ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

THOMAS KARLSSON

Hölder continuity and thin obstacle problems for vector valued functions

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 13, nº 2 (1986), p. 281-297

<http://www.numdam.org/item?id=ASNSP_1986_4_13_2_281_0>

Hölder Continuity and Thin Obstacle Problems for Vector Valued Functions.

THOMAS KARLSSON

1. – Introduction.

The object of this paper is to study the regularity of solutions u, defined in a set $\Omega \subset \mathbb{R}^n$, of a thin obstacle problem for vector valued functions. We are concerned with the Hölder continuity for solutions of a nonlinear diagonal system of variational inequalities. The convex set of admissible variations is given by an obstacle which is active on a smooth $n-1$-dimensional manifold $E \subset \Omega$.

Our main result is that if the obstacle function ψ has distributional derivatives of order one which are of class L^p for some $p > n$ and is Hölder continuous with exponent μ, $0 < \mu \leqslant 1$, then solutions are Hölder continuous with the same exponent μ on the manifold E. In order to achieve this we first prove the following, a result of interest in itself. If the set E, where the constraint is introduced, is a hyperplane then solutions are Hölder continuous with exponent μ in directions parallel to the hyperplane. In the special case $\mu = 1$ we get Lipschitz continuity in those directions and this is one of the results in the author's doctoral thesis [7].

In [8] the author studied a, in some sense, more general problem. It follows from the results there that bounded weak solutions of our problem are Hölder continuous with some exponent, at least if the bound is sufficiently small.

There are few existing papers dealing with thin obstacle problems for vector valued functions. However, we can compare our situation with that in [11]. There Kinderlehrer was concerned with continuity properties for the solution of a boundary obstacle problem with a variational inequality pertaining to a non-diagonal system with smooth coefficients, a problem

Pervenuto alla Redazione il 28 Dicembre 1984.

which arises in the theory of linear elasticity. Let us also mention [13] where Nečas proved $W^{2,2}$-regularity for solutions of a class of non-linear boundary obstacle problems.

The scalar valued case has been more extensively studied and this was initiated by Lewy [12]. He considered the problem of minimizing the Dirichlet integral in two dimensions and showed that if the thin obstacle function has a continuous derivative then the solution, restricted to the set where the constraint is introduced, has a continuous derivative. After that this kind of interior thin obstacle problems as well as boundary obstacle problems have been studied by several authors. Here we mention Kinderlehrer [9] and [10], Beirão da Veiga and Conti [1], Giaquinta and Modica [5], Frehse [3], Frehse and Mosco [4] and Caffarelli [2].

Acknowledgements. I am much obliged to Dr. Bengt Winzell for his careful reading and for his many suggestions for improvements of the manuscript. I am also very grateful to Mrs. Birgitta Arnsby for her efficient typing of the manuscript.

2. – Notations.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n\geqslant 3$, of points $x=(x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$, and put $B_r(x_0)=\{x\in\mathbb{R}^n\colon |x-x_0|<r\}$. Denote by $w^{1,p}(\Omega)$, $p\geqslant 1$, the Sobolev space of functions η such that

$$\|\eta\|_{w^{1,p}(\Omega)}=\left\{\int_\Omega(|\eta|^p+|\nabla\eta|^p)\,dx\right\}^{1/p}<\infty,$$

and by $w_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the $w^{1,p}$-norm. In the notation for a function space we add the symbol $\mathbb{R}^N$ to denote the corresponding space of $\mathbb{R}^N$-valued functions. For instance $C^{0,\mu}(S,\mathbb{R}^N)$ stands for the space of $\mathbb{R}^N$-valued functions with components in $C^{0,\mu}(S)$, the space of locally Hölder continuous functions with exponent μ. We also use the notations $u=(u^1, \ldots, \ldots, u^{N-1}, u^N)=(Pu, u^N)$, $D_\alpha=\partial/\partial x_\alpha$ and $\nabla u=(\ldots, D_\alpha u^i, \ldots)$, where $1\leqslant\alpha\leqslant n$ and $1\leqslant i\leqslant N$. For positive h and for unit vector $e=(e', 0)$, $e'\in\mathbb{R}^{n-1}$, we let $u_{\pm h}(x)=u(x\pm he)$ and

$$\Delta_{\pm h}^\mu u(x)=\pm h^{-\mu}\big(u_{\pm h}(x)-u(x)\big)\,, \qquad 0<\mu\leqslant 1\,.$$

When $\mu=1$ we write $\Delta_h u$ instead of $\Delta_h^1 u$. Moreover, we use a summation convention such that

$$\int a^{\alpha\beta}\, D_\alpha u\, D_\beta(v-u)\, dx \geqslant \int f(v-u)\, dx$$

means that

$$\sum_{i=1}^{N} \sum_{\alpha,\beta=1}^{n} \int a^{\alpha\beta} D_\alpha u^i D_\beta(v^i - u^i)\, dx \geqslant \sum_{i=1}^{N} \int f^i(v^i - u^i)\, dx$$

where u^i, v^i and f^i are the components of u, v and f, respectively. Finally, different constants appearing in the text will mostly be denoted by the same letter c.

3. – Results.

We will look at solutions $u = (u^1, \ldots, u^N)$ of the variational inequality

$$(1) \qquad u \in \mathbf{K} \quad \text{and} \quad \int_\Omega a^{\alpha\beta}(x, u)\, D_\alpha u\, D_\beta(v-u)\, dx \geqslant \int_\Omega f(x, u, \nabla u)(v-u)\, dx$$

for all $v \in \mathbf{K} = \{v \in w^{1,2}(\Omega, \mathbb{R}^N) \colon (v - \psi)(x) \in F(x)$ for $x \in E$ and $v - \varphi \in w_0^{1,2}(\Omega, \mathbb{R}^N)\}$, where φ is a prescribed $\mathbb{R}^N$-valued function, $\{F(x)\}_{x\in\Omega}$ is a set of half spaces in $\mathbb{R}^N$ and $E \subset \Omega$ is an $n-1$-dimensional manifold without boundary in Ω. The obstacle function ψ is of class $w^{1,p} \cap C^{0,\mu}(\Omega, \mathbb{R}^N)$ for some $p > n$ and some μ, $0 < \mu \leqslant 1$. Furthermore, the coefficients $a^{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, give an elliptic operator, that is there exists a positive constant λ such that

$$\lambda |\xi|^2 \leqslant a^{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \quad \text{for all } \xi \in \mathbb{R}^n,\ x \in \Omega \text{ and } u \in \mathbf{K}\,.$$

In the three first theorems we consider the situation where the sets $F(x)$, $x \in \Omega$, all are the same, namely the upper half space. This means that the constraint $(u - \psi)(x) \in F(x)$ for $x \in E$ takes the form: $u^N(x) \geqslant \psi^N(x)$ for $x \in E$. Moreover, the right hand side is of the form

$$f(x, u, \nabla u) = g_0(x, u, \nabla u) - D_\alpha g_\alpha\big(x, u(x)\big) + g(x, u, \nabla u)\,.$$

For g, which grows linearly in ∇u, only measurability is assumed. On the other hand for g_0, which grows quadratically in ∇u, we need some regularity condition. More precisely, g_0 together with g_α and $a^{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, are continuous functions satisfying the following conditions. For bounded $u \in \mathbf{K}$

and with $u_j = u(x_j)$ and $\nabla u_j = \nabla u(x_j)$, $j = 1, 2$, we suppose that

$$(2) \qquad |a^{\alpha\beta}(x_1, u_1) - a^{\alpha\beta}(x_2, u_2)| \leqslant C\big(|x_1 - x_2|^{\mu} + |Pu_1 - Pu_2|\big)\,,$$

$$(3) \qquad |g_{\alpha}(x_1, u_1) - g_{\alpha}(x_2, u_2)| \leqslant C\big(|x_1 - x_2|^{\mu} + |u_1 - u_2|^{\mu}\big) \qquad \text{and for } 1 \leqslant i \leqslant N$$

$$\begin{aligned}(4) \qquad & |g_0^i(x_1, u_1, \nabla u_1) - g_0^i(x_2, u_2, \nabla u_2)| \\ & \leqslant C\big(|x_1 - x_2|^{\mu} + |\delta_i u|\big)\big(1 + |\nabla u_1|^2 + |\nabla u_2|^2\big) \\ & + C|u_1 - u_2|^{\mu'}\big(1 + |\nabla u_1|^{2-\mu'_+} + |\nabla u_2|^{2-\mu'}\big) \\ & + C|\nabla \delta_i u|\big(1 + |\nabla u_1| + |\nabla u_2|\big)\end{aligned}$$

for some μ', $\mu \leqslant \mu \leqslant 1$, and for almost every $x_1, x_2 \in \Omega$. Here $\delta_N u = u_1 - u_2$, $\nabla \delta_N u = \nabla u_1 - \nabla u_2$ and $\delta_i u = Pu_1 - Pu_2$, $\nabla \delta_i u = \nabla Pu_1 - \nabla Pu_2$ for $1 \leqslant i \leqslant N - 1$. For the function $g = g(x, u, \nabla u)$ we assume measurability in Ω for $u \in \mathbb{K}$ and the existence of positive functions $a \in L^{\infty}(\Omega)$ and $b \in L^{p}(\Omega)$ such that

$$(5) \qquad |g(x, u, p)| \leqslant a|p| + b$$

for $x \in \Omega$, $u \in \mathbb{K}$ and $p \in \mathbb{R}^{nN}$.

Note that, except for the restriction that g_0^i, $1 \leqslant i \leqslant N - 1$, are independent of ∇u^N, condition (4) allows the function $g_0 = (g_0, \ldots, g_0^{N-1}, g_0^N)$ to grow quadratically in ∇u.

The first theorem treats the case when $E = \Omega' = \{x = (x', x_n) \in \Omega\colon x_n = 0\}$, and states that in directions parallel to E solutions are Hölder continuous with the exponent μ, $0 < \mu \leqslant 1$, given by the obstacle function ψ, From [8] we know that bounded weak solutions, with sufficiently small L^{∞}-norm, are Hölder continuous with some exponent so we will consider only Hölder continuous solutions of (1).

THEOREM 1. *If $u \in c^{0,\gamma}(\bar{\Omega}, \mathbb{R}^N)$, $0 < \gamma < \mu \leqslant 1$, is a solution of* (1) *with $E = \Omega'$ then for every $\Omega_0 \subset\subset \Omega$,*

$$\sup \frac{|u(x', x_n) - u(y', x_n)|}{|x' - y'|^{\mu}} < \infty$$

where the supremum is taken over all (x', x_n) and $(y', x_n) \in \Omega_0$, $x' \neq y'$.

REMARK. Let us for a moment assume that $0 < \mu < 1$. Then it follows from Theorem 1 that u is Hölder continuous with exponent μ in all directions if we for the system of differential equations pertaining to the variational inequality have relevant Schauder estimates.

The proof of Theorem 1 also gives the following theorem dealing with the corresponding thick obstacle problem. Define $\mathbb{K}_\Omega = \{v \in w^{1,2}(\Omega, \mathbb{R}^N)\colon (v - \psi)(x) \in F$ for $x \in \Omega$ and $v - \varphi \in w_0^{1,2}(\Omega, \mathbb{R}^N)\}$.

THEOREM 2. *If* $u \in C^{0,\gamma}(\bar{\Omega}, \mathbb{R}^N)$, $0 < \gamma < \mu \leqslant 1$ *is a solution of* (1) *with* $\mathbb{K}_\Omega$ *instead of* $\mathbb{K}$ *then* u *is Hölder continuous with exponent* μ *in* Ω.

Next we consider $n-1$-dimensional manifolds E with the following smoothness property.

PROPERTY E. For every $x_0 \in E$ there is a neighbourhood U and a bijection θ of U onto $B_R(0)$, $R > 0$, such that the image of $U \cap E$ is $B'_R(0) = B_R(0) \cap \{y \in \mathbb{R}^n\colon y_n = 0\}$. Moreover, θ and its invers θ^{-1} are of class $C^{1,\mu}$ and $c(U)$ denotes a positive constant for which the Jacobian matrix $(d\theta/dx)(x)$ satisfies

$$c(U)^{-1}|\xi| \leqslant \left|\frac{d\theta}{dx}(x)\,\xi\right| \leqslant c(U)|\xi| \quad \text{for all } x \in U \text{ and } \xi \in \mathbb{R}^n .$$

THEOREM 3. *Let the set* E *in the definition of* $\mathbb{K}$ *have Property* E *and let* $u \in c^{0,\mu}(\bar{\Omega}, \mathbb{R}^N)$, $0 < \gamma < \mu$, *be a solution of* (1). *Then* $u \in c^{0,\mu}(E, \mathbb{R}^N)$.

Finally, we are concerned with the regularity of solutions of (1) when the sets $F(x)$, $x \in \Omega$, are defined by

$$F(x) = \{\xi \in \mathbb{R}^N\colon \nu(x)\cdot\xi \leqslant 0\},$$

with $\nu(x) \in c^{1,\mu}(\Omega, \mathbb{R}^N)$ and $|\nu(x)| = 1$ for $x \in \Omega$. Also in this case we may have quadratic growth in the right hand side, but here the restriction on this growth depends on the directions $\nu(x)$, $x \in \Omega$. There is a similar restriction on the coefficients $a^{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, and their dependence on u. For simplicity, we dispense with the details here and assume that $a^{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, are independent of u and that $g_0 \equiv 0$. That is, we look at solutions u of

$$\text{(6)} \qquad u \in \mathbb{K} \quad \text{and} \quad \int_\Omega a^{\alpha\beta}(x)\, D_\alpha u\, D_\beta(v - u)\, dx \geqslant \int_\Omega g_\alpha(x, u)\, D_\alpha(v - u) + g(x, u, \nabla u)(v - u)\, dx$$

for all $v \in \mathbb{K} = \{v \in w^{1,2}(\Omega, \mathbb{R}^N)\colon (v - \psi)(x) \in F(x)$ for $x \in E$ and $v - \varphi \in w_0^{1,2}(\Omega, \mathbb{R}^N)\}$, where $a^{\alpha\beta} \in c^{0,\mu}(\bar{\Omega}, \mathbb{R}^N)$ and the functions g_α, $1 \leqslant \alpha \leqslant n$, and g are as before.

THEOREM 4. *Let E satisfy Property* E. *If* $u\in c^{0,\gamma}(\bar{\Omega},\mathbb{R}^N)$, $0<\gamma<\mu$, *is a solution of* (6) *then* $u\in c^{0,\mu}(E,\mathbb{R}^N)$.

4. – Auxiliary lemmata.

When mentioning Cauchy's inequality and Young's inequality we mean the inequalities

$$2st\leqslant\varepsilon s^2+1/\varepsilon t^2 \quad\text{and}$$

$$st\leqslant 1/ps^p+1/p't^{p'}\,, \quad\text{respectively}\,.$$

Here $\varepsilon>0$, $s,t\geqslant 0$ and $1/p+1/p'=1$, $p>1$. We write Young's inequality in the form

$$s^\mu t^{2-\mu}\leqslant\mu s^2/2+(2-p)t^2/2\,, \qquad 0<\mu<2\,,$$

which is more convenient for our purposes.

We will utilize the Green function G for the operator $L=-D_\alpha(b^{\alpha\beta}D_\beta)$, where $b^{\alpha\beta}$ are Hölder continuous in a ball B, and its mollification G^ϱ defined by

$$G^\varrho(x,z)=\frac{1}{\operatorname{meas}(B_\varrho(z))}\int_{B_\varrho(z)}G(x,y)\,dy=\fint_{B_\varrho(z)}G(x,y)\,dy\,.$$

We list some useful properties of G and G^ϱ cf. Widman [14] and [15].

LEMMA 1. *For* $z\in B$ *and* $\varrho>0$ *small enough* $G^\varrho(\cdot,z)\in w_0^{1,2}(B)$ *and satisfies*

$$\int_B b^{\alpha\beta}D_\alpha\eta D_\beta G^\varrho\,dx=\fint_{B_\varrho(z)}\eta\,dx \qquad \text{for all } \eta\in L^\infty\cap w_1^{1,2}(B)$$

and

$$\lim_{\varrho\to 0}G^\varrho(x,z)=G(x,z) \qquad \text{for } x\neq z\,,\ x\in B\,.$$

Moreover, since $b^{\alpha\beta}$ *are Hölder continuous* G *will be regular and we have the inequalities*

$$G(x,z)\leqslant C|x-z|^{2-n} \quad \textit{and}$$

$$|\nabla G(x,z)|\leqslant C|x-z|^{1-n} \quad \textit{for } x\neq z,\ x\in B\,.$$

Finally, if $B_{2r}\subset B$ *and* B_r *is the concentric ball of radius* r *then*

$$G(x,z)\geqslant c|x-z|^{2-n} \quad \textit{for } x\neq z,\ x,z\in B_r\,.$$

Next we generalize a lemma from Hildebrandt and Widman [6], p. 203, which will be needed in the proof of Theorem 1.

LEMMA 2. *Let $x_0 \in \Omega$ and suppose that $u \in C^{0,\gamma}(\bar{\Omega}, \mathbb{R}^N)$, $0 < \gamma \leqslant \mu$, is a solution of the inequality* (1). *Then for $0 \leqslant \varkappa < 2\gamma'$, $\gamma' = \min(\gamma, 1 - n/p)$,*

$$\int\limits_{B_r(x_0)} |\nabla u(x)|^2 |x - x_0|^{2-n-\varkappa}\, dx \leqslant C r^{2\gamma' - \varkappa}$$

for all r, $0 < r \leqslant \frac{1}{2}$ dist $(x_0, \partial\Omega)$. Here the constant C depends on parameters of the problem, on $\varkappa$ and on the $C^{0,\gamma}$-norm of u.

PROOF. Fix $x_0 \in \Omega$ and let r be such that $B_{2r}(x_0) \subset \Omega$. Moreover, let $\eta \in C_0^\infty(B_{2r}(x_0))$ be a function which satisfies $\eta(x) = 1$ for $x \in B_r(x_0)$, $0 \leqslant \eta \leqslant 1$ and $|\nabla \eta| \leqslant c/r$. If $B_{2r}(x_0) \cap E \neq \emptyset$ choose an x_1 in $B_{2r}(x_0) \cap E$ and put

$$v = \psi + \big(1 - \varepsilon G^\varrho(\cdot, x_0)\eta^2\big)(u - \psi) + \varepsilon G^\varrho(\cdot, x_0)\eta^2 (u - \psi)(x_1)\,,$$

where $\varepsilon > 0$ and G^ϱ, $0 < \varrho < r/2$, is the mollification of the Green function G for the operator $L = -D_\alpha(a^{\alpha\beta} D_\beta)$. Here, the Hölder continuous coefficients $a^{\alpha\beta} = a^{\alpha\beta}(x, u)$ are extended to Hölder continuous functions defined in an open ball $B \supset \bar{\Omega}$ such that the ellipticity property still holds, at least for a slightly smaller ellipticity constant λ. It is readily seen that v is a test function if ε is sufficiently small. Now, use this test function in (1), cancel ε and rearrange terms to obtain

$$\int\limits_\Omega a^{\alpha\beta} D_\alpha u\, D_\beta\{G^\varrho \eta^2 (u - u(x_1))\}\, dx \leqslant \int\limits_\Omega a^{\alpha\beta} D_\alpha u\, D_\beta\{G^\varrho \eta^2(\psi - \psi(x_1))\}\, dx + \int\limits_\Omega f(x, u, \nabla u) G^\varrho \eta^2 \{u - u(x_1) - (\psi - \psi(x_1))\}\, dx\,. \tag{7}$$

Standard calculation now gives

$$\int\limits_{B_r(x_0)} |\nabla u|^2\, dx \leqslant C r^{n-2+2\gamma'} \quad \text{for all } r\,,\ 0 < r \leqslant \tfrac{1}{2} \operatorname{dist}(x_0, \partial\Omega)\,. \tag{8}$$

For convenience of the reader we sketch the proof of (8). Expand the left hand side in (7), use the ellipticity condition, the continuity of u, known properties of G^ϱ and Cauchy's inequality to obtain

$$\int_\Omega a^{\alpha\beta} D_\alpha u\, D_\beta\{G^\varrho \eta^2(u-u(x_1))\}\, dx \geq (\lambda - \lambda/4)\int_\Omega |\nabla u|^2 G^\varrho \eta^2\, dx$$
$$+ \tfrac{1}{2}\int_\Omega a^{\alpha\beta} D_\alpha(|u-u(x_1)|^2\eta^2)\, D_\beta G^\varrho\, dx - Cr^{2\gamma}\,.$$

Treating the first integral on the right hand side in (7) in a similar manner we find

$$\int_\Omega a^{\alpha\beta} D_\alpha u\, D_\beta\{G^\varrho \eta^2(\psi-\psi(x_1))\}\, dx \leq \int_\Omega a^{\alpha\beta} D_\alpha\{(u-u(x_1))(\psi-\psi(x_1))\eta^2\}\, D_\beta G^\varrho\, dx$$
$$+ \lambda/4\int_\Omega |\nabla u|^2 G\text{-}\eta^2\, dx + Cr^{2\gamma'}.$$

From (4) it follows that there are constants c_1 and c_2 depending on the L^∞-norm of u such that $|g_0(x, u, \nabla u)| \leq c_1|\nabla u|^2 + c_2$ for almost every $x \in \Omega$. Using this, the fact that the functions g_α, $1 \leq \alpha \leq n$, are bounded, condition (5) and as before suitable properties of u, ψ and G^ϱ we see that the second integral on the right hand side in (7) is bounded from above by

$$\left(\frac{\lambda}{4} + Cr^\gamma\right)\int_\Omega |\nabla u|^2 G^\varrho \eta^2\, dx + Cr^{2\gamma'}.$$

Summing up and using Lemma 1 we get

$$\left(\frac{\lambda}{4} - Cr^\gamma\right)\int_\Omega |\nabla u|^2 G^\varrho \eta^2\, dx + \frac{1}{2}\;\fint_{B_\varrho(x_0)} |u-u(x_1)|^2\eta^2\, dx \leq Cr^{2\gamma'},$$

and together with Fatou's lemma this gives the estimate (8).

Next we consider the case when $B_{2r}(x_0) \cap E = \emptyset$. Here we can use the same test function, where x_1 now is an arbitrary point in $B_{2r}(x_0)$, for instance x_0. The same procedure as above yields the inequality (8) also in this case. It is not hard to see that the statement in the lemma follows from (8). In fact,

$$\int_{B_r(x_0)} |\nabla u|^2 |x-x_0|^{2-n-\varkappa}\, dx = \sum_{i=0}^{\infty} \int_{B_{r/2^i}(x_0)\setminus B_{r/2^{i+1}}(x_0)} |\nabla u|^2 |x-x_0|^{2-n-\varkappa}\, dx$$
$$\leq C\sum_{i=0}^{\infty} (r/2^{i+1})^{2-n-\varkappa}(r/2^i)^{n-2+2\gamma'} \leq Cr^{2\gamma'-\varkappa}\,,$$

and we are done.

5. – Proofs of the results.

PROOF OF THEOREM 1. We will use a rather special test function and in order to construct this we need some notations. Introduce the sets $\Omega_\tau = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \tau\}$ and $I = \{x \in \Omega' \cap \Omega_{\tau/2}\colon u^N(x) = \psi^N(x)\}$, and let $0 < h < \tau/4$. Choose $l > 0$ such that

$$\Delta_h^\mu \psi^N(x) + l \geqslant \delta > 0 \text{ for some } \delta \text{ independent of } x \in \Omega_{\tau/2}, h \text{ and } e = (e', 0).$$

Here e' is a unit vector in $\mathbb{R}^{N-1}$. Put

$$u_\Delta(x) = \begin{cases} \big(P\Delta_\mu^h u(x), u_\Delta^N(x)\big) & \text{for } x \in \Omega_{\tau/2}, \\ 0 & \text{for } x \in \Omega' \setminus \Omega_{\tau/2}, \end{cases}$$

where

$$u_\Delta^N(x) = \min\big(\Delta_h^\mu u^N(x) + l, 0\big).$$

Fix $x_0 \in \Omega_\tau$ and let $0 < r \leqslant \delta_\tau(x_0)/4$, where $\delta_\tau(x_0) = \max\big(\operatorname{dist}(x_0, \partial\Omega) - \tau, 0\big)$. As a test function we use

$$v = u + \varepsilon\Delta_{-h}^\mu\big(u_\Delta G^\varrho(\cdot, x_0)\zeta^2\big) = u + \varepsilon\Delta_{-h}^\mu w,$$

where $\varepsilon > 0$ and $\zeta = \delta_\tau(x_0)\eta$. Here η, ϱ, G^ϱ are as in the proof of Lemma 2. We will comment on this choice in an appendix at the end of the paper. This v is an admissible test vector since $v \in w^{1,2}(\Omega, \mathbb{R}^N)$ and $v - \varphi \in w_0^{1,2}(\Omega, \mathbb{R}^N)$, and since it is possible to show that $(v^N - \psi^N)(x) \geqslant 0$ for $x \in \Omega'$ if ε is sufficiently small. In fact, if $x \in I \subset \Omega'$ we have

$$\Delta_h^\mu u^N(x) + l = h^{-\mu}(u_h^N - \psi_h^N)(x) - h^{-\mu}(u^N - \psi^N)(x) + \Delta_h^\mu \psi^N(x) + l \geqslant \delta > 0$$

and it follows that $\Delta_h^\mu u^N + l > 0$ in some $n - 1$-dimensional neighbourhood $N(I) \subset \Omega'$ of I. Consequently,

$$u_\Delta^N(x) = 0 \quad \text{in } N(I).$$

This implies that for $x \in N(I)$,

$$(v^N - \psi^N)(x) = (u^N - \psi^N)(x) - \varepsilon h^{-\mu}\big(w_{-h}^N(x) - w^N(x)\big) \geqslant 0,$$

since $w_{-h}^N(x) \leqslant 0$ and $w^N(x) = 0$ if x belongs to $N(I)$.

For $x \in \Omega' \setminus \Omega_{\tau/2}$, $v = u$ so trivially $v^N(x) \geqslant \psi^N(x)$. Finally, for $x \in \Omega' \cap \Omega_{\tau/2} \setminus N(I)$ there is a positive constant c such that $(u^N - \psi^N)(x) \geqslant c$ and

$$(v^N - \psi^N)(x) = (u^N - \psi^N)(x) + \varepsilon \Delta^{\mu}_{-h} w^N \geqslant 0$$

if ε is chosen small enough. By use of this v in the variational inequality (1) we can show that

$$|u_{\Delta}(x)| \leqslant c \quad \text{for } x \in \Omega_{2\tau}, \tag{9}$$

with a constant c which is independent of e and h. The proof of (9) is postponed to somewhat later.

From (9) and the definition of u_{Δ} we obtain that

$$|P\Delta^{\mu}_h u| \leqslant c \quad \text{and} \quad |u^N_{\Delta}| \leqslant c. \tag{10}$$

According to the definition of u^N_{Δ}, $|u^N_{\Delta}| \leqslant c$ yields that

$$\text{either} \quad |\Delta^{\mu}_h u^N + l| \leqslant c \quad \text{or } \Delta^{\mu}_h u^N + l \geqslant 0$$

from which the following estimate from below is deduced:

$$\Delta^{\mu}_h u^N \geqslant -l - c = -c_1 \quad \text{for all } x \in \Omega_{2\tau}. \tag{11}$$

However, due to the fact that the constant c_1 is independent of h, e and x, we are allowed to reverse the direction of e in (11), whence

$$|\Delta^{\mu}_h u^N| \leqslant c_1 \quad \text{for } x \in \Omega_{3\tau}.$$

This together with (10) gives the statement in Theorem (1). It remains to prove (9). To do so we exploit the technique used by Hildebrandt-Widman in [6], pp. 205-207.

First we assume that $g \equiv 0$. Now, insert the proposed test function in (1) move the operator Δ^{μ}_{-h} to the other factor and delete ε to arrive at

$$\int_{\Omega} \Delta^{\mu}_h (a^{\alpha\beta} D_{\alpha} u) D_{\beta}(u_{\Delta} G^{\varrho} \zeta^2)\, dx \leqslant \int_{\Omega} \Delta^{\mu}_h f(x, u, \nabla u) u_{\Delta} G^{\varrho} \zeta^2\, dx. \tag{12}$$

By use of the identity

$$\Delta^{\mu}_h (a^{\alpha\beta} D_{\alpha} u) = a^{\alpha\beta} D_{\alpha} \Delta^{\mu}_h u + \Delta^{\mu}_h a^{\alpha\beta} D_{\alpha} u_h$$

we split the integral on the left hand side in (12) into two parts, move the second one to the right hand side and exploit the conditions (2)-(4) on $a^{\alpha\beta}$ and on the right hand side to obtain

$$\tag{13} \int_\Omega a^{\alpha\beta} D_\alpha \Delta_h^\mu u \, D_\beta(u_\Delta G^\varrho \zeta^2)\, dx$$

$$\leq c\int_\Omega (1 + |P\Delta_h^\mu u|)|\nabla u_h|\big(|\nabla u_\Delta| G^\varrho \zeta^2 + |u_\Delta||\nabla G^\varrho|\zeta^2 + |u_\Delta|G^\varrho|\nabla\zeta^2|\big)\, dx$$

$$+ c\int_{\Omega_l} \{(1 + |\Delta_h^\mu u|)(1 + |\nabla u_h|^2 + |\nabla u|^2) + |\nabla \Delta_h^\mu u|(1 + |\nabla u_h| + |\nabla u|)\}|u_\Delta^N| G^\varrho \zeta^2\, dx$$

$$+ c\int_\Omega \{(1 + |P\Delta_h^\mu u|)(1 + |\nabla u_h|^2 + |\nabla u|^2) + |\Delta_h u|^{\mu'}(1 + |\nabla u_h|^{2-\mu'} + |\nabla u|^{2-\mu'})$$

$$+ |\nabla P\Delta_h^\mu u|(1 + |\nabla u_h| + |\nabla u|)\}|u_\Delta| G^\varrho \zeta^2\, dx$$

$$+ c\int_\Omega (1 + |\Delta_h u|^\mu)\big(|\nabla u_\Delta| G^\varrho \zeta^2 + |u_\Delta||\nabla G^\varrho|\zeta^2 + |u_\Delta|G^\varrho|\nabla\zeta^2|\big)\, dx\,,$$

where $\Omega_l = \{x \in \Omega^\tau\colon \Delta_h^\mu u^N(x) + l < 0\} = \{x \in \Omega_\tau\colon u_\Delta^N \neq 0\}$.

By use of Cauchy's and Young's inequalities and the facts that in Ω_l, $|\Delta_h^\mu u| \leq (l+1)(|u_\Delta| + 1)$ and $|\nabla \Delta_h^\mu u| = |\nabla u_\Delta|$ the right hand side in (13) can be estimated by

$$c\int_\Omega (1 + |u_\Delta|^2)|\nabla u_h|^2 G^\varrho \zeta^2\, dx + \frac{\lambda}{3}\int_\Omega |\nabla u_\Delta|^2 G^\varrho \zeta^2\, dx$$

$$+ c\int_\Omega (1 + |u_\Delta|^2)|\nabla u_h|\,|\nabla G^\varrho|\,\zeta^2\, dx + c\int_\Omega (1 + |u_\Delta|^2)|\nabla u_h|\, G^\varrho\, |\nabla\zeta^2|\, dx$$

$$+ c\int_\Omega (1 + |u_\Delta|^2)(1 + |\nabla u_h|^2 + |\nabla u|^2) G^\varrho \zeta^2\, dx + c\int_\Omega |\Delta_h u|^2 |u_\Delta| G^\varrho \zeta^2\, dx$$

$$+ c\int_\Omega (1 + |\nabla u_h|^2 + |\nabla u|^2)|u_\Delta|\, G^\varrho \zeta^2\, dx + c\int_\Omega (1 + |\Delta_h u|^2) G^\varrho \zeta^2\, dx$$

$$+ c\int_\Omega (1 + |\Delta_h u|(\,|u_\Delta|\, |\nabla G^\varrho|\, \zeta^2\, dx + c\int_\Omega (1 + |\Delta_h u|)|u_\Delta|\, G^\varrho\, |\nabla\zeta^2|\, dx\,.$$

When expanding the left and side in (13) the integral

$$\int_\Omega a^{\alpha\beta} D_\alpha \Delta_h^\mu u u_\Delta D_\beta G^\varrho \zeta^2 \, dx$$

will show up. Since

$$D_\alpha u_\Delta^N = \begin{cases} D_\alpha \Delta_h^\mu u^N & \text{if } \Delta_h^\mu u^N + l < 0 , \\ 0 & \text{if } \Delta_h^\mu u + l \geqslant 0 , \end{cases}$$

and

$$D_\alpha (u_\Delta^N)^2 = 2u_\Delta^N \, D_\alpha u_\Delta^N = 2u_\Delta^N \, D_\alpha \Delta_h^\mu u^N$$

we find

$$\int_\Omega a^{\alpha\beta} D_\alpha \Delta_h^\mu u |u_\Delta| D_\beta G^\varrho \zeta^2 \, dx = \frac{1}{2} \int_\Omega a^{\alpha\beta} D_\alpha \big(|u_\Delta|^2 \zeta^2 \big) D_\beta G^\varrho \, dx$$
$$- \frac{1}{2} \int_\Omega a^{\alpha\beta} |u_\Delta|^2 D_\alpha \zeta^2 D_\beta G^\varrho \, dx = \frac{1}{2} \fint_{B_\varrho(x_0)} |u_\Delta|^2 \zeta^2 \, dx - \frac{1}{2} \int_\Omega a^{\alpha\beta} |u_\Delta|^2 D_\alpha \zeta^2 D_\beta G^\varrho \, dx .$$

Now, we split the integral on the left hand side in (13) into three parts the first of which is estimated from below by use of the ellipticity condition, the second of which is treated as indicated above and the third of which is treated in the following way. Utilizing Cauchy's inequality we obtain that

$$\int_\Omega a^{\alpha\beta} D_\alpha \Delta_h^\mu u \, u_\Delta \, G^\varrho 2\zeta D_\beta \zeta \, dx \geqslant -\frac{\lambda}{3} \int_\Omega |\nabla u_\Delta|^2 G^\varrho \zeta^2 \, dx - c \int_\Omega |u_\Delta|^2 G^\varrho \, |\nabla \zeta|^2 \, dx .$$

Summarizing and using $|u_\Delta| \leqslant |\Delta_h^\mu u|$ and $|u_\Delta| \leqslant 1 + |u_\Delta|^2$ we get

$$(14) \quad \frac{\lambda}{3} \int_\Omega |\nabla u_\Delta|^2 G^\varrho \zeta^2 \, dx + \fint_{B_\varrho(x_0)} |u_\Delta|^2 \zeta^2 \, dx \leqslant c \int_\Omega |\Delta_h^\mu u|^2 \big(|\nabla \zeta^2| \, |\nabla G^\varrho| + G^\varrho \, |\nabla \zeta|^2 \big) \, dx$$
$$+ c \int_\Omega (1 + |u_\Delta|^2)(1 + |\nabla u_h|^2 + |\nabla u|^2 + |\Delta_h u|^2) G^\varrho \zeta^2 \, dx$$
$$+ c \int_\Omega (1 + |u_\Delta|^2)(1 + |\nabla u_h| + |\Delta_h u|)\big(|\nabla G^\varrho| \zeta^2 + G^\varrho \, |\nabla \zeta^2| \big) \, dx .$$

Since $r \leqslant \delta_\tau(x_0)/4$ we have $\delta_\tau(x_0) \leqslant 2\delta_\tau(x)$ for $x \in B_{2r}(x_0)$ and this implies that

$$|u_\Delta(x)| \delta_\tau(x_0) \leqslant 2 \sup_{X \in \Omega_\tau} |u_\Delta(x)| \delta_\tau(x) = 2 M_{h,\tau}$$

for all $x \in B_{2r}(x_0)$. Combine this and known inequalities for G^ϱ and ∇G_ϱ with (14) whereupon

$$(15)\qquad \fint_{B_\varrho(x_0)} |u_\Delta|^2 \zeta^2\,dx \leqslant c\delta_\tau(x_0)^2 r^{-n} \int_{\operatorname{supp}\eta} |\Delta_h^\mu u|^2\,dx$$

$$+ c\big(\delta_\tau(x_0)^2 + M_{h,\tau}^2\big) \int_{\operatorname{supp}\eta} (1 + |\nabla u_h|^2 + |\nabla u|^2 + |\Delta_h u|^2)|x - x_0|^{2-n}\,dx$$

$$+ c\big(\delta_\tau(x_0)^2 + M_{h,\tau}^2\big) \int_{\operatorname{supp}\eta} (1 + |\nabla u_h| + |\Delta_h u|)|x - x_0|^{1-n}\,dx\,.$$

Furthermore, if h is so small that independently of e $\operatorname{supp}\eta + he \subset B_{2r}(x_0)$ then

$$\int_{\operatorname{supp}\eta} |\Delta_h^\mu u|^2\,dx \leqslant h^{1-\mu} \int_{B_{2r}(x_0)} |\nabla u|^2\,dx$$

and, taking Lemma 2 into account,

$$\int_{\operatorname{supp}\eta} |\Delta_h u|^2 |x - x_0|^{2-n-\varkappa}\,dx \leqslant \int_0^1 \int_{B_{2r}(x_0+the)} |\nabla u|^2 |x - (x_0 + the)|^{2-n-\varkappa}\,dx\,dt \leqslant cr^{2\gamma' - \varkappa}\,.$$

Combine these inequalities and the inequality

$$\int_{B_{2r}(x_0)} |\nabla u|\,|x - x_0|^{1-n}\,dx \leqslant \Big(\int_{B_{2r}(x_0)} |\nabla u|^2\,|x - x_0|^{2-n-\varkappa}\,dx\Big)^{\frac12} \Big(\int_{B_{2r}(x_0)} |x - x_0|^{-n+\varkappa}\,dx\Big)^{\frac12} \leqslant Cr^\gamma$$

with (15) it is easily seen that

$$\fint_{B_\varrho(x_0)} |u_\Delta|^2 \zeta^2\,dx \leqslant C\delta_\tau(x_0)^2 r^{-2+2\gamma'} + C_0\big(\delta_\tau(x_0)^2 + M_{h,\tau}^2\big) r^{\gamma'}.$$

Now, let $\varrho \to 0$ and infer

$$|u_\Delta(x_0)|^2 \delta_\tau(x_0)^2 \leqslant C\delta_\tau(x_0)^2 r^{-2+2\gamma'} + C_0 M_{h,\tau}^2 r^{\gamma'}.$$

If we choose $x_0 \in \Omega_\tau$ such that

$$|u_\Delta(x_0)|\,\delta_\tau(x_0) = M_{h,\tau}$$

and let $r = \min\left(\delta_\tau(x_0)/4,\ (2C_0)^{-1/\gamma'}\right)$ we find

$$M_2^{h,\tau} \leqslant C$$

with a constant C independent of h and e. This implies that

$$|u_\Delta(x)| \leqslant C \qquad \text{for all } x \in \Omega_{2\tau}$$

which is the desired inequality (9) and this completes the proof for the case where $g \equiv 0$. If this is not the case the term

$$\int_\Omega g(x, u, \nabla u) h^{1-\mu} \Delta_{-h}(u_\Delta G^\varrho \zeta^2)\, dx$$

will show up. If we as before exchange the difference operator Δ_{-h} for the gradient operator ∇ and use condition (5) it is not hard to see that this integral is bounded from above by

$$\frac{\lambda}{3}\int_\Omega |\nabla u_\Delta|^2 G^\varrho \zeta^2\, dx + C\left(\delta_\tau(x_0)^2 + M_{h,\tau}^2\right) r^{\gamma'}.$$

Now looking at (14) we see that it is possible to have also these terms in the right hand side and still get the desired inequality. This completes the proof of Theorem 1.

By inspection it is readily seen that Theorem 2 follows from this proof if we let e in the definition of $\Delta_h^\mu u$ be an arbitrary unit vector in $\mathbf{R}^n$, instead of being of the form $(e', 0)$.

Proof of Theorem 3. Fix $x_0 \in E$. Let U be a neighbourhood and θ be a transformation such that Property E holds. Observe that our solution u of (1) is also a solution of the inequality

$$u \in \mathbf{K}_U \quad \text{and} \quad \int_U a^{\alpha\beta}(x, u)\, D_\alpha u\, D_\beta(v - u)\, dx \geqslant \int_U f(x, u, \nabla u)(v - u)\, dx$$

$$\text{for all } v \in \mathbf{K}_U = \{v \in w^{1,2}(U, \mathbf{R}^N) \colon v^N(x) \geqslant \psi^N(x)$$

$$\text{for } x \in U \cap E \text{ and } v - u \in w_0^{1,2}(U, \mathbf{R}^N)\}.$$

After an elementary calculation we see that $\tilde{u} = u \circ \theta^{-1}$ is a solution of

$$\tilde{u} \in \mathrm{K}_{B_R} \quad \text{and} \quad \int_{B_R(0)} \tilde{a}^{\alpha'\beta'}(y, \tilde{u})\, D_{\alpha'}\tilde{u}\, D_{\beta'}(\tilde{v} - \tilde{u})\, dy \geqslant \int_{B_R(0)} \tilde{f}(y, \tilde{u}, \nabla\tilde{u})(\tilde{v} - \tilde{u})\, dy$$

for all $\tilde{v} \in \mathrm{K}_{B_R} = \{\tilde{v} \in w^{1,2}(B_R(0), \mathbf{R}^N) : \tilde{v}^N(y) \geqslant \tilde{\psi}^N(y)$ for $y \in B'_R(0)$ and $\tilde{v} - \tilde{u} \in w_0^{1,2}(B_R(0), \mathbf{R}^N)\}$, where $\tilde{v} = v \circ \theta^{-1}$ and $\tilde{\psi} = \psi \circ \theta^{-1}$. It is not hard to see that the new coefficients $\tilde{a}^{\alpha'\beta'}$ and the new right hand side $\tilde{f}$ satisfy conditions like those in (2)-(5). Note especially that the functions

$$\tilde{a}^{\alpha'\beta'}(y, \tilde{u}(y)) = a^{\alpha\beta}(x, u(x)) D_\alpha \theta^{\alpha'}(x) D_\beta \theta^{\beta'}(x) \left| \det \frac{\mathrm{d}\theta^{-1}}{\mathrm{d}y}(y) \right|$$

give an elliptic operator with an ellipticity constant depending on λ and $C(U)$, the constant appearing in Property E. Since $\tilde{\psi}$ is Hölder continuous with exponent μ Theorem 1 implies that $\tilde{u}$ is Hölder continuous with the same exponent on B'_R and consequently u is Hölder continuous on $U \cap E$. Now the collection of sets $U \cap E$, $x_0 \in E$, constitute an open covering of E and since every compact subset of E is covered by a finite subcollection this proves the theorem.

PROOF OF THEOREM 4. We will show that it is possible to introduce functions $\tilde{u}$ and $\tilde{\psi}$ such that the constraint $(u - \psi)(x) \in F(x)$ for $x \in E$ takes the form $\tilde{u}^N(x) \geqslant \tilde{\psi}(x)$ for $x \in E$. In fact, let $\tilde{u} = Qu$ where Q is an orthogonal matrix with entries of class $C^{1,\mu}$ such that $\tilde{u}^N(x) = \gamma(x) u(x)$, and where u is our solution of (6). Then it is not hard to show that $\tilde{u}$ is a solution of a similar problem where all the relevant conditions still hold and where the constraint is given by $\tilde{u}^N(x) \geqslant \tilde{\psi}(x) = \gamma(x)\psi(x)$ for $x \in E$. From Theorem 3 we conclude that $\tilde{u} \in C^{0,\mu}(E, \mathbf{R}^N)$, and consequently $u \in C^{0,\mu}(E, \mathbf{R}^N)$, which is the statement in Theorem 4.

6. – Appendix.

In [6], Hildebrandt and Widman considered a similar problem but with a thick obstacle, that is when the constraint is of the form $(u - \psi)(x) \in F$ for all $x \in \Omega$. There ψ was assumed to be three times continuously differentiable and then a simple argument shows that one can suppose that $\psi \equiv 0$, at least if the conditions on the right hand side are less restrictive than those in [6]. In this case $v = u + \varepsilon\Delta_{-h}(\Delta_h u G^\varrho \zeta^2)$ is a test function if ε is sufficiently small, since it can be written as a convex combination of elements in **K**. If the obstacle function ψ is only Hölder continuous the argument above is not available and a natural choice in this case would have seemed to be $v = u + \varepsilon\Delta^\mu_{-h}(\Delta^\mu_h(u - \psi)G^\varrho\zeta^2)$, which certainly is an admissible test function if $\varepsilon > 0$ is small enough. Unfortunately, terms of the form $D_\beta \Delta^\mu_h \psi$ will show up and we have no uniform bound for these kind of terms. By introducing the function u_Δ and using

$$v = u + \varepsilon\Delta^\mu_{-h}(u_\Delta G^\varrho \zeta^2)$$

as a test vector we eliminate this difficulty.

REFERENCES

[1] H. Beirão da Veiga - F. Conti, *Equazioni ellittiche non lineari con ostacoli sottili*, Ann. Scuola Norm. Super. Pisa, Cl. Sci. III, **26** (1972), pp. 533-562.

[2] L. Caffarelli, *Further regularity for the Signorini problem*, Comm. Pure Appl. Math., **4** (1979), pp. 1067-1075.

[3] J. Frehse, *On Signorini's problem and variational problems with thin obstacles*, Ann. Scuola Norm. Super. Pisa Cl. Sci., **4** (1977), pp. 343-362.

[4] J. Frehse - U. Mosco, *Sur la continuité ponctuelle des solutions locales faibles du problème d'obstacle*, C. R. Acad. Sci. Paris, **295** (1982), pp. 571-574.

[5] M. Giaquinta - G. Modica, *Regolarità Lipschitziana per la soluzione di alcuni problemi di minimo con vincolo*, Ann. Mat. Pura Appl., **106** (1975), pp. 95-117.

[6] S. Hildebrandt - K.-O. Widman, *Variational inequalities for vector valued functions*, J. Reine Angew. Math., **309** (1979), pp. 191-220.

[7] T. Karlsson, *Regularity results for solutions of some obstacle problems*, Dissertation No. **92**, Dept. Math., Linköping University (1983).

[8] T. Karlsson, *Wiener's criterion and obstacle problems for vector valued functions*, Arkiv för Mat., **23** (1985), pp. 315-325.

[9] D. Kinderlehrer, *Variational inequalities with lower dimensional obstacles*, Israel J. Math., **10** (1971), pp. 339-348.

[10] D. Kinderlehrer, *The smoothness of the solution of the boundary obstacle problem*, J. Math. Pures Appl., **60** (1981), pp. 193-212.

[11] D. Kinderlehrer, *Remarks about Signorini's problem in linear elasticity*, Ann. Scuola Norm. Super. Pisa Cl. Sci., **4** (1981), pp. 605-645.

[12] H. Lewy, *On a refinement of Evans' law in potential theory*, Atti Accad. Naz. Lincei (1970), pp. 1-9.

[13] J. Nečas, *On regularity of solutions to nonlinear variational inequalities for second-order elliptic systems*, Rend. Mat., **8** (1975), pp. 481-498.

[14] K.-O. WIDMAN, *The singularity of the Green function for nonuniformly elliptic partial differential equations with discontinuous coefficients*, Uppsala University Report (1970).

[15] K.-O. WIDMAN, *Inequalities for the Green function of second order elliptic operators*, Report 8-1972, Dept. Math., Linköping University (1972).

Linköping University
Department of Mathematics
S-581 83 Linköping, Sweden