partial \nu} < -\delta \text{ on } \overline{\Gamma},$$

where δ is a positive constant and Γ is an open connected neighborhood of $\partial\Omega$ in Ω (since $\partial\Omega$ is connected, we can choose Γ to be connected.). Of course, $|\nabla u_1| \geq \delta$ and $|\nabla u_2| \geq \delta$ on Γ. Therefore it follows from the regularity theory of the elliptic partial differential equation (see [9]) that u_1 and u_2 belong to $C^{\infty}(\Gamma)$. Furthermore we have from (A.9)

(A.10) $$t\ \frac{\partial u_1}{\partial \nu} + (1-t)\ \frac{\partial (bu_2)}{\partial \nu} \leq -\min\,(\delta, b\delta) < 0 \text{ on } \Gamma,$$

for all numbers t $(0 \le t \le 1)$. Thus we obtain from (A.8) and the mean value theorem

$$0 \le -\operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) - \{-\operatorname{div}(|\nabla bu_2|^{p-2}\nabla bu_2)\}$$
$$= -\sum_{i,j} \frac{\partial}{\partial x_i}\left[a^{ij}(x)\frac{\partial}{\partial x_j}(u_1 - bu_2)\right] \text{ in } \Gamma,$$

where $a^{ij}(x) = \int_0^1 \frac{\partial a^i}{\partial q_j}[t\nabla u_1 + (1-t)\nabla(bu_2)]\,dt$, and $a^i(q) = |q|^{p-2}q_i$ $(i = 1, \dots, n)$ for $q = (q_1, \dots, q_n) \in \mathbb{R}^n$. Put $L\cdot = \sum_{i,j} \frac{\partial}{\partial x_i}[a^{ij}(x)\frac{\partial}{\partial x_j}\cdot]$. From (A.10) we see that L is a uniformly elliptic operator on Γ. Consequently, we have

$$\text{(A.11)} \qquad \begin{cases} -L(u_1 - bu_2) \ge 0 \text{ in } \Gamma, \\ u_1 - bu_2 > 0 \text{ in } \Gamma \text{ and } u_1 - bu_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, by using Hopf's boundary point lemma for uniformly elliptic operators (see [9, Lemma 3.4, p. 33]), we get

$$\frac{\partial}{\partial\nu}(u_1 - bu_2) < 0 \text{ on } \partial\Omega.$$

Therefore, in view of the continuity, combining this and the assumption that $u_1 - bu_2 > 0$ in Ω, we have

$$u_1 - (b+\eta)u_2 > 0 \text{ in } \Omega,$$

for some positive number η. This contradicts the definition of the number b (see (A.6)). Thus we see that there exists a point $z \in \Omega$ where $u_1 - bu_2$ vanishes.

Next we show that there exists a point $z^{\sim} \in \Gamma$ where $u_1 - bu_2$ vanishes. Here Γ is the open connected neighborhood of $\partial\Omega$ in Ω in (A.9). Choose a bounded subdomain $\Omega^{\sim}$ of Ω with smooth boundary $\partial\Omega^{\sim}$ which satisfies

$$\overline{\Omega^{\sim}} \subset \Omega, \partial\Omega^{\sim} \subset \Gamma \text{ and } z \in \Omega^{\sim}.$$

Then we have a point $z^{\sim} \in \partial\Omega^{\sim}$ where $u_1 - bu_2$ vanishes.
Indeed, suppose $u_1 - bu_2 > 0$ on $\partial\Omega^{\sim}$. By the continuity we get

$$u_1 - bu_2 \ge \tau > 0 \text{ on } \partial\Omega^{\sim},$$

for some $\tau > 0$. Since the function $w = bu_2 + \tau$ satisfies

$$-\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) = \lambda(bu_2)^{p-1} \text{ in } \Omega^{\sim}$$

in the weak sense, we have

$$\begin{cases} -\operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \ge -\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) \text{ in } \Omega^{\sim}, \\ \qquad \text{and } u_1 \ge w \text{ on } \partial\Omega^{\sim}. \end{cases}$$

Then it follows from Lemma A.2 that

$$u_1 \geq w \text{ in } \Omega^{\sim}.$$

Since $z \in \Omega^{\sim}$, so $u_1(z) \geq w\ (z) = bu_2(z)+r$. This contradicts $u_1(z)-bu_2(z) = 0$. Thus we have a point $z^{\sim} \in \partial\Omega^{\sim} \subset \Gamma$ where $u_1 - bu_2$ vanishes.

Observing that

$$\begin{cases} -L\ (u_1 - bu_2) \geq 0 \text{ in } \Gamma \text{ (see (A.11))}, \\ u_1 - bu_2 \geq 0 \text{ in } \Gamma \text{ and } u_1 - bu_2 = 0 \text{ at } z^{\sim} \in \Gamma, \end{cases}$$

we obtain by the strong maximum principle for uniformly elliptic operators (see [9, Theorem 3.5, p. 34])

$$u_1 - bu_2 = 0 \text{ in } \Gamma. \tag{A.12}$$

Here we define the number $b^{\sim}$ by

$$b^{\sim} = \sup \{\mu \in \mathbb{R}; u_2 - \mu u_1 > 0 \text{ in } \Omega\}. \tag{A.13}$$

It follows from the same argument as above that

$$u_2 - b^{\sim} u_1 = 0 \text{ in } \Gamma. \tag{A.14}$$

Since u_1 is positive in Γ, combining (A.12) and (A.14) we get $bb^{\sim} = 1$. Therefore, observing that $u_1 - bu_2 \geq 0$ and $u_2 - (1/b)u_1 \geq 0$ in Ω, we obtain $u_1 = bu_2$ in Ω. This shows that u_1 and u_2 are proportional. The proof is completed.

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