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Introduction

Let Y be a real C^∞ submanifold of $\mathbb{C}^n$ and Ω a pseudo-convex bounded domain in $\mathbb{C}^n$ with smooth C^∞ boundary.

Let $X = \Omega \cap Y$ and suppose that X is a holomorphic subspace of Ω; can X be defined by the annulation of holomorphic functions in Ω and C^∞ up to the boundary?

This very natural question (all hypothesis are C^∞) was asked to me by professor Forstneric.

In this work we study only the case where $\dim_R Y = 2n - 2$ i.e. the condimension one for X in C^n.

We show:

THEOREM 1: *Let Y be a real submanifold of $\mathbb{C}^n$, C^∞ and of real dimension $2n-2$; let Ω a pseudo-convex bounded domain in $\mathbb{C}^n$, with smooth boundary, let $X = \Omega \cap Y$. If X is a holomorphic subspace of Ω and if Y and $\overline{\Omega}$ are regularly situated then:*

A) $\forall \mathbf{z}_0 \in \overline{X}$, $\exists U_{z_0}$ *neighbourhood of* $\mathbf{z}_0$ *in* $\mathbb{C}^n$, $\exists u_{z_0} \in A^\infty(\Omega \cap U_{z_0})$ *s.t.* $X \cap U_z = \{u_{z_0} = 0\}$

B) *if moreover,* $H^2(\Omega, Z) = 0$ *then* $\exists u \in A^\infty(\Omega)$ *s.t.:* $X = \{u = 0\}$.

Let me recall the notion regular situation (R.S.) introduced by Lojaciewicz [6]:

- if A and B are closed sets in R^n, then A and B are R. S.
- if $A \cap B = \phi$ or $\forall K \subset\subset R^n$, $\exists C > 0$, $\exists \alpha > 0$ s.t:

$$\forall z \in K, d(z, A) + d(z, B) \geq Cd(z, A \cap B)^\alpha.$$

We can drop this R.S. condition, at least in $\mathbb{C}^2$, if we add a stronger convexity condition:

THEOREM 2: *Let Y be a real submanifold of $\mathbb{C}^2$, C^∞ and with real dimension 2; let Ω be a pseudo-convex domain in $\mathbb{C}^2$, bounded with C^∞ smooth boundary and $X = Y \cap \Omega$.*

If X is a holomorphic subspace of Ω and if all points of $Y \cap \partial\Omega$ are points of strict pseudo-convexity of $\partial\Omega$ then A) and B) of theorem 1 hold.

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1. - Local results

§1 - Let Ω be an open pseudo-convex, bounded set in $\mathbb{C}^n$, with C^∞ smooth boundary

Let u in $C^\infty(\mathbb{C}^n)$ and let:

$$Y = \{z \in \mathbb{C}^n \text{ s.t. } u(z) = 0\}; X = \overline{\Omega} \cap Y. \tag{1.1}$$

Let us suppose that u verifies:

$$\begin{cases} 1 - \overline{\partial} u \text{ is flat on } X \\ 2 - u \text{ is flat in no point of } X. \end{cases} \tag{1.2}$$

Then we have:

PROPOSITION 1.1: *If Y and $\overline{\Omega}$ are regularity situated at $0 \in \overline{X}$, there is a neighbourhood U of 0 in $\mathbb{C}^n$, and a function v in $A^\infty(U \cap \Omega)$ such that:*

$$X \cap U = \{v = 0\}.$$

PROOF. Because of the R.S. (regular situation) we get:

$$\exists C > 0, \exists \alpha > 0, \text{ s.t. } \forall z \in \overline{\Omega} \cap U, d(z, Y) \geq C d(z, X)^\alpha. \tag{1.3}$$

But u is flat in no point of X so we get;

$$\exists C > 0, \exists \beta > 0 \text{ s.t. } \forall z \in U, |u(z)| \geq C d(z, Y)^\beta. \tag{1.4}$$

Now, using the fact that $\overline{\partial} u$ is flat on X:

$$\forall k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \exists C_{k,\mu} \text{ s.t. } \forall z \in U \quad |D^\mu(\overline{\partial} U)(z)| \leq C_{k,\mu} d(z, X)^k. \tag{1.5}$$

where, as usual, we used;

$$D^\mu f = \frac{\partial^{|\mu|} f}{\partial z_1^{\mu_1} \ldots \partial z_n^{\mu_n} \partial \overline{z}_1^{\mu_{n+1}} \ldots \partial \overline{z}_n^{\mu_{2n}}}.$$

Using Leibnitz's formula we get:

$$\exists E_{k,\mu} \text{ s.t. } : |D^\mu \left(\frac{\overline{\partial} u}{u}\right)| \leq E_{k,\mu} \frac{d(z, X)^k}{d(z, Y)^{\beta|\mu|}}; \forall k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \forall z \in U. \tag{1.6}$$

With (1.3) it becomes:

$$\left| D^\mu \left(\frac{\overline{\partial} u}{u}\right) \right| \leq \frac{E_{k,\mu}}{C^\beta} \frac{d(z, X)^k}{D(z, X)^{\alpha\beta|\mu|}}, \forall_k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \forall z \in U \cap \overline{\Omega} \tag{1.7}$$

Taking $k \geq \alpha\beta|\mu|$ we have:

$$\left| D^{\mu}\left(\frac{\overline{\partial} u}{u}\right)(z)\right| \text{ is bounded in } U \cap \overline{\Omega}.$$

Now, using Sobolev's theorem we get:

$$\omega = \frac{\overline{\partial} u}{u} \text{ is } C^{\infty} \text{ in } U \cap \overline{\Omega}. \tag{1.9}$$

Let U' be an admissible neighbourhood of 0 in $\overline{\Omega}$ [1], $U' \subset U \cap \Omega$ i.e.:

(1.10) U' is pseudo-convex with C^{∞} smooth boundary and $\partial U' \cap \partial\Omega$ contains a neighbourhood of 0 in $\partial\Omega$

$$\overline{\partial} G = \omega, \text{ with } G \in C^{\infty}(\overline{U}'). \tag{1.11}$$

Let now $h = e^{-G}$, we have:

$$h \in C^{\infty}\left(\overline{U}'\right), h \neq 0 \text{ in } \overline{U}', \tag{1.12}$$

and:

$$\overline{\partial}(hu) = h\overline{\partial} u + u\overline{\partial} h \equiv 0 \text{ in } \overline{U}. \tag{1.13}$$

So the function $v - hu$ is the solution asked for in proposition 1.1.

2. - Manifold

Let Y be a real submanifold of $\mathbb{C}^n (= R^{2n})$ of real dimension $2n-2$. Let X be an open set in Y.

DEFINITION 2.1 : *We call X a holomorphic submanifold of $\mathbb{C}^n$ if:*

$$\forall z \in X, T_z Y \textit{ is a } \mathbb{C}\textit{-linear hyperplane of } \mathbb{C}^n \tag{2.1}$$

where $T_z Y$ is the tangent space of Y at z.

The continuity of the complex structure of $\mathbb{C}^n$ implies:

$$\text{if } X \text{ is holomorphic, then } \forall z \in \overline{X}, T_z Y \text{ is } \mathbb{C}\text{-linear.} \tag{2.2}$$

Let now z_0 be a point of $\overline{X}$; using a $\mathbb{C}$-linear change of coordinates we may suppose:

$$z_0 = 0 \quad \text{and} \quad T_0 Y = \{z_n = 0\} \tag{2.3}$$

Then we have:

LEMMA 2.1: *There is a neighbourhhood U of 0 in $\mathbb{C}^n$ such that:*

$$Y \cap U = \{z_1, \dots, z_{n-1}\}$$

where $f \in C^\infty(\mathbb{C}^{n-1})$ and f is holomorphic in $\pi_n(X \cap U)$.

Here π_n is the canonical projection from $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$

$$(z_1, \dots, z_n) \rightsquigarrow (z_1, \dots, z_{n-1}, 0).$$

PROOF: Using the implicit function theorem we get:

There is U_1, neighbourhood of 0 in $\mathbb{C}^{n-1}$, and $f \in C^\infty(U_1)$ such that:

$$Y \cap U = \{z_n = f(z_1, \dots, z_{n-1})\}, \tag{2.4}$$

where U is a neighbourhood of 0 in $\mathbb{C}^n$.

Let us write the tangent hyperplane in z to Y:

$$(Z_1 - z_n) - \sum_{i=1}^{n-1} (Z_i - z_i) \frac{\partial f}{\partial z_i}(z') - \sum_{i=1}^{n-1} \left(\overline{Z}_i - \overline{z}_i\right) \frac{\partial f}{\partial \overline{z}_i}(z') = 0 \tag{2.5}$$

with $z' = (z_1, \dots, z_{n-1}) \in U_1$.

But, if $z' \in \pi_n(X \cap U)$, then this hyperplane must be $\mathbb{C}$-linear so:

$$\forall z' \in \pi_n (X \cap U), \frac{\partial f}{\partial \overline{z}_i}(z') = 0, i = 1, \dots, n-1 \tag{2.6}$$

and f must be holomorphic in $\pi_n(x \cap U)$.

REMARK 1: π_n is a C^∞ diffeomorphism of Y on $\pi_n(y)$ near 0, because at 0 we have $d\pi_n$ = identity; so $\pi_n(X \cap U)$ is an open set in $\mathbb{C}^{n-1}$.

REMARK 2: Using directly the finer results of G. Galusinsky [3] we can shown that $\overline{X}$ is a regular holomorphic retract of an open set in $\mathbb{C}^n$.

3. - Proof of Theorem 1

Let Y be a submanifold (real) of $\mathbb{C}^n$, $\dim_R Y = 2n - 2$, and Ω a pseudo-convex domain in $\mathbb{C}^n$, bounded with smooth boundary.

Let $X = \Omega \cap Y$ and let us suppose that X is a holomorphic submanifold of $\mathbb{C}^n$.

Using the results of §2, with $u = z_n - f(z')$ we get: $\forall z_0 \in \overline{X}$, $\exists U_{z_0}$ and $u \in C^\infty(U_{z_0})$ with:

$$Y \cap U_{z_0} = \{z \in U_{z_0} \text{ s.t. } u(z) = 0\} \tag{3.1}$$

and, more over:

$$\left.\begin{array}{l}\overline{\partial}u = 0\\ \partial u \neq 0\end{array}\right\} \text{ on } \overline{X} \cap U_z. \tag{3.2}$$

Shrinking U_{z_0} if necessary, if Y and $\overline{\Omega}$ are R.S. in z_0, we can apply the results of §1 and get the A) of theorem 1.

II - Strictly pseudo-convex case

Now we suppose $n = 2$ i.e. Ω is a pseudo-convex domain in $\mathbb{C}^2$, smoothly bounded.

Y is a (real) submanifold of $\mathbb{C}^2$, $X = Y \cap \Omega$, and we suppose that: $0 \in \overline{X} \cap \partial\Omega$ is a point of strict pseudo-convexity of $\partial\Omega$.

Using again the notations of I.2, let us make the change of variables:

$$\begin{cases} Z_1 = z_1 \\ Z_2 = z_2 - f(z_1). \end{cases} \tag{4.1}$$

This is a C^∞ change of variables in a neighbourhood of 0, which is "holomorphic" in 0, i.e. $\overline{\partial} Z_2$ is flat in 0.

Let us denote Ω' the image the Ω by this change of variables, then $\partial\Omega'$ is still strictly pseudo-convex in 0.

Y', image of Y, has for equation:

$$Y' = \left\{ Z \in \mathbb{C}^2 \text{ s. t. } Z_2 = 0 \right\}. \tag{4.2}$$

Now, using the results in [2], we know that the points making obstruction to the regular situation of Y' and $\overline{\Omega}'$ are localized on a curve $\Gamma' \subset \partial\Omega'$, C^∞ and smooth.

So, going back to Y and Ω, we get that the points making obstruction to the regular situation of Y and $\overline{\Omega}$ are localizable on a curve $\Gamma \subset \partial\Omega$, C^∞ and smooth near 0.

Using again the method introduced in [2], we used the projection of Γ onto $\{z_2 = 0\}$:

$$\Gamma'' = \text{ proj } \{z_2 = 0\}\,\Gamma. \tag{4.3}$$

And, because a C^∞ smooth curve is always totally real, we can modify f on $\Gamma'' \cap \pi_2(\overline{\Omega})^c$ in such a way that ([2]):

$$\overline{\partial} f \text{ is flat on } \Gamma. \tag{4.4}$$

Going on reproducing the methods used in [2], we get, again because of the strict pseudo-convexity:

$$(4.5) \qquad \frac{\overline{\partial} f}{z_2 - f} \text{ is in } C^\infty\left(\overline{\Omega}\right) \text{ near } 0.$$

Now, using §1 we get the theorem 2, locally.

III - Globalization

Using the results of I or II we have:

$$(5.1) \qquad \begin{array}{l} \forall z_0 \in \overline{X}, \quad \exists U_{z_0}, \in A^\infty(U_{z_0} \cap \Omega) \text{ s.t.} \\ X \cap U z_0 = \{u_{z_0} = 0\} \text{ and } \partial u_{z_0} \neq 0 \end{array}$$

This gives us a divisor on Ω: let us cover $\overline{\Omega}$ by open sets U_i, with associated functions u_i such that:

(5.2) if $U_i \cap X \neq \emptyset$, then U_i is a U_z and u_i is a u_z by (5.1)

(5.3) if $U_i \cap \overline{X} = \emptyset$ then $u_i = 1$.

We then get the transitions functions:

(5.4) on $U_i \cap U_j$, $g_{ij} = u_i u_j^{-1} \in A^\infty(U_i \cap U_j \cap \Omega)$ and $g_{ij} \neq 0$.

Shrinking U_i if necessary we can assume that:

(5.5) $U_i \cap \Omega$ and $U_i \cap U_j \cap \Omega$ are admissible [1] and simply connected.

Using then Hörmander's method [4, chap V-5], we take logarythms to go to an additive cousin problem, we solve it in $A^\infty(\Omega)$ using J. Kohn's estimates as soon as $H^2(\Omega, Z) = 0$. This ends the proof of theorem 1 and of theorem 2, part B.

REMARK: A. Sebbar [7] proved that a fiber bundle in $A^\infty(\Omega)$ which is topologically trivial is also $A^\infty(\Omega)$ trivial under some hypothesis on Ω.

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