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Some existence results on noncoercive variational inequalities

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Some Existence Results on Noncoercive Variational Inequalities (*).

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1. – Introduction.

The aim of this paper is to extend and unify some well known results on noncoercive variational inequalities due to Fichera (see [5], [6]) and to Lions and Stampacchia (see [9]).

Our framework is this: we are given V, a, $\boldsymbol{K}$, L such that the following *structural hypotheses* are verified.

(1.1) *V is a real Hilbert space with scalar product* $(\cdot, \cdot)$ *and norm* $\|\cdot\|$;

(1.2) $a\colon \{u, v\} \to a(u, v)$ *is a bilinear continuous form on* $V \times V$ *with values in* $\boldsymbol{R}$;

(1.3) $a(v, v) \geqslant 0 \qquad \forall v \in V$;

(1.4) $\boldsymbol{K} \subset V$ *is a closed, non-empty convex set*;

(1.5) $L\colon v \to \langle L, v\rangle$ *is a real continuous functional on* V .

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In (1.5) the symbol $\langle\cdot,\cdot\rangle$ denotes the pairing between V' (the dual space of V) and V. It will not be restrictive to require also that

$$V = \overline{\operatorname{span}(\boldsymbol{K} - \boldsymbol{K})}\,. \tag{1.6}$$

Here and in the following, for all subsets A, B of V we set

$$A \pm B = \{a \pm b\colon a \in A,\, b \in B\}\,. \tag{1.7}$$

Let $\boldsymbol{A}$, $\boldsymbol{A}^*$ be respectively the operator associated to a and its adjoint, say

$$\langle \boldsymbol{A}u, v\rangle = a(u, v) \qquad \forall u, v \in V\,, \tag{1.8}$$

$$\langle \boldsymbol{A}^*u, v\rangle = a(v, u) \qquad \forall u, v \in V\,; \tag{1.9}$$

both $\boldsymbol{A}$ and $\boldsymbol{A}^*$ belong to $\mathcal{L}(V, V')$, the space of linear continuous operators from V into V'. Let

$$Y = \{v \in V\colon a(v, v) = 0\} \tag{1.10}$$

be the kernel of a.

We consider the problem, henceforth denoted by pb $(a, \boldsymbol{K}, L)$: to find u such that

$$u \in \boldsymbol{K} \quad \textit{and} \quad a(u, u - v) \leqslant \langle L, u - v\rangle \qquad \forall v \in \boldsymbol{K}\,. \tag{1.11}$$

It is well known that this *variational inequality* is equivalent to a minimum problem, when a is symmetric, say

$$a(u, v) = a(v, u) \qquad \forall u, v \in V\,. \tag{1.12}$$

In this case, let us define the functional

$$F(v) = \tfrac{1}{2}a(v, v) - \langle L, v\rangle \qquad \forall v \in V\,; \tag{1.13}$$

then u solves pb $(a, \boldsymbol{K}, L)$ if and only if

$$F(u) = \inf_{v \in \boldsymbol{K}} F(v)\,. \tag{1.14}$$

It is also well known that if the requirement (1.3) is strengthened by imposing the coerciveness of a:

$$\exists \alpha > 0 \ \textit{such that} \ a(v, v) \geqslant \alpha \|v\|^2 \qquad \forall v \in V\,, \tag{1.15}$$

then Stampacchia's theorem (see [11]) guarantees existence and uniqueness of the solution of pb $(a, \boldsymbol{K}, L)$. Still better, under the assumption (1.15), the map $L \to u$ is Lipschitz continuous from V' into V.

If (1.15) fails to be true, some conditions of compactness and compatibility are necessary for the existence of a solution of pb $(a, \boldsymbol{K}, L)$. This happens even when $\boldsymbol{K} = V$: in this case, pb $(a, \boldsymbol{K}, L)$ is equivalent to

$$\text{(1.16)} \qquad \textit{to find } u \in V \textit{ such that } Au = L\,;$$

existence of a solution of this problem is possible only if

$$\text{(1.17)} \qquad A^* w = 0 \Rightarrow \langle L, w \rangle = 0\,.$$

On the other hand, solvability of (1.16) *for all* L satisfying (1.17) is equivalent to the request that $A(V)$ is closed, while (1.17) becomes a sufficient condition if $A = J - T$, with $T =$ compact and $J =$ Riesz operator ($\langle Jv, w \rangle = (v, w)$ for all v and w of V).

In the sequel, we will not assume (1.15) and seek for necessary and/or sufficient conditions for the solvability of pb $(a, \boldsymbol{K}, L)$. The study will be carried out under the following *compactness-coerciveness* assumption:

(1.18)
there exist two linear continuous operators $\Pi_i \colon V \to V$, $i = 0, 1$ *and a real number* $\alpha > 0$ *such that*

(i) $\Pi_0(\boldsymbol{K})$ *is bounded*;

(ii) Π_1 *is compact*;

(iii) $a(v, v) + \|\Pi_0 v\|^2 + \|\Pi_1 v\|^2 \geqslant \alpha \|v\|^2 \qquad \forall v \in V\,.$

In particular, (iii) together with the continuity of a, Π_0 and Π_1, says that

$$\text{(1.19)} \qquad [a(\cdot, \cdot) + \|\Pi_0 \cdot\|^2 + \|\Pi_1 \cdot\|^2]^{\frac{1}{2}}$$

is a norm on V, equivalent to the natural norm $\|\cdot\|$.

It is worthwhile to note that the assumption (1.18) is satisfied if (1.15) holds true (with the choice $\Pi_0 = \Pi_1 \equiv 0$) or if

$$\text{(1.20)} \qquad \boldsymbol{K} \textit{ is bounded}$$

(with the choice $\Pi_0 =$ identity, $\Pi_1 \equiv 0$, $\alpha = 1$). Indeed, without assuming

(1.15), (1.20) is itself a sufficient condition for the solvability of pb $(a, \boldsymbol{K}, L)$, without any compatibility assumption (see [9], Theorem 4.1).

In [9] (Theorem 5.1) an existence result for pb $(a, \boldsymbol{K}, L)$ is given assuming $0 \in \boldsymbol{K}$, a compactness hypothesis analogous to (1.18) (1) and

$$w \in \boldsymbol{K} \cap Y \quad \textit{and} \quad w \neq 0 \Rightarrow \langle L, w\rangle < 0 \tag{1.21}$$

(Y is defined in (1.10)).

On the other hand, in [9] (Remark 5.1) it is proved that, when $\boldsymbol{K}$ is a closed cone (2) and a is symmetric, a necessary condition for the existence of a solution of pb $(a, \boldsymbol{K}, L)$ is that

$$w \in \boldsymbol{K} \cap Y \Rightarrow \langle L, w\rangle \leqslant 0 \,. \tag{1.22}$$

Our goal is to extend these results and to reduce the gap between the necessary condition (1.22) and the sufficient one (1.21). The main abstract results are the following (see Theorems 3.1, 4.1 and 4.2).

ASSUME (1.1)-(1.5). Then a necessary condition for the solvability of pb $(a, \boldsymbol{K}, L)$ is that

$$w \in \operatorname{rc} \boldsymbol{K} \quad (^3) \Rightarrow \langle L, w\rangle \leqslant \sup_{v \in \boldsymbol{K}} a(v, w) \,. \tag{1.23}$$

Provided that (1.18) and (1.23) hold, either of the following is a sufficient condition:

(1.24) *the set* $\{w \in Y \cap \operatorname{rc} \boldsymbol{K} \colon a(v, w) \leqslant \langle L, w\rangle, \ \forall v \in \boldsymbol{K}\}$ *is a subspace*;

(1.25) $\boldsymbol{K}$ *is a cone and* $\boldsymbol{K} - \{w \in \boldsymbol{K} \cap Y \colon a(v, w) \leqslant \langle L, w\rangle, \ \forall v \in \boldsymbol{K}\}$ *is closed*;

(1.26) $\boldsymbol{K}$ *is a cone,* $\boldsymbol{K} \cap \ker \boldsymbol{A} \cap \ker L$ *is a subspace and*

$$\forall w \in \boldsymbol{K} \cap Y \ \textit{with} \ \boldsymbol{A}w \neq 0, \ \exists v = v(w) \in \boldsymbol{K} \ \textit{s.t.} \ \langle L, w\rangle < a(v, w) \,.$$

We note that, when $\boldsymbol{K} = V$, (1.23) turns out to be (1.17), and (1.24) trivially holds.

(1) See hypotheses (5.1)-(5.4) of the quoted theorem. We note that these are special cases of our assumption (1.18), with $\Pi_0 \equiv 0, \Pi_1 =$ projection on $\ker P_1$ with respect to the norm p_0.

(2) By *cone* we mean any subset Q of V such that $x, y \in Q$, $\alpha, \beta \geqslant 0 \Rightarrow \alpha x + \beta y \in Q$ (the convexity of Q is contained in the definition).

(3) rc $\boldsymbol{K}$ denotes the *recession cone* of $\boldsymbol{K}$, that is the set of unbounded directions contained in $\boldsymbol{K}$ (see section 2 for the precise definition).

The interest of the result lies in that (1.24) (which actually contains (1.23)) requires only a finite number of verifications.

Actually, the gap between (1.23) and (1.24) has not been filled yet: indeed we will exhibit an example which the sufficient condition does not apply to, though existence holds. Even more, (1.24) is not necessary for the existence, whilst (1.23) alone is not even sufficient, when a is symmetric, to give that

$$\inf_{v\in \boldsymbol{K}} F(v) > -\infty , \tag{1.27}$$

(F defined in (1.13)) which is of course a necessary condition for the solvability of pb $(a, \boldsymbol{K}, L)$.

The spirit of the procedure we will follow is that of the mentioned work of Lions and Stampacchia, yet our scheme (see Theorems 4.1 and 4.2) provides also a new proof of Fichera's abstract results in a more general setting. This we will show in particular for the Signorini problem in linear elasticity and for the problem of the partially supported plate (or beam).

The main results of our work have been reported in [4].

Here is an out line of the paper.

In section 2 we give some definitions and properties of the geometrical tools we will need in the following. The *compatibility* of the triplet $\{a, \boldsymbol{K}, L\}$ is defined and necessary conditions for the solvability of pb $(a, \boldsymbol{K}, L)$ are introduced.

In section 3 we prove the existence theorem (see Theorem 3.1) in a constructive way which can be useful in numerical approximation. The procedure follows that of [9]: the final result is stronger because of a thorough exploitation of the properties of the recession cone. The method consists in getting a solution (precisely the one of minimal norm) of pb $(a, \boldsymbol{K}, L)$ as a limit of a family of solutions of approaching problems pb $(a_\varepsilon, \boldsymbol{K}, L)$, with a_ε coercive on V. As an abstract application, section 3 contains also a sufficient condition in order that the algebraic difference between two closed convex subsets of a Hilbert space is closed (see Theorem 3.2).

In section 4 the sufficient condition of Theorem 3.1 is weakened, by introducing suitable enlargements of the convex $\boldsymbol{K}$. A comparison is made with an analogous method due to Fichera (see [6], Theorems 1.II and 2.I). Instead of taking projections as in Fichera's, we work with « *cylindrations* » of the convex, say we consider the cylinder generated by sliding $\boldsymbol{K}$ along a suitable direction of V. This permits greater flexibility, due also to the abstract result of Theorem 3.2. Further necessary conditions for the solvability of pb $(a, \boldsymbol{K}, L)$ are also proved, which are stronger than those introduced in section 2.

Section 5 is devoted to mechanical applications: we prove the existence of an equilibrium configuration for a supported plate (or beam) when the data are compatible. Moreover, the abstract setting applies to the plate problem when the support degenerates, for instance is a segment. Consideration of this special case allows us to study problems for which the abstract theorem is not directly exploitable, since the compatibility condition is not satisfied. For instance, we can give a satisfactory scheme of existence and nonexistence results even for the « ambiguous » case of a supported plate subject to external forces whose center belongs to the boundary of the convex hull of the supporting set. Eventually, an application to the Signorini problem in elasticity in mentioned, that gives the classical results proved by Fichera (see [5], [6]).

We end this introduction by mentioning a further application. Let V be a subspace of the usual Sobolev space $\boldsymbol{H}^1(\Omega)$, where Ω is an open, bounded subset of $\boldsymbol{R}^N$ with smooth boundary. Set

$$a(v, w) \equiv \int_\Omega \nabla v \, \nabla w \, dx \,, \tag{1.28}$$

$$a_\lambda(v, w) \equiv a(v, w) - \lambda \int_\Omega vw \, dx \,, \qquad v, w \in V \,, \ \lambda \in \boldsymbol{R} \,. \tag{1.29}$$

Consider pb $(a_{\lambda_1}, \boldsymbol{K}, L)$, where λ_1 is the first eigenvalue (4) of the operator $\boldsymbol{A}$ defined by (1.28), (1.8) and $\boldsymbol{K}$, L are chosen as in (1.4), (1.5). To this problem our abstract existence theorem applies, since the kernel of $\boldsymbol{A}$ is finite dimensional hence the compactness-coerciveness assumption is fulfilled. For results about the problem pb $(a_\lambda, \boldsymbol{K}, L)$ when $\lambda > \lambda_1$ see, for instance, a recent work of Szulkin [13] and the references quoted there.

(4) By « first eigenvalue » we mean the quantity

$$\lambda_1 = \max \{\lambda \in \boldsymbol{R} \colon a_\lambda(v, v) \geqslant 0 \ \ \forall v \in \boldsymbol{K} - \boldsymbol{K}\} \,.$$

Assuming (1.6), λ_1 is the smallest value of λ for which the *eigenvalue problem associated to* pb $(a, \boldsymbol{K}, L)$

$$\begin{cases} v \in V \,, \quad v \neq 0 \\ a(v, \varphi) - \lambda(v, \varphi)_{L^2(\Omega)} = 0 \,, \qquad \forall \varphi \in V \end{cases}$$

is solvable.

2. – Notations and preliminary results.

We begin by introducing the set $\boldsymbol{S}(a, \boldsymbol{K}, L)$ of solutions of pb $(a, \boldsymbol{K}, L)$ (in short $\boldsymbol{S}$, when not ambiguous):

$$\boldsymbol{S}(a, \boldsymbol{K}, L) = \{u \in \boldsymbol{K}\colon\ u \textit{ solves } \text{pb}\,(a, \boldsymbol{K}, L)\}\ . \tag{2.1}$$

We have that

(2.2) $\boldsymbol{S}(a, \boldsymbol{K}, L)$ *is a closed convex set, possibly empty*:

this is a consequence (see [9]) of Minty's lemma, which states that pb $(a, \boldsymbol{K}, L)$ is equivalent to:

(2.3) *to find* $u \in \boldsymbol{K}$ *such that* $a(v, u - v) \leqslant \langle L, u - v\rangle \qquad \forall v \in \boldsymbol{K}$.

Let $T \subset V$ be a closed, nonempty convex set. Using Rockafellar's terminology (see [10], p. 62), we call *recession cone of* T (*asymptotic cone*, according to Bourbaki's book [1], p. 125) the set

$$\text{rc}\, T \equiv \bigcap_{\lambda > 0} \lambda(T - t_0)\ , \qquad t_0 \in T\ . \tag{2.4}$$

This definition turns out to be independent of t_0. Immediate properties are:

(2.5) $\text{rc}\, T$ *is always a cone, contained in* V ;

(2.6) *if* T *is a cone, then* $\text{rc}\, T = T$;

(2.7) *if* $0 \in T$, *then* $\text{rc}\, T \subset T$.

Moreover, it can be easily seen that $w \in \text{rc}\, T$ if and only if $w \in V$ and either of the following conditions is satisfied:

$$t + w \in T\ , \qquad \forall t \in T\ , \tag{2.8}$$

$$t + \lambda w \in T\ , \qquad \forall t \in T\ ,\ \forall \lambda \geqslant 0\ , \tag{2.9}$$

$$\exists t_0 \in T\colon t_0 + \lambda w \in T\ , \qquad \forall \lambda \geqslant 0\ . \tag{2.10}$$

DEFINITION 2.1. We call *resolvent cone* of pb $(a, \boldsymbol{K}, L)$ the set

$$\boldsymbol{C} = \boldsymbol{C}(a, \boldsymbol{K}, L) = \{w \in \text{rc}\, \boldsymbol{K}\colon a(v, w) \leqslant \langle L, w\rangle,\ \forall v \in \boldsymbol{K}\}\ . \tag{2.11}$$

An immediate remark is that $\boldsymbol{C}(a, \boldsymbol{K}, L)$ is always a cone. Moreover, the following result holds, which justifies the name.

LEMMA 2.1. Assume $\boldsymbol{S}(a, \boldsymbol{K}, L) \neq \emptyset$. Then

$$\boldsymbol{C}(a, \boldsymbol{K}, L) = \mathrm{rc}\, \boldsymbol{S}(a, \boldsymbol{K}, L) . \tag{2.12}$$

PROOF. Since $\boldsymbol{S}$ is nonempty, (2.2) entails that $\mathrm{rc}\, \boldsymbol{S}$ is well defined. That $\boldsymbol{C} \subset \mathrm{rc}\, \boldsymbol{S}$ follows from (2.8) and (2.3). Conversely, we have $\mathrm{rc}\, \boldsymbol{S} \subset \boldsymbol{C}$. Indeed, if w belongs to $\mathrm{rc}\, \boldsymbol{S}$ then it lies also in $\mathrm{rc}\, \boldsymbol{K}$. If u is in $\boldsymbol{S}$ and λ is a positive real number, using Minty's formulation for $u + \lambda w$ (which is a solution) we get

$$\lambda[a(v, w) - \langle L, w\rangle] \leqslant \langle L, u - v\rangle - a(v, u) + a(v, v) \qquad \forall \lambda > 0 \, , \ \forall v \in \boldsymbol{K} .$$

Hence, $a(v, w) \leqslant \langle L, w\rangle$, $\forall v \in \boldsymbol{K}$ and $w \in \boldsymbol{C}$. ■

We claim that

$$\boldsymbol{C}(a\, \boldsymbol{K}, L) = \{w \in Y \cap \mathrm{rc}\, \boldsymbol{K} \colon a(v, w) \leqslant \langle L, w\rangle, \ \forall v \in \boldsymbol{K}\} \tag{2.13}$$

(Y is the kernel of a, as in (1.10)). In fact, the inclusion $\supset$ is trivial and for the opposite we just need to prove that any w of $\boldsymbol{C}(a, \boldsymbol{K}, L)$ belongs to Y. This follows taking in (2.11) $v = v_0 + \lambda w$ with $\lambda > 0$ and v_0 fixed in $\boldsymbol{K}$ (this is possible since w belongs to $\boldsymbol{C}(a, \boldsymbol{K}, L)$), then letting λ go to $+\infty$. This proves (2.13): thus we have at our disposal two equivalent ways of representing $\boldsymbol{C}(a, \boldsymbol{K}, L)$.

DEFINITION 2.2. We say that $\{a, \boldsymbol{K}, L\}$ is a *compatible set of data* if $C(a, \boldsymbol{K}, L)$ is a subspace.

Since $\boldsymbol{C}(a, \boldsymbol{K}, L)$ is a cone, the compatibility is clearly equivalent to

$$w \in \boldsymbol{C}(a, \boldsymbol{K}, L) \Rightarrow - w \in \boldsymbol{C}(a, \boldsymbol{K}, L) . \tag{2.14}$$

Another way of expressing the compatibility is given by the following

LEMMA 2.2. Compatibility for $\{a, \boldsymbol{K}, L\}$ is equivalent to the pair of conditions

$$w \in \mathrm{rc}\, \boldsymbol{K} \Rightarrow \langle L, w\rangle \leqslant \sup_{v \in \boldsymbol{K}} a(v, w) , \tag{2.15}$$

$$w \in \mathrm{rc}\, \boldsymbol{K} \, ; \ \langle L, w\rangle = \sup_{v \in \boldsymbol{K}} a(v, w) \Rightarrow - w \in \mathrm{rc}\, \boldsymbol{K} \, ; \ a(v, w) = \langle L, w\rangle \quad \forall v \in \boldsymbol{K} . \tag{2.16}$$

Furthermore, (2.15) is a necessary condition for the solvability of pb $(a, \boldsymbol{K}, L)$ and it is equivalent to

$$w \in Y \cap \operatorname{rc} \boldsymbol{K} \Rightarrow \langle L, w\rangle \leqslant \sup_{v\in \boldsymbol{K}} a(v, w) . \tag{2.17i}$$

Eventually, (2.16) is equivalent to

$$w \in Y \cap \operatorname{rc} \boldsymbol{K} \; ; \; \langle L, w\rangle = \sup_{v\in \boldsymbol{K}} a(v, w) \Rightarrow - w \in \operatorname{rc} \boldsymbol{K} \; ; \; a(v, w) = \langle L, w\rangle \qquad \forall v \in \boldsymbol{K} . \tag{2.17ii}$$

PROOF. Let (2.14) hold. If w belongs to $\operatorname{rc} \boldsymbol{K}$, then either $\exists v_0 \in \boldsymbol{K}$ such that $\langle L, w\rangle \leqslant a(v_0, w)$ or

$$\langle L, w\rangle \geqslant a(v, w) \qquad \forall v \in \boldsymbol{K} . \tag{2.18}$$

In the former case, (2.15) is obviously verified; but it does also in the latter, for (2.18) means that w is actually in $\boldsymbol{C}(a, \boldsymbol{K}, L)$: so it does $-w$ by hypothesis. Hence $\langle L, -w\rangle \geqslant a(v, -w)$, $\forall v \in \boldsymbol{K}$. Then (2.15) is again true. By the way, the previous argument shows that, if (2.18) holds, it actually is $\langle L, w\rangle = a(v, w)$, $\forall v \in \boldsymbol{K}$ and $-w \in \boldsymbol{C}(a, \boldsymbol{K}, L)$, so (2.16) is true. That (2.15) and (2.16) imply (2.14) is obvious. Let us prove that (2.15) is a necessary condition. If u solves pb $(a, \boldsymbol{K}, L)$ and w belongs to $\operatorname{rc} \boldsymbol{K}$, we can plug $v = u + w$ into (1.11). We get

$$\langle L, w\rangle \leqslant a(u, w) \leqslant \sup_{v\in \boldsymbol{K}} a(v, w) ,$$

so (2.15) is necessary.

Since (2.15) (respectively (2.16)) obviously implies (2.17i) (respectively (2.17ii)), we only have to prove the converse. This is very easy, since if w belongs to $\operatorname{rc} \boldsymbol{K}$ and $a(w, w) \neq 0$, then

$$\sup_{v\in \boldsymbol{K}} a(v, w) = \sup_{\substack{v\in \boldsymbol{K}\\ \lambda>0}} a(v + \lambda w, w) = + \infty .$$

This means that (2.15) is fulfilled once (2.17i) is and nothing has to be checked in (2.16). ■

Note that (2.17i) and (2.17ii) are more convenient in the applications than (2.15) and (2.16).

Let us introduce the *symmetric part* of a:

$$a_s(u, v) \equiv \tfrac{1}{2}[a(u, v) + a(v, u)] \qquad \forall u, v \in V , \tag{2.19}$$

and its related operator $\boldsymbol{A}_s \in \mathcal{L}(V, V')$:

$$\langle \boldsymbol{A}_s u, v\rangle = a_s(u, v) \qquad \forall u, v \in \boldsymbol{V}\,. \tag{2.20}$$

REMARK 2.1. We have

$$\ker \boldsymbol{A} = \ker \boldsymbol{A}^* \subseteq \ker \boldsymbol{A}_s = \ker a_s = Y\,. \tag{2.21}$$

In particular, Y is a closed subspace of V (see (1.8), (1.9), (1.10) for the definitions of $\boldsymbol{A}$, $\boldsymbol{A}^*$, Y).

REMARK 2.2. The inclusion in (2.21) may be proper. However, if a is symmetric, all kernels in (2.21) coincide.

REMARK 2.3. The compactness assumption (1.18) entails that

$$\dim (Y \cap \ker \Pi_0) < +\infty\,. \tag{2.22}$$

Since $\operatorname{rc} \boldsymbol{K}$ is obviously a subset of $\ker \Pi_0$, we have that

(2.23) $\boldsymbol{C}(a, \boldsymbol{K}, L)$ *is contained in a finite dimensional space*.

It follows that, once the necessary condition (2.17i) is satisfied, only a finite number of verifications is needed to check whether or not the compatibility condition (2.14) holds true.

Now we give a more explicit formulation of our conditions in some particular cases, say when a is symmetric or $\boldsymbol{K}$ is a cone, omitting the easy proofs.

LEMMA 2.3. If a is symmetric, the necessary condition (2.17i) becomes

$$\langle L, w\rangle \leqslant 0 \qquad \forall w \in Y \cap \operatorname{rc} \boldsymbol{K}\,. \tag{2.24}$$

Compatibility is equivalent to assuming (2.24) and

(2.25) $Y \cap \ker L \cap \operatorname{rc} \boldsymbol{K}$ *is a subspace*. ■

LEMMA 2.4. Let $\boldsymbol{K}$ be a cone. Then the necessary condition (2.17i) becomes

$$w \in \boldsymbol{K} \cap Y\,, \quad a(v, w) \leqslant 0 \quad \forall v \in \boldsymbol{K} \;\Rightarrow\; \langle L, w\rangle \leqslant 0\,. \tag{2.26}$$

Compatibility is equivalent to assuming (2.26) and

$$(2.27)\qquad w \in \boldsymbol{K} \cap Y \cap \ker L\,;$$

$$a(v, w) \leqslant 0 \quad \forall v \in \boldsymbol{K} \Rightarrow -w \in \boldsymbol{K}\,; \qquad a(v, w) = 0 \quad \forall v \in \boldsymbol{K}\,. \qquad \blacksquare$$

Let us point out that when a is symmetric *and* $\boldsymbol{K}$ is a cone, we find the necessary condition (1.22) of [9].

REMARK 2.4. If $\boldsymbol{K} = V$, conditions of type « theorem of the alternative » are obtained. In particular, the necessary condition reads

$$(2.28)\qquad w \in V\,; \quad a(v, w) = 0 \qquad \forall v \in V \Rightarrow \langle L, w\rangle = 0\,.$$

Once (2.28) is satisfied, the compatibility condition (2.27) is also true, so that both (2.17i) and (2.17ii) reduce to (1.17). The compactness-corciveness hypothesis (1.18) takes the following form ([5]):

(2.29) *there exist a coercive isomorphism* $B \in \mathcal{L}(V, V')$ *and a compact operator* $\Pi_1 \in \mathcal{L}(V, V')$ *such that* $A = B - \Pi_1^* J \Pi_1$,

where $J \in \mathcal{L}(V, V')$ is the Riesz operator defined in the introduction.

We will see (Theorem 3.1) that, assuming (2.29), (2.28) is a (necessary and) sufficient condition in order that pb (a, V, L) is solvable.

3. – Existence theorem and first applications.

Since (1.3) holds, we can recover coerciveness by setting

$$(3.1)\qquad a_\varepsilon(u, v) = a(u, v) + \varepsilon(u, v) \qquad \forall u, v \in V\,,\ \forall \varepsilon > 0\,.$$

Indeed, a_ε satisfies (1.15) with $\alpha = \varepsilon$. Then Stampacchia's theorem states existence and uniqueness of u_ε in $\boldsymbol{K}$ such that

$$(3.2)\qquad a_\varepsilon(u_\varepsilon, u_\varepsilon - v) \leqslant \langle L, u_\varepsilon - v\rangle \qquad \forall v \in \boldsymbol{K}\,,$$

or equivalently in Minty's form

$$(3.3)\qquad a_\varepsilon(v, u_\varepsilon - v) \leqslant \langle L, u_\varepsilon - v\rangle \qquad \forall v \in \boldsymbol{K}\,.$$

([5]) More generally, if $\boldsymbol{K}$ is a cone the only linear continuous operator Π_0 which is bounded on $\boldsymbol{K}$ is the trivial one, for $\Pi_0 v \neq 0 \Rightarrow \|\Pi_0(\lambda v)\| = \lambda \|\Pi_0 v\| \xrightarrow[\lambda \to +\infty]{} +\infty$.

LEMMA 3.1. The function $\varepsilon \to \|u_\varepsilon\|$ is non increasing on $]0, +\infty[$. Furthermore, if $S(a, K, L)$ is not empty, then

$$\|u_\varepsilon\| \leqslant \|u\| \qquad \forall u \in S(a, K, L)\,, \ \forall \varepsilon > 0\,. \tag{3.4}$$

PROOF. Let ε_1, ε_2 be such that $\varepsilon_1 > \varepsilon_2 > 0$. Plugging $v = u_{\varepsilon_2}$ (resp. $v = u_{\varepsilon_1}$) into (3.2) written for ε_1 (resp. ε_2), then adding, we get

$$\varepsilon_1(u_{\varepsilon_1}, u_{\varepsilon_1} - u_{\varepsilon_2}) + \varepsilon_2(u_{\varepsilon_2}, u_{\varepsilon_2} - u_{\varepsilon_1}) \leqslant a(u_{\varepsilon_1} - u_{\varepsilon_2}, u_{\varepsilon_2} - u_{\varepsilon_1}) \leqslant 0\,.$$

Hence,

$$\varepsilon_1(u_{\varepsilon_1}, u_{\varepsilon_1} - u_{\varepsilon_2}) \leqslant \varepsilon_2(u_{\varepsilon_2}, u_{\varepsilon_1} - u_{\varepsilon_2})$$
$$= -\varepsilon_2\|u_{\varepsilon_1} - u_{\varepsilon_2}\|^2 + \varepsilon_2(u_{\varepsilon_1}, u_{\varepsilon_1} - u_{\varepsilon_2}) \leqslant \varepsilon_2(u_{\varepsilon_1}, u_{\varepsilon_1} - u_{\varepsilon_2})\,.$$

This gives that

$$(\varepsilon_1 - \varepsilon_2)(u_{\varepsilon_1}, u_{\varepsilon_1} - u_{\varepsilon_2}) \leqslant 0\,,$$

say

$$\|u_{\varepsilon_1}\|^2 \leqslant (u_{\varepsilon_1}, u_{\varepsilon_2}) \leqslant \|u_{\varepsilon_1}\|\,\|u_{\varepsilon_2}\|\,,$$

whence the first assertion follows. The second holds trivially, since we can repeat the same argument as in the first part taking formally $\varepsilon_2 = 0$, whenever S in not empty. ■

As a particular case of Theorem 3.3 of [9], the following lemma holds.

LEMMA 3.2. $S(a, K, L)$ is not empty if and only if

$$\exists C\colon\ \|u_\varepsilon\| \leqslant C \qquad \forall \varepsilon > 0\,. \tag{3.5}$$

Moreover, if (3.5) holds, we have that u_ε converges strongly to the element u_0 of S of minimal norm (6). ■

THEOREM 3.1. Assume (1.1)-(1.5) and (1.18). If $\{a, K, L\}$ is compatible, then pb (a, K, L) has a solution. In this case, the family $\{u_\varepsilon\}$ of solutions of (3.2) converges to the solution u_0 of minimal norm.

(6) Such a u_0 exists and is unique thanks to (2.2). Since the convergence of u_ε to u_0 depends upon the strong topology in V, the limit point may vary when changing the scalar product in V. However, once this is fixed our results hold true.

PROOF. The proof will be carried out by contradiction. Assume that (3.5) is not satisfied (say, there exists no solution). Then there exists ε_n, w such that

$$\varepsilon_n \to 0 \; ; \qquad \|u_{\varepsilon_n}\| \to +\infty \tag{3.6}$$

and, possibly taking subsequences,

$$w_n \equiv \frac{u_{\varepsilon_n}}{\|u_{\varepsilon_n}\|} \rightharpoonup w \text{ (weak convergence)}. \tag{3.7}$$

Let us prove that

$$(3.6) \textit{ and } (2.14) \Rightarrow w = 0 \, . \tag{3.8}$$

Dividing both sides of (3.2) (resp. (3.3)) with $\varepsilon = \varepsilon_n$ by $\|u_{\varepsilon_n}\|^2$ (resp. $\|u_{\varepsilon_n}\|$), using (3.6) we get that

$$\lim_{n\to\infty} a(w_n, w_n) = 0 \, , \tag{3.9}$$

$$a(v, w) \leqslant \langle L, w\rangle \qquad \forall v \in \boldsymbol{K} \, . \tag{3.10}$$

Again from (3.6), if v is fixed in $\boldsymbol{K}$ and n is large enough, we have that

$$\left(1 - \frac{1}{\|u_{\varepsilon_n}\|}\right) v + \frac{1}{\|u_{\varepsilon_n}\|} u_{\varepsilon_n} = \left(1 - \frac{1}{\|u_{\varepsilon_n}\|}\right) v + w_n$$

belongs to $\boldsymbol{K}$ and it converges weakly to $v + w$. This means that $w \in \operatorname{rc} \boldsymbol{K}$. Due to (3.10), we derive that

$$w \in \boldsymbol{C}(a, \boldsymbol{K}, L) \, . \tag{3.11}$$

Since we assume that (2.14) is true, (3.11) gives that $-w$ belongs to $\boldsymbol{C}(a, \boldsymbol{K}, L)$, hence $a(u_\varepsilon, w) = \langle L, w\rangle$, $\forall \varepsilon > 0$. So, $v = u_\varepsilon \pm w$ is permitted in (3.2), say

$$a(u_\varepsilon, \pm w) + \varepsilon(u_\varepsilon, \pm w) \leqslant \pm \langle L, w\rangle = a(u_\varepsilon, \pm w) \qquad \forall \varepsilon > 0 \, .$$

As a consequence of this, we get successively

$$(u_\varepsilon, w) = 0 \quad \forall \varepsilon > 0 \; ; \qquad (w_n, w) = 0 \quad \forall n \textit{ integer} \; ; \qquad \|w\|^2 = \lim_{n\to\infty} (w_n, w) = 0$$

and (3.8) is proved.

So, w_n converges weakly to 0, then (1.18ii) gives that $\|\Pi_1 w_n\|$ converges to 0. So $\|w_n\| \to 0$ too, according to (3.9) and (1.18iii), once we note that from (1.18i) it follows that

$$\|\Pi_0 w_n\| = \frac{\|\Pi_0 u_{\varepsilon_n}\|}{\|u_{\varepsilon_n}\|} \leqslant \frac{C}{\|u_{\varepsilon_n}\|}$$

for some constant C. Thus, w_n converges strongly to 0, but this is impossible, since $\|w_n\| = 1$, $\forall n$ integer. So, the proof of the theorem is complete. ■

REMARK 3.1. Assume that either of (1.15), (1.20) is satisfied. We have seen in section 1 that (1.18) is fulfilled. Still more, in both cases compatibility is true, since $\boldsymbol{C}(a, \boldsymbol{K}, L) \equiv \{0\}$. Hence Theorem 3.1 includes Stampacchia's theorem and Theorem 4.1 of [9].

As an application of the existence Theorem 3.1 we are going to give sufficient conditions in order that the algebraic difference between two closed convex sets is closed itself. This (not trivial) property will be used in section 4 to give some extensions of the existence theorem for pb $(a, \boldsymbol{K}, L)$.

THEOREM 3.2. Let H be a Hilbert space and A, B be two nonempty closed convex subsets of H. Assume that any of the following is satisfied:

(3.12) *either A or B is bounded*;

(3.13) *either A or B is contained in a finite dimensional subspace of H and* $\mathrm{rc}\, A \cap \mathrm{rc}\, B$ *is a subspace of H*.

Then, $A - B$ is closed.

PROOF. Since $A - B$ is convex, it is also closed if and only if there exists the projection u in $A - B$ of any w of H, say

$$\frac{1}{2}\|u\|_H^2 - (w, u)_H = \min_{z \in A-B}\left[\frac{1}{2}\|z\|_H^2 - (w, z)_H\right], \quad u \in A - B. \tag{3.14}$$

So, for any $w \in H$ we must investigate the solvability of the problem:

$$\text{to find} \min_{a \in A,\, b \in B}\left[\frac{1}{2}(a-b, a-b)_H - (w, a-b)_H\right], \qquad u \in A - B. \tag{3.15}$$

First, we note that (3.15) can be written in the following way:

$$\text{to find} \min_{v \in \boldsymbol{K}}\left[\frac{1}{2}a(v, v) - \langle L, v\rangle\right], \tag{3.16}$$

where $a\colon (H\times H)\times(H\times H)\to \boldsymbol{R}$, $L\in(H\times H)'$ and $\boldsymbol{K}\subset H\times H$ are defined as follows:

$$(3.17)\quad a(u, v)\equiv(u_1-u_2, v_1-v_2)_H \quad \forall u=[u_1, u_2]\in H\times H,\ \forall v=[v_1, v_2]\in H\times H;$$

$$(3.18)\quad \langle L, v\rangle\equiv(w, v_1-v_2)_H \qquad \forall v=[v_1, v_2]\in H\times H\ ;$$

$$(3.19)\quad \boldsymbol{K}\equiv A\times B\ .$$

Closedness of $A-B$ is then equivalent to solving (3.16) *for any* L satisfying (3.18). We claim that this actually possible, as soon as (3.12) or (3.13) are fulfilled. In fact, with our choice of a and L, defined on $V=H\times H$, structural hypotheses of Theorem 3.1 are satisfied. Further,

$$Y=\{v=[v_1, v_2]\in H\times H \ \textit{s.t.}\ v_1=v_2\}$$

and

$$\operatorname{rc}(A\times B)\cap Y=(\operatorname{rc}A\times\operatorname{rc}B)\cap Y=(\operatorname{rc}A\cap\operatorname{rc}B)\times(\operatorname{rc}A\cap\operatorname{rc}B)\,.$$

So, the set $\{a, A\times B, L\}$ is compatible, since a is symmetric and (2.24), (2.25) are fulfilled: $\operatorname{rc}(A\times B)\cap Y\subset\ker L$ and $\operatorname{rc}(A\times B)\cap Y\cap\ker L$ is a subspace by hypothesis (in fact, if (3.12) is satisfied, then $\operatorname{rc}A\cap\operatorname{rc}B$ reduces to the origin). The compactness-coerciveness assumption is satisfied, once one of (3.12) or (3.13) holds. So, we can apply Theorem 3.1, hence (3.18) has a solution. This achieves the proof. ■

4. – Extensions.

Our aim in this section is to study pb $(a, \boldsymbol{K}, L)$ when the compactness-coerciveness assumption (1.18) and the necessary condition (2.17*i*) are satisfied, without requiring compatibility. Then, we will seek for sufficient conditions weaker than (2.14).

The idea is to modify the convex set $\boldsymbol{K}$ in order to write a new problem which Theorem 3.1 applies to, and then to come back to a solution for pb $(a, \boldsymbol{K}, L)$. Modification of $\boldsymbol{K}$ can be done in many ways without losing equivalence with the original problem, yet some care is required, as we shall see a little later.

LEMMA 4.1. Let W be any subset of V such that

$$(4.1)\qquad W\subset\ker A\cap\ker L\,.$$

Then pb $(a, \boldsymbol{K}, L)$ and pb $(a, \boldsymbol{K}-W, L)$ (7) are equivalent, in the following sense: for any solution u of pb $(a, \boldsymbol{K}, L)$ there exists a w in W such that $u-w$ solves pb $(a, \boldsymbol{K}-W, L)$ and conversely for any solution z of pb $(a, \boldsymbol{K}-W, L)$ there exists w in W such that $z+w$ solves pb $(a, \boldsymbol{K}, L)$.

PROOF. Since W does not affect a nor L, the proof is obvious. We just note that if $z=u-w$ $(u\in\boldsymbol{K},\ w\in W)$ is a solution of pb $(a, \boldsymbol{K}-W, L)$, then u solves pb $(a, \boldsymbol{K}, L)$. ■

A different modification of $\boldsymbol{K}$ is the convex

$$\boldsymbol{K}_1=\boldsymbol{K}-\boldsymbol{C}(a, \boldsymbol{K}, L)\,. \tag{4.2}$$

LEMMA 4.2. (i) Any solution of pb $(a, \boldsymbol{K}, L)$ solves also pb $(a, \boldsymbol{K}_1, L)$ (7).

(ii) Conversely, if z solves pb $(a, \boldsymbol{K}_1, L)$, then there exists w in $\boldsymbol{C}(a, \boldsymbol{K}, L)$ such that $z+w$ solves pb $(a, \boldsymbol{K}, L)$.

PROOF. (i) Let u solve pb $(a, \boldsymbol{K}, L)$. Since $\boldsymbol{C}(a, \boldsymbol{K}, L)$ contains 0, then u belongs to $\boldsymbol{K}_1$ and

$$\begin{aligned}\forall w\in\boldsymbol{C}(a, \boldsymbol{K}, L)\,,\ \forall v\in\boldsymbol{K}\,,\qquad & a\big(u, u-[v-w]\big)-\langle L, u-[v-w]\rangle\\ &=a(u, u-v)-\langle L, u-v\rangle+a(u, w)-\langle L, w\rangle\leqslant 0\,.\end{aligned}$$

So, u solves pb $(a, \boldsymbol{K}_1, L)$.

(ii) Since $z\in\boldsymbol{K}_1\equiv\boldsymbol{K}-\boldsymbol{C}(a, \boldsymbol{K}, L)$, there exist u in $\boldsymbol{K}$ and w in $\boldsymbol{C}(a, \boldsymbol{K}, L)$ such that $z=u-w$. Now, z solves pb $(a, \boldsymbol{K}_1, L)$, so we can write, for $v\in\boldsymbol{K}$,

$$\begin{aligned}a(u, u-v)-\langle L, u-v\rangle&=a(u-w, u-w-v)\\ &\quad-\langle L, u-w-v\rangle+a(w, u-w-v)+a(u-w, w)-\langle L, w\rangle\\ &\leqslant 2a_s(u, w)-2a_s(v, w)+a(v, w)-\langle L, w\rangle=a(v, w)-\langle L, w\rangle\,.\end{aligned}$$

The last quantity is nonpositive, since w belongs to $\boldsymbol{C}(a, \boldsymbol{K}, L)$. This proves that u solves pb $(a, \boldsymbol{K}, L)$. ■

REMARK 4.1. It goes without saying that the element w in Lemma 4.2 (ii) must satisfy the only requirement that $z+w$ belongs to $\boldsymbol{K}$.

(7) pb $(a, \boldsymbol{K}-W, L)$ is not well posed, in general, since $\boldsymbol{K}-W$ is not even closed. Yet, the statement of the lemma remains true. This holds also for

$$\text{pb}\,(a, \boldsymbol{K}-\boldsymbol{C}(a, \boldsymbol{K}, L), L)\,.$$

This is true, in particular, if z is itself a solution of pb $(a, \boldsymbol{K}, L)$. Hence $\boldsymbol{K} \cap \boldsymbol{S}(a, \boldsymbol{K}_1, L) \subset \boldsymbol{S}(a, \boldsymbol{K}, L)$. Lemma 4.2 (i) provides the opposite inclusion, so that

$$\boldsymbol{S}(a, \boldsymbol{K}, L) = K \cap \boldsymbol{S}(a, \boldsymbol{K}_1, L).$$

REMARK 4.2. The statements of the Lemmas 4.1 and 4.2 may be unified in the following way:

(4.3) *if* $Z \subset \ker \boldsymbol{A} \cap \ker L + \boldsymbol{C}(a, \boldsymbol{K}, L)$ *then* pb $(a, \boldsymbol{K}, L)$ *and* pb $(a, \boldsymbol{K} - Z, L)$ *are equivalent*

(the equivalence is in the sense of the quoted lemmas). The proof of this assertion can be obtained easily by combining those of the two lemmas.

Lemmas 4.1 and 4.2 allow us to transform pb $(a, \boldsymbol{K}, L)$ into equivalent problems. Yet, the question of existence of a solution is shifted to the new problems: the widened convexes might not be closed and either the necessary condition or the compatibility might not hold for the modified problems. Although pb $(a, \boldsymbol{K}_1, L)$ seems to be the natural candidate for which the compatibility requirement is automatically true ($\boldsymbol{C}(a, \boldsymbol{K}, L)$ has been added its missing directions), yet, if $\boldsymbol{K}_1$ is not closed, $\boldsymbol{C}(a, \boldsymbol{K}_1, L)$ is not even defined. Furthermore, $\boldsymbol{K}_1$ may not be closed whilst $\boldsymbol{K} - W$ does, for some W satisfying (4.1).

Extensions of Theorem 3.1 may be obtained by a careful balance between *little* modifications of $\boldsymbol{K}$ (in order to get closedness only by a few verifications) and *large* modifications of $\boldsymbol{K}$ (which can lead to more general existence theorems). Lemmas 4.1 and 4.2 tell us in some sense where the subtracted set has to be contained in order to come back to a solution of pb $(a, \boldsymbol{K}, L)$.

Let us focus the question of the solvability of the modified problems. Our aim is to identify on the one hand the minimal set to subtract in order to get the closure of the extended convexes and the compatibility for the related problems and on the other hand the maximal set we can subtract in order to maintain compactness-coerciveness and the necessary condition.

First, we study the compactness-coerciveness condition.

LEMMA 4.3. Let W be any subset of a finite dimensional subspace of V (alternatively, let W be any subset of $\ker \Pi_0$). Assume that (1.18) is fulfilled for pb $(a, \boldsymbol{K}, L)$. Then it is satisfied also for pb $(a, \boldsymbol{K} - W, L)$.

PROOF. A change of the mappings Π_0 and Π_1 is needed in the first case. Precisely, denoting by P the operator defined as

$$Px = \begin{cases} \Pi_0 x & \textit{if } x \in \operatorname{span} W, \\ 0 & \textit{if } x \in (\operatorname{span} W)^{\perp}, \end{cases}$$

we have that P is linear and continuous, since W is finite dimensional. So, $\Pi_0 - P$ is bounded on $\boldsymbol{K} - W$ and $\Pi_1 + P$ is compact (again because P maps onto a finite dimensional subspace). With these two operators it is easy to check that (1.18) is fulfilled for $\mathrm{pb}\,(a, \boldsymbol{K} - W, L)$ [8].

In the alternative hypothesis, it is immediate to check that (1.18) holds for $\mathrm{pb}\,(a, \boldsymbol{K} - W, L)$ with the same Π_0, Π_1 and α as in $\mathrm{pb}\,(a, \boldsymbol{K}, L)$. ∎

It will be useful to introduce a symbol for the necessary condition (2.17i): we say that

$$(4.4) \qquad \boldsymbol{N}(a, \boldsymbol{K}, L) \textit{ holds, if and only if } w \in \mathrm{rc}\,\boldsymbol{K} \Rightarrow \langle L, w\rangle \leqslant \sup_{v \in \boldsymbol{K}} a(v, w)\,.$$

We are going to prove an existence theorem more general than Theorem 3.1 in the case $\boldsymbol{K} =$ cone.

LEMMA 4.4. Assume that $\boldsymbol{K}$ is a cone and that:

$$(4.5) \qquad \begin{cases} \boldsymbol{N}(a, \boldsymbol{K}, L) \textit{ holds}\,; \\ \boldsymbol{K}_1 \textit{ is closed } (^9)\,. \end{cases}$$

Then $\boldsymbol{N}(a, \boldsymbol{K}_1, L)$ holds and $\{a, \boldsymbol{K}_1, L\}$ is compatible.

PROOF. Since $\boldsymbol{K}$ is a cone, also $\boldsymbol{K}_1$ is a cone: it is closed (see (4.5)), hence $\mathrm{rc}\,\boldsymbol{K}_1$ is defined and coincides with $\boldsymbol{K}_1$ itself (see (2.6)). So, any w of $\mathrm{rc}\,\boldsymbol{K}_1$ may be written as $w = k - c$ for some k in $\boldsymbol{K}$ and c in $\boldsymbol{C}(a, \boldsymbol{K}, L)$. Using (2.11) and (2.17i) we have that

$$(4.6) \qquad \langle L, w\rangle = \langle L, k\rangle - \langle L, c\rangle \leqslant \sup_{z \in \boldsymbol{K}} a(z, k) - a(v, c) \qquad \forall v \in \boldsymbol{K}.$$

Hence, for any $\varepsilon > 0$ there exists z_ε in $\boldsymbol{K}$ such that

$$(4.7) \qquad \langle L, w\rangle \leqslant a(z_\varepsilon, k) - a(z_\varepsilon, c) + \varepsilon = a(z_\varepsilon, w) + \varepsilon \leqslant \sup_{v \in \boldsymbol{K}_1} a(v, w) + \varepsilon\,.$$

Since ε is arbitrary, we derive that

$$w \in \mathrm{rc}\,\boldsymbol{K}_1 \Rightarrow \langle L, w\rangle \leqslant \sup_{v \in \boldsymbol{K}_1} a(v, w)\,,$$

say $\boldsymbol{N}(a, \boldsymbol{K}_1, L)$ holds.

(8) Of course, in (1.18) (which is valid for $\mathrm{pb}\,(a, \boldsymbol{K}, L)$) it is not restrictive to assume on one hand that Π_0 maps orthogonal subspaces onto orthogonal subspaces and on the other hand that $\Pi_1 x = 0$ for those x of V such that $\Pi_0 x \neq 0$.

(9) $\boldsymbol{K}_1$ is defined in (4.2).

If our w satisfies $\langle L, w\rangle = \sup_{v\in \boldsymbol{K}_1} a(v, w)$, then for any h in $\boldsymbol{K}$ we have that

$$\langle L, k\rangle - \langle L, c\rangle \geqslant a(h, k-c) = a(h, k) - a(h, c)\ .$$

Hence, recalling that c belongs to $\boldsymbol{C}(a, \boldsymbol{K}, L)$,

$$\langle L, k\rangle - a(h, k) \geqslant \langle L, c\rangle - a(h, c) \geqslant 0\ , \qquad \forall h\in \boldsymbol{K}\ .$$

This means that $k\in \boldsymbol{C}(a, \boldsymbol{K}, L)$, then $-w = c - k \in \boldsymbol{C} - \boldsymbol{C}$. Since $\boldsymbol{C}\subset \boldsymbol{K}$, we have that $-w\in \boldsymbol{K} - \boldsymbol{C}$, and then $\{a, \boldsymbol{K}_1, L\}$ is compatible. ■

Let us consider a different way of extending $\boldsymbol{K}$.

LEMMA 4.5. Assume that $\boldsymbol{K}$ is a cone and that

(4.8) $\boldsymbol{K}^1 \equiv \boldsymbol{K} - \boldsymbol{K}\cap \ker \boldsymbol{A} \cap \ker L$ *is closed*,

(4.9) $\langle L, w\rangle \leqslant 0$ *for any* $w\in \boldsymbol{K}\cap \ker \boldsymbol{A}$ (10),

(4.10) *for any* $w\in \boldsymbol{K}\cap Y$ *with* $Aw\neq 0$ *there exists* $v = v(w)\in \boldsymbol{K}$ *such that* $\langle L, w\rangle < a(v, w)$.

Then $\boldsymbol{N}(a, \boldsymbol{K}^1, L)$ holds and $\{a, \boldsymbol{K}^1, L\}$ is compatible.

PROOF. As in the proof of Lemma 4.4, we have

$$\mathrm{rc}\,(\boldsymbol{K} - \boldsymbol{K}\cap \ker \boldsymbol{A}\cap \ker L) = \boldsymbol{K} - \boldsymbol{K}\cap \ker \boldsymbol{A}\cap \ker L\ . \tag{4.11}$$

Therefore any w of $Y\cap \mathrm{rc}\,(\boldsymbol{K} - \boldsymbol{K}\cap \ker \boldsymbol{A}\cap \ker L)$ can be represented as $w = k - l$ for some k of $\boldsymbol{K}\cap Y$ and some l of $\boldsymbol{K}\cap \ker \boldsymbol{A}\cap \ker L$. So, we have

$$\begin{cases} \langle L, w\rangle = \langle L, k\rangle \leqslant 0 = a(v, w) & \forall v\in \boldsymbol{K}^1 \text{ if } k\in \ker \boldsymbol{A}\ ,\\ \langle L, w\rangle = \langle L, k\rangle < a(v(w), w) & \text{if } k\in Y\setminus \ker \boldsymbol{A}\ . \end{cases} \tag{4.12}$$

In any case,

$$\langle L, w\rangle \leqslant \sup_{v\in \boldsymbol{K}^1} a(v, w)\ . \tag{4.13}$$

That is, $\boldsymbol{N}(a, \boldsymbol{K}^1, L)$ is satisfied. Furthermore, if w is such that

$$\langle L, w\rangle = \sup_{v\in \boldsymbol{K}^1} a(v, w)\ , \tag{4.14}$$

(10) Note that (4.9) is fulfilled as soon as the condition $\boldsymbol{N}(a, \boldsymbol{K}, L)$ holds.

then

$$\langle L, k\rangle = \sup_{h\in K} a(h, k)\,. \tag{4.15}$$

Now, it cannot be $Ak \neq 0$, since this contradicts (4.10). So, it is $k \in \ker A$. Due to (4.15), k must belong to $K \cap \ker A \cap \ker L$, hence

$$-w = l - k \in K \cap \ker A \cap \ker L - K \cap \ker A \cap \ker L \subset K^1 \cap \ker A \cap \ker L\,. \tag{4.16}$$

In particular, w belongs to $\ker A \cap \ker L$, that is

$$\langle L, w\rangle = a(v, w) \qquad \forall v \in K^1\,. \tag{4.17}$$

We derive that $\{a, K^1, L\}$ is compatible. ∎

Analogous result holds when extending K in a further way: we report it in the following lemma, whose proof is similar to the previous one (hence we omit it).

LEMMA 4.6. Let K be a cone and assume that

(4.18) $K - \ker A \cap \ker L$ *is closed*,

(4.19) $\langle L, w\rangle \leq 0$ *for any* $w \in K \cap \ker A$,

(4.20) *for any* $w \in K \cap Y$ *with* $Aw \neq 0$ *there exists* $v = v(w) \in K$ *such that* $\langle L, w\rangle < a(v, w)$.

Then $N(a, K - \ker A \cap \ker L, L)$ holds and $\{a, K - \ker A \cap \ker L, L\}$ is compatible. ∎

Now we can state an existence theorem for pb (a, K, L).

THEOREM 4.1. Assume (1.1)-(1.5) and (1.18). Assume further that

(4.21) K *is a cone*,

(4.22) $N(a, K, L)$ *holds*,

(4.23) $K_1 \equiv K - C(a, K, L)$ *is closed*.

Then pb (a, K, L) is solvable.

PROOF. Owing to Remark 2.3, to the Lemmas 4.3 and 4.4 and to Theorem 3.1, pb $(a, \boldsymbol{K}_1, L)$ has a solution. Lemma 4.2 yields that pb $(a, \boldsymbol{K}, L)$ has a solution as well. ∎

Again, we can state an analogous existence result for a different modification of $\boldsymbol{K}$.

THEOREM 4.2. Assume (1.1)-(1.5), (1.18) and (4.21). Assume further that

(4.24) $\boldsymbol{K} \cap \ker \boldsymbol{A} \cap \ker L$ *is a subspace*,

(4.25) $\langle L, w\rangle \leqslant 0$ *for any* $w \in \boldsymbol{K} \cap \ker A$,

(4.26) *for any* $w \in \boldsymbol{K} \cap Y$ *with* $Aw \neq 0$ *there exists* $v = v(w) \in \boldsymbol{K}$ *such that* $\langle L, w\rangle < a(v, w)$.

Then pb$(a, \boldsymbol{K}, L)$ is solvable.

PROOF. Since $\boldsymbol{K}$ is a cone, from Remark 2.3 and footnote (5) we derive that Y has finite dimension. So, we can apply Theorem 3.2 and get that $\boldsymbol{K} - \boldsymbol{K} \cap \ker \boldsymbol{A} \cap \ker L$ is closed. Due to Lemmas 4.3 and 4.5 and to Theorem 3.1, pb $(a, \boldsymbol{K}^1, L)$ (11) has a solution. Lemma 4.1 yields that pb $(a, \boldsymbol{K}, L)$ is solvable as well. ∎

REMARK 4.3. We notice that one could substitute (4.24) with either of the following

(4.27) $\boldsymbol{K} - \boldsymbol{K} \cap \ker \boldsymbol{A} \cap \ker L$ *is closed*,

(4.28) $\boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L$ *is closed*,

and Theorem 4.2 would still hold. Yet, these hypotheses are hard to verify, though more general than (4.24).

REMARK 4.4. Theorem 4.2 is useful in the non-symmetric case. Indeed, if a is symmetric, condition (4.26) is empty and the statement reduces to the one of Theorem 3.1.

Removing the restriction that $\boldsymbol{K}$ is a cone carries some troubles. In fact, for a general closed, nonempty convex set $\boldsymbol{K}$, even if Q is a cone and $\boldsymbol{K} - Q$ is closed, the equality $\mathrm{rc}(\boldsymbol{K} - Q) = \mathrm{rc}\, \boldsymbol{K} - Q$ does not hold, as it is shown by the following example.

(11) $\boldsymbol{K}^1$ is defined in (4.8).

EXAMPLE 4.1. Let $V = \boldsymbol{R}^2$, $\boldsymbol{K} = \{(x, y): y \geqslant x^2\}$, $Q = \{(x, y): x = 0\}$. Then, $\operatorname{rc}(\boldsymbol{K} - Q) = \boldsymbol{R}^2$ while $\operatorname{rc}\boldsymbol{K} - Q = Q$.

In general we have only that $\operatorname{rc}(\boldsymbol{K} - Q) \supset \operatorname{rc}\boldsymbol{K} - Q$ and it is not possible to characterize $\operatorname{rc}(\boldsymbol{K} - Q)$ in an abstract way without imposing conditions which are hard to verify. Yet, for general convexes we can give the following two theorems, whose proof follows immediately from Lemmas 4.1, 4.2 and 4.3 and from Theorem 3.1.

THEOREM 4.3. Let (1.1)-(1.5) and (1.18) hold. Assume that

$$\boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L \cap \operatorname{rc}\boldsymbol{K} \textit{ is closed}, \tag{4.29}$$

$$\{a, \boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L \cap \operatorname{rc}\boldsymbol{K}, L\} \textit{ is compatible}. \tag{4.30}$$

Then pb $(a, \boldsymbol{K}, L)$ is solvable. ■

THEOREM 4.4. Let (1.1)-(1.5) and (1.18) hold. Assume that

$$\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L) \textit{ is closed}, \tag{4.31}$$

$$\{a, \boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L), L\} \textit{ is compatible}. \tag{4.32}$$

Then pb $(a, \boldsymbol{K}, L)$ is solvable. ■

We notice that Theorems 4.3 and 4.4 are different only when a is not symmetric (see Lemma 2.3).

Remark also that only a finite number of verifications is needed to check whether or not compatibility for the extended problems holds (see Remark 2.3).

REMARK 4.5. The enlargement of the convex by subtracting a suitable set may be iterated again and again. In Appendix a we will detail this procedure, at least for symmetric bilinear forms. We will show that after a finite number of steps the convexes do not change any more: at that point, the transformed problem is *always* solvable. Moreover, the intersection between the set of its solution and $\boldsymbol{K}$ gives all the solutions of pb $(a, \boldsymbol{K}, L)$.

So far, we have given several sufficient conditions for the solvability of pb $(a, \boldsymbol{K}, L)$. It is worthwhile to mention the relationship between Fichera's results (see [6]) and ours.

A preliminary step consists in transforming pb $(a, \boldsymbol{K}, L)$ into a problem with a convex containing the origin. This we are going to do as follows.

Since $\boldsymbol{K}$ is not empty, it must contain some z_0: put

$$\boldsymbol{K}_0 \equiv \boldsymbol{K} - \{z_0\}\,, \tag{4.33}$$

$$\langle L_0, v\rangle \equiv \langle L, v\rangle - \langle \boldsymbol{A} z_0, v\rangle \qquad \forall v \in V\,. \tag{4.34}$$

pb $(a, \boldsymbol{K}, L)$ and pb $(a, \boldsymbol{K}_0, L_0)$ are equivalent. Precisely, we have the following lemma whose proof we omit.

LEMMA 4.7. Let u solve pb $(a, \boldsymbol{K}, L)$. Then

$$u_0 \equiv u - z_0 \tag{4.35}$$

solves pb $(a, \boldsymbol{K}_0, L_0)$. Conversely, if u_0 solves pb $(a, \boldsymbol{K}_0, L_0)$, then

$$u \equiv u_0 + z_0 \tag{4.36}$$

solves pb $(a, \boldsymbol{K}, L)$. ∎

We claim that the conditions imposed in the existence theorems by Fichera ([6], Theorems 1.II and 2.I) are sufficient to achieve the closedness of $\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0$, as well as compatibility and compactness-coerciveness for pb $(a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0)$: we refer to the Appendix b for the proof of this assertion. Hence, pb $(a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0)$ is solvable, then Lemma 4.1 gives the solvability of pb $(a, \boldsymbol{K}_0, L_0)$ and Lemma 4.7 shows that pb $(a, \boldsymbol{K}, L)$ is solvable.

We end this section giving a number of necessary conditions for the existence of a solution of pb $(a, \boldsymbol{K}, L)$ and stating the relations among them.

LEMMA 4.8. If pb $(a, \boldsymbol{K}, L)$ has a solution, then

(4.37) *there exist* $v_0 \in V$, $M \in V'$ *such that* M *is bounded from above on* $\boldsymbol{K}$ *and* $\langle L, v\rangle = a(v_0, v) + \langle M, v\rangle$, $\forall v \in V$.

PROOF. It is enough to take as v_0 any solution of pb $(a, \boldsymbol{K}, L)$: indeed, defining M as

$$\langle M, v\rangle \equiv \langle L, v\rangle - a(v_0, v)\,, \tag{4.38}$$

we derive from (1.11)

$$\langle M, v\rangle \leqslant \langle M, v_0\rangle \qquad \forall v \in \boldsymbol{K}\,. \tag{4.39}$$

That is, M is bounded from above on $\boldsymbol{K}$. ∎

We will derive further necessary conditions, beginning with the symmetric case: recalling (1.13), let us set

$$i(\boldsymbol{K}) = \inf_{v\in\boldsymbol{K}} F(v)\,. \tag{4.40}$$

LEMMA 4.9. Let a be symmetric. If pb $(a, \boldsymbol{K}, L)$ has a solution, then

$$i(\boldsymbol{K}) \textit{ is finite} \tag{4.41}$$

and

$$L \textit{ is bounded from above on } \boldsymbol{K}\cap Y\,. \tag{4.42}$$

Moreover,

$$(4.37) \Rightarrow (4.41) \Rightarrow (4.42)\,. \tag{4.43}$$

PROOF. Since (4.37) is necessary (see Lemma 4.8), it is enough to prove (4.43) and necessity of (4.41), (4.42) will follow. So, assume (4.37). Then,

$$\begin{aligned} F(v) &= \frac{1}{2}a(v-v_0, v-v_0) - \frac{1}{2}a(v_0, v_0) - \langle M, v\rangle \\ &\geqslant -\frac{1}{2}a(v_0, v_0) - \langle M, v\rangle \geqslant -\frac{1}{2}a(v_0, v_0) - \sup_{v\in\boldsymbol{K}}\langle M, v\rangle > -\infty \end{aligned} \tag{4.44}$$

and this proves (4.41). That this implies (4.42) is obvious, since

$$F(v) = -\langle L, v\rangle \qquad \forall v\in\boldsymbol{K}\cap Y\,. \tag{4.45}$$ ■

We notice that (4.41) and (4.42) are not equivalent, as we will show in the Example a.1 of the Appendix a.

For the nonsymmetric case, we obtain necessary conditions that are fairly close to the sufficient ones of Theorem 4.2. Precisely, we have the following

LEMMA 4.10. If pb $(a, \boldsymbol{K}, L)$ has a solution, then

(4.46) *for any* $w\in \operatorname{rc}\boldsymbol{K}\cap Y$ *there exists* $v = v(w)\in\boldsymbol{K}$ *such that*

$$\langle L, w\rangle \leqslant a(v, w)\,.$$

If

$$0\in\boldsymbol{K}\,, \tag{4.47}$$

then

$$(4.48) \qquad (4.37) \Rightarrow (4.46)\ (^{12}) .$$

PROOF. If pb $(a, \boldsymbol{K}, L)$ is solved by u, then it is easy to check (4.46) with $v(w) \equiv u$ for all $w \in Y \cap \operatorname{rc} \boldsymbol{K}$.

On the other hand, assume *only* (4.37). From (4.47) it follows that $\operatorname{rc} \boldsymbol{K} \subset \boldsymbol{K}$. So, $\langle M, w\rangle \leqslant 0$, $\forall w \in \operatorname{rc} \boldsymbol{K}$, hence

$$(4.49) \qquad \langle L, w\rangle \leqslant a(v_0, w) ,$$

where v_0 is found as in (4.37). Then, (4.46) holds with $v(w) \equiv v_0$ for any w. Note that if w belongs to $\ker A \cap \operatorname{rc} \boldsymbol{K}$, then (4.46) obviously reduces to

$$\langle L, w\rangle \leqslant 0 \ \textit{for any} \ w \in \ker A \cap \operatorname{rc} \boldsymbol{K} . \qquad \blacksquare$$

5. – Applications to unilateral problems.

In this section we consider some equilibrium problems in linear elasticity. Our goal is on one hand to re-discover the well known existence results when classical compatibility is satisfied. On the other hand, we will apply our abstract existence Theorem 3.1 to some limit cases particularly interesting.

We begin with the problem of equilibrium of a beam or of a plate in presence of a rigid support. Since our first results will hold for both cases, we will consider them at conce, as long as distinction is not necessary.

Let Ω be a bounded, connected open subset of $\boldsymbol{R}^N$, $N = 1, 2$ (13). Let

$$(5.1) \qquad V = \boldsymbol{H}^2(\Omega)\ (^{14}) :$$

we assume that Ω is smooth enough so that

$$(5.2) \qquad V \subset \boldsymbol{C}^0(\bar{\Omega}) .$$

(12) Note that (4.46) follows *directly* from (4.37), even if a solution does not exists! On the other hand, the fact that (4.46) is necessary for the existence of a solution coincides with lemma 2.III of [6].

(13) Actually, our procedure would work for $N = 3$ as well, but this case does not seem to have physical interpretation.

(14) Here and in the following, $\boldsymbol{H}^s(\Omega)$ denotes the Sobolev space of order s (s real) (see [8]).

Denoting with $v_{,j}$ the derivative of v with respect to x_j, $j = 1, \dots, N$, we get from (5.2) that

$$(5.3)\qquad |v| = \left[\sum_{i,j=1}^{N}\int_{\Omega}|v_{,ij}|^2\,dx + \sum_{j=0}^{N}|v(P_j)|^2\right]^{\frac{1}{2}}$$

makes sense and is an equivalent norm on V, provided $P_0, \dots, P_N$ are points of $\bar{\Omega}$ affinely independent.

Let $a(\cdot,\cdot)$ be a bilinear symmetric form on $V\times V$ with values on $\mathbf{R}$ such that

$$(5.4)\qquad \exists M, \alpha > 0\colon\ \alpha\sum_{i,j=1}^{N}\|v_{,ij}\|^2_{L^2(\Omega)} \leqslant a(v,v) \leqslant M\sum_{i,j=1}^{N}\|v_{,ij}\|^2_{L^2(\Omega)}\qquad \forall v\in V.$$

In particular, the beam problem is obtained with

$$(5.5)\qquad a(u,v) = D\int_{\Omega}u''v''\,dx\qquad \forall u, v\in V,$$

and the plate problem with

$$(5.6)\qquad a(u,v) = D\int_{\Omega}[u_{,11}v_{,11} + u_{,22}v_{,22} + \nu(u_{,11}v_{,22} + u_{,22}v_{,11}) + 2(1-\nu)u_{,12}v_{,12}]\,dx\qquad \forall u, v\in V:$$

D denotes the stiffness coefficient ($D > 0$) and ν the Poisson coefficient ($0 < \nu < \frac{1}{2}$). Indeed, (5.4) is obviously satisfied by (5.5); as for (5.6), we just note that one can write

$$(5.7)\qquad a(v,v) = D\int_{\Omega}[(1-\nu)(v_{,11}^2 + v_{,22}^2) + \nu(v_{,11} + v_{,22})^2 + 2(1-\nu)v_{,12}^2]\,dx\qquad \forall v\in V.$$

Once (5.4) is satisfied, the kernel of a is given by the linear affine functions, say

$$(5.8)\qquad Y = \{y\in V\colon \exists y_j\in\mathbf{R},\ j = 0, \dots, N\ \ s.t.\ y(x) = y_0 + y_jx_j,\ \forall x\in\Omega\}.$$

Let L satisfy (1.5). It remains to choose $\mathbf{K}$, which we will do in a moment. Let E be a nonempty subset of $\bar{\Omega}$. We set

$$(5.9)\qquad \mathbf{K} = \{v\in \mathbf{H}^2(\Omega)\colon v(x) \geqslant 0,\ \forall x\in E\}.$$

Since (5.2) holds, the inequality defining $\boldsymbol{K}$ may be imposed indifferently on E or on $\bar{E}$. So, it is not restrictive to assume E *closed*, which we will do henceforth. It is important to note that $\boldsymbol{K}$ is a cone. First consequence is that (1.18) simplifies as shown in footnote (4); it reduces to check whether or not there exists a compact operator $\Pi \in \mathcal{L}(V, V)$ such that

$$[a(\cdot, \cdot) + \|\Pi \cdot\|^2]^{\frac{1}{2}}$$

is a norm on V. Such an operator may be defined, for instance, from

$$v \to [v(P_0), \ldots, v(P_N)] ,$$

where P_j are the points appearing in (5.3). Since $\boldsymbol{K}$ is a cone, verifying that $\{a, \boldsymbol{K}, L\}$ is compatible means checking that (2.26), (2.27) are true. Whether or not they are, verification requires only a finite number of steps.

REMARK 5.1. With the definition (5.5) (resp. (5.6)), pb $(a, \boldsymbol{K}, L)$ is the mathematical formulation of the equilibrium problem for a horizontal beam (resp. plate) subject to vertical load L and constrained to lie on or above the obstacle E placed at zero level. Referring to (1.13), $\frac{1}{2}a(v, v)$ represents the elastic energy and $\langle L, v\rangle$ the work of external forces.

As it is natural, pb $(a, \boldsymbol{K}, L)$ fails to have a solution if the resultant of the external forces points upward. Indeed,

LEMMA 5.1. pb $(a, \boldsymbol{K}, L)$ can have a solution only if

$$\langle L, 1\rangle \leqslant 0 . \tag{5.10}$$

PROOF. One just writes (2.26) with $w = 1$. ■

Assuming (5.10), we distinguish the admissible loads according to whether $\langle L, 1\rangle = 0$ or $\langle L, 1\rangle < 0$. In the latter case, we can define the *center of external forces* $c \equiv (c_1, \ldots, c_N)$, through

$$c_j = \frac{\langle L, x_j\rangle}{\langle L, 1\rangle} \qquad \forall j = 1, \ldots, N . \tag{5.11}$$

With this definition, we can completely exploit the necessary condition (2.26), obtaining the following lemma.

LEMMA 5.2. Provided (5.10) holds, the necessary condition (2.26) is equivalent to:

$$\langle L, x_j\rangle = 0 , \qquad j = 1, \ldots, N , \text{ when } \langle L, 1\rangle = 0 ; \tag{5.12}$$

and

$$c \in \operatorname{conv}(E) \text{ [15]}, \qquad \textit{when } \langle L, 1\rangle < 0 . \tag{5.13}$$

PROOF. Assume that $\langle L, 1\rangle = 0$. Since Ω is bounded, there exist a_j, $b_j \in \boldsymbol{R}$ such that $a_j \leqslant x_j \leqslant b_j$ for all $j = 1, \ldots, N$, for all x in Ω. Writing (2.26) with $w = x_j - a_j$ and $w = b_j - x_j$, we get (5.12). Conversely, (5.12) implies that $\langle L, y\rangle = 0$ for all y in Y, hence (2.26).

If $\langle L, 1\rangle < 0$, let y be any element of Y: then $y(x) = y_0 + \sum_{j=1}^{N} y_j x_j$ for some $y_j \in \boldsymbol{R}$, $j = 0, \ldots, N$. We have

$$\langle L, y\rangle = \langle L, 1\rangle \left[y_0 + \sum_{j=1}^{N} y_j \frac{\langle L, x_j\rangle}{\langle L, 1\rangle} \right] = \langle L, 1\rangle y(c) \text{ [16]} .$$

Hence, (2.26) is equivalent to

$$y \in Y \textit{ and } y \geqslant 0 \textit{ on } E \Rightarrow y(c) \geqslant 0 . \tag{5.14}$$

Using Hahn-Banach theorem, this means that $c \in \operatorname{conv}(E)$ and the proof is complete. ■

Besides the necessary condition, also compatibility may be interpreted in geometrical terms involving c. Recall the

DEFINITION 5.1. Let B be a convex subset of $\boldsymbol{R}^N$. We define *algebraic interior of* B the set $\operatorname{int} B$ equal to the union of the internal points in the topology of the affine hull of B.

The algebraic interior (*relative interior* in the terminology of [10]) of B may be different from the topological interior B^0: they coincide when B is truly N dimensional.

LEMMA 5.3. Assume $\langle L, 1\rangle < 0$. Then the sufficient condition (2.27) is equivalent to

$$c \textit{ belongs to } \operatorname{int}(\operatorname{conv}(E)) . \tag{5.15}$$

PROOF. A procedure analogous to the proof of Lemma 5.2 shows that (2.27) is equivalent to

$$y \in Y,\ y \geqslant 0 \textit{ on } E,\ y(c) \text{ [16]} = 0 \Rightarrow -y \geqslant 0 \textit{ on } E \text{ (hence } y \equiv 0 \text{ on } E) .$$

[15] conv (E) denotes the *(closed) convex hull* of E.

[16] Since c may lie outside Ω, $y(c)$ is not defined, strictly speaking. However, y is a linear function, hence it can be extended in a natural way outside Ω; abusing notations, we will still indicate with y this extension.

This is equivalent to (5.15), since Y is constitued by linear affine functions. ■

REMARK 5.2. Lemma 5.3 shows that Theorem 3.1 is a proper extension of Theorem 5.1 of [9]. This one gives existence only if $\langle L, 1\rangle < 0$ *and* c belongs to $(\text{conv}(E))^0$. The latter condition may be weakened as in (5.15); the former too may be removed, as the following theorem states.

THEOREM 5.1. If $\langle L, 1\rangle = 0$, then pb $(a, \boldsymbol{K}, L)$ has a solution if and only if

$$\langle L, x_j\rangle = 0 \qquad \forall j = 1, \ldots, N\,. \tag{5.16}$$

PROOF. The *only if* part is proved in Lemma 5.2. For the *if* part, we note that when (5.16) holds L satisfies the «alternative» (2.28), hence we can solve pb (a, V, L). Let u_0 be a solution of this problem: if γ is any constant, we have that

$$a(u_0+\gamma, u_0+\gamma-v) = \langle L, u_0+\gamma-v\rangle\,, \qquad \forall v \in \boldsymbol{K} \subset V\,. \tag{5.17}$$

Since u_0 is continuous on $\bar{\Omega}$, there exists a constant λ such that $u_0+\lambda \geqslant 0$ on E, say $u_0+\lambda$ belongs to $\boldsymbol{K}$; it satisfies (5.17), then it solves pb $(a, \boldsymbol{K}, L)$. ■

Again, we notice that in general there is a gap between the necessary and sufficient conditions, as soon as $\langle L, 1\rangle < 0$. The ambiguous region is the *algebraic boundary* of conv (E) (i.e. $\text{conv}(E)\setminus\text{int conv}(E)$). Anyway, ambiguity can be removed for the one-dimensional problem (partially supported beam), giving a complete description of the phenomenon (a different approach in the study of limit cases for this problem may be found in [3]).

THEOREM 5.2. If $N = 1$ and $\langle L, 1\rangle < 0$, then pb $(a, \boldsymbol{K}, L)$ has a solution if and only if c belongs to conv (E).

PROOF. Necessity has been proved in Lemma 5.2. Sufficiency has got to be proved only when $c \in \text{conv}(E)\setminus\text{int}(\text{conv}(E))$, say c is an endpoint of conv (E). Set

$$\boldsymbol{K}_c \equiv \{v \in V\colon v(c) \geqslant 0\}\,. \tag{5.18}$$

We claim that the hypotheses of Theorem 4.1 are satisfied. In fact, $\boldsymbol{K}$ is actually a cone. Moreover, $\langle L, 1\rangle < 0$ implies that $N(a, \boldsymbol{K}, L)$ holds (see

(5.13)). Hence, (4.21) and (4.22) are fulfilled. We have that $\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L)$ is closed. For, $\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L) = \boldsymbol{K} - \{v = \beta(x - c),$ *whenever* $\beta(x-c) \geqslant 0$ *on* $E\} = \boldsymbol{K}_c$. For, any $v \in \boldsymbol{K}_c$ belongs to $C^1(\bar{\Omega})$, hence v' is bounded on E and there exists β such that $\beta(x-c) \geqslant 0$ on E and $v(x) + \beta(x-c)$ belongs to $\boldsymbol{K}$. So, $\boldsymbol{K}_c$ is a subset of $\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L)$: since the opposite inclusion is obvious, we have the equality. Now, $\boldsymbol{K}_c$ is closed, hence also $\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L)$ is closed and (4.23) is satisfied. Theorem 4.1 allows us to deduce that pb $(a, \boldsymbol{K}, L)$ has a solution ([17]). ■

Things are not so plain for $N = 2$: the region covered by the necessary condition but not by the sufficient one must be investigated in every single case. The results we are going to prove show that, in general, existence depends on the geometry of the supporting set and on the distribution of the external forces.

As in the one dimensional case, the idea is to consider a problem with reduced obstacle which Lemma 5.3 applies to. Hence, existence for the original problem will be derived. We will detail the procedure under the following assumptions:

(5.19) $\quad$ $\operatorname{conv}(E)$ *is a (convex) polygon, contained in* Ω ;

(5.20) $\quad$ c *belongs to* $\partial \operatorname{conv}(E)$;

(5.21) $\quad$ *there exists* $\varepsilon > 0$ *such that* $L \in \boldsymbol{H}^{-2+\varepsilon}(\Omega)$.

Let us remark that the hypothesis (5.21), though restrictive, allows distributed loads as well as concentrated ones.

Unfortunately, the argument of the proof of Theorem 5.2 does not apply to our present case. To see this, assume that

(5.22) $\quad$ c *is internal to a (closed) side* S *of* $\partial \operatorname{conv}(E)$:

choose the reference frame such that $c = 0$, the x_1 axis coincides with the direction of S and E lies on the positive side of the x_2 axis. We have that

$$\boldsymbol{C}(a, \boldsymbol{K}, L) = \{y \in Y\colon\ y(x) = \beta x_2,\ \textit{with}\ \beta \geqslant 0\} .$$

Then

$$\boldsymbol{K}_1 = \boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L) = \boldsymbol{K} - \{y \in Y\colon\ y(x) = \beta x_2,\ \textit{with}\ \beta \geqslant 0\} .$$

([17]) Let us point out that so far the symmetry of a has never been used. Note that our results hold also for nonhomogeneous and anysotropic plates or beams.

The argument of the proof of Theorem 5.2 leads to consider pb $(a, \boldsymbol{K}_S, L)$, where

$$(5.23) \qquad \boldsymbol{K}_S = \{v \in \boldsymbol{H}^2(\Omega) \colon v \geqslant 0 \ \ on \ \ S\} .$$

But now $\boldsymbol{K}_1 \neq \boldsymbol{K}_S$, because $\boldsymbol{K}_S$ is closed in $\boldsymbol{H}^2(\Omega)$, while $\boldsymbol{K}_1$ is not. In particular, Theorem 4.1 cannot be applied and a different argument is needed.

LEMMA 5.4. Assume $\langle L, 1\rangle < 0$ and (5.19)-(5.22). Then pb $(a, \boldsymbol{K}, L)$ is solvable.

PROOF. Due to Lemma 5.3, pb $(a, \boldsymbol{K}_S, L)$ has a solution u_S which must vanish at some point of S. Now, there exists a nonnegative distribution on Ω, say a measure, R_S with support in S such that

$$(5.24) \qquad \Delta^2 u_S = L + R_S \colon$$

the equality is intended in the sense of $\boldsymbol{H}^{-2+\varepsilon}(\Omega)$. Such an R_S belongs to $(\boldsymbol{C}_0^0(\Omega))' \subset \boldsymbol{H}^{-1-\sigma}(\Omega)$ $(\sigma > 0$ arbitrary), hence the standard regularity theory applies to the equation (5.24) to give

$$(5.25) \qquad u_S \in \boldsymbol{H}^{2+\varepsilon}(\Omega) \subset \boldsymbol{C}^1(\bar{\Omega}) .$$

Using (5.19), (5.22) and (5.25) it is easy to see that

(5.26) *there exists* $\gamma \geqslant 0$ *such that* $u \equiv u_S + \gamma x_2 \geqslant 0$ *on* E .

This proves that pb $(a, \boldsymbol{K}, L)$ has a solution. ■

Still assuming (5.19)-(5.21), let us suppose that

(5.27) *c is a vertex* (*say the origin*) *of* $\partial \operatorname{conv}(E)$.

The argument of the proof of Lemma 5.4 still applies, provided we substitute $\boldsymbol{K}_S$ with

$$(5.28) \qquad \boldsymbol{K}_0 = \{v \in V \colon v(0) \geqslant 0\}$$

and consider pb $(a, \boldsymbol{K}_0, L)$ instead of pb $(a, \boldsymbol{K}_S, L)$. Hence we have

LEMMA 5.5. Assume $\langle L, 1\rangle < 0$ and (5.19)-(5.21), (5.27). Then

pb $(a, \boldsymbol{K}, L)$ is solvable . ■

Lemmas 5.3, 5.4, 5.5 and Theorem 5.1 give the following

THEOREM 5.3. Assume that the necessary condition $\langle L, 1\rangle \leqslant 0$ holds and that $\operatorname{conv}(E)$ is a (convex) polygon contained in Ω.

If $\langle L, 1\rangle = 0$ then $\operatorname{pb}(a, \boldsymbol{K}, L)$ is solvable if and only if $\langle L, x_i\rangle = 0$ for $i = 1, 2$.

If $\langle L, 1\rangle < 0$ then $\operatorname{pb}(a, \boldsymbol{K}, L)$ can have a solution only if $c \in \operatorname{conv}(E)$; $\operatorname{pb}(a, \boldsymbol{K}, L)$ is actually solvable if either of the following conditions holds:

(*a*) c is internal to $\operatorname{conv}(E)$;

(*b*) c belongs to $\partial \operatorname{conv}(E)$ and (5.21) holds. ■

REMARK 5.3. We point out that the assumption $\operatorname{conv}(E) \subset \Omega$ is made just in order to ease the study of the case when c is internal to a side of $\operatorname{conv}(E)$.

Under the assumptions (5.20) and (5.21), the study of the equilibrium problem for a plate supported by a polygon is by now complete.

REMARK 5.4. Let us consider the case of a set E reduced to a finite number of points. Assume (5.10): then a necessary and sufficient condition for the solvability of $\operatorname{pb}(a, \boldsymbol{K}, L)$ is that either (5.12) or (5.13) holds. Note that in this case the study of existence could be carried out using just the *continuity* of the solution u_{ro} of the problem with *r*educed *o*bstacle [18]. So, the assumption (5.21) is no longer necessary: in fact, u is continuous (once (5.2) holds), and this property holds for any L in $(\boldsymbol{H}^2(\Omega))'$.

Now we want to give some results of different kind on an equilibrium problem for the plate when (5.20) is satisfied but $\operatorname{conv}(E)$ is no longer a polygon. We will consider only the case of a uniformly loaded plate (for instance, homogeneous plate subject to its own weight), i.e. assuming that

$$L \text{ is a negative constant}. \tag{5.29}$$

A preliminary step is the study of the behavior of the solution of the problem with obstacle reduced to a point. Precisely, let x_0 be a point of Ω and set

$$\boldsymbol{K}_{x_0} = \{v \in V : v(x_0) \geqslant 0\}. \tag{5.30}$$

Recalling Lemma 5.3, the problem $\operatorname{pb}(a, \boldsymbol{K}_{x_0}, L)$ admits a solution u_0 (note that $\{x_0\} = \operatorname{int}\{x_0\}$, since $\{x_0\}$ coincides with its affine hull).

[18] According to whether (5.22) or (5.27) is satisfied, u_{ro} indicates the solution of either $\operatorname{pb}(a, \boldsymbol{K}_S, L)$ (see (5.23)) or $\operatorname{pb}(a, \boldsymbol{K}_0, L)$ (see (5.28)).

LEMMA 5.6. We have that

$$\eta(x) \equiv u_0(x) + \frac{L}{8\pi}\,\text{meas}\,(\Omega)\,|x - x_0|^2 \log|x - x_0| \tag{5.31}$$

is a C^∞ function near x_0. Moreover, η is biharmonic, hence analytic.

PROOF. Again, u_0 must vanish at x_0 and there exists a nonnegative distribution on Ω (i.e. a measure) R_0 with support reduced to x_0 such that

$$\Delta^2 u_0 = L + R_0\,, \tag{5.32}$$

in the sense of $H^{-2}(\Omega)$. Now, a well known theorem states that R_0 must be as bad as the Dirac mass δ_{x_0} at x_0, times a constant. Using (5.32), one can find that

$$R_0 = -L\,\text{meas}\,(\Omega)\,\delta_{x_0}\,, \tag{5.33}$$

where the coefficient of δ_{x_0} has the required value in order that the resultant of all forces acting on the plate (external and constraint reaction) vanishes. So, equation (5.32) can be solved explicitly in a circular neighborhood of x_0 by a separation of variables and one obtains the asserted behavior. ■

THEOREM 5.4. Assume that $\langle L, 1\rangle < 0$. Assume also (5.20), (5.29) and

(5.34) ∂ conv (E) *is strictly convex at* c [19] ; $\qquad c \in \Omega$.

Then pb $(a, \boldsymbol{K}, L)$ is solvable if and only if there exist $\beta > 0$, a neighborhood U of c and a frame of orthogonal coordinates $\{0, x_1, x_2\}$ such that $c \equiv 0$ and

$$U \cap \text{conv}\,(E) \subset \Big\{(x_1, x_2) : x_2 \geqslant \beta x_1^2\,\big|\log|x_1|\big|\Big\}\,. \tag{5.35}$$

PROOF. After choosing a coordinate frame with origin at c and recalling (5.28), we claim that pb $(a, \boldsymbol{K}, L)$ has a solution if and only if there exists $\gamma > 0$ such that $u_0 + \gamma x_2 \geqslant 0$ on E $\big(u_0$ solution of pb $(a, \boldsymbol{K}_0, L)\big)$. In fact, if u solves pb $(a, \boldsymbol{K}, L)$, then for any $\gamma > 0$, $u_\gamma \equiv u + \gamma x_2$ is still a solution (γx_2 does not affect L, since $c \equiv 0$). Moreover, u_γ is strictly posi-

[19] Hence, c belong to $\bar{\Omega}$ (see remark 5.3).

tive on $E \setminus \{0\}$ (in particular, there is no constraint reaction outside the origin). So, u_γ solves pb $(a, \boldsymbol{K}_0, L)$. The converse is obvious. An elementary calculation yields the expected result. ■

In particular, *when ∂E is an arc of circle near c*, pb $(a, \boldsymbol{K}, L)$ *has no solution*. On the contrary, existence holds if, for instance, $\partial \operatorname{conv}(E)$ has a strictly convex corner at c, providing in this special case a new proof of Lemma 5.5.

The discussion of the beam-plate problem is now complete.

As a final remark, it is worthwhile to note that Theorem 3.1 furnishes the existence result for the Signorini problem via an immediate verification. In this case, existence was already well known: we quote the two works by Fichera [5], [6] (20), where it is also proved that the sufficient condition becomes necessary, at least in some cases. More precisely, assume that the convex hull C of the potential contact area (say, the part of the boundary on which the unilateral condition is given) has nonempty topological interior and is suitably smooth. In this case, Fichera exhibits a general counterexample showing that solutions cannot exist if the center of external forces belongs to the boundary of C.

We just point out that the abstract sufficient condition of our Theorem 3.1, when interpreted in this concrete case, reduces *exactly* to Fichera's condition on bilateral rigid displacements. Without recalling the (heavy) terminology of the problem, we give a mechanical formulation of the results contained in Theorem 3.1, as follows.

THEOREM 5.5. If the center of external forces belongs to the interior of the convex hull of the potential contact area, then the Signorini problem in linear elasticity has a solution. ■

In [7] a Signorini-like problem is studied, imposing a unilateral condition along the whole boundary. When the *initial* contact area has empty topological interior, the sufficient condition of Theorem 3.1 leads to more general existence theorems, of the following type.

THEOREM 5.6. If the center of external forces belongs to the algebraic interior of the convex hull of the *initial* contact area, then the Signorini-like problem of [7] has a solution. ■

We refer to a forthcoming paper for a more detailed study of contact problems in elasticity, either in the linear approximation or in the nonlinear approach.

(20) About this subject we mention also [12].

Appendix a.

This appendix is devoted to the study of the iterated enlargements of $\boldsymbol{K}$, under the assumptions:

(a.1) *a is symmetric*,

(a.2) *there exist $v_0 \in V$, $M \in V'$ such that M is bounded from above on $\boldsymbol{K}$ and $\langle L, v\rangle = a(v_0, v) + \langle M, v\rangle$, $\forall v \in V$*,

(a.3) $0 \in \boldsymbol{K}$ [21].

We remind that (a.2) is a necessary condition for the solvability of pb $(a, \boldsymbol{K}, L)$ (see Lemma 4.8). Since we assume (a.1) and (a.2), we have that

$$\boldsymbol{C}(a, \boldsymbol{K}, L) = Y \cap \ker L \cap \operatorname{rc} \boldsymbol{K} \tag{a.4}$$

and pb $(a, \boldsymbol{K}, L)$ is solvable if $Y \cap \ker L \cap \operatorname{rc} \boldsymbol{K}$ is a subspace (see Theorem 3.1 and Lemma 2.3). When this condition does not hold, we have seen in section 4 that the method of widening $\boldsymbol{K}$ can help. Among all possible ways of enlarging $\boldsymbol{K}$, we are going to iterate the one defined in (4.2). Precisely, set

$$\begin{cases} \boldsymbol{K}_0 \equiv \boldsymbol{K}, \\ \boldsymbol{K}_{i+1} \equiv \overline{\boldsymbol{K}}_i - \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L), \qquad i \geq 1. \end{cases} \tag{a.5}$$

LEMMA a.1. The following statements are true for all $i \geq 0$:

(a.6) (a.2) *and* (a.4) *hold when $\boldsymbol{K}$ is substituted with $\overline{\boldsymbol{K}}_i$*;

(a.7) $\boldsymbol{S}(a. \boldsymbol{K}, L) = \boldsymbol{K} \cap \boldsymbol{S}(a, \overline{\boldsymbol{K}}_i, L)$;

(a.8) $\overline{\boldsymbol{K}}_{i+1} = \overline{\boldsymbol{K}}_i$ *if and only if* $\operatorname{span}[\boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L)] \subset \overline{\boldsymbol{K}}_i$.

PROOF. (a.6) is proved by induction, using the vanishing of M on $Y \cap \ker L$ and taking (a.3) into account.

(a.7) can be shown with the same argument as in Remark 4.1: presently we have to take also closures, but this does not affect the proof.

[21] In the following, $\boldsymbol{K}$ will not be required to be a cone, as in section 4. Note that the assumption (a.3) is not restrictive thanks to lemma 4.7.

(a.8) is obtained as follows. Assume $\overline{\boldsymbol{K}}_{i+1} = \overline{\boldsymbol{K}}_i$. From (a.5) we get that

$$\boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L) - \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L)] \subset \overline{\boldsymbol{K}}_{i+1} = \overline{\boldsymbol{K}}_i .$$

This proves the « only if » part, since $\boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L)$ is a cone.

Conversely, if span $[\boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L)] \subset \overline{\boldsymbol{K}}_i$, then

$$\text{(a.9)} \quad \boldsymbol{K}_{i+1} = \overline{\boldsymbol{K}}_i - \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L) = \overline{\boldsymbol{K}}_i + \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L) - \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L) \subset \overline{\boldsymbol{K}}_i .$$

So, $\overline{\boldsymbol{K}}_{i+1} \subset \overline{\boldsymbol{K}}_i$. Since the opposite inclusion is trivial, (a.8) is proved. ■

LEMMA a.2. Under the assumptions (a.1) and (a.2),

$$\text{(a.10)} \qquad \boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L) \subset Y \cap \ker \Pi_0 \qquad \forall i \geq 0 .$$

PROOF. Assume for a moment that Π_0 is bounded on $\overline{\boldsymbol{K}}_i$ for all $i \geq 0$. Then, for any v in $\overline{\boldsymbol{K}}_i$, w in $Y \cap \ker L \cap \operatorname{rc} \overline{\boldsymbol{K}}_i$ and $\lambda > 0$, it is $\Pi_0(v + \lambda w) = \Pi_0 v + \lambda \Pi_0 w$, which has to be bounded when λ goes to infinity. Hence, $\Pi_0 w = 0$ and (a.10) holds. We need only prove that Π_0 is bounded on $\overline{\boldsymbol{K}}_i$ for all $i \geq 0$. This is easily done by induction. In fact, (1.18) gives the initial step. Moreover, if we assume that Π_0 is bounded on $\overline{\boldsymbol{K}}_i$, then we derive from the first part that Π_0 vanishes on $\boldsymbol{C}(a, \overline{\boldsymbol{K}}_i, L)$, hence $\Pi_0(\boldsymbol{K}_{i+1}) = \Pi_0(\overline{\boldsymbol{K}}_i)$, which is bounded. Taking the closure of $\boldsymbol{K}_{i+1}$ we have the boundedness of Π_0 on $\overline{\boldsymbol{K}}_{i+1}$. ■

With the aid of these results, we can prove that after a finite number (say, N) of steps the convex $\boldsymbol{K}_N$ coincides with x_{N+j}, $j \geq 1$ and pb $(a, \boldsymbol{K}_N, L)$ is solvable. This will entail solvability for pb $(a, \boldsymbol{K}, L)$ as soon as there is a solution of pb $(a, \boldsymbol{K}_N, L)$ that belongs to $\boldsymbol{K}$.

THEOREM a.1. Assume (a.1), (a.2) and (a.3). Then there exists an integer N such that

$$\text{(a.11)} \qquad \boldsymbol{K}_{N+j} = \boldsymbol{K}_N , \qquad \textit{for all } j \geq 1 ;$$

$$\text{(a.12)} \qquad \boldsymbol{S}(a, \boldsymbol{K}_N, L) \neq \emptyset ;$$

$$\text{(a.13)} \qquad \boldsymbol{S}(a, \boldsymbol{K}, L) = \boldsymbol{K} \cap \boldsymbol{S}(a, \boldsymbol{K}_N, L) .$$

PROOF. (a.11) is a consequence of Lemma a.2: when passing from $\overline{\boldsymbol{K}}_i$ to $\boldsymbol{K}_{i+1}$, only a finite number of directions are added, all of them included into $Y \cap \ker \Pi_0$. After a finite number of steps, it is impossible to add new directions. In particular, $\boldsymbol{K}_N$ is closed.

(a.13) was actually valid for every $\overline{\boldsymbol{K}}_i$, as shown in (a.7).

To prove (a.12), we just note that the necessary condition in the form (a.2) holds at each step (see (a.6)). The compactness-coerciveness assumption does the same (see the proof of Lemma a.2). Finally, $\{a, \boldsymbol{K}_N, L\}$ is compatible: in fact, from (a.8) and (a.11) we get that

$$w \in Y \cap \ker L \cap \operatorname{rc} \boldsymbol{K}_N \Rightarrow \pm \mu w \in \boldsymbol{K}_N , \qquad \textit{for all } \mu \geqslant 0 ,$$

hence $-w \in \operatorname{rc} \boldsymbol{K}_N$, say (2.16). So, we can apply Theorem 3.1 and get (a.12). ∎

REMARK a.1. In the pratical applications given so far, the procedure described in this appendix always stopped at the first step, say $N = 1$ and pb $(a, \boldsymbol{K}_1, L)$ was solvable. Nevertheless, further steps are actually needed in the general case, as the following finite dimensional example shows.

EXAMPLE a.1. Let $V \equiv \boldsymbol{R}^3$. (1.18) holds independently of $\boldsymbol{K}$ and a with $\Pi_0 \equiv 0$, $\Pi_1 = \textit{identity}$ and $\alpha = 1$. Let $a(u, v) \equiv u_3 v_3$ for u and v in V. Then

$$Y = \{v \in \boldsymbol{R}^3 \colon v_3 = 0\} .$$

We take the following as convex $\boldsymbol{K}$

$$\boldsymbol{K} = \{v \in \boldsymbol{R}^3 \colon v_i \geqslant 0 \textit{ for all } i \textit{ and } v_1 v_3 \geqslant v_2^2\} :$$

note that $\boldsymbol{K}$ is actually a cone. Since a is symmetric, $\boldsymbol{N}(a, \boldsymbol{K}, L)$ reads

$$\langle L, v \rangle \leqslant 0 \textit{ for all } v = (x, 0, 0) \textit{ with } x \geqslant 0 .$$

In particular, if $\langle L, v \rangle = v_2$, (4.42) is valid while (4.41) is not satisfied ([22]), showing that in Lemma 4.9 there is no equivalence between (4.41) and (4.42). Now, let us choose $\langle L, v \rangle = -v_3$. Then, (a.2) is satisfied and we have

$$\boldsymbol{C}(a, \boldsymbol{K}, L) = \{v \in \boldsymbol{R}^3 \colon v_1 \geqslant 0, v_2 = v_3 = 0\} ,$$

$$\boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L) = \{v \in \boldsymbol{R}^3 \colon v_i > 0 \textit{ for } i = 2, 3\} \cup \{v \in \boldsymbol{R}^3 \colon v_3 \geqslant 0 \textit{ and } v_2 = 0\} .$$

Note that $\boldsymbol{K}_1 \equiv \boldsymbol{K} - \boldsymbol{C}(a, \boldsymbol{K}, L)$ is not closed, so we have to take the clo-

([22]) To see this, it is enough to evaluate $F(v)$ with $v = (\beta, \sqrt{\beta v_3}, v_3)$, for v_3 fixed and $\beta \geqslant 0$ variable.

sure, getting $\overline{\boldsymbol{K}}_1 = \{v \in \boldsymbol{R}^3 \colon v_i \geqslant 0 \textit{ for } i = 2, 3\}$. We have successively

$$\boldsymbol{C}(a, \overline{\boldsymbol{K}}_1, L) = \{v \in \boldsymbol{R}^3 \colon v_3 = 0, v_2 \geqslant 0\}, \qquad \boldsymbol{K}_2 \equiv \{v \in \boldsymbol{R}^3 \colon v_3 \geqslant 0\}.$$

Now, $\boldsymbol{K}_2$ is closed and $\boldsymbol{C}(a, \boldsymbol{K}_2, L) = Y$, which is a subspace. So, pb $(a, \boldsymbol{K}_2, L)$ is solvable and the minimal value of N is 2. Moreover

$$\boldsymbol{S}(a, \boldsymbol{K}_2, L) = \{v \in \boldsymbol{R}^3 \colon v_3 = 0\}, \qquad \boldsymbol{S}(a. \boldsymbol{K}, L) = \{v \in \boldsymbol{R}^3 \colon v_1 \geqslant 0, v_2 = v_3 = 0\}.$$

In particular, $\boldsymbol{S}(a, \boldsymbol{K}, L)$ is non-empty.

Appendix b.

Here we give some conditions that are sufficient to guarantee the well posedness of pb $(a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0)$. $\boldsymbol{K}_0$ and L_0 depend on a fixed element z_0 of $\boldsymbol{K}$ and are defined in (4.33) and (4.34).

We need some notations. Let G and T be two subspaces of V, such that

$$\ker \boldsymbol{A} = (\ker \boldsymbol{A} \cap \ker L) \oplus G, \tag{b.1}$$

$$Y = \ker \boldsymbol{A} \oplus T, \tag{b.2}$$

($\oplus$ denotes orthogonal decomposition). Let P (respectively, Q) be the orthogonal projection operator into G (resp. into $Z \equiv G \oplus T$). Of course, $T = \emptyset$ and $P \equiv Q$ if a is symmetric.

In general, the following lemma holds.

LEMMA b.1. Let k be any element of $Y \cap \operatorname{rc}(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L)$. Then

$$Q(k) \in \operatorname{rc} Q(\boldsymbol{K}_0 \cap Y), \tag{b.3}$$

whenever the right hand side makes sense. If in addition $k \in \ker \boldsymbol{A}$, then

$$P(k) \in \operatorname{rc} P(\boldsymbol{K}_0 \cap \ker \boldsymbol{A}) \ (^{23}). \tag{b.4}$$

([23]) The right hand side *always* makes sense. In fact, G (see (b.1)) is at most one dimensional, then $P(\boldsymbol{K}_0 \cap \ker \boldsymbol{A})$ is a convex set of a one dimensional space: even if it is not closed, its recession cone is well defined and satisfies (2.5)-(2.10).

PROOF. In our hypothesis, for any $\lambda > 0$ it is

$$\text{(b.5)}\qquad \lambda k \in Y \cap \operatorname{rc}(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L) \subset (^{24})\ (\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L) \cap Y\ (^{25}) = (\boldsymbol{K}_0 \cap Y) - \ker \boldsymbol{A} \cap \ker L\,.$$

Taking the projections, we have that

$$\text{(b.6)}\qquad \lambda Q(k) = Q(\lambda k) \in Q[(\boldsymbol{K}_0 \cap Y) - \ker \boldsymbol{A} \cap \ker L] = Q(\boldsymbol{K}_0 \cap Y) \qquad \forall \lambda > 0\,.$$

In other words, (b.3) is satisfied. Now, assume that k belongs to $\ker \boldsymbol{A}$. To prove (b.4), we proceed as in (b.5) and (b.6), obtaining in turn

$$\lambda k \in (\boldsymbol{K}_0 \cap \ker \boldsymbol{A}) - (\ker \boldsymbol{A} \cap \ker L) \qquad \forall \lambda > 0$$

and

$$\lambda P(k) = P(\lambda k) \in P[(\boldsymbol{K}_0 \cap \ker \boldsymbol{A}) - (\ker \boldsymbol{A} \cap \ker L)] = P(\boldsymbol{K}_0 \cap \ker \boldsymbol{A}) \quad \forall \lambda > 0\,. \blacksquare$$

We just mention that, when a is symmetric, (b.3) and (b.4) coincide.

LEMMA b.2. Let W be any subspace of V. If z belongs to $\operatorname{rc}(\boldsymbol{K}_0 - W)$, then

(i) $z \in \boldsymbol{K}_0 - W$,

(ii) if $z = k - x$ for some k in $\boldsymbol{K}_0$ and some x in W, then

$$k \in \operatorname{rc}(\boldsymbol{K}_0 - W)\,.$$

PROOF. (i) is obvious, since $0 \in \boldsymbol{K}_0$ (recall (2.7)). Now, let d be any element of $\boldsymbol{K}_0 - W$: we have $d + k = (d + z) - x \in \boldsymbol{K}_0 - W - W = \boldsymbol{K}_0 - W$, say (ii). $\blacksquare$

Now we are able to prove the main result of this section: under the hypotheses of Theorems 1.II and 2.I of Fichera [6], we will show that $\{a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0\}$ is compatible. As we will see, this will entail that pb $(a, \boldsymbol{K}, L)$ is solvable.

For the sake of clarity, we report the two Fichera's theorems mentioned above grouped in one which, with our notations, can be stated in the following way.

(^{24}) The inclusion holds since $0 \in \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L$, then (2.7) is used.

(^{25}) The equality is a consequence of the following result: if A, B, D are subsets of V, such that D is a subspace and $B \subset D$, then $(A - B) \cap D = (A \cap D) - B$.

THEOREM b.1. Assume

(b.7) $\dim Y < +\infty\,;\qquad \sqrt{a(\cdot,\cdot)}$ *is a norm on* $Y^{\perp}$;

(b.8) $\boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L$ *is closed*.

Assume also that there exists z_0 in $\boldsymbol{K}$ such that

(b.9) *for any* $k \in \boldsymbol{K}_0 \cap \ker \boldsymbol{A}$ *with* $\langle L, k\rangle \neq 0$ *and* $P(k) \in \operatorname{rc} P(\boldsymbol{K}_0 \cap \ker \boldsymbol{A})$ *it is* $\langle L, k\rangle < 0$,

(b.10) *for any* $k \in \boldsymbol{K}_0 \cap Y$ *with* $\boldsymbol{A}k \neq 0$ *and* $Q(k) \in \operatorname{rc} Q(\boldsymbol{K}_0 \cap Y)$ *there exists* $v = v(k) \in \boldsymbol{K}$ *such that* $\langle L, k\rangle < a\big(v(k), k\big)$ [26].

Then, pb $(a, \boldsymbol{K}, L)$ is solvable. ■

Now, let us prove the following

THEOREM b.2. Assume that the hypotheses (b.7)-(b.10) hold. Then the triplet $\{a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0\}$ is compatible.

PROOF. First we notice that in (b.9) $\operatorname{rc} P(\boldsymbol{K}_0 \cap \ker \boldsymbol{A})$ makes sense (see footnote [23]). We also claim that in (b.10) $\operatorname{rc} Q(\boldsymbol{K}_0 \cap Y)$ is well defined. Indeed, recalling (b.2) and the definition of Z, we have that

$$Q(\boldsymbol{K}_0 \cap Y) = Z \cap [(\boldsymbol{K}_0 \cap Y) - (\ker \boldsymbol{A} \cap \ker L)]\ {}^{(27)}$$
$$= Z \cap Y \cap (\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L)$$

is closed, thanks to (b.8). Then also $\operatorname{rc} Q(\boldsymbol{K}_0 \cap Y)$ makes sense.

Let us prove that $\{a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0\}$ is compatible (see (2.14)). Let w be such that

(b.11) $$w \in Y \cap \operatorname{rc}(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0)$$

and

(b.12) $$a(v, w) \leqslant \langle L_0, w\rangle \qquad \forall v \in \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0 :$$

then we must prove that

(b.13) $$-w \in Y \cap \operatorname{rc}(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0)$$

[26] Of course, (b.10) is trivially satisfied if a is symmetric, or more generally $Y = \ker \boldsymbol{A}$.

[27] See footnote [25].

and

$$a(v, w) = \langle L_0, w\rangle \qquad \forall v \in \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0 . \tag{b.14}$$

Due to Lemma b.2, if w satisfies (b.11), then there exist k in $\boldsymbol{K}_0 \cap Y \cap \mathrm{rc}\,(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0)$ and l in $\ker \boldsymbol{A} \cap \ker L_0$ such that

$$w = k - l . \tag{b.15}$$

We distinguish the cases:

1) $w \in \ker \boldsymbol{A} \cap \mathrm{rc}\,(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0)$;

2) $w \in Y \cap \mathrm{rc}\,(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0)$ and $\boldsymbol{A}w \neq 0$.

1) In this case, k belongs to $\ker \boldsymbol{A}$ and (b.12) implies that $\langle L_0, k\rangle = \langle L, k\rangle = \langle L, w\rangle \geqslant 0$ (remember that L and L_0 coincide on $\ker \boldsymbol{A}$). Now, Lemma b.1 and assumption (b.9) yield that $\langle L, k\rangle$ cannot be strictly positive, hence w must belong to $\ker L$. So,

$$\begin{aligned} -w &\in \ker \boldsymbol{A} \cap \ker L \\ &= \ker \boldsymbol{A} \cap \ker L_0 \subset \mathrm{rc}\,\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0 \subset \mathrm{rc}(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0) \end{aligned}$$

and (b.13) and (b.14) are true.

2) In this case, referring again to (b.15), it is $\boldsymbol{A}k \neq 0$. Using Lemma b.1, we derive from (b.10) that there exists a $v(k)$ in $\boldsymbol{K}$ such that $\langle L, k\rangle < a\big(v(k), k\big)$. In other words, the element $v_0(k) \equiv v(k) - z_0$ belongs to $\boldsymbol{K}_0$ and satisfies

$$0 < a\big(v_0(k), k\big) + a(z_0, k) - \langle L, k\rangle = a\big(v_0(k), k\big) - \langle L_0, k\rangle .$$

This contradicts (b.12), hence this case can never occur.

So, the proof is complete. ■

At this point it is very easy to obtain the solvability of pb $(a, \boldsymbol{K}, L)$ under the assumptions of Theorem b.1. The first step consists in proving that pb $(a, \boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L, L)$ is solvable. Indeed, from (b.8) we derive obviously that $\boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L$ is closed. Furthermore, since $\ker \boldsymbol{A} \cap \ker L_0 = \ker \boldsymbol{A} \cap \ker L$, we have that

$$\mathrm{rc}\,(\boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0) = \mathrm{rc}\,(\boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L) ,$$

and it is easy to check that

$$\text{(b.16)} \qquad \boldsymbol{C}(a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0) = \boldsymbol{C}(a, \boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L, L) .$$

Hence, $\{a, \boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L, L\}$ is compatible. Thanks to (b.7), this triplet satisfies also (1.18) with $\Pi_0 \equiv 0$ and $\Pi_1 =$ *orthogonal projection onto* Y. The Theorem 3.1 can be applied and pb $(a, \boldsymbol{K} - \ker \boldsymbol{A} \cap \ker L, L)$ is solvable.

Using Lemma 4.1, we derive that also pb $(a, \boldsymbol{K}, L)$ has a solution. ■

REMARK b.1. We have proved that if there exists a z_0 satisfying the assumptions of Theorem b.1, then (b.16) holds. By the way, (b.16) shows that the set $\boldsymbol{C}(a, \boldsymbol{K}_0 - \ker \boldsymbol{A} \cap \ker L_0, L_0)$ is *independent of* z_0: On this subject we notice that for *any* z_0 in V it is

$$\boldsymbol{C}(a, \boldsymbol{K}_0, L_0) = \boldsymbol{C}(a, \boldsymbol{K}, L)$$

(we refer to the definitions (4.33) and (4.34)). Hence, for $\{a, \boldsymbol{K}_0, L_0\}$, the property of being compatible is intrinsic, that is, independent of z_0.

REFERENCES

[1] N. BOURBAKI, *Eléments de mathématique. Espaces vectoriels topologiques*, Ch. 1 et 2, Act. Sci. Ind., 1189, Hermann, Paris, 1966.

[2] C. BAIOCCHI - A. CAPELO, *Variational and quasivariational inequalities: applications to free boundary problems*, J. Wiley and Sons, Chichester, 1984.

[3] P. BOIERI - F. GASTALDI, *Deflection of a partially supported elastic beam subjected to nonuniform load*, Meccanica, **20** (1985), pp. 59-70.

[4] C. BAIOCCHI - F. GASTALDI - F. TOMARELLI, *Inéquations variationnelles non coercives*, C. R. Acad. Sci. Paris, **299** (1984), pp. 647-650.

[5] G. FICHERA, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei Mem. Sez. I, (8) **7** (1964), pp. 71-140.

[6] G. FICHERA, *Boundary value problems in elasticity with unilateral constraints*, Handbuch der Physik, Band VI*a*/2, Springer, Berlin (1972), pp. 347-389.

[7] D. KINDERLEHRER, *Estimates for the solution and its stability in Signorini's problem*, Appl. Math. Optim., **8** (1982), pp. 159-188.

[8] J. L. LIONS - E. MAGENES, *Nonhomogeneous boundary value problems and applications*, Vol. 1, Springer, Berlin, 1972.

[9] J. L. LIONS - G. STAMPACCHIA, *Variational inequalities*, Comm. Pure Appl. Math., **20** (1967), pp. 493-519.

[10] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[11] G. STAMPACCHIA, *Formes bilinéaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258** (1964), pp. 4413-4416.

[12] M. SCHATZMAN, *Problèmes aux limites non linéaires, non coercifs*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (3) **27** (1973), pp. 641-686.

[13] A. SZULKIN, *Positive solutions of variational inequalities: a degree-theoretical approach*, to appear.

Added in proof. Recently, H. Brézis has called our attention to a partial overlapping of our results and those contained in

[14] H. BRÉZIS - A. HARAUX, *Image d'une somme d'opérateurs monotones et applications*, Israel J. Math., **23** (1976), pp. 165-186.

[15] H. BRÉZIS - L. NIRENBERG, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) **5** (1978), pp. 225-326.

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