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The Spectral Distribution of a Globally Elliptic Operator

FERNANDO CARDOSO - RAMON MENDOZA

0. - Introduction

The aim of this paper is to extend some results about the spectral distribution of the harmonic oscillator to more general self-adjoint positive globally elliptic pseudodifferential operators Q, of order two, in $\mathbb{R}^n$, as considered by Helffer in [10].

We start by recalling, in Section 1, a few facts about the positive square root of the laplacian in S^1 and the harmonic oscillator in $\mathbb{R}$. Although these are, in some sense, the simplest examples we may think of, they already give us a hint of the main features and results that can be obtained in the general compact (without boundary) and non-compact contexts, respectively. In Section 2, we use the approximation of the unitary group $\exp(-itQ)$ by a global Fourier integral operator (see, [10]) to show that if the hamiltonian flow of the principal symbol q of Q is completely periodic with minimal positive common period T and the average of the subprincipal symbol of Q, sub$Q$, over these H_q solution curves is equal to a constant γ, then the principal symbol of the pseudodifferential operator $\exp(-iTQ)$ is given by

$$\sigma(e^{-iTQ})(x, \xi) = i^{-\alpha} e^{-i\gamma T}, \tag{1}$$

where α is the Maslov index of the "lifted" bicharacteristic of $\tau + q(x, \xi)$ which passes trough the point $(0, -q(x, \xi); x, \xi; x, -\xi)$.

Using (1), we can give a geometric meaning to formula (3.3.8) in Helffer, [10].

We know (see [11]) that the singularities of the spectral distribution of Q

$$S_Q(t) = \sum_{\lambda \in Sp(Q)} e^{-it\lambda} = \text{trace } e^{-itQ}, \tag{2}$$

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where $Sp(Q)$ denotes the countable set of the eigenvalues of Q, are located in the set $\mathcal{L}_Q$ of periods of periodic bicharacteristics of q, of energy one, i.e.,

(3) $$\text{sing supp } S_Q(t) \subset \mathcal{L}_Q.$$

In section 3, we study the behavior of S_Q at a singular point T under the assumptions that T is an isolated point of $\mathcal{L}_Q$ and the set of bicharacteristics of q of period T is a "good" manifold. We also compute the principal symbol of S_Q at $[T, -1]$. This is the analogue of Poisson's formula of Chazarain, [5], and Duistermaat-Guillemin, [8]. These results were also established by V. Guillemin-S. Stenberg, [9], and by L. Boutet de Monvel, [4], by indirect methods which consist of making transformations that lead to the case of elliptic operators on a compact manifold, [9], or the case of Toeplitz operators, [4]. However, whereas our proof seems to be adaptable to quasi-homogeneous operators, such as the anharmonic oscillator, their methods do not lead in this case to operators whose spectrum has already been studied.

In Section 4, we discuss the geometrical interpretation of the principal symbols of $\exp(-iTQ)$ and of S_Q. We compare them with that of Duistermaat, [7].

1. - The spectral distribution of P_0 and Q_0

We denote by P_0 the positive square root of the laplacian in S^1 and let $Q_0 = \frac{1}{2}(-\partial_x^2 + x^2)$ be the harmonic oscillator in $\mathbb{R}$; in both cases the eigenvalues and the respective eigenfunctions are well-known.

Let us consider the spectral distribution of P_0

(1.1) $$S_{P_0}(t) = \sum_{\lambda \in Sp(P_0)} e^{-it\lambda}$$

where $Sp(P_0)$ is the set of all nonnegative integers. We use the following identities in $\mathcal{D}'(\mathbb{R})$:

(1.2) $$\begin{aligned} 1 + \cos t + \cos 2t + \ldots &= \frac{1}{2} + \pi \sum_{j \in \mathbb{Z}} \delta(t - 2\pi j) \\ \sin t + \sin 2t + \ldots &= \frac{1}{2} \cot \frac{t}{2} \end{aligned}$$

Substituting (1.2) in (1.1), we get

(1.3) $$S_{P_0}(t) = \frac{1}{2} + \pi \sum_{j \in \mathbb{Z}} \delta(t - 2\pi j) - \frac{i}{2} \cot \frac{t}{2}.$$

From formula (1.3), it is clear that the restriction of S_{P_0} to a small neighborhood of the singular point $T_j = 2\pi j$, j an arbitrary integer, is a Fourier integral distribution belonging to $I^{1/4}(\mathbb{R}, \Lambda_{T_j})$, where $\Lambda_{T_j} = \{(T_j, \tau);\ \tau > 0\}$. Of course, T_j is the period of a closed geodesic in S^1 and we obtain, after localization, the following residue formula

$$\lim_{t \to 2\pi j} (t - 2\pi j) S_{P_0}(t) = i^3 \tag{1.4}$$

The eigenvalues $\lambda_j = j$, $j = 0, 1, 2, \dots$ are equal to the sequence of numbers

$$\nu_j = \frac{2\pi}{T}\left(j + \frac{\alpha}{4}\right) + \gamma, \quad j = 0, 1, 2, \dots \tag{1.5}$$

where T is the minimal positive geodesic period and α and γ are as in the Introduction. Of course, in our example, $T = 2\pi$, $\gamma = 0$, and $\alpha = 0$.

In the case of the square root of the laplacian in a compact riemannian manifold or, still more generally, of a positive elliptic selfadjoint operator P of first order on a compact manifold (see [8]), one can not get a formula like (1.5) for its eigenvalues. In fact no general formula is known at all. Nevertheless, under the same assumptions as above, about the bicharacteristic flow of P being completely periodic, the eigenvalues of P will cluster, in a certain sense, around the ν_j, that is, they will be given "approximately" by formula (1.5).

We consider now the harmonic oscillator Q_0; it is well-known that its eigenvalues, all of multiplicity one, are of the form $j + \frac{1}{2}$, j any natural number.

Therefore

$$S_{Q_0}(t) = \sum_{j=1}^{+\infty} e^{-it(j+1/2)}. \tag{1.6}$$

If we note that

$$S_{Q_0}(t) = e^{-it/2}(S_{P_0}(t) - 1),$$

then we obtain

$$S_{Q_0}(t) = \pi \sum_{j \in \mathbb{Z}} (-1)^j \delta(t - 2\pi j) + \frac{e^{-it/2}}{e^{it} - 1} \tag{1.7}$$

One can see directly from (1.7) that the singularities of S_{Q_0} are located at the periods $T_j = 2\pi j$, $j \in \mathbb{Z}$, of the closed bicharacteristics of the principal symbol of Q_0; here we consider $q_0 = \frac{1}{2}(\xi^2 + x^2)$ as the principal symbol of Q_0. It is also clear that S_{Q_0}, restricted to a small neighborhood of T_j, belongs to $I^{1/4}(\mathbb{R}, \Lambda_{T_j})$. After localization we can prove that

$$\lim_{t \to 2\pi j} (t - 2\pi j) S_{Q_0}(t) = i^{2j+1}. \tag{1.8}$$

Formula (1.8) allows us to distinguish between the odd and even periods of the closed bicharacteristics. In fact, (1.8) is equal to i^3 for j odd and to i for j even.

The eigenvalues of Q_0 still satisfy (1.5) since now $T = 2\pi$, $\gamma = 0$ and $\alpha = 2$ (see the appendix). It can be shown that, for general Q, its eigenvalues λ_j cluster around the ν_j, defined by (1.5), in the same way as already observed in the compact case (see [11]).

2. - The unitary group $\exp(-itQ)$

Let Q be a globally elliptic pseudodifferential operator, in R^n, of order two, classical, selfadjoint positive, whose symbol q verifies (see [10], for definitions):

(2.1) $$q(x,\xi) \sim \sum_{j \in \mathbb{N}\cup\{0\}} q_{2-2j}(x,\xi),$$

with q_{2-2j} homogeneous of degree $(2-2j)$ in $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

It is shown in [10], how one can approximate $\exp(-itQ)$, for all $t \in \mathbb{R}$, by a classical global Fourier integral operator whose class is also introduced there.

We shall make the following hypothesis:

(2.2) *The flow* Φ^t *associated to the hamiltonian*

$$H_{q_2} = \left(\frac{\partial q_2}{\partial \xi}, -\frac{\partial q_2}{\partial x}\right)$$

is completely periodic with minimal positive common period $T > 0$, *i.e., we have*:

$$\Phi^T(y,\eta) = (y,\eta), \textit{for all } (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

It is then known, [10], that $\exp(-itQ)$ is a pseudodifferential operator in $G_1^{0,c\ell}(\mathbb{R}^n)$, whose symbol is of the form:

$$a_T(x,\xi) \sim \sum_{j \in \mathbb{N}\cup\{0\}} a_{-2j}^T(x,\xi),$$

where $a_{-2j}^T(x,\xi)$ is homogeneous of degree $(-2j)$ for $|x| + |\eta| \geq 1$. The aim of this Section is to compute its principal symbol $a_0^T(x,\xi)$ and put into evidence its geometric meaning.

The starting point in the approximation alluded to above is the following fact

(2.3) $$\begin{cases} (i^{-1}\partial_t + Q)e^{-itQ} = 0 \text{ in } C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)), \\ e^{-itQ}\big|_{t=0} = I \end{cases}$$

It is well-known (see [10], [11]) from the theory of Hamiltonian-Jacobi that the Schwartz kernel, $U = U(t,x,y)$, of $\exp(-itQ)$, may be represented (locally in t, e.g., for $|t| < T_0$) as a classical global Fourier integral depending in a C^∞ way on the parameter t :

$$U(t,x,y) = (2\pi)^{-n} \int e^{i\phi(t,x,y,\eta)} a(t,x,y) d\eta, \tag{2.4}$$

where the phase

$$\phi_t = \phi(t,x,y,\eta) = S(t,x,\eta) - y\cdot\eta,$$
$$S(t,\lambda x,\lambda\eta) = \lambda^2 S(t,x,\eta), \text{ for } |x|+|\eta| \geq 1, \ \lambda > 0,$$

and the amplitude

$$a(t,x,\eta) \sim \sum_{j\in\mathbb{N}\cup\{0\}} a_{-2j}(t,x,\eta),$$

with $a_{-2j}(t,x,\eta)$ homogeneous of degree $-2j$ in (x,η), $|x|+|\eta| \geq 1$. Furthermore, $S(t,x,\eta)$ is a solution of the eikonal equation:

$$\begin{aligned} &(\partial_t S)(t,x,\eta) + q_2(x,\partial_x S(t,x,\eta)) = 0, \quad |t| < T_0 \\ &S(0,x,\eta) = x\cdot\eta, \end{aligned} \tag{2.5}$$

whereas $a_{-2j}(t,x,\eta)$, $j = 0,1,\ldots$, are solutions, for $|t| < T_0$, of the transport equations with initial conditions 1 if $j = 0$ and 0 if $j > 0$. In particular, we obtain

$$a_0(t,x,\eta) = [\det \partial_x\partial_\eta S(t,x,\eta)]^{1/2} \exp\left[-i\int_0^t \operatorname{sub}Q(\phi^s(y,\eta))ds\right] \tag{2.6}$$
$$\text{with } y = (\partial_\eta S)(t,x,\eta), \quad |t| < T_0.$$

Just as in [2], we put, for $|t| < T_0$,

$$C_t = C_{\phi_t} = \{(x,y,\eta) \in \mathbb{R}^{3n} | \frac{\partial}{\partial\eta}\phi(t,x,y,\eta) = 0\} \tag{2.7}$$

and let $\Lambda_t = \Lambda_{\phi_t}$ be the image of C_t under the mapping

$$C_t \ni (x,y,\eta) \xrightarrow{i} \left(x, \frac{\partial\phi}{\phi x}(t,x,y,\eta), y, -\frac{\partial\phi}{\partial y}(t,x,y,\eta)\right) \in T^*(\mathbb{R}^n\times\mathbb{R}^n).$$

The set $C_t \subset R^{3n}$ is a C^∞ submanifold of codimension n and $\Lambda_t \subset T^*(\mathbb{R}^n\times\mathbb{R}^n)$ is a regularly embedded submanifold under the mapping i of dimension $2n$, such that for $|t| < T_0$,

$$\Lambda_t = \operatorname{graph}(\phi^t)^{-1} \text{ i.e., the set of all points} \tag{2.8}$$
$$(x,\xi,y,\eta) \in T^*(\mathbb{R}^n\times\mathbb{R}^n) \text{ with } (x,\xi) = \phi^t(y,\eta), (y,\eta) \neq 0).$$

We have a volume element v_0^t on C_t such that

$$v_0^t \wedge d\left(\frac{\partial\phi}{\partial\eta_1}\right) \wedge \ldots \wedge d\left(\frac{\partial\phi}{\partial\eta_n}\right) = dx_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge dy_1 \wedge \ldots \wedge dy_n \wedge d\eta_1 \wedge \ldots \wedge d\eta_n.$$

Consider the following diagram (always for $|t| < T_0$)

$$(2.9)\qquad \begin{array}{c} (x,y,\eta) \in C_t \\ p_t \swarrow \qquad \searrow q_t \\ T^*(\mathbb{R}^n) \ni \left(x, \dfrac{\partial\phi}{\partial x}(t,x,y,\eta)\right) \overset{\chi_t}{\longleftarrow} \left(y, -\dfrac{\partial\phi}{\partial y}(t,x,y,\eta)\right) \in T^*(\mathbb{R}^n) \end{array}$$

We choose T_0 small enough so that the matrix $S_{x\eta}(t,x,\eta)$ is invertible for $|t| < T_0$. Consequently, all arrows in (2.9) are diffeomorphisms and $\chi_t \overset{\text{def}}{=} p_t q_t^{-1}$ is a symplectomorphism whose graph is Λ_t. We may, therefore, use $y = (y_1, \ldots, y_n)$ and the dual coordinates, $\eta = (\eta_1, \ldots, \eta_n)$ as coordinates in C_t. We see that

$$(2.10)\qquad v_0^t = \det(S_{x\eta})^{-1} \mathrm{d}y \wedge \mathrm{d}\eta.$$

This never vanishes on C_t. We know (see [2] and [12]) that the principal symbol, $\sigma(U(t))$, of $U(t)$, is a section of the half-density bundle of Λ_t tensored with a section of the Maslov bundle, L_{Λ_t} of Λ_t, corresponding to $a_0|_{C_t}\sqrt{v_t^0}$ under the diffeomorphism i. We consider intrinsic trivializations of both of these bundles. In fact, for the half-density bundle, the projection

$$\pi\colon \Lambda_t \longrightarrow T^*(\mathbb{R}^n) \setminus \{0\}, \quad \pi(x,\xi;y,-\eta) = (x,\xi)$$

is a diffeomorphism, hence

$$(2.11)\qquad \omega = \pi^*\left(|\mathrm{d}x \wedge \mathrm{d}\xi|^{1/2}\right)$$

is a nowhere vanishing half-density. As for L_{Λ_t}, we are interested in finding a trivialization given by a constant section. To see that such a section exists (it is unique up to scalar multiples), we note that the subset Λ_0 of $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ is identical with

$$N^*(\Delta(\mathbb{R}^{2n})) = \{(y,\eta;y,-\eta),(y,\eta) \in T^*(\mathbb{R}^n) \setminus \{0\}\}.$$

Since $N^*(\Delta(\mathbb{R}^{2n}))$ is a normal bundle, $L_{N^*(\Delta(\mathbb{R}^{2n}))}$ possesses a canonical constant section s, corresponding to the identity operator $I\colon C^\infty(R^n) \to C^\infty(\mathbb{R}^n)$ (i.e., $\sigma(I)(y,\eta;y,-\eta) = |\mathrm{d}y\mathrm{d}\eta|^{1/2} \otimes s$. Now extend s to a global section, denoted by σ, by requiring it to be constant along each bicharacteristic

$$(2.12)\qquad (x,\xi;y,-\eta),(x,\xi) = \Phi^t(y,\eta), \quad -\infty < t < \infty.$$

We conclude then, from (2.6) and (2.10) that, for $|t| < T_0$,

$$\sigma(U(t))(\lambda(t)) = \exp\left[-i\int_0^t \operatorname{sub} Q(\Phi^s(y,\eta)\mathrm{d}s\right]\omega\otimes\sigma \tag{2.13}$$

where $\lambda(t) = (\Phi^t(y,\eta); y, -\eta) \in \Lambda_t$,

$$\lambda\colon [0,T] \to T^*(\mathbb{R}^n)\setminus\{0\} \times T^*(\mathbb{R}^n)\setminus\{0\},$$

being a continuous closed curve. Observe that at $t = 0$ the left and right-hand side of (2.13) are equal since both are equal to the symbol of the identity map. Moreover $\omega\otimes\sigma$ is invariant under the Hamilton flow so the right-hand side satisfies the transport equation for $\sigma(U(t))$ induced by the equation $(i^{-1}\dfrac{\partial}{\partial t} + Q)U(t) = 0$. This proves the equality for all $|t| < T_0$.

Denote by $M_1(\lambda(t))$ the vertical space at $\lambda(t)$, i.e., the tangent space at $\lambda(t)$ to the fiber of $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ over $\pi_1(\lambda(t))$, where π_1 is the base projection from $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ onto $\mathbb{R}^n \times \mathbb{R}^n$, and by $M_2(\lambda(t))$ the tangent space to Λ_t at $\lambda(t)$. Choose a continuous curve in the lagrangian-grassmannian $\Lambda(T^*(\mathbb{R}^n\times\mathbb{R}^n))$ over $T^*(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\hat\lambda(t) = (\lambda(t), L^t), \quad 0 \le t \le T,$$

where the lagrangian subspace $L^t \in \Lambda(T_{\lambda(t)}(T^*(\mathbb{R}^n \times \mathbb{R}^n)))$, is transverse to $M_j(\lambda(t))$, $j = 1, 2$. The meaning of (2.13) is, for $|t| < T_0$,

$$\begin{aligned} &\sigma(U(t))(\lambda(t))(L^t)(w_1^t \wedge \ldots \wedge w_{2n}^t) = \\ &\exp\left[-i\int_0^t \operatorname{sub} Q(\Phi^s(y,\eta))\mathrm{d}s\right]\omega(w_1^t\wedge\ldots\wedge w_{2n}^t)s(\lambda(0))(L^0),\end{aligned} \tag{2.14}$$

where $w_i^t = (\mathrm{d}\Phi^t v_i, v_i)$, $i = 1,\ldots,2n$, is a basis of $M_2(\lambda(t))$, (v_i) being a basis of $T(T^*(\mathbb{R}^n))$ at $(y, -\eta)$. Observe that the set of vectors $\{w_1^t,\ldots,w_{2n}^t\}$ is a basis of the tangent space to Λ_t at the point $\lambda(t)$. We remark that because of hypothesis (2.2), (2.14) remains valid for t near T. If we write

$$\gamma(y,\eta)T = \int_0^T \operatorname{sub} Q(\Phi^s(y,\eta))\mathrm{d}s,$$

and choose $v_i = \partial/\partial y_i$, $v_{i+n} = \partial/\partial\eta_i$, $i = 1,\ldots,n$, we obtain from (2.14) and the hypothesis (2.2):

$$\sigma(U(T))(\lambda(0))(L^T) = e^{-i\gamma(y,\eta)T}\sigma(I)(\lambda(0))(L^0). \tag{2.15}$$

Recalling the definition of the Maslov bundle and denoting by $s(M_1^0, M_2^0, L^0, L^T)$ the Hörmander index, where $M_j^0 = M_j(\lambda(0))$, $j = 1, 2$, (2.15) yields

$$\sigma(U(T))(\lambda(0))(L^T) = e^{-i\gamma(y,\eta)T} i^{s(M_1^0, M_2^0, L^0, L^T)} \sigma(I)(\lambda(0))(L^T). \tag{2.16}$$

Since Φ^t is a symplectic transformation in $T^*(\mathbb{R}^n)$, we may consider the closed curve of lagrangian subspaces:

$$\begin{aligned} \overline{\lambda}\colon [0,T] &\to \Lambda(T^*(\mathbb{R}^n \times (\mathbb{R}^n)) \\ t &\to \text{graph}[(\mathrm{d}\Phi^t)(y,\eta)]^{-1}. \end{aligned} \tag{2.17}$$

We can equip $\Lambda(T^*(\mathbb{R}^n \times \mathbb{R}^n))$ with manifold structure by using local charts in $\mathbb{R}^{2n}$. Then we can subdivide $\overline{\lambda}$ into a succession of arcs $\overline{\lambda}_1, \ldots, \overline{\lambda}_r$ each contained in some domain of local coordinates. We assume that the endpoint of $\overline{\lambda}_j$ is the starting point of $\overline{\lambda}_{j+1}$, and that $\overline{\lambda_{r+1}} = \overline{\lambda}_1$. The local charts enable us to transfer each $\overline{\lambda}_j$ as a smooth arc of curve, $\check{\lambda}_j$, in $\Lambda(2n)$, the lagrangian-grassmannian of $\mathbb{R}^{4n}$. We connect by a smooth curve the endpoint of $\check{\lambda}_j$ to the starting point of $\check{\lambda}_{j+1}$, for each $j = 1, \ldots, r$. This yields a closed curve $\check{\lambda}$ in $\Lambda(2n)$ whose Maslov index is, by definition, the Maslov index α of the curve $\overline{\lambda}$. It is easy to show that (see Section 4)

$$s(M_1^0, M_2^0, L^0, L^T) = \alpha \tag{2.18}$$

It is shown in [7] that α is also the Maslov index of the curve

$$\begin{aligned} [0,T] &\to \Lambda_{(y,\eta)}(T^*(R^n)) \\ t &\mapsto [d(\Phi^t)(y,\eta)]^{-1}(V), \end{aligned} \tag{2.19}$$

where V is the vertical space of $T^*(\mathbb{R}^n)$ at $\Phi^t(y,\eta)$. Moreover, if

$$C' = \{(t, -q_2(y,\eta); \Phi^t(y,\eta); y, -\eta), (y,\eta) \neq 0\} \subset T^*(\mathbb{R}) \times T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n),$$

and

$$\begin{aligned} [0,T] &\xrightarrow{\tilde{\lambda}} C' \\ t &\mapsto (t, -q_2(y,\eta); \Phi^t(y,\eta); y, -\eta), \end{aligned} \tag{2.20}$$

one can show (identifying $T \equiv 0$) that the Maslov index of $\tilde{\lambda}$ with respect to C', $\mu_{C'}(\tilde{\lambda})$, is equal to

$$\mu_{C'}(\tilde{\lambda}) = \alpha. \tag{2.21}$$

Indeed, at each point $\tilde{\lambda}(t) \in C'$, $0 \leq t \leq T$, consider the homotopy between the lagrangian subspaces

$$\langle H_\tau \rangle \oplus \text{graph } [\mathrm{d}\Phi^t(y,\eta)]^{-1} = H(0,t)$$

and

$$T_{\tilde{\lambda}(t)}C' = H(1,t),$$

given by the space generated by the following $2n+1$ vectors

$$H(u,t) = \langle H_{\tau+uq_2}, W_1, \ldots, W_{2n}\rangle, \quad 0 \le u \le 1,$$

where $W_i = -u\mathrm{d}q_2(v_i)\partial_\tau + \mathrm{d}\Phi^t(v_i) + v_i$, $1 \le i \le 2n$, (v_i) being a basis of $T(T^*(\mathbb{R}^n))$ at (y,η). The Maslov index of $H(0,t)$ is clearly equal to the Maslov index of $\overline{\lambda}(t) = \text{graph } [\mathrm{d}\Phi^t(y,\eta)]^{-1}$. This is a consequence of the fact that we may choose the fundamental cycle $\Lambda^1(\tilde{M})$ in $\Lambda(2n+1)$, with $\tilde{M} = m \oplus M$, in such a way that $m \cap \langle H_\tau\rangle = 0$. Hence (2.21) holds.

It is possible to relate α with the Morse index of the bicharacteristic of q_2, through (y,η) and with the reduced (mod. 4) Maslov index $\dot{\mu}_{C'}(\tilde{\lambda})$ as defined in Treves, [14].

Finally, from (2.16) and after the obvious identifications, we may state:

THEOREM 2.1. *Under the hypothesis* (2.2),

$$\sigma(e^{-iTQ})(y,\eta) = e^{-i(\gamma(y,\eta)T + \frac{\pi}{2}\alpha)}. \tag{2.22}$$

Note that sub$Q$ is real, since $Q = Q^*$. Hence γ is a real function and, consequently, $|\sigma(e^{-iTQ})| = 1$, as it should.

Assuming that

$$\gamma(y,\eta) = \gamma \textit{ is independent of } (y,\eta) \in \mathbb{R}^{2n} \setminus \{0\}, \tag{2.23}$$

we can derive from (2.22), using the argument given in [11] (see for example [10]), the following result:

THEOREM 2.2. *Under the hypothesis of Theorem* 2.1 *and* 2.23, *let*

$$\nu_k = \frac{2\pi}{T}\left(k + \frac{\alpha}{4}\right) + \gamma, \quad k \ge 1. \tag{2.24}$$

Then there exists $R > 0$, independent of k, such that the spectrum of Q is contained in the union of the intervals centered at ν_k and radius R/k.

3. - Poisson's formula

In this Section we study the singularities of

$$S_Q(t) = \sum_{\lambda \in Sp(Q)} e^{-it\lambda} \tag{3.1}$$

and compute its principal symbol at the points $(T, -1)$ where T belongs to $\mathcal{L}_Q$, the set of periods of the closed bicharacteristics of q_2, of energy one, that is

$$\mathcal{L}_Q = \{t \in \mathbb{R},\ \exists (y,\eta) \in \mathbb{R}^{2n},\ q_2(y,\eta) = 1 \text{ and } \Phi^t(y,\eta) = (y,\eta)\} \tag{3.2}$$

We shall assume throughout that:

$$T \text{ is an isolated point of } \mathcal{L}_Q. \tag{3.3}$$

Using our knowledge of U, from Section 2, we can give an interpretation of $S_Q = \text{Trace } U$. If Δ denotes the diagonal map

$$(t,x) \mapsto (t,x,x): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

and π denotes the projection $(t,x) \mapsto t: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, then we get

$$S_Q = \pi_* \Delta^* U, \tag{3.4}$$

where Δ^* is the pull-back of functions on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ to functions on $\mathbb{R} \times \mathbb{R}^n$, suitably extended to certain distributions, and π_* is the pushforward, i.e., the integration over x.

We recall (see [11]) that the singularities of S_Q occur at $\mathcal{L}_Q$. If the H_{q_2} solution curves of energy one and period T form a "nice" submanifold we can obtain more precise information on the singularity at T. We first need a definition (due to Bott, [3]):

DEFINITION 3.1. *Let M be a manifold and let $\Phi: M \to M$ be a diffeomorphism. A submanifold $Z \subset M$ of fixed points of Φ is called clean if for each $z \in Z$, the set of fixed points of $\mathrm{d}\Phi_z: T_z(M) \to T_z(M)$ equals the tangent space to Z at z.*

It can be shown that if M and Φ are symplectic, Z possesses an intrinsic positive measure $\mathrm{d}\mu_Z$ (see [8]).

If we want to write S_Q as a Fourier integral operator, near T, we must choose first a description of $\exp(-itQ)$ by means of the classical *generating function*:Any point $(x_0, \xi^0, y_0, \eta^0)$ of $\Lambda_T = \text{graph}(\Phi^T)^{-1}$ has a neighborhood of the form $\Gamma_X^0 \times \Gamma_Y^0$, with Γ_X^0 (resp., Γ_Y^0) a conic open neighborhood of (x_0, ξ^0) (resp., of (y_0, η^0)) in $\mathbb{R}^{2n}_{x,\xi} \setminus 0$ (resp., in $\mathbb{R}^{2n}_{y,\eta} \setminus 0$) such that $\Lambda_T \cap (\Gamma_X^0 \times \Gamma_Y^0)$ is the graph of $(\Phi^T)^{-1}: \Gamma_X^0 \overset{\text{onto}}{\longrightarrow} \Gamma_Y^0$. We remind that, here, to say that Γ_X^0 is conic means that

$$(x,\xi) \in \Gamma_X^0 \Longrightarrow (\tau x, \tau\xi) \in \Gamma_X^0 \text{ for all } \tau > 0.$$

By homogeneity and compactness, we can select a finite open conic covering, $\{\Gamma_Y^k\}_{0 \le k \le m}$, of $\mathbb{R}^{2n}_{(y,\eta)} \setminus 0$ such that $\{\Phi^T(\Gamma_Y^k) \times \Gamma_Y^k\}_{0 \le k \le m}$ covers Λ_T. Let $\chi_k(y,\eta)$, $0 \le k \le m$, be a C^∞ function with conically compact support contained in Γ_Y^k, positive-homogeneous of degree zero with respect to (y, η) such that $\Sigma\chi_k \equiv 1$ in $\mathbb{R}^{2n}_{(y,\eta)} \setminus 0$. Of course,

$$e^{-iTQ} = e^{-iTQ} \sum_{k=0}^{m} \chi_k(y, D). \tag{3.5}$$

We consider the operator $\exp(-iTQ)\chi_k(y, D)$. The wave front set of its Schwartz kernel is contained in the set $\Phi^T(\Gamma_Y^k) \times (\Gamma_Y^k)'$ where $(\Gamma_Y^k)' = \{(y, \eta) \in \mathbb{R}^{2n} \setminus 0 \colon (y, -\eta) \in \Gamma_Y^k\}$. By eventually shrinking Γ_Y^k (this might of course force us to increase the number m) we can make symplectic changes of coordinates in Γ_Y^k and $\Phi^T(\Gamma_Y^k)$ respectively, in such a way that (see [1]) there exists a function $S_k(x, \eta)$ defined in the $x\eta$-projection of $\Phi^T(\Gamma_Y^k) \times \Gamma_Y^k$ which is homogeneous of degree two with respect to (x, η), such that $\Phi^T|_{\Gamma_Y^k}$ is given by $(S'_{k\eta}, \eta) \to (x, S'_{kx})$ and $\det S''_{kx\eta} \neq 0$, that is $S_k(x, \eta)$ is locally the generating function for the canonical transformation Φ^T. Next, we take S_k, for t near T, as the solution of the Cauchy problem

$$\mathrm{d}_t S_k(t, x, \eta) + q_2(x, \mathrm{d}_x S_k(t, x, \eta)) = 0, \quad S_k|_{t=T} = S_k(x, \eta). \tag{3.6}$$

The symplectic changes of coordinates alluded to above can be generated respectively by mappings of the form:

$$\begin{aligned} &(y_j, \eta_j) \to (\eta_j, -y_j), \quad j \text{ fixed}, \quad 1 \le j \le n \\ &[(x_i, \xi_i) \to (\xi_i, -x_i), \quad i \text{ fixed}, \quad 1 \le i \le n], \end{aligned}$$

the other coordinates being left fixed, which by a particular case of a general result of *I*. Segal (see [13]) on the action of the metaplectic group by conjugation over the pseudodifferential operators, corresponds to quantizations by Fourier transformations with respect to y_j and x_i respectively. Hence, if $\mathcal{F}_k$ denotes some partial Fourier transformation with respect to (y, x), and the superscript "w" indicates that we are using the Weyl calculus, the Schwartz kernel of the operator

$$U_k(t) = \mathcal{F}_k^{-1} e^{-itQ} \chi_k^w(y, D) \mathcal{F}_k, \quad t \in]T - \varepsilon, T + \varepsilon[, \quad \varepsilon > 0 \text{ small} \tag{3.7}$$

can be represented in the form:

$$U_k(t, x, y) = (2\pi)^{-n} \int e^{i[S_k(t,x,\eta) - y\cdot\eta]} b^k(t, x, \eta) \mathrm{d}\eta, \tag{3.8}$$

where b^k admits an asymptotic expansion (see [10]):

$$b^k(t, x, \eta) \sim \sum_{\alpha \in \mathbb{N}} b^k_{-2\alpha}(t, x, \eta), \tag{3.9}$$

with $b^k_{-2\alpha}(t,x,\eta)$ homogeneous of degree -2α with respect to (x,η), $|x|+|\eta|\geq 1$. Since, evidently,

$$S_Q(t)=\sum_{k=0}^{m} S_Q^k(t) \text{ with } S_Q^k(t)=\mathrm{tr}U_k(t), \tag{3.10}$$

we may focus our attention on $U_k(t,x,y)$.

In order to compute the principal symbol of S_Q^k at $(T,-1)$, we may choose the function $g(t)=T-t$ which evidently satisfies $g'(t)=-1$, $g(T)=0$, and take a cut-off function $\theta(t)\in C_0^\infty(\mathbb{R})$, $\theta(T)=1$, supp $\theta\subset]T-\varepsilon,T+\varepsilon[$, and supp $\theta\cap\mathcal{L}_Q=\{T\}$. We obtain

$$\langle S_Q^k,\theta e^{-i\rho g}\rangle=\left(\frac{\rho}{2\pi}\right)^n\cdot \iiint e^{i\rho(S_k(t,x,\eta)-x\cdot\eta+t-T)}\theta(t)b^k(t,\rho^{\frac12}x,\rho^{\frac12}\eta)\mathrm{d}t\mathrm{d}x\mathrm{d}\eta. \tag{3.11}$$

We recall here an extension of the theorem of the stationary phase, due to Colin de Verdière, [6].

THEOREM 3.1. *Let Y be a riemannian manifold. Let $a\in C_0^\infty(Y)$ and ϕ in $C^\infty(Y)$ be a real-valued phase function. We assume that the critical points of ϕ in the support of a, make up a compact connected submanifold W of Y of codimension ν and that W is a non-degenerated critical manifold for ϕ (i.e., for all $y\in W$, the hessian $\phi''(y)$, restricted to the normal space $N_y=T_yY/T_yW$, is a non-degenerated quadratic form. We denote by σ its signature). We get the following asymptotic behavior:*

$$I(\rho)=\int_Y e^{i\rho\phi(y)}a(y)\mathrm{d}y=\left(\frac{2\pi}{\rho}\right)^{\frac{\nu}{2}} e^{i\frac{\pi}{4}\mathrm{sgn}\sigma}e^{i\rho\phi(w)}p(\rho), \tag{3.12}$$

where $p(\rho)$ admits for $\rho\to\infty$ an asymptotic development of the form:

$$p(\rho)\sim\sum_{k\geq 0} b_k\rho^{-k}, \tag{3.13}$$

with

$$b_0=\int a(y)\left|\frac{\det\phi''(y)}{N_Y}\right|^{-\frac12}\mathrm{d}_W y, \tag{3.14}$$

where $\mathrm{d}_W y$ is the measure induced by the riemannian structure over W.

We apply Theorem 3.1 to (3.11), where the phase

$$\phi_k(t,x,\eta)=S_k(t,x,\eta)-x\cdot\eta+t-T$$

and the amplitude is $\theta(t)b^k(t, \rho^{\frac{1}{2}}x, \rho^{\frac{1}{2}}\eta)$.

The main contribution to (3.11) comes from the critical points of the phase function ϕ_k, contained in the support of the amplitude. These critical points are given by the equations

$$\begin{cases} S_{kx} = \eta \\ S_{k\eta} = x \\ S_{kt} = -1 \end{cases} \tag{3.15}$$

They are in bijection with the points of C' of the form

$$(t, -1; x, \eta; x, -\eta) \tag{3.16}$$

and, according to the definition of C', this means that $t \in \mathcal{L}_Q$; but $t \in \operatorname{supp} \theta$ and, consequently, $t = T$.

We denote by Z' the set of fixed points of Φ^T and by Z the intersection of Z' with the cosphere $q_2 = 1$. It is obvious that Z is in bijection with the set

$$\Sigma_{\phi_k} = \{(T, x, \eta) \colon \Phi^T(x, \eta) = (x, \eta), q_2(x, \eta) = 1\}, \tag{3.17}$$

of critical points of ϕ_k. We shall decompose Z into its connected components:

$$Z = \bigcup_{j=1}^{r} Z_j, \quad d_j = \dim Z_j. \tag{3.18}$$

We can now state:

LEMMA 3.1. *Σ_{ϕ_k} is a non-degenerated critical manifold for ϕ_k if and only if each Z_j, $j = 1, \ldots, r$, is a clean fixed point submanifold of $T^*(R^n)$.*

Proof. The hessian of ϕ_k restricted to the tangent space of $T^*(\mathbb{R}^n)$ is given by the matrix

$$\begin{pmatrix} S_{kxx} & S_{kx\eta} - I \\ S_{k\eta s} - I & S_{k\eta\eta} \end{pmatrix} \tag{3.19}$$

evaluated at points $(T, x, \eta) \in \Sigma_{\phi_k}$. Using formula (4.12) of [7], we can prove that the bilinear symmetric form defined by (3.19) is similar to the bilinear symmetric form (see [7] for its definition)

$$Q(\text{graph } d\Phi_k^T(x, \eta), H \times V, \Delta),$$

where H is the "horizontal space" $\{\binom{\delta y}{\delta \eta}; \delta\eta = 0\}$, V is the "vertical space" $\{\binom{\delta x}{\delta \xi}; \delta x = 0\}$ and Δ is the diagonal of $T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n)$. From its definition we derive that

$$\text{nullspace } Q = \text{graph } d\Phi^T(x, \eta) \cap \Delta, \tag{3.20}$$

which shows that the nullspace of (3.19), at the point (T, x, η), is exactly the set of fixed points of $d\Phi_k^T(x, \eta)$, that is, the tangent space to Z at (x, η), and thus the lemma holds.

It can also be shown that ϕ_k is a clean phase function with excess d_j in a neighborhood of points (T, x, η) with $(x, \eta) \in Z_j$.

In order to simplify the notation, we shall assume temporarily that Z is connected and, consequently, we omit the subscript j. Under the hypothesis of Lemma 3.1., we may apply Theorem 3.1 to (3.11) and obtain

$$(3.21) \qquad \langle S_Q^k, \theta e^{-i\rho g}\rangle = \left(\frac{2\pi}{\rho}\right)^{\frac{1-d}{2}} e^{i\frac{\pi}{4}\operatorname{sgn}\phi_k''} \cdot \int b_0^k(T, x, \eta)\left|\det \phi_k''/N(\Sigma_{\phi_k})\right|^{-\frac{1}{2}} dm + 0\left(\rho^{\frac{d-3}{2}}\right),$$

where dm is the riemannian measure induced from $\mathbb{R}^{2n}$ on Σ_{ϕ_k}.

At this point, we use the results and notations of [10] (see also (2.4) and the arguments which follow it). Let $T_1 < T_0/2$ (one may eventually diminish T_1) and let

$$(3.22) \qquad I_h =]hT_1, (h+2)T_1[, \quad h \in \mathbb{Z},$$

such that $]T-\varepsilon, T+\varepsilon[\subset I_h$. One can then approximate, $\exp(-itQ)$, for $t \in I_h$, by the Fourier integral operator $I(b_h^t, \phi_t^{(h)})$, that is,

$$e^{-itQ} - I(b_h^t, \phi_t^{(h)}) \in C^\infty(I_h, \mathcal{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)),$$

where the phase $\phi_t^{(h)}$ is given by:

$$(3.23) \qquad \begin{cases} \phi_t^{(h)}(x, \xi, y) = S(t - hT_1, x, \eta_{|h|+1}) - y_{|h|+1} \cdot \eta_{|h|+1} + \\ \qquad\qquad + \sum_{j=1}^{|h|} (S(T_1, y_{j+1}, \eta_j) - y_j \cdot \eta_j) \\ y_1 = y \\ \xi = (\eta_1, y_2, \eta_2, \ldots, y_{|h|+1}, \eta_{|h|+1}) \in \mathbb{R}^{(2|h|+1)n}, \end{cases}$$

and the amplitude b_h^t is given by:

$$(3.24) \qquad b_h^t(x, \xi, y) = (2\pi)^{-nh} a(t - hT_1, x, \eta_{|h|+1}) \prod_{j=1}^{|h|} a(T_1, y_{j+1}, \eta_j).$$

Moreover, by Lemma 3.3.1 of [10],

$$(3.25) \qquad \Lambda_{\phi_t^{(h)}} = \operatorname{graph}(\Phi^t)^{-1}, \quad \text{for all } t \in I_h.$$

The composition theorem (see [10], Theorem 2.5.2) yields:

$$I(b_h^T, \phi_T^{(h)})\chi_k(., D) = I(c_h^T, \phi_T^{(h)}) \tag{3.26}$$

where

$$c_h^T \sim \sum_\alpha c_{h,\alpha}^T$$

with

$$c_{h,0}^T = \chi_k(y, -\partial\phi_T^h/\partial y) b_{h,0}^T(x, \xi, y), \tag{3.27}$$

and

$$\begin{aligned} b_{h,0}^T &= (2\pi)^{-n|h|} a_0(T - hT_1, x, \eta_{|h|+1}) \prod_{j=1}^{|h|} a_0(T_1, y_{j+1}, \eta_j), \\ b_h^T &\equiv b_{h,0}^T \quad (\text{modulo } \Gamma^{-2,cl,\phi_T^{(h)}}), \end{aligned} \tag{3.28}$$

where a_0 is given by (2.6). Let

$$\psi_k^t(x, \eta, y) = S_k(t, x, \eta) - y \cdot \eta. \tag{3.29}$$

The definition of S_k, (3.9), (3.25), (3.27), (3.28), (3.29) and formula 2.10.21 of [10], imply that, in Σ_{ϕ_k},

$$b_0^k(T, x, \eta) = i^{-\alpha} \chi_k(x, \eta) e^{-i\gamma(x,\eta)T} \, |\det S_{kx\eta}|^{\frac{1}{2}} \tag{3.30}$$

where α and γ are defined in Section 2. We have used the fact that (see [10], formulas 3.3.13 to 3.3.15):

$$(2\pi)^{n|h|} b_{h,0}^T \left|\det D(\phi_T^{(h)})\right|^{-\frac{1}{2}} = e^{-i\gamma(x,\eta)T}. \tag{3.31}$$

The factor $i^{-\alpha}$ in (3.30) is the Maslov factor picked up by the amplitude as in [8], page 68. On the other hand, a simple computation shows that

$$\left|\det \phi_k'' \Big/ N(\Sigma_{\phi_k})\right| = |\det S_{kx\eta}| \left|\det(I - \mathrm{d}\Phi^T) \Big/ N(Z)\right|, \tag{3.32}$$

and an argument similar to that which led to formula (6.16), in [8], yields:

$$\frac{1}{2}\operatorname{sgn} \phi_k'' = \alpha - \sigma - \frac{1}{2}(d - 1). \tag{3.33}$$

Substituting (3.30), (3.32) and (3.33) in (3.21) and taking into account Lemma (4.4) in [8], we obtain that the principal symbol of S_Q at $(T,-1)$ is equal to (recall that we are assuming that Z is connected)

$$\sigma(S_Q)(T,-1)=\left(\frac{1}{2\pi i}\right)^{\frac{d-1}{2}} i^{-\sigma}\int e^{-i\gamma T}\mathrm{d}\mu_Z|\mathrm{d}\tau|^{\frac{1}{2}}. \tag{3.34}$$

REMARK 3.1. When Z is not connected, we evidently get:

$$\sigma(S_Q)(T,-1)=\sum_{j=1}^{r}\left(\frac{1}{2\pi i}\right)^{\frac{d_j-1}{2}} i^{-\sigma_j}\int e^{-i\gamma_j T}\mathrm{d}\mu_{Z_j}, \tag{3.35}$$

where $\gamma_j = \gamma_j(y,\eta)$ denotes the average of sub Q over the periodic bicharacteristic curve of q_2 through $(y,\eta)\in Z_j$ and $\sigma_j=\alpha_j-\frac{1}{2}\mathrm{sgn}\ \phi_j''-\frac{1}{2}(\mathrm{d}_j-1)$, with sgn $\phi''/N(\{T\}\times Z_j)$. Furthermore, in case Z is connected it follows that $\sigma(S_Q)(T,-1)\neq 0$ and, consequently,

$$(T,-1)\in WF(S_Q). \tag{3.36}$$

To sum up we have proved the following theorem, analogous to Theorem 4.5 in [8].

THEOREM 3.2. *Assume that the set of periodic H_{q_2} solutions curves of period T is a union of connected submanifolds $Z_1, Z_2, \ldots, Z_r$ in the cosphere $\{q_2=1\}$, each Z_j being a clean fixed point set of Φ^T of dimension d_j. If T is an isolated point of $\mathcal{L}_Q$, then there is an interval around T in which no other periods occur and on such interval we have*

$$S_Q(t)=\sum_{j=1}^{r}\beta_j(t-T) \tag{3.37}$$

where

$$\beta_j(t)=\int_{-\infty}^{\infty}\delta_j(s)e^{-ist}\mathrm{d}s,$$

with

$$\delta_j(s)\sim\left(\frac{s}{2\pi i}\right)^{\frac{(d_j-1)}{2}} i^{-\sigma_j}\sum_{k=0}^{\infty}\delta_{j,k}s^{-k} \ \ \textit{as}\ s\to\infty. \tag{3.38}$$

and

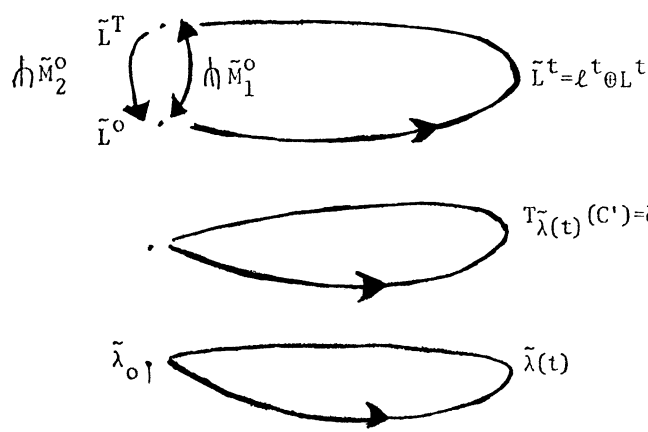
$$\delta_{j,0}=\int e^{-i\gamma_j T}\mathrm{d}\mu_{Z_j}. \tag{3.39}$$

From Theorem 3.2 we can obtain, as in [8], the residue formula generalizing (1.8).

4. - Geometrical interpretation of the principal symbol of exp $(-iTQ)$ and $S_Q(t)$

We refer to the comments and notations of Section 2. We shall relate the Hörmander index $s(M_1^0, M_2^0; L^0, L^T)$ (see formula (2.16)) with the Maslov index of the closed bicharacteristic $\tilde{\lambda}$, through the point $\tilde{\lambda}_0 = (0, -q_2(y,\eta); y, \eta; y, -\eta)$ of C' (here we identify 0 with T), and also with the Maslov index of $t \mapsto ((d\Phi)^t(y,\eta))^{-1}(V)$ where V is the vertical space of $T^*(\mathbb{R}^n)$ at $\Phi^t(y,\eta)$. Let us consider the following picture:

(4.1)



We recall that $\tilde{L}^t$ is a curve of lagrangian subspaces connecting $\tilde{L}^0$ to $\tilde{L}^T$, which is transverse to both the vertical $\tilde{M}_1^t$ of $T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ and to the tangent space $\tilde{M}_2^t$ to C' at $\tilde{\lambda}(t)$. Consider the closed curve starting at $\tilde{L}^0$, given by $\tilde{L}^t$, followed by the segment from $\tilde{L}^T$ to $\tilde{L}^0$ transverse to $\tilde{M}_2^0$ at $\tilde{\lambda}_0$. The latter is contained in the set of lagrangian subspaces over $\tilde{\lambda}_0$. We obtain in this way a curve that over each point $\tilde{\lambda}(t)$ is transverse to $\tilde{\sigma}(t)$. It is not difficult to see that the Maslov index of these two curves must be equal; in fact, we may choose continuous vector fields $e_i(t)$ and $f_i(t)$, $i = 1, \ldots, 2n+1$, such that these curves are given by

$$t \mapsto \langle e_1(t), \ldots, e_{2n+1}(t) \rangle = \tilde{L}^t$$
$$t \mapsto \langle f_1(t), \ldots, f_{2n+1}(t) \rangle = \tilde{\sigma}(t),$$

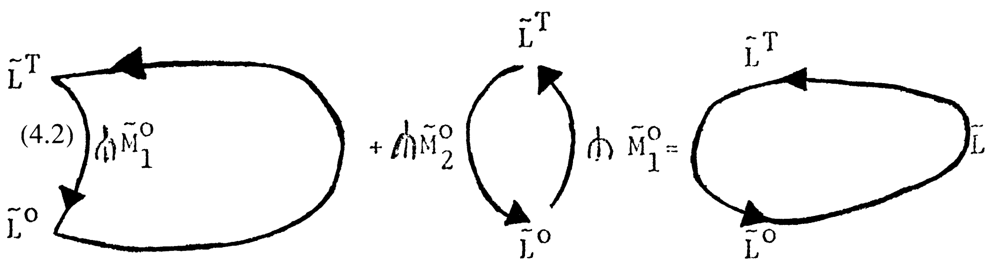
where $\{e_i(t), f_i(t)\}$ is a symplectic basis of the tangent space to $T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ at $\tilde{\lambda}(t)$. We define:

$$H(t,s) = \langle \cos s e_1(t) + \sin s f_1(t), \dots, \cos s e_{2n+1}(t) + \sin s f_{2n+1}(t)\rangle,$$
$$0 \le t \le 1, \quad 0 \le s \le \pi/2.$$

This is a homotopy in $\Lambda(T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n))$ between $H(t,0) = \tilde{L}^t$ and $H(t,\pi/2) = \tilde{\sigma}(t)$.

We shall look now at the following picture:

(4.2)



We observe that the first curve from left to right is transverse to the vertical space $\tilde{M}_1^t$ at each t; since the latter is identified, in any coordinate system in $\Lambda(T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n))$ coming from a coordinate system in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, with a fixed lagrangian subspace (actually $i\mathbb{R}^{2n+1}$), we conclude that the first curve from left to right in (4.2) is contained in $\Lambda^0(i\mathbb{R}^{2n+1})$. Because this set is simply connected, it follows at once that the Maslov index of the alluded curve equals zero. We then get that the Maslov index of the middle curve is equal to the Maslov index of the third curve and hence to the Maslov index of $\tilde{\sigma}$. Recalling the definition of $\mu_{C'}(\tilde{\lambda})$, we have shown that

$$s(\tilde{M}_1^0, \tilde{M}_2^0, \tilde{L}^0, \tilde{L}^T) = s(M_1^0, M_2^0, L^0, L^T) = \mu_{C'}(\tilde{\lambda}), \tag{4.3}$$

which together with (2.21), proves (2.18).

Appendix

Consider the harmonic oscillator $Q_0 = \frac{1}{2}(-\partial_x^2 + x^2)$ and denote (after identifying $\mathbb{R}^2$ with $\mathbb{C}$) the wave front set of the Schwartz kernel $U(t,x,y)$ of $\exp(-itQ_0)$ by

$$\Lambda = \left\{ \lambda = \left(t, -\frac{1}{2}|z|^2; e^{-it}z; \overline{z} \right), \ t \in \mathbb{R}, z = y + i\eta \right\}.$$

The tangent space at $\lambda \in \Lambda$ is generated by the vectors

$$e_1 - ie^{-it}z, \quad -f_1 + e^{-it} + e_3 \text{ and } -f_1 - ie^{-it} - f_3,$$

where

$$e_1 = (1, 0; 0, 0; 0, 0), \quad f_1 = (0, 1; 0, 0; 0, 0), \quad e_2 = (0, 0; 1, 0; 0, 0) \text{ etc.,}$$

$e^{-it} = \cos t e_2 - \sin t f_2$ and similarly for ie^{-it} and $ie^{-it}z$. We remind that the symplectic form ω_n in C^n is

$$\omega_n(z, z') = -Im \sum_{k=1}^{n} z_k \overline{z}'_k, \quad z = (z_1, \dots, z_n), \quad z' = (z'_1, \dots, z'_n).$$

Let

$$\text{(A.1)} \qquad \begin{gathered} [0, 2\pi] \xrightarrow{\rho} \Lambda \\ \rho(s) = (s, -1; e^{-is}(1 + i); 1 - i) \end{gathered}$$

be the lifted bicharacteristic curve of $i^{-1}\partial_t + Q_0$ through the point $(0, -1; 1 + i; 1 - i) \in \Lambda$ and consider the curve $\tilde{\rho}(s)$ of lagrangian subspaces:

$$\tilde{\rho}(s) = T_{\rho(s)}(\Lambda) = \langle e_1 - ie^{-is}(1 + i), -f_1 + e^{-is} + e_3, -f_1 - ie^{-is} - f_3 \rangle.$$

It can be shown that

$$\tilde{\rho}(s) = \langle a, b + e^{-is}, c - ie^{-is} \rangle,$$

where

$$a = e_1 - e_3 - f_3, \quad b = -f_1 + e_3 \text{ and } c = -f_1 - f_3.$$

Let

$$H(u, s) = \langle a, ub + e^{-is}, c - uie^{-is} \rangle, \quad 0 \le u \le 1.$$

We easily verify that

$$\omega(ub + e^{-is}, c - uie^{-is}) = 0$$

Since $H(u, s)$ defines an homotopy of lagrangian curves between

$$H(0, s) = \langle a, e^{-is}, c \rangle \text{ and } H(1, s) = \tilde{\rho}(s),$$

and the curve $s \mapsto H(0, s)$ can be symplectically transformed into the curve:

$$[0, 2\pi] \ni s \overset{\sigma}{\longmapsto} \langle \cos s e_1 - \sin s f_1, e_2, e_3 \rangle,$$

the Maslov index α of $\rho(s)$ is equal to the intersection number of σ. In order to find this intersection number, we choose a fixed lagrangian subspace, say

$$i\mathbb{R}^3 = \langle f_1, f_2, f_3 \rangle,$$

and consider the fundamental cycle $\Lambda^1(i\mathbb{R}^3) = \{N \in \Lambda(3); \dim N \cap i\mathbb{R}^3 = 1\}$. It is easy to see that σ intersects $\Lambda^1(i\mathbb{R}^3)$ at $\langle f_1, e_2, e_3 \rangle$, i.e., precisely for $s = \dfrac{\pi}{2}$

and $\frac{3\pi}{2}$. Take L such that $L \cap \sigma\left(\frac{\pi}{2}\right) = \{0\}$ and $L \cap i\mathbb{R}^3 = \{0\}$, for instance, L may be chosen equal to $\langle e_1, f_2 - e_2, f_3 - e_3\rangle$. A straight computation gives (we use the notation of [7])

$$A(f_1) = -\frac{\cos s}{\sin s} e_1, \quad A(f_2) = e_2 - f_2, \quad A(f_3) = e_3 - f_3,$$

where $A: i\mathbb{R}^3 \to L$ is the unique linear mapping such that

$$\sigma\left(\frac{\pi}{2}\right) = \{u + Au; u \in i\mathbb{R}^3\}.$$

Therefore,

$$Q(\rho(s))(f_1, f_1) = \omega_3(Af_1, f_1) = -\frac{\cos s}{\sin s}.$$

The derivative of this function at both $s = \frac{\pi}{2}, \frac{3\pi}{2}$ is equal to 1, which allows us to conclude the Maslov index α is equal to 2.

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