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Introduction

Recently Matano [14] proved that if $\underline{v}$ and $\overline{v}$ are time-independent strict sub- and supersolutions of an order-preserving nonlinear semigroup $T(t)$ and if $\underline{v} \leq \overline{v}$, then there is a stable stationary solution w between $\underline{v}$ and $\overline{v}$. (Of course there are some technical assumptions as well.) This result is very useful as it is one of the few general techniques for producing stable solutions. In applications these are by far the most important stationary solutions. Matano [13,15] and Matano and Mimura [16] have applied these ideas to a number of weakly nonlinear second order elliptic equations and certain systems of such equations. In this paper we use related arguments to prove a corresponding theorem for the existence of stable T-periodic solutions of a single quasi-linear second order parabolic equation having T-periodic coefficients. For such equations we prove a natural analogue of the result of Matano. There are two ways this problem could be attacked: one can work with the Poincaré map or one can proceed more directly. Either approach can be used, but we prefer the second one. We remain entirely within the framework of sub- and supersolutions and the monotone iteration schemes they induce. It seems that for quasilinear problems the employed iteration schemes were not known before; in the elliptic case the setup of these schemes has been suggested by the result of Hofer [10].

We also deduce the existence of a stable stationary solution in the autonomous case by showing that, in this case, a time-dependent periodic solution must be unstable. The stability proved in Matano tends to be a stability in very strong norms. We show certain autonomous cases that this implies stability in much weaker norms. This depends upon work of Weissler [21]. It seems probable that this result can be extended to apply to much more general situations.

We further prove a variant Matano's result where we do not require that $\underline{v}$ and $\overline{v}$ be *strict* sub- and supersolutions and obtain a solution stable with respect to perturbations of the initial conditions in the order interval $[\underline{v}, \overline{v}]$. This

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is sometimes useful because it may happen that stability relative to the order interval is the natural stability in a particular situation. We show this in our application to a type of Fisher's equation occurring in population genetics.

1. - Statement of the results

Let $\Omega \in \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2+\mu}$ $(\mu \in]0,1[)$, and let $\mathcal{L} := \frac{\partial}{\partial t} + A\left(x, t, \frac{\partial}{\partial x}\right)$ be a uniformly parabolic linear differential expression with

$$(1.1) \qquad A\left(x,t,\frac{\partial}{\partial x}\right)u = -\sum_{j,k=1}^{N} a_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + a_0(x,t)u.$$

We assume that, for fixed $T > 0$, the coefficient functions $a_{jk} = a_{kj}$ and a_0 are in the real Banach space $E := \{w \in C^{\mu,\mu/2}(\overline{\Omega}\times\mathbb{R}) : w \text{ is } T\text{-periodic in } t\}$. Further let $\beta \in C^{1+\mu}(\partial\Omega, \mathbb{R}^N)$ be an outward pointing, nowhere tangent vector field on $\partial\Omega$ and $b \in C^{1+\mu}(\partial\Omega)$, $b \geq 0$. Define the boundary operator $B = B\left(x, \frac{\partial}{\partial x}\right)$ either by $Bu = u$ (implying Dirichlet boundary conditions, abbreviated DBC), or by $Bu = \frac{\partial u}{\partial \beta} + bu$ (implying Neumann or regular oblique derivative boundary conditions, abbreviated NBC). Finally, denote by (x, t, ξ, η) a generic point of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and let the continuous function $g: \overline{\Omega} \times \mathbb{R}^{N+2} \to \mathbb{R}$ be such that $g(x, t, \xi, \eta)$ is T-periodic in t and of class $C^{\mu,\mu/2}(\overline{\Omega} \times \mathbb{R})$ in (x, t) uniformly for (ξ, η) in bounded subsets of $\mathbb{R} \times \mathbb{R}^N$, and such that $\frac{\partial g}{\partial \xi}$ and $\frac{\partial g}{\partial \eta_i}$ $(i = 1, \ldots, N)$ exist and enjoy the same properties as g. Moreover suppose there exists a function $c: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(1.2) \qquad |g(x, t, xi, \eta)| \leq c(\rho)\left(1 + |\eta|^2\right)$$

for every $\rho \geq 0$ and $(x, t, \xi, \eta) \in \overline{\Omega} \times \mathbb{R} \times [-\rho, \rho] \times \mathbb{R}$.

We consider the quasilinear periodic-parabolic boundary value problem

$$(\star) \qquad \begin{cases} Lu = g(x, t, u, \nabla u) & \text{in } \Omega \times \mathbb{R} \\ Bu = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u(\bullet, t) = u(\bullet, t+T) & \text{on } \overline{\Omega}, \quad \forall t \in \mathbb{R}. \end{cases}$$

Problems of this type arise naturally, e.g. in population dynamics, if one looks at the population density in a non-homogeneous medium and assumes that both diffusion and growth rate are subject to seasonal variations. In such situations stable periodic solutions are of particular interest. Here we define the solution

u of $(\star)$ to be stable provided to arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that, if $v_0 \in C_B^{2+\mu}(\overline{\Omega})$ satisfies

$$\text{a)} \quad \|v_0 - u(\bullet, 0)\|_{C_0^1(\overline{\Omega})} < \delta \text{ for DBC,}$$
$$\text{b)} \quad \|v_0 - u(\bullet, 0)\|_{C(\overline{\Omega})} < \delta \text{ for NBC,}$$

then for the solution v of the associated initial boundary value problem with $v(\bullet, D) = v_0$ we have $\|v - u\|_{C(\overline{\Omega}\times[0,\infty[)} < \varepsilon$ (where $C_B^{2+\mu}(\overline{\Omega} := \{w \in C^{2+\mu}(\overline{\Omega}) : Bw = 0\})$. By a solution of $(\star)$ we mean a function u belonging to $C^{2+\mu,1+\mu/2}(\overline{\Omega}\times\mathbb{R})$.

DEFINITION. The function $\overline{v} \in C^{2,1}(\overline{\Omega}\times]0,\tilde{T}]) \cap C^{1,0}(\overline{\Omega}\times[0,\tilde{T})$, where $\tilde{T} > T$, is called *supersolution* for problem $(\star)$ provided

$$\begin{cases} L\overline{v} \geq g(\bullet,\bullet,\overline{v},\nabla\overline{v}) & \text{in } \Omega\times]0,T] \\ B\overline{v} \geq 0 & \text{on } \partial\Omega\times]0,T] \\ \overline{v}(\bullet,0) \geq \overline{v}(\bullet,T) & \text{on } \overline{\Omega}. \end{cases}$$

A supersolution is a *strict supersolution* if it is not a solution. Correspondingly subsolution and strict subsolution are defined by reversing the inequality signs.

It is well-known that between given order-related sub- and supersolutions $\underline{v} \leq \overline{v}$ of $(\star)$ there exist a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ in the order-interval $[\underline{v}, \overline{v}]$ ([2, Thm. 1.2]).

THEOREM 1. *Suppose $\underline{v}$ and $\overline{v}$ are strict sub- and supersolutions of $(\star)$, respectively, with $\underline{v} \leq \overline{v}$ in $\overline{\Omega}\times[0,T]$. Then there exists at least one stable solution u of $(\star)$ with $\underline{v} \leq u \leq \overline{v}$.*

With minor modifications in the proof we also get

THEOREM 2. *Suppose u_1 and u_2 are solutions of $(\star)$, $u_1 < u_2$. Then there exists at least one solution u with $u_1 \leq u \leq u_2$, which is stable with respect to the order interval $[u_1, u_2]$.*

(This means that we only admit initial conditions $v_0 \in C_B^{2+\mu}(\overline{\Omega})$ satisfying $u_1(\bullet,0) \leq v_0 \leq u_2(\bullet,0)$.)

Next we consider the quasilinear elliptic boundary value problem

$$(\star\star) \qquad \begin{cases} Au = g(x,u,\nabla u) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A = A\left(x, \frac{\partial}{\partial x}\right)$ is a uniformly linear differential expression of the form (1.1) with time-independent coefficient functions belonging to $C^{\mu}(\overline{\Omega})$, and $g : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a function as above, but independent of t. The following result are consequences of Theorems 1 and 2.

THEOREM 3. *If $\underline{v}$ and $\overline{v}$ are strict sub- and supersolutions of* $(\star\star)$, *respectively, with $\underline{v} \leq \overline{v}$ in $\overline{\Omega}$, there exists at least one stable solution $u : \underline{v} \leq u \leq \overline{v}$, of* $(\star\star)$.

THEOREM 4. *Suppose u_1 and u_2 are solutions of* $(\star\star)$, *$u_1 < u_2$. Then there exists at least one solution u of* $(\star\star)$ *with $u_1 \leq u \leq u_2$, which is stable with respect to $[u_1, u_2]$.*

The paper is organized as follows. In Section 2 we construct the iteration schemes needed in Section 3 for the proof of Theorems 1 and 2. In Section 4 we discuss the autonomous case, while in Section 5 we obtain a better stability result. In Section 6 we give an application to the following model equation in population genetics, a spatially inhomogeneous periodic version of Fisher's equation.

$$(\star\star\star) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - a(x,t)\Delta u = s(x,t)h(u) & \text{in } \Omega \times \mathbb{R} \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u(\bullet, t) = u(\bullet, t+T) & \text{on } \overline{\Omega}, \quad \forall t \in \mathbb{R}. \end{cases}$$

Here a and s are functions of the space E, a is positive on $\overline{\Omega} \times \mathbb{R}$, and n denotes the normal to $\partial\Omega$. It is assumed that $h \in C^1(\mathbb{R})$ satisfies $h(0) = h(1) = 0$, $h(\xi) > 0$ for $0 < \xi < 1$, and $h'(0) > 0$, $h'(1) < 0$. Moreover the function s is assumed to change sign in $\overline{\Omega} \times [0, T]$ (selective advantage/disadvantage). Problem $(\star\star\star)$ admits the two trivial solutions 0 and 1, and only solutions $u : 0 \leq u \leq 1$ are of practical interest. From Theorem 2 it follows that $(\star\star\star)$ has always such a solution which is stable with respect to $[0, 1]$. We study the stability question for $(\star\star\star)$ in more detail by employing the notion of "principal eigenvalue", as it has been introduced for periodic-parabolic problems in [4,12] and in particular [3]. (For a discussion of the existence of nontrivial equilibrium solution of $(\star\star\star)$ if $a = 1$ and s is independent of t, cf. [5,19,20].).

2. - A monotone iteration scheme for quasilinear periodic-parabolic problems

Besides E, we employ in the following the real Banach space $F_\mu := \{w \in C^{2+\mu, 1+\mu/2}(\overline{\Omega} \times \mathbb{R}) : Bw = 0 \text{ on } \partial\Omega \times \mathbb{R} \text{ and } w \text{ is } T\text{-periodic in } t\}$, $0 < \mu < 1$, ordered by the cone of pointwise nonnegative functions. We use the standard notations for positivity in an ordered Banach space with positive cone $P : w \geq 0$ iff $w \in P$, $w > 0$ iff $w \geq 0$ but $w \not\equiv 0$, and $w \gg 0$ iff $w \in \text{int}(P)$.

Without loss of generality we may assume that the coefficient function of order zero of A, $a_0 \geq \varepsilon_0 > 0$ on $\overline{\Omega} \times \mathbb{R}$. It is proved in [3] (at least for DBC, for NBC the proof is similar) that the realization L in E of the differential expression $\mathcal{L}$ subject to the boundary- and periodic condition, has domain $D(L) = F_\mu$ and is bijective: $F_\mu \to E$. Since $L \in \mathcal{L}(F_\mu, E)$, it is a closed operator in E with compact inverse. Moreover $L^{-1} : E \to F_\mu$ (in case of DBC) and $L^{-1} : E \to E$ (in case NBC) is strongly positive by the parabolic maximum principle. We set $Q_T := \Omega \times]0, T[$.

LEMMA 2.1. *For every $b \in E$ there exists a unique solution $u \in F_\mu$ of the equation*

$$Lu = b\left(1 + |\nabla u|^2\right) \tag{2.1}$$

in E, and $\|u\|_{C^{1+\mu,(1+\mu)/2}(\overline{Q}_T)} \le d(\|b\|_{C(\overline{Q}_T)})$.

PROOF. Note that the constants $\pm M$, where $M = \varepsilon_0^{-1}\|b\|_{C(\overline{Q}_T)}$, are sub- and supersolutions of problem (2.1), respectively. Hence by [2, Thm. 1.2] there exists at least one solution $u \in F_\mu$ of (2.1), with $\|u\|_{C(\overline{Q}_T)} \le M$. Now [1, Thm. 2.4] implies that

$$\begin{aligned}\|u\|_{C^{1+\mu,(1+\mu)/2}(\overline{Q}_T)} &= \|u\|_{C^{1+\mu,(1+\mu)/2}(\overline{\Omega}\times[T,2T])}\\ &\le \delta(T, \|u\|_{C(\overline{Q}_T)}) =: d(\|b\|_{C(\overline{Q}_T)}).\end{aligned}$$

In order to prove the uniqueness of a solution of (2.1), suppose u_1 and u_2 are solutions and set $w : u_1 - u_2$. Then $w \in F_\mu$ and $Lw = b\nabla(u_1 + u_2) \cdot \nabla w$. Thus $\tilde{L}w = 0$, where $\tilde{L} := L - b\nabla(u_1 + u_2) \cdot \nabla$. The maximum principle and the periodicity of w imply $w = 0$.

Let the function $g : \overline{\Omega} \times \mathbb{R}^{N+2} \to \mathbb{R}$ satisfy the hypotheses formulated in Section 1. By $G(u, z)$ we denote the composition operator corresponding to g, depending on $u \in C(\overline{\Omega} \times \mathbb{R})$ and $z \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^N)$, that is, $G(u,z)(x,t) := g(x,t,u(x,t),z(x,t))$. Obviously u is a solution of $(\star)$ having the desired regularity properties iff $Lu = G(u, \nabla u)$.

To g we associate a function $\gamma : \overline{\Omega} \times \mathbb{R}^{N+2} \to \mathbb{R}$ defined by

$$\gamma(x,t,\xi,\eta) := \int_0^\xi \left|\frac{\partial g}{\partial \xi}(x,t,s,\eta)\right| \mathrm{d}s + \xi, \tag{2.2}$$

and set

$$h(x,t,\xi,\eta) := g(x,t,\xi,\eta) + \gamma(x,t,\xi,\eta). \tag{2.3}$$

Then both functions

$$\xi \mapsto \gamma(x,t,\xi,\eta)$$

and

$$\xi \mapsto h(x,t,\xi,\eta)$$

are strictly monotonically increasing in ξ (for fixed (x,t,η)), and enjoy the same smoothness – and growth properties as g. Let H and Γ denote the composition operators induced by h and γ.

LEMMA 2.2. *For given $(v,r) \in E \times E$ there exists a unique solution $u = S(v,r)$ of*

$$Lu + \Gamma(u, \nabla u) = H(v, \nabla u) + r. \tag{2.4}$$

and $S : E \times E \to F_\mu$ is strictly increasing in both arguments.

PROOF. For $M > 0$ sufficiently large, the constant function $\pm M$ are sub- and supersolutions of problem (2.4). Thus [2, Thm. 1.2] implies the existence of a solution of (2.4) with $\|u\|_{C(\overline{Q}_T)} \leq M$.

The uniqueness of a solution and the positivity property of the solution operator S follow from the subsequent consideration. Let v_i and r_i $(i = 1,2)$ be given elements of E, with $v_2 \geq v_1$ and $r_2 \geq r_1$, and let u_1, u_2 be the associated solutions:

$$Lu_i + \Gamma(u_i, \nabla u_i) = H(v_i, \nabla u_i) + r_i \quad (i = 1,2).$$

Then $w := u_1 - u_2$ satisfies

$$Lw + (\Gamma(u_1, \nabla u_1) - \Gamma(u_2, \nabla u_1)) + (\Gamma(u_2, \nabla u_1) - \Gamma(u_2, \nabla u_2)) = \\ (H(v_1, \nabla u_1) - H(v_2, \nabla u_1)) + (H(v_2, \nabla u_1) - H(v_2, \nabla u_2)) + (r_1 - r_2).$$

Since

$$\Gamma(u_1, \nabla u_1) - \Gamma(u_2, \nabla u_1) = b_0 w$$

and

$$\Gamma(u_2, \nabla u_1) - \Gamma(u_2, \nabla u_2) = \sum_{j=1}^{N} b_j \frac{\partial w}{\partial x_j}$$

with (Hölder-)continuous coefficient functions $b_0 \gg 0$ and b_j, and similarly

$$H(v_1, \nabla u_1) - H(v_2, \nabla u_1) = c_0(v_1 - v_2) \text{ with } c_0 \gg 0,$$
$$H(v_2, \nabla u_1) - H(v_2, \nabla u_2) = \sum_{j=1}^{N} c_j \frac{\partial w}{\partial x_j},$$

we conclude that

$$\tilde{L}w = c_0(v_1 - v_2) + (r_1 - r_2), \tag{2.5}$$

where $\tilde{L} := L + \sum_{j=1}^{N} (b_j - c_j) \frac{\partial}{\partial x_j} + b_0$. It follows from (2.5) and the assumptions on v_i and r_i that $w \leq 0$ and $w \ll 0$ if either $v_2 > v_1$ or $r_2 > r_1$.

Since sub- and supersolutions to problem ($\star$) need not be periodic functions, we have to use a separate argument in the first step of the iteration schemes. For that we need

LEMMA 2.3. *Let v and r be functions in $C^{\mu,\mu/2}(\overline{Q}_T)$. There exists a unique solution $u \in C^{2+\mu,1+\mu/2}(\overline{\Omega}\times]0,T]) \cap C^{1+\sigma,(1+\sigma)/2}(\overline{Q}_T)$ (for some $0<\sigma<1$) of*

$$(2.6)\qquad \begin{cases} \mathcal{L}u + \Gamma(u,\nabla u) = H(v,\nabla u) + r & in\ \Omega\times]0,T] \\ Bu = 0 & on\ \partial\Omega\times]0,T] \\ u(\bullet,0) = u(\bullet,T) & on\ \overline{\Omega}, \end{cases}$$

and $u = \tilde{S}(v,r)$ depends strongly increasingly on both arguments.

PROOF. The existence follows similarly as in Lemma 2.2; we employ the argument of [2, proof of Prop. 5.1] to show that the Poincaré operator Π is a compact self-map of the order interval $[-M,M]$ in $C_B^{2+\nu}(\overline{\Omega})$, $0<\nu<\mu$. A fixed point of Π is a solution of (2.6), and its regularity follows by [2, Lemma 4.2 and Remark 4.3]. The monotonicity of $\tilde{S}$ – and hence the uniqueness of a solution of (2.6) – is proved in the same way as in Lemma 2.2, using the parabolic maximum principle [17, pp. 173-174 and the Remark on p. 174].

Let now $\underline{v} < \overline{v}$ be strict sub- and supersolution of problem ($\star$), respectively. We set up the iteration scheme as follows:

$v^0 := \overline{v}$

$v^1 :=$ the (by Lemma 2.3 unique) solution of

$$\begin{cases} \mathcal{L}v^1 + \Gamma(v^1,\nabla v^1) = H(v^0,\nabla v^1) \\ Bv^1 = 0 \\ v^1(\bullet,0) = v^1(\bullet,T). \end{cases}$$

By the regularity properties of v^1 we can extend v^1 to a function (again denoted by v^1) in E. We then define v^n $(n\geq 2)$ by

$$(2.7)\qquad Lv^n + \Gamma(v^n,\nabla v^n) = H(v^{n-1},\nabla v^n);$$

$v^n \in F_\mu$ is uniquely defined by Lemma 2.2.

LEMMA 2.4. (i) *The sequence $(v^n)_{n\geq 2}$ is (pointwise) strictly monotonically decreasing in $\Omega\times\mathbb{R}$ and convergent in F_ν $(0<\nu<\mu)$ to the minimal solution $\overline{u}$ of ($\star$) in $[\underline{v},\overline{v}]$.*

(ii) *The functions v^n $(n\geq 1)$ are strict supersolutions of ($\star$).*

PROOF. (i) By Lemmas 2.3 and 2.2 the sequence $(v^n)_{n\geq 0}$ is strictly monotonically decreasing and bounded below by $\underline{v}$, and thus converges (pointwise) to a function $\overline{u} \geq \underline{v}$. We write

$$Lv^n = b_n \left(1 + |\nabla v^n|^2\right) =: f^n,$$

where

$$b_n := \frac{H(v^{n-1}, \nabla v^n) - \Gamma(v^n, \nabla v^n)}{1 + |\nabla v^n|^2}.$$

Since $\|b_n\|_{C(\overline{Q}_T)} \leq$ const., Lemma 2.1 implies that $\|v^n\|_{C^{1+\mu,(1+\mu)/2}(\overline{Q}_T)} \leq$ const. By compactness of the embedding $C^{1+\mu,(1+\mu)/2}(\overline{Q}_T) \subset C^{1,0}(\overline{Q}_T)$ we infer that $v^n \to \overline{u}$ in $C^{1,0}(\overline{Q}_T)$. Hence the sequence (f^n) converges in $C(\overline{Q}_T)$ – and thus in $L_p(Q_T)$ with $p > N$ – to some element f. By [10, p. 342] and a standard argument (e.g. [1, proof of Thm. 2.4]) we conclude that $v^n \to \overline{u}$ in $W_p^{2,1}(Q_T)$ and hence (by continuous embedding) in $C^{1+\mu,(1+\mu)/2}(\overline{Q}_T)$. By continuity of the composition operators (*) it follows that $f^n \to f$ in $C^{\nu,\nu/2}(\overline{Q}_T)$ for $0 < \nu < \mu$, and consequently $v^n \to \overline{u}$ in F_ν. By passing to the limit $n \to \infty$ in the defining equation (2.7) we see that $\overline{u}$ solves ($\star$). That $\overline{u}$ is the maximal solution in $[\underline{v}, \overline{v}]$ follows by well-known arguments.

Assertion (ii) is an immediate consequence of the positivity results in Lemmas 2.3 and 2.2.

If we start the iteration similarly at the strict subsolution $\underline{v}$, we get an increasing sequence (v_n) of strict subsolutions converging to the minimal solution $\underline{v}$ in $[\underline{v}, \overline{v}]$.

3. - Proofs of Theorems 1 and 2

For the *proof of Theorem* 1 let $\underline{v} < \overline{v}$ be strict sub- and supersolutions of ($\star$). With the iteration schemes introduced in Section 2 we have

$$\underline{v} < v_n \nearrow \underline{u} \leq \overline{u} \swarrow v^n < \overline{v}, \tag{3.1}$$

where the v_n and v^n are strict sub- and supersolutions lying in F_μ and converging in F_ν $(0 < \nu < \mu)$ to the minimal and maximal solutions of ($\star$) in $[\underline{v}, \overline{v}]$, respectively.

DEFINITION. We say a solution u of ($\star$) is *strongly stable from above* if there exists a strictly decreasing sequence (v^n) of strict supersolutions $v^n \in F_{\mu/2}$ converging to u in $F_{\mu/2}$. Strong stability from below is defined analogously.

(*) Though claimed in various publications, it is not correct that a Nemytskii operator $G{:}G(u)(x)=g(x,u(x))$, where $g(x,\xi)$ is uniformly Lipschitz in ξ and C^μ in x, is a continuous mapping of $C^\mu(\overline{\Omega})$ into itself (a simple counterexample has been constructed by A. Kennington). It is however continuous: $C^\mu(\overline{\Omega}) \to C^\nu(\overline{\Omega})$ for $0 \leq \nu < \mu$. This weaker result suffices for our purposes.

The minimal and maximal solutions $\underline{u}$ and $\overline{u}$ are thus strongly stable from below and above, respectively.

REMARK. Our notion of strong stability differs slightly from others (e.g. Matano's). This is however only of technical interest here. Since $v^n(\bullet, 0) \gg u(\bullet, 0)$ in $C^1_B(\overline{\Omega})$ by the strong maximum principle, the comparison theorem guarantees that strong stability from above implies stability from above.

Let $S := \{u : u = \text{solution of } (\star) \text{ with } \underline{u} \leq \overline{u}\}$, and consider the subset $S_1 := \{u \in S : u \text{ is strongly stable from below}\}$, provided with the natural ordering in $C(\overline{\Omega} \times \mathbb{R})$. Clearly $\underline{u} \in S_1$.

LEMMA 3.1. *S_1 is inductively ordered: every totally ordered subset $T \subset S_1$ has an upper bound in S_1.*

PROOF. Let $T \subset S_1$ be totally ordered. The assertion is clear if T contains only finitely many elements. In the other case set $u := \sup_{v \in \mathcal{T}} v$ (pointwise). Since $\mathcal{T}$ is totally ordered, we can select a sequence (v_n) in $\mathcal{T}$ converging pointwise increasingly to u, hence converging in $L_p(Q_T)$ and therefore in $F_{\mu/2}$ (cf. the proof of Lemma 2.4). We infer that $u \in S$. In order to show the strong stability of u from below, we note first that $v_{n-1} \ll v_n$ in $F_{\mu/2}$ by the strong maximum principle. To each v_n we choose a strict subsolution $\phi_n : v_{n-1} < \phi_n < v_n$, $\|v_n - \phi_n\|_{F_{\mu/2}} < \dfrac{1}{n}$. Then $\phi_n \nearrow u$ in $F_{\mu/2}$, which proves that $u \in S_1$.

By Zorn's lemma S_1 has a maximal element u_1. Let now $S_2 := \{u \in S : u \geq u_1, u \text{ is strictly stable from above}\}$. Since $\overline{u} \in S_2$, we conclude by the same argument that S_2 has a minimal element u_2. Obviously

$$(3.2) \qquad \underline{v} < \underline{u} \leq u_1 \leq u_2 \leq \overline{u} < \overline{v}.$$

We distinguish between two cases.

Case A: If $u_1 = u_2 =: u$, then u is strongly stable from above and from below, and hence stable. In this case Theorem 1 is proved.

Case B: $u_1 < u_2$. We first observe that there exist neither strict sub- nor strict supersolutions of $(\star)$ in $[u_1, u_2]$. In fact, assuming for example the existence of a strict subsolution ψ, we could construct a solution $w : u_1 < \psi < w \leq u_2$, by iteration from the subsolution ψ. Thus w would be strongly stable from below, contradicting the maximality of u_1.

LEMMA 3.2. *Let $u_1 < u_2$ be solutions of $(\star)$ such that there is no strict sub- and supersolution in $[u_1, u_2]$. Then there exists at least one solution $u \in]u_1, u_2[$ of $(\star)$.*

PROOF. Assume there is no solution of $(\star)$ in $]u_1, u_2[$. Let $k : \overline{\Omega} \times \mathbb{R}^{N+2} \to \mathbb{R}^+$ be a function satisfying the same smoothness – and growth assumptions as g, T-periodic in t, and such that $k(u, \nabla u) = 0$ for all u in a $F_{\mu/2}$ – neighbourhood of u_2, while $k(u_1, \nabla u_1) > 0$ (note that $u_1 \ll u_2$ in $F_{\mu/2}$). Consider the equation

$$Lu = G(u, \nabla u) + k(u, \nabla u). \tag{3.3}$$

Then u_2 is a solution of (3.3), while u_1 is a strict subsolution of (3.3). Hence there exists a solution w of (3.3), $u_1 < w \leq u_2$, obtained from u_1 by monotone iteration $v_n \nearrow w$ in $F_{\mu/2}$.

If $w < u_2$, w can not be a solution of $(\star)$ by assumption. Thus w is a strict supersolution of $(\star)$ since $k(w, \nabla w) \geq 0$.

If $w = u_2$, $k(v_n, \nabla v_n) = 0$ for large n. The v_n are strict subsolutions of (3.3) and hence of $(\star)$.

In both cases we arrive at a contradiction to the hypothesis of the lemma.

LEMMA 3.3. *Let again $u_1 < u_2$ be a solution of $(\star)$ such that there is neither a strict sub- nor a strict supersolution in $[u_1, u_2]$. Set $\tilde{S} := \{u: u =$ solution of $(\star)$ with $u \in [u_1, u_2]\}$. Then $\tilde{S}$ is totally ordered and connected in $C(\overline{Q}_T)$, and $u \in \tilde{S} \setminus \{u_1\}$ is stable from below, $u \in \tilde{S} \setminus \{u_2\}$ stable from below.*

PROOF. (i) $\tilde{S}$ is totally ordered: let u, $u^\star \in \tilde{S}$, and assume they are not order-related. Define $w \in E$ by $w(x,t) := \max\{u(x,t), u^\star(x,t)\}$. Then $u < w \leq u_2$ and $Lu + \Gamma(u, \nabla u) = H(u, \nabla u) < H(w, \nabla u) \leq H(u_2, \nabla u)$. Let $\tilde{u}$ be the solution of $L\tilde{u} + \Gamma(\tilde{u}, \nabla\tilde{u}) = H(w, \nabla\tilde{u})$. By Lemma 2.2, $u \ll \tilde{u} \leq u_2$ in F_μ, and similarly $u^\star \ll \tilde{u} \leq u_2$. Hence $w < \tilde{u}$ in E. We conclude that $L\tilde{u} + \Gamma(\tilde{u}, \nabla\tilde{u}) = H(w, \nabla\tilde{u}) < H(\tilde{u}, \nabla\tilde{u})$ and thus $L\tilde{u} < G(\tilde{u}, \nabla\tilde{u})$, i.e. $\tilde{u} \in [u_1, u_2]$ is a strict subsolution of $(\star)$. This contradicts the hypothesis.

(ii) $\tilde{S}$ is compact in $C(\overline{Q}_T)$: this follows by the same arguments as in the proof of Lemma 2.4.

(iii) $\tilde{S}$ is connected in $C(\overline{Q}_T)$: suppose not. Since $\tilde{S}$ is totally ordered and compact, then there exist $v_1 < v_2$ in $\tilde{S}$ such that there is no solution in between. This contradicts Lemma 3.2.

(iv) We can now identify $\tilde{S} := \{u_\theta : u_\theta =$ solution of $(\star)$, $1 \leq \theta \leq 2\}$. Let us look specifically at the case of DBC, for NBC the proof runs similarly. Note that $\theta_1 < \theta_2$ implies $u_{\theta_1} \ll u_{\theta_2}$ in F_μ and hence $u_{\theta_1}(\bullet, 0) \ll u_{\theta_2}(\bullet, 0)$ in $C_0^1(\overline{\Omega})$. Let $1 \leq \theta \leq 2$; we show the stability of u_θ from above. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $\|u_{\theta+\delta_1} - u_\theta\|_{C(\overline{Q}_T)} < \varepsilon$. We find $\delta > 0$ such that $v_0 \in C_0^{2+\mu}(\overline{\Omega})$, $v_0 \geq u_\theta(\bullet, 0)$, and $\|v_0 - u_\theta(\bullet, 0)\|_{C_0^1(\overline{\Omega})} < \delta$ imply $u_\theta(\bullet, 0) \leq v_0 \leq u_{\theta-\delta_1}(\bullet, 0)$. By the comparison theorem and the periodicity of the solutions u_θ, the stability follows.

Since in case B the solutions u_1 and u_2 are by definition stable from below and above, respectively, all elements of $\tilde{S}$ are stable solutions of $(\star)$. This proves Theorem 1 also in this case.

The *Proof of Theorem* 2 necessitates only minor modifications: we set $\underline{u} := u_1$, $\overline{u} := u_2$ and define the sets S_1 and S_2 using strong stability from below and from above only with respect to $[u_1, u_2]$. Then trivially $u_1 \in S_1$ and $u_2 \in S_2$. The rest of the proof remains unchanged.

REMARK. If G is real analytic as a mapping $C^{1,0}(\overline{Q}_T) \to C(\overline{Q}_T)$, it can be shown that, under the assumptions of Theorem 1, there is an asymptotically stable solution u such that $\underline{v} \leq u \leq \overline{v}$. The idea here is to use the real analyticity and bifurcation theory to show that there cannot be a totally ordered connected set $\tilde{S}$ of periodic solutions in $[\underline{v}, \overline{v}]$. Matano's result [14] can be similarly improved if the equation for the stationary solutions is real analytic between suitable spaces.

4. - The autonomous case

In this short section we prove Theorem 3 and 4. We assume that $\mathcal{A}$ and g are independent of time and prove that the stable periodic solutions we have constructed are stationary states, that is, time-independent solutions.

Let $\underline{v}$ and $\overline{v}$ be time-independent strict sub- and supersolutions of problem $(\star)$ such that $\underline{v} \leq \overline{v}$ (though the time-independence is not really necessary). We show that any stable solution $\tilde{u}$ of $(\star)$ with $\underline{v} \leq \tilde{u} \leq \overline{v}$ is in fact an equilibrium solution. This obviously suffices to prove Theorem 3. Theorem 4 can be proved by similar arguments.

It is convenient to work in this section in the space $X := L_p(\Omega)$ with $p > N$. It is well-known (e.g. [6, p. 101]) that $\mathcal{A}$:

$$\mathcal{A}u = -\sum_{j,k=1}^{N} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + a_0(x) u$$

generates a sectorial operator A in X with domain $D(A) = W_B^{2,p}(\Omega)$ being continuously inbedded in $C^1(\overline{\Omega})$. Solutions $\tilde{u}$ of our problem are classical solutions and hence they are mild solutions of our parabolic equation in X (in the sense of Browder, e.g. [7, p. 55]). In particular, it follows from [7, Thm. 3.5.2] that $\frac{\partial \tilde{u}}{\partial t}(\bullet, t) \in X^\alpha$ for each $t > 0$, and that $t \mapsto \frac{\partial \tilde{u}}{\partial t}(\bullet, t) \in X^\alpha$ is locally Hölder continuous (here we choose $\alpha \in]0, 1[$ such that $X^\alpha \subset C^1(\overline{\Omega})$). Thus, using [7, Lemma 3.3.2] we can deduce as in [7, Lemma 8.2.2] that $w = \frac{\partial \tilde{u}}{\partial t}$ is a T-periodic solution of the linearized equation. In other words, w is a solution of the linear periodic-parabolic boundary value problem

$$\begin{cases} \mathcal{L}'w := \dfrac{\partial w}{\partial t} + \mathcal{A}'\left(x, t, \dfrac{\partial}{\partial x}\right) w = 0 & \text{in } \Omega \times \mathbb{R} \\ Bw = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ w(\bullet, t) = w(\bullet, t+T) & \text{on } \overline{\Omega}, \quad \forall t \in \mathbb{R}, \end{cases}$$

where

$$\mathcal{A}'w = \mathcal{A}w - \frac{\partial g}{\partial \xi}(\bullet, \tilde{u}, \nabla\tilde{u})w - \sum_{i=1}^{N} \frac{\partial g}{\partial \eta_i}(\bullet, \tilde{u}, \nabla\tilde{u})\frac{\partial w}{\partial x_i}.$$

Note that, since $\tilde{u}$ is T-periodic in t, w must change sign in $\Omega \times \mathbb{R}$ provided $\tilde{u}$ is not constant in t. It is known [3, Thm. 1] that the eigenvalue problem

$$\text{(4.1)} \qquad \begin{cases} \mathcal{L}'w = \lambda w & \text{in } \Omega \times \mathbb{R} \\ Bw = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ w(\bullet, t) = w(\bullet, t+T) & \text{on } \overline{\Omega}, \quad \forall t \in \mathbb{R} \end{cases}$$

has a real (principal) eigenvalue λ_1 such that all the eigenvalues of (4.1) lie in $\{\lambda : \text{Re } \lambda \geq \lambda_1\}$, and such that the eigenspace corresponding to λ_1 is one-dimensional and spanned by a positive eigenfunction.Now suppose $\tilde{u}$ is not constant in t. Since $w = \dfrac{\partial \tilde{u}}{\partial t}$ changes sign and corresponds to the eigenvalue zero, $\lambda_1 < 0$. Let $z > 0$ be the (principal) eigenfunction of problem (4.1)corresponding to λ_1, and let $\mathcal{U} = \mathcal{U}(t, s)$ be the evolution operator associated with the linear initial value problem

$$\begin{cases} \dfrac{du}{dt}(t) + A'(t)u(t) = f(t), & t > 0 \\ u(0) = u_0 \end{cases}$$

in X (where A' is induced by $\mathcal{A}'$). It is well-known (e.g. [2, p. 25]) that $\mathcal{U}(t) := \mathcal{U}(t+T, t)$ is a compact positive irreducible operator in $C(\overline{\Omega})$. Since

$$z(t) = z(t+T) = \mathcal{U}(t)z(t) + \int_t^{t+T} \mathcal{U}(t+T, \tau)\lambda_1 z(\tau)d\tau,$$

and since the integral on the right is a negative element of $C(\overline{\Omega})$ as $\lambda_1 < 0$, it follows that the spectral radius spr $\mathcal{U}(t) > 1$ by the theory of positive compact operators in ordered Banach spaces. Hence $\sigma(\mathcal{U}(t)) \cap \{|\lambda| > 1\} \neq \phi$, and we can apply [7, Thm. 8.2.4] to deduce that there exist (mild) solutions v of the original nonlinear initial value problem such that $\|v(\bullet, 0) - \tilde{u}(\bullet, 0)\|_\alpha$ is small (where $\|\bullet\|_\alpha$ is the natural norm in $X^\alpha = D(A^\alpha)$), but $v(\bullet, t)$ is not close to the set $\Gamma := \{\tilde{u}(\bullet, s),\ s \in [0, T]\}$ in the $L_p(\Omega)$-norm for some large t. (To see that the solution $v(\bullet, s)$ is not close to Γ in the L_p-norm rather that in the X^α-norm, we use that the L_p-norm and the X^α-norm are equivalent on a finite-dimensional space and examine the proof of [7, Thm. 8.2.4].) Now, since the embeddings $X^\alpha \subset C^1(\overline{\Omega})$ and $C(\overline{\Omega}) \subset L_p(\Omega)$ are continuous, this implies that $\tilde{u}$ is not stable in the original norms. Thus, if $\tilde{u}$ is stable, it must be independent of t as claimed.

REMARK. 1. The above argument actually shows more, namely that orbitally stable periodic solutions $\tilde{u}$ of the autonomous problem $(\star)$ are stationary states.

2. It is possible to give a direct stationary proof of the existence of a stable solution of problem $(\star\star)$ by using a time-independent iteration scheme. Alternatively (and this applies to much of this paper) one could work in Sobolev spaces rather than in Hölder spaces and obtain our results under slightly different regularity assumptions.

5. - An improved stability result

In this section we concentrate on the semilinear elliptic equation $(\star\star)$ with Dirichlet boundary conditions. The solution we have constructed in Theorem 3 is stable only in a very weak sense in that $C_0^1(\overline{\Omega})$-small changes of initial data imply that the solutions of the initial value problem are $C(\overline{\Omega})$-close for the later time. In this section we show that one can sometimes deduce a much better stability property, namely stability with respect to small $L_\infty(\Omega)$-changes of the initial values. Our assumptions are rather strong and it seems likely that they can be considerably weakened.

We thus assume that the equation in autonomous, and that g is independent of ∇u. By a solution of the initial value problem satisfying $u(\bullet, 0) = u_0$ (where $u_0 \in L_\infty(\Omega)$) we mean here a function $u \in L_\infty([0, T'] \times \overline{\Omega})$ for some $T' > 0$, u satisfies the equation for $0 < t \leq T'$ and such that the map $t \mapsto u(\bullet, t)$ is weak$\star$-continuous in $L_\infty(\overline{\Omega})$ and norm-continuous in $L_1(\Omega)$ at $t = 0$, with $u(\bullet, 0) = u_0$. One readily deduces that $t \mapsto u(\bullet, t)$ is then norm-continuous in $L_p(\Omega)$ at $t = 0$, $\forall 1 \leq p \leq \infty$. (This map can however not be norm-continuous in $L_\infty(\overline{\Omega})$ at $t = 0$ if u_0 is continuous in $\overline{\Omega}$ and $u_0(x) \neq 0$ on $\partial\Omega$.) It is now easy to see that u is a mild solution of the initial value problem in the sense of Weissler [21], and we can argue as in [21] to show that there can be at most one solution in this sense. (Note that our solutions are uniformly bounded and thus we do not need the growth conditions employed in [21] to ensure uniqueness.)

Next we prove that there exists a solution in our sense. We can reduce the problem to the case where our given stationary solution is the zero function. Hence $g(\bullet, 0) = 0$ on $\overline{\Omega}$. Choose $M > \|u_0\|_\infty$. By truncating g for $|\xi| \geq M$, we can use the results in [21] to obtain a solution u of our initial value problem with g replaced by its truncation g_M, which is norm-continuous at $t = 0$ in $L_p(\Omega)$ with the correct initial condition. Suppose we can prove that this solution u has the property that there exists $\varepsilon > 0$ such that $\sup\{|u(x,t)| : x \in \overline{\Omega},\ t \in [0, \varepsilon]\} \leq M$.

Then the solution satisfies our original equation on $]0,\varepsilon]\times\overline{\Omega}$ and is weak$\star$-continuous in $L_\infty(\overline{\Omega})$ at $t=0$ (by the dominated convergence theorem). This function can then be extended for $t\geq\varepsilon$ as a solution of the original equation, by standard results (since $u(\bullet,\varepsilon)\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$).

In order to prove the existence of such an $\varepsilon>0$ we first note that the solution mapping Weissler constructs in order-preserving. Thus it suffices to assume that u_0 is a positive constant, without loss of generality. Moreover, it suffices to prove the above estimate for the function $g_M(x,\xi)$ replaced by $K\xi$, where the constant K is chosen such that $g_M(x,\xi)\leq K\xi$, $\forall x\in\overline{\Omega}$ and $\xi\geq 0$. (The point here is that, if we solve the initial value problem by the obvious iteration on the integral equation, then each of the iterates becomes larger.) Thus if $\overline{u}$ is the solution when $g_M(x,\xi)$ is replaced by $K\xi$, then $0\leq u(x,t)\leq\overline{u}(x,t)$. Hence we have reduced our estimate to the linear case. Now we have only to show that $\overline{u}(x,t)\leq u_0e^{\alpha t}$ on $\overline{\Omega}$ if $0\leq t$ and $\alpha>K$. Suppose $u_{0,n}$ are non-negative C^∞-functions with compact support in Ω, such that $u_{0,n}\leq u_0$ in Ω and $u_{0,n}\to u_0$ in $L_2(\Omega)$. Since the linear initial value problem generates a positive C_0-semigroup on $L_p(\Omega)$ if $1<p<\infty$, it suffices to prove that $u_n(x,t)\leq u_0e^{\alpha t}$ on $\overline{\Omega}\times[0,\infty[$, where u_n is the (classical) solution of our linear problem with initial value $u_{0,n}$. Since $u_0e^{\alpha t}$ is a supersolution of this equation and $u_n(\bullet,0)\leq u_0$ in Ω, the result follows by the classical maximum principle.

To prove the stability in $L_\infty(\overline{\Omega})$ of the given solution (which is reduced here to the zero function), we see from the positivity of the corresponding semiflow $W(t)$ (which follows from [21]) and from the known stability from $C_0^1(\overline{\Omega})$ to $C(\overline{\Omega})$ that it suffices to prove that, if v_0 is a small positive constant, then $W(1)\ v_0\leq w$ in $\overline{\Omega}$ where w is small in $C_0^1(\overline{\Omega})$. By the same comparison arguments as in the previous paragraph we see that it suffices to establish this assertion when $g(x,\xi)$ is replaced by $K\xi$. In this case the required result follows from standard analytic semigroup theory since the evolution operator is a continuous map of $L_p(\Omega)$ into $W_0^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and thus into $C_0^1(\overline{\Omega})$ if $p>N$.

Hence we have the claimed stability in $L^\infty(\overline{\Omega})$.

REMARK. If $\dfrac{\partial g}{\partial\xi}$ has polynomial growth in ξ, we even get stability from $L_p(\Omega)$ to $C(\overline{\Omega})$ for p large.

6. - A model equation in population genetics

We now turn to problem $(\star\star\star)$. Theorem 2 implies the following.

PROPOSITION 6.1. *Problem* $(\star\star\star)$ *admits a solution* $u:0\leq u\leq 1$ *which is stable with respect to* $[0,1]$.

In order to be more precise we first investigate the stability of the trivial solutions. Appealing to a somewhat more general situation, let $\mathcal{L}$ be a periodic-parabolic differential expression as in Section 1, with $a_0 = 0$, and let $B = \frac{\partial}{\partial \beta}$. Then $L1 = 0$. Suppose the periodic (in t) function g is independent of ∇u and satisfies $g(x,t,0) = 0 \ \forall (x,t) \in \overline{\Omega} \times \mathbb{R}$. Let $m(x,t) := \frac{\partial g}{\partial \xi}(x,t,0)$, and let $M \in \mathcal{L}(E)$ denote the multiplication operator by the function m. For $\sigma \in \mathbb{R}$ we look at the eigenvalue problem

$$(L - \sigma M)x = \lambda z, \quad z > 0$$

in E. To each σ there exists a unique principal eigenvalue $\lambda_1 = \lambda_1(\sigma) \in \mathbb{R}$ with positive eigenfunction $z = z(\sigma) \in F_\mu$. By the principle of linearized stability the trivial solution $u = 0$ of $(\star)$ is asymptotically exponentially stable if $\lambda_1(1) > 0$ and unstable if $\lambda_1(1) < 0$.

We list the properties of the function $\sigma \mapsto \lambda_1(\sigma)$ (cf. [3,8] for proofs under DBC).

1. $\lambda_1(0) = 0$, with $z(\sigma = 0) = 1$.
2. λ_1 is an analytic function of σ, and also $z : \mathbb{R} \to F_\mu$ can be chosen to depend analytically on σ (by the implicit function theorem).
3. λ_1 is concave.

We introduce the quantities

$$P(m) := \int_0^T \max_{x \in \overline{\Omega}} m(x,t) \mathrm{d}t, \quad N(m) := \int_0^T \min_{x \in \overline{\Omega}} m(x,t) \mathrm{d}t.$$

4. $\lim_{\sigma \to +\infty} \lambda_1(\sigma) = -\infty$ if $P(m) > 0$,

$\lim_{\sigma \to -\infty} \lambda_1(\sigma) = -\infty$ if $N(m) < 0$,

and moreover

$$\lambda_1(\sigma) \geq -\sigma T^{-1} P(m) \text{ for } \sigma > 0,$$

$$\lambda_1(\sigma) \geq -\sigma T^{-1} N(m) \text{ for } \sigma < 0,$$

with strict inequalities provided m depends nontrivially on $x \in \Omega$.

In order to derive and expression for the derivative $\frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0)$, we consider the operators $A_1(t) := A(t) + I$ in $X = L_p(\Omega)$ and let $\mathcal{U}_1(t,s)$ denote the evolution operator for the initial value problem

$$\begin{cases} \frac{\mathrm{d}w}{\mathrm{d}t}(t) + A_1(t)w(t) = f(t) \\ w(0) = w_0 \end{cases}$$

in X. Set $\mathcal{U}_1 := \mathcal{U}_1(T,0)$, and observe that $\mathcal{U}_1$ is a compact positive irreducible operator in the space $C := C(\overline{\Omega})$ with $\mathrm{spr}(\mathcal{U}_1) = e^{-T}$ ([2, p. 25]). By the Krein-Rutman theorem there is a unique $\phi^\star \in C^\star$, $\phi^\star > 0$, normalized by $\|\phi^\star\|_{C^\star} = 1$, such that $\mathcal{U}_1^\star \phi^\star = e^{-T}\phi^\star$. Let $J : E \to \mathbb{R}$ be the linear functional defined by

$$J(m) := \left\langle \phi^\star, \int_0^T \mathcal{U}_1(T,\tau)e^{-\tau}m(\bullet,\tau)\mathrm{d}\tau \right\rangle .$$

LEMMA 6.2. $\dfrac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0) = -T^{-1}e^{-T}\langle \phi^\star, 1\rangle^{-1}J(m).$

PROOF. Differentiating the equation

$$(L - \sigma M)z(\sigma) = \lambda_1(\sigma)z(\sigma) \tag{6.1}$$

at $\sigma = 0$, we obtain (with $p := \left.\dfrac{\mathrm{d}z}{\mathrm{d}\sigma}\right|_{\sigma=0}$)

$$Lp = m + \frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0).$$

We perform the transformation $q(t) := e^{-t}p(t)$ to get

$$\frac{\mathrm{d}q}{\mathrm{d}t}(t) + A_1(t)q(t) = e^{-t}\left(m(\bullet,t) + \frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0)\right);$$

thus

$$q(T) = \mathcal{U}_1 q(0) + \int_0^T \mathcal{U}_1(T,\tau)e^{-\tau}\left(m(\bullet,\tau) + \frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0)\right)\mathrm{d}\tau.$$

Since $q(T) = e^{-T}q(0)$, we arrive at

$$(e^{-T} - \mathcal{U}_1)q(0) = Te^{-T}\frac{\mathrm{d}\lambda_1}{\mathrm{d}\sigma}(0) + \int_0^T \mathcal{U}_1(T,\tau)e^{-\tau}m(\bullet,\tau)\mathrm{d}\tau. \tag{6.2}$$

The application of $\phi^\star$ to (6.2) gives the result.

As a consequence of the above listed properties of λ_1 we have

PROPOSITION 6.3. *Suppose* $N(m) < 0 < P(m)$. *Then:*

(i) *if* $J(m) < 0$, *there exists besides* $\sigma = 0$ *a unique* $\sigma_1 = \sigma_1(m) > 0$ *such that* $\lambda_1(\sigma_1) = 0$, *and* $\lambda_1(\sigma) < 0$ *for* $\sigma < 0$ *and* $\sigma > \sigma_1$;

(ii) *if* $J(m) > 0$, *there exists besides* $\sigma = 0$ *a unique* $\sigma_1 = \sigma_1(m) < 0$ *such that* $\lambda_1(\sigma_1) = 0$, *and* $\lambda_1(\sigma) < 0$ *for* $\sigma < \sigma_1$ *and* $\sigma > 0$; (iii) *if* $J(m) = 0$, $\sigma = 0$ *is the only zero of* λ_1, *and* $\lambda_1(\sigma) < 0$ *for* $\sigma \neq 0$.

We now apply Proposition 6.3 to the model problem $(\star\star\star)$. For the nonlinearity $g(x, t, \xi) := s(x, t)h(\xi)$ we have

$$m_0(x,t) := \frac{\partial g}{\partial \xi}(x,t,0) = h'(0)s(x,t)$$

$$m_1(x,t) := \frac{\partial g}{\partial \xi}(x,t,1) = h'(1)s(x,t)$$

and recall that $h'(0) > 0$, $h'(1) < 0$.

Assuming that $N(s) < 0 < P(s)$, we conclude:

(i) if $J(s) < 0$, the solution 0 is stable if $\sigma_1(m_0) > 1$ and unstable if $\sigma_1(m_0) < 1$; the solution 1 is always unstable;

(ii) if $J(s) > 0$, the solution 1 is stable if $\sigma_1(m_1) > 1$ and unstable if $\sigma_1(m_1) < 1$; the solution 0 is always unstable;

(iii) if $J(s) = 0$, both trivial solutions are unstable.

In these three cases, if both trivial solutions are unstable, we can construct strict sub- and supersolutions $\underline{v} < \overline{v}$ in $]0,1[$ (cf. [18, p. 991]). Theorem 1 then guarantees the existence of a stable periodic solution between.

If $N(s) < P(s)$ but $P(s) \leq 0$ or $N(s) \geq 0$, either the trivial solution 0 or 1 is stable. The same is true if $N(s) = P(s) \neq 0$. (Then $\lambda_1(\sigma)$ is a linear function and $\lambda_1 \not\equiv 0$.) If $N(s) = P(s) = 0$, we are in an exceptional situation since $\lambda_1(\sigma) \equiv 0$, for both weights m_0 and m_1. In this case we search for spatially constant solution $u = w(t)1$ of $(\star\star\star)$ and reduce the problem to the ODE

$$\frac{dw}{dt}(t) = s(t)h(w(t)) \tag{6.3}$$

in $\mathbb{R}$, with w being T-periodic. It is easily seen that for each initial condition $w_0 \in]0,1[$ for $t = 0$, the solution w of (6.3) is T-periodic, and all the corresponding solutions $u = w1$ are stable solutions of $(\star\star\star)$ by the comparison theorem.

The full force of Theorem 2 is needed if $N(s) < 0 < P(s)$ and $\sigma_1(m_0) = 1$ or $\sigma_1(m_1) = 1$.

REMARKS 1. Looking at the nonlinear eigenvalue problem $Lu = \gamma sh(u)$ ($\gamma \in \mathbb{R}$) associated with $(\star\star\star)$, if $h \in C^3$ and $h''(0) < 0$, we have bifurcation to the right of an unbounded continuum of positive solutions (γ, u), from the line $\mathbb{R} \times \{0\}$ of trivial solutions, at $(\sigma_1(m_0), 0)$. The continuum lies entirely in the strip $]0, \infty[\times]0,1[$ of $\mathbb{R} \times F_\mu$. In the neighbourhood of $(\sigma_1(m_0), 0)$ the nontrivial solutions are stable. A corresponding result holds for the line $\mathbb{R} \times \{1\}$. (See [19] for a proof in the stationary case.)

2. If Ω is convex, $a = 1$ and $N(s) = P(s)$ (i.e., s independent of $x \in \Omega$), it follows from the result of [9] that stable solutions of $(\star\star\star)$ are spatially homogeneous.

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