ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, nº 1 (1987), p. 165-167

<http://www.numdam.org/item?id=ASNSP_1987_4_14_1_165_0>

Existence Results for Embedded Minimal Surfaces of Controlled Topological Type, III (*)

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The present note can be considered as an appendix to the second part of our investigations on embedded minimal surfaces, hereafter referred to as [2].

We shall show by an approximation argument that the considerations of [2] can be extended from supporting surfaces of positive mean curvature to surfaces of nonnegative mean curvature.

Also, we shall relax the regularity assumption on the supporting surface.

THEOREM: *Suppose A is a bounded open subset of a three-dimensional Riemannian manifold X, $\overline{A}$ being diffeomorphic to the unit ball, and suppose ∂A is a class C^2 and has nonnegative mean curvature w.r.t. the interior normal. Moreover, assume that $\overline{A}$ contains no embedded minimal two-sphere.*

Then there exists an embedded minimal disk M in A which meets ∂A orthogonally.

PROOF. It was shown in [5] and [6; § 1] that A can be approximated by a sequence of bounded open manifolds A_k with boundary $\partial A_k \in C^4$ of positive mean curvature, in the sense that the metrics and their derivatives of A_k converge to the corresponding ones of A, and that (in our case) ∂A_k converges to ∂A in C^2.

From Theorem 4.1 in [2], we know that A_k contains an embedded minimal two-sphere or an embedded minimal disk meeting ∂A_k orthogonally. Let us denote this minimal surface by Σ_k. Furthermore, there exists a conformal harmonic map $f_k: B \to \overline{A}_k$ with $f_k(B) = \Sigma_k$ where B is S^2 or the closed unit disk, resp. From the construction of [2], it is clear that there exist two fixed positive constants, K_1, K_2 with

$$0 \leq K_1 \leq |\Sigma_k| \leq K_2 < \infty. \tag{1}$$

As Struwe did in [8], we can use the argument of Sacks-Uhlenbeck [7] to

Pervenuto alla redazione il 9 Luglio 1985 ed in forma definitiva il 23 Aprile 1986.

(*) Supported by SFB 72 at the University of Bonn

produce an embedded minimal surface Σ in A as a limit of the Σ_k's. The fact that Σ_k's are already minimal surfaces actually allows a considerable simplification of the argument.

In the simplest case $\sup_B |\nabla f_k|$ is bounded independently of k. Otherwise we let

$$c_k = \sup_B |\nabla f_k|$$

and assume that

$$|\nabla f_k(w_k)| = c_k$$

for a suitable $w_k \in B$.

If $B = S^2$, we project B stereographically onto $\mathbb{R}^2$, w_k corresponding to the origin.

Then, if $c_k \to \infty$ as $k \to \infty$

$$B = \left\{ w \colon w_k + \frac{1}{c_k} w \in B \right\}$$

either converges to $\mathbb{R}^2$ or some half-plane in $\mathbb{R}^2$, w.l.o.g.: $H = \{(u, v) \in \mathbb{R}^2 \colon v > 0\}$. We put

$$\tilde{f}_k(w) = f_k \left(w_k + \frac{1}{c_k} w \right)$$

in case $c_k \to \infty$, otherwise

$$\tilde{f}_k(w) = f_k(w).$$

Therefore, in any case

$$\sup_B |\nabla \tilde{f}_k|$$

is bounded independently of k.

In particular, the $\tilde{f}_k$ are equicontinuous, conformal, harmonic maps. Therefore, they converge uniformly on compact sets to a conformal, harmonic map

$$f \colon B \to \overline{A},$$

i.e. $f(B)$ is a minimal surface. It is clear from the construction that $f(B)$ is not a point (using (1)).

Also, f is of class C^2 on the interior of B.

In order to conclude that f is of class $C^{1,\alpha}$ at the boundary and that $f(B)$ meets ∂A orthogonally, we only have to observe that because ∂A_k converges to ∂A in C^2, $\tilde{f}_k$ converges to f in $C^{1,\beta}$ $(0 < \beta < 1)$ on compact subsets of B_k. This in turn follows because by equicontinuity of the $\tilde{f}_k$ we can localize the problem in the image and then straighten out ∂A_k by diffeomorphisms thereby trasforming the equations for $\tilde{f}_k$ into a nonlinear elliptic system on a half space, where the nonlinearity is quadratic in the gradient of the solution with bounded coefficients (this is the point where we use $\partial A \in C^2$), cf. [3] for details.

Since, by (1), f has finite Dirichlet integral, it extends as a weak solution of the free boundary problem (in the sense of [4]) to either S^2 or D, the closed unit disk.

It follows from [4] that f actually is of class $C^{1,\alpha}$ on S^2 or D, respectively.

On the other hand, the image of f is a limit of embedded minimal surfaces in three-dimensional manifolds and hence cannot have selfintersections. The maximum principle excludes that different sheets touch each other. Hence we obtain an embedded minimal surface.

Since, by assumption the case of an embedded minimal 2-sphere is excluded, we obtain an embedded minimal disk meeting ∂A orthogonally as desired.

BIBLIOGRAPHY

[1] J. JOST, *Existence results for embedded minimal surfaces of controlled topological type, I*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **13** (1986), pp. 15-50.

[2] J. JOST, dto., *II*, **13** (1986), pp. 401-426.

[3] M. GRÜTER, S. HILDEBRANDT and J.C.C. NITSCHE, *On the boundary behaviour of minimal surfaces with a free boundary which are not minima of the area*, Manuscripta Math. **35** (1981), pp. 387-410.

[4] J. JOST, *On the regularity of minimal surfaces with free boundaries in Riemannian manifolds*, Manuscripta Math. **56** (1986), pp. 279-291.

[5] W. MEEKS and S.T. YAU, *The classical Plateau problem and the topology of three-dimensional manifolds*, Top. **21** (1982), pp. 409-442.

[6] W. MEEKS and S.T. YAU, *The existence of embedded minimal surfaces and the problem of uniqueness*, M.Z. **179** (1982), pp. 151-168.

[7] J. SACKS and K. UHLENBECK, *The existence of minimal immersion of 2-spheres*, Ann. Math. **113** (1981), pp. 1-24.

[8] M. STRUWE, *On a free boundary problem for minimal surfaces*, Inv. Math. **75** (1984), pp. 547-560.

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