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On some parabolic variational problems with quadratic growth

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1. – Introduction and results.

In this work, we consider weak solutions of parabolic systems of the type

$$u_t - \sum_{\alpha,\beta=1}^{n} \frac{d}{dx_\beta}\{\gamma_{\alpha\beta}\cdot u_{x_\alpha}\} = f(x,t,u,\nabla u) - \sum_{\alpha=1}^{n} g_{x_\alpha}(x,t) \tag{1.1}$$

and weak solutions of the corresponding variational inequalities. We prove the Hölder-continuity of the weak solutions, in the interior, and up to the boundary and up to the initial data, if the weak solutions have Hölder-continous initial and boundary values and if the boundary satisfies the condition (A) of Ladyzhenskaya-Uraltseva [14]. Apart from conditions on the obstacle, we need the same assumptions for our Hölder-estimates as Hildebrandt-Widman [9] for the corresponding elliptic systems. Once one has obtained a-priori bounds, it is rather easy to obtain existence results. Therefore, we take the opportunity to give an existence proof for the Cauchy-Dirichlet problem for variational inequalities and systems.

Our Hölder-estimates and existence results for systems generalize those due to M. Struwe [15], Giaquinta-Struwe [4], R. S. Hamilton [7] and W. v. Wahl [21], in some respects. Applied to elliptic vector valued variational inequalities, they are slightly stronger than those due to Hildebrandt-Widman [10]. For vector valued and scalar two-sided parabolic variational inequalities with quadratic growth, no corresponding results are known to us. For one-sided scalar variational inequalities with quadratic growth, Struwe-Vivaldi [17] obtained more general results, recently. There are also

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new existence results for parabolic systems and parabolic vector valued variational inequalities due to Alt-Luckhaus [1]. The variational problems considered by them, however, are quite different from our ones.

Another aim of this paper is to present a new and simpler approach to the regularity of such variational problems. It is quite similar to the one used by the author in [18] and [19], for elliptic systems and harmonic mappings between Riemannian manifolds. This, in turn, has been inspired by the work [2] of L. A. Caffarelli. Our regularity proofs are based on a Strong Maximum Principle and some refinements of the techniques due to De Giorgi [3], Ladyzhenskaya-Uraltseva [14] and Ladyzhenskaya-Solonnikov-Uraltseva [13] which we prove in this paper, too. Thus, our regularity proofs are self-contained, apart from some elementary facts on Sobolev-spaces.

A weak solution u of (1.1) belongs to the class $L^\infty(Q) \cap L^2(]0, T[; H^{1,2}(\Omega))$ and satisfies

$$\int_0^T \int_\Omega - u \cdot \varphi_t + \sum_{\alpha,\beta=1}^n \gamma_{\alpha\beta}(x, t) \cdot u_{x_\alpha} \cdot \varphi_{x_\beta} \, dx \, dt = \int_0^T \int_\Omega f(x, t, u, \nabla u) \cdot \varphi + g(x, t) \cdot \nabla\varphi \, dx \, dt \,, \tag{1.2}$$

for all $\varphi \in C_c^\infty(Q)$. Here, Ω is an open bounded subset of $\mathbf{R}^n$, $T > 0$ and

$$Q = \Omega \times]0, T[\,.$$

In addition to that, we use the notations

$$B_R(x_0) = \{x \in \mathbf{R}^n \colon |x - x_0| < R\} \,,$$

$$Q_R(x_0, t_0) = B_R(x_0) \times]t_0 - R^2, t_0[\,.$$

The domain Ω is supposed to satisfy the condition (A) of Ladyzhenskaya-Uraltseva [14], i.e., there is a $\theta < 1$ such that

$$\text{meas}\, \{\Omega \cap B_R(x_0)\} \leqq \theta \cdot \text{meas}\, B_R(x_0) \,, \tag{1.3}$$

for all $x_0 \in \partial\Omega$ and all $R \in]0, 1]$. The coefficients $\gamma_{\alpha\beta}$ are measurable, with respect to (x, t), and f is a Carathéodory-function, i.e., it is measurable, with respect to (x, t), and continous, with respect to u and ∇u. Moreover, there are positive constants a and λ such that

$$\lambda \cdot |\eta|^2 \leqq \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot \eta_\alpha \cdot \eta_\beta \leqq \lambda^{-1} \cdot |\eta|^2 , \tag{1.4}$$

$$|f(z, \xi, \eta)| \leqq a \cdot \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot \eta_\alpha \cdot \eta_\beta + b(z) , \tag{1.5}$$

for all $z \in Q$, $\xi \in \mathbf{R}^N$ and $\eta \in \mathbf{R}^{N \cdot n}$ and some function $b \in L^p(Q)$, where

$$p > n . \tag{1.6}$$

The function g is supposed to belong to $L^{2p}(Q)$.

THEOREM 1. *Let u be a weak solution of* (1.1) *and let* $M \geqq 0$, $a^* \in \mathbf{R}$, $z_0 = (x_0, t_0) \in \bar{Q}$ *and* $R > 0$ *be such that*

$$u(z) \cdot f(z, u, \nabla u) \leqq a^* \cdot \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot u_{x_\alpha}(z) \cdot u_{x_\beta}(z) + b(z) , \tag{1.7}$$

$$|u(z)| \leqq M , \tag{1.8}$$

for all $z \in Q_R(z_0) \cap Q$, *and that*

$$a \cdot M + a^* < 2 . \tag{1.9}$$

Then, u is Hölder-continous, in $\overline{Q_{R/2}(z_0)} \cap \bar{Q}$, *under the following conditions.*

1-st *case.*

$$Q_R(z_0) \subset Q . \tag{1.10}$$

In this case,

$$|u(z) - u(z_0)| \leqq c \cdot |z - z_0|^\mu , \tag{1.11}$$

for all $z \in Q_{R/2}(z_0)$, *where the constants* $c_0 \geqq 0$ *and* $\mu > 0$ *can be determined only in dependence on* $n, N, \lambda, a, a^*, M, R, p$ *and upper bounds for* $|b|_{L^p(Q)}$ *and* $|g|_{L^{2p}(Q)}$.

2-nd *case.*

$$(1.12)\qquad u - u_D \in L^2(]0, T[; H_0^{1,2}(\Omega)), \qquad \text{if } B_R(x_0) \not\subset \Omega,$$

$$(1.13)\qquad \lim_{t\to 0} |u(\cdot, t) - u_D(\cdot, 0)|_{L^2(\Omega)} = 0, \qquad \text{if } t_0 < R^2,$$

$$(1.14)\qquad |u_D(z) - u_D(z')| \leqq c_D \cdot |z - z'|^{\mu_D}, \qquad \forall z, z' \in \overline{Q_R(z_0)} \cap \bar{Q},$$

for some function $u_D \in H^{1/2}(Q)$, *some* $c_D \geqq 0$ *and some* $\mu_D > 0$. *In this case,* (1.11) *holds for all* $z \in Q_{R/2}(z_0) \cap \bar{Q}$ *and some constants* $c > 0$ *and* $\mu > 0$ *which depend only on* $n, N, \lambda, a, a^*, M, \theta, R, p, c_D, \mu_D$ *and upper bounds for* $|b|_{L^p(Q)}$ *and* $|g|_{L^{2p}(Q)}$.

REMARKS. Our interior regularity result is sharp, because the function $u(x) = |x|^{-1}\cdot x$ is a weak solution of

$$u_t - \Delta u = u \cdot |\nabla u|^2,$$

for $n \geqq 3$. Here, $a = a^* = M = 1$. This counterexample can be found already in [8]. Moreover, Struwe [16] showed that there exists a weak solution of the Cauchy-Dirichlet problem corresponding to a system of the type (1.1), where (1.3)-(1.6) are satisfied, where the initial and boundary data are smooth and where the solution develops a discontinuity in a finite time.

Higher interior regularity of the weak solutions can be derived from the $C^{1,\mu}$-estimates due to Giaquinta-Struwe [5] and the regularity results for linear equations.

THEOREM 2. *For any function* $u_D \in C^{\mu_D}(\bar{Q}) \cap H^{1,2}(Q)$ $(\mu_D > 0)$, *there exist constants* $\mu > 0$, $T^* \in]0, T]$ *and a weak solution* $u \in C^\mu(\bar{\Omega} \times]0, T^*[)$ *of* (1.1) *which satisfies*

$$(1.15)\qquad u = u_D, \qquad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times]0, T^*]).$$

Moreover, there are lower bounds for T^* *and* μ *and upper bounds for* $|u|_{C^\mu(\bar{\Omega}\times[0,T^*])}$ *which depend only on* $n, N, \lambda, \Omega, a, T, \mu_D$ *and upper bounds for* $|b|_{L^p(Q)}$, $|g_{L^{2p}(Q)}$ *and* $|u_D|_{C^{\mu_D}(\bar{Q})}$.

Combining Theorem 1 and 2 and the Strong Maximum Principle (Proposition 1), one obtains

THEOREM 3. *Suppose that*

$$(1.16)\qquad g = 0, \qquad \text{in } Q,$$

and let $a^* \in \mathbb{R}$, $M \geqq 0$, $\mu_D > 0$ *and* $u_D \in C_{\mu_D}(\bar{Q}) \cap H^{1,2}(Q)$ *such that*

$$a \cdot M + a^* < 2\,, \tag{1.17}$$

$$|u_D(z)| \leqq M\,, \tag{1.18}$$

$$\xi \cdot f(z, \xi, \eta) \leqq a^* \cdot \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot \eta_\alpha \cdot \eta_\beta\,, \tag{1.19}$$

for all $z \in Q$, *all* $\xi \in B_M(0)$ *and all* $\eta \in \mathbb{R}^{n \cdot N}$. *Then, there exists a weak solution* $u \in C^0(\bar{Q})$ *of* (1.1) *which satisfies*

$$u = u_D\,, \quad \text{on } \partial Q \setminus (\Omega \times \{T\})\,, \tag{1.20}$$

$$|u| \leqq M\,, \quad \text{in } Q\,. \tag{1.21}$$

REMARKS. This theorem states that, under conditions (1.16)-(1.19), the Cauchy-Dirichlet problem corresponding to (1.1) is solvable, for all $T > 0$. Here, we leave open the question whether the solution converges, if T tends to infinity, and whether the limit is a solution of the corresponding elliptic system. Moreover, we do not treat the uniquess and stability problem. Therefore, we refer the reader interested in those questions to the works of W. v. Wahl [21], J. Jost [11] and N. Kilimann [12].

Theorem 1 and 2 are easy consequences of more general results concerned with variational inequalities. In order to formulate them, we need some *additional notations*. For any $z \in \bar{Q}$, $K(z)$ is a closed convex nonempty subset of $\mathbb{R}^N$ and

$$K = \{v \in L^\infty(Q) \cap L^2(]0, T[; H^{1,2}(\Omega)) : v(z) \in K(z),\ \forall z \in Q\}\,.$$

A function u solves the variational inequality corresponding to (1.1) and K if and only if

$$u \in K \tag{1.22}$$

and

$$\int_Q -\frac{1}{2} \cdot |u - v|^2 \cdot \psi_t + \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta} \cdot u_{x_\alpha} \cdot \frac{d}{dx_\beta}\{(u - v) \cdot \psi\}\, dx\, dt$$
$$\leqq \int_Q f \cdot (u - v) \cdot \psi + g \cdot \nabla\{(u - v) \cdot \psi\} - v_t \cdot (u - v) \cdot \psi\, dx\, dt\,, \tag{1.23}$$

for all $v \in K \cap H^{1,2}(Q)$ and all nonnegative $\psi \in C_0^\infty(Q)$. For our regularity and existence results, we need a regularity condition on K, namely, that there are constants $c_K \geqq 0$ and $\mu_K > 0$ such that, for every $z_0 = (x_0, t_0) \in \bar{Q}$, every $u_0 \in K(z_0)$ and every $R \in]0, \mu_K]$, there is a $u_K \in \mathbb{R}^N$ satisfying

$$(1.24) \qquad u_R \in K(z)\,, \quad \forall z \in \{z \in \bar{Q}\colon |z - z_0| \leqq R\}\,,$$

$$(1.25) \qquad |u_0 - u_R| \leqq c_K \cdot R^{\mu_K}\,.$$

THEOREM 4. *Let u be a solution of the variational inequality corresponding to* (1.1) *and K. Moreover, suppose that*

$$(1.26) \qquad 0 \in K(z)\,, \quad \forall z \in \overline{Q_R(z_0)} \cap \bar{Q}\,.$$

Then, the Hölder-estimates of Theorem 1 *hold also for u, apart from the fact that the bounds depend additionally on* c_K *and* μ_K.

THEOREM 5. *Let* $a^* \in \mathbb{R}$, $M \geqq 0$, $\mu_D > 0$ *and* $u_D \in C^{\mu_D}(\bar{Q}) \cap H^{1,2}(Q)$ *be such that*

$$(1.27) \qquad a \cdot M + a^* < 2\,,$$

$$(1.28) \qquad u_D \in K\,,$$

$$(1.29) \qquad |u_D(z) - \xi| \leqq M\,,$$

$$(1.30) \qquad \big(\xi - u_D(z)\big) \cdot f(z, \xi, \eta) \leqq a^* \cdot \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot \eta_\alpha \cdot \eta_\beta + b(z)\,,$$

for all $z \in Q$, *all* $\xi \in K(z)$ *and all* $\eta \in \mathbb{R}^{n \cdot N}$. *Then, there is a solution* $u \in C^0(\bar{Q})$ *of the variational inequality corresponding to* (1.1) *and K. Moreover,*

$$(1.31) \qquad u = u_D\,, \quad on\ \partial Q \setminus (\Omega \times \{T\})\,.$$

In order to see that Theorems 1-3 follow from Theorem 4 and 5, we need the following lemmas.

LEMMA 1. *Suppose that u is a solution of the variational inequality corresponding to* (1.1) *and K. Moreover, assume that there are open nonempty sets* $U_0 \subset \mathbb{R}^N$, $Q_0 \subset \mathbb{R}^{n+1}$ *and a closed convex set* $K_0 \subset \mathbb{R}^N$ *such that*

$$(1.32) \qquad u(z) \in K_0 \subset U_0 \subset K(z)\,, \quad \forall z \in Q_0\,.$$

Then, u solves (1.2), *for all* $\varphi \in C_c^\infty(Q_0)$.

LEMMA 2. *A function* $u \in L^\infty(Q) \cap L^2(]0, T[; H^{1,2}(\Omega))$ *is a weak solution of* (1.1) *if and only if it solves* (1.23), *for all* $v \in L^\infty(Q) \cap H^{1,2}(Q)$ *and all nonnegative* $\psi \in C_c^\infty(Q)$.

Theorem 1 is an easy consequence of Lemma 2 and Theorem 4. Hence, let us illustrate how one derives Theorem 2 from Theorem 4 and 5 and Lemma 1. For this, one sets

$$K(z) = \{w \in \mathbf{R}^N\colon\ |w - u_D(z)| \leqq (2\cdot a)^{-1}\}\ .$$

Theorem 5 provides us with a solution $u \in C^0(\bar{Q})$ of the variational inequality corresponding to (1.1) and K which satisfies (1.31). As u is continous up to the initial data, there is a $T^* > 0$ such that

$$|u(x, t) - u_D(x, t)| < (2\cdot a)^{-1}\ , \qquad \forall x \in \bar{\Omega},\ \forall t \in [0, T^*]\ .$$

This, Lemma 1 and a simple partition argument show that u is a weak solution of (1.1), in $\Omega \times]0, T^*[$. The rest of Theorem 2, namely the a-priori bounds for T^* and the Hölder-continuity of u, follow from the Hölder-estimates of Theorem 4.

Before we go into the proofs, let us introduce two notations. For any $c \in \mathbf{R}$,

$$c_+ = \max\{c, 0\}\ , \qquad c_- = \max\{-c, 0\}\ .$$

Moreover, we will frequently use the following

REMARK. Let u be a weak solution of the variational inequality corresponding to (1.1) and K, $Q_0 \subset Q$ be an open set and $\psi \in L^\infty(Q_0) \cap H_0^{1,2}(Q_0)$ be a nonnegative function. Suppose that $v \in H^{1,2}(Q_0)$ satisfies

$$v(z) \in K(z)\ , \qquad \forall z \in Q_0\ . \tag{1.33}$$

Then, (1.23) holds for ψ and v.

PROOF OF THE REMARK. It is sufficient to prove the remark for nonnegative functions

$$\psi \in C_c^\infty(Q_0)\ . \tag{1.34}$$

From (1.24), (1.25) and the assumption that $K(z)$ is nonempty, for all $z \in \bar{Q}$, one derives that there is a $u^* \in K \cap H^{1,2}(Q)$. By (1.34), there is a function

$\varrho \in C_c^\infty(Q_0)$ satisfying

$$\varrho = 1\,, \quad \text{in supp}\,\psi\,. \tag{1.35}$$

Now, we see that (1.23) holds for ψ and

$$v^* = (1-\varrho)\cdot u^* + \varrho\cdot v\,.$$

This and (1.35) show that the remark holds.

PROOF OF LEMMA 1. Let Q_1 be an open subset of Q and let $\psi \in C_c^\infty(Q_1)$ be a nonnegative function such that

$$\text{supp}\,\varphi \subset Q_1 \subset \bar{Q}_1 \subset Q_0\,, \tag{1.36}$$

$$\psi = 1\,, \quad \text{in supp}\,\varphi\,. \tag{1.37}$$

For $(x, t) \in Q_1$, $\delta > 0$ and sufficiently small $\varepsilon > 0$, we set

$$v_1(x,t) = \delta\cdot\varphi(x,t) + \varepsilon^{-1}\cdot\int_t^{t+\varepsilon} u(x,s)\,ds\,,$$

$$v_2(x,t) = \delta\cdot\varphi(x,t) + \varepsilon^{-1}\cdot\int_{t-\varepsilon}^{t} u(x,s)\,ds\,,$$

and note that

$$\begin{aligned}\int_Q v_{1,t}(x,t)\cdot\{u(x,t) - v_1(x,t)\}\cdot\psi(x,t) \,+ \\ + v_{2,t}(x,t)\cdot\{u(x,t) - v_2(x,t)\}\cdot\psi(x,t-\varepsilon)\,dx\,dt \\ = \int_Q u^2(x,t)\cdot\varepsilon^{-1}\cdot\{\psi(x,t-\varepsilon) - \psi(x,t)\}\,dx\,dt \\ + \int_Q \delta\cdot\varphi_t(x,t)\cdot u(x,t)\cdot\{\psi(x,t) + \psi(x,t-\varepsilon)\}\,dx\,dt \\ + \int_Q \tfrac{1}{2}\cdot\{(v_1(x,t))^2\cdot\psi_t(x,t) + (v_2(x,t))^2\cdot\psi_t(x,t-\varepsilon)\}\,dx\,dt\,,\end{aligned} \tag{1.38}$$

By (1.32), we may insert $v_1(x,t)$ together with $\psi(x,t)$ and $v_2(x,t)$ together with $\psi(x,t-\varepsilon)$, into (1.23), provided that $\varepsilon > 0$ and $\delta > 0$ are sufficiently small. This, (1.37), (1.38) and a passage to the limit $\varepsilon \to 0$ show that (1.2) holds for φ.

PROOF OF LEMMA 2. Let u be a weak solution of (1.1) and pick a $v \in L^\infty(Q) \cap H^{1,2}(Q)$ and a nonnegative $\psi \in C_c^\infty(Q)$. For sufficiently small $\varepsilon > 0$, we insert

$$\varphi_\varepsilon(x, t) = \varepsilon^{-2} \cdot \int_{t-\varepsilon}^{t} \int_{s}^{s+\varepsilon} (u-v)(x, s') \cdot \psi(x, s)\, ds'\, ds$$

into (1.2). We note that

$$\int_Q (u - v) \cdot \varphi_{\varepsilon,t}\, dz = \int_Q \left\{ \varepsilon^{-1} \cdot \int_t^{s+\varepsilon} (u - v)(x, s)\, ds \right\}^2 \cdot \psi_t(x, t)\, dx\, dt$$

and let ε tend to zero. In this way, we obtain (1.23). Thus, we have shown one conclusion of Lemma 2. The other one follows from Lemma 1.

2. – Variational inequalities satisfying a one-sided condition.

In this section, we consider a solution u of the variational inequality corresponding to (1.1) and K. In addition to (1.4)-(1.6), (1.24) and (1.25), we suppose that

$$(2.1) \qquad 0 \in K(z), \qquad \forall z \in \overline{Q_R(z_0)} \cap \bar{Q},$$

$$(2.2) \qquad |u| \leq M, \qquad \text{in } Q_R(z_0) \cap Q,$$

$$(2.3) \qquad (u - u_D) \in L^2(]0, T[; H_0^{1,2}(\Omega)), \qquad \text{if } \partial\Omega \cap B_R(x_0) \neq \emptyset,$$

for some $z_0 = (x_0, t_0) \in \bar{Q}$, some $R > 0$, some $M > 0$ and some $u_D \in H^{1/2}(Q)$. Moreover, we assume that a one-sided condition is satisfied. More precisely,

$$(2.4) \qquad u(z) \cdot f(z, u, \nabla u) \leq (1 - \delta^* - a \cdot M \cdot \delta^*) \cdot \sum_{\alpha,\beta=1}^{x} \gamma_{\alpha\beta}(z) \cdot u_{x_\alpha}(z) \cdot u_{x_\beta}(z) + b^*(z),$$

for some $\delta^* > 0$, some $b^* \in L^p(Q)$ and all $z \in Q_R(z_0) \cap Q$. We set

$$k_0 = \begin{cases} 0, & \text{if } \partial\Omega \cap B_R(x_0) = \emptyset, \\ \operatorname*{ess\,sup}_{\partial Q \cap Q_R(z_0)} |u_D|^2, & \text{if } \partial\Omega \cap B_R(x_0) \neq \emptyset. \end{cases}$$

LEMMA 3. *There is a constant $c^* > 0$ depending only on n, N, λ, p and*

δ^* such that

$$
\begin{aligned}
(2.5)\qquad & c^{*-1}\cdot \int_{t_1+\varepsilon_1}^{t_2-\varepsilon_2} \int_{A(k,r_1 t,u_0)} |\nabla|u-u_0|^2|^2 + |\nabla u|^2\cdot\big(|u-u_0|^2-k\big)\,dx\,dt \\
& + \varepsilon_2^{-1}\cdot \int_{t_2-\varepsilon_2}^{t_2} \int_{A(k,r_1,t,u_0)} \big(|u-u_0|^2-k\big)^2\,dx\,dt \\
& \leqq \varepsilon_1^{-1}\cdot \int_{t_1}^{t_1+\varepsilon_1} \int_{A(k,r_2,t,u_0)} \big(|u-u_0|^2-k\big)^2\,dx\,dt \\
& + c^*\cdot M^2\cdot\{|b^*|^2_{L^p(Q)} + M\cdot|b|^2_{L^p(Q)} + |g|^2_{L^{2p}(Q)}\}\cdot\left\{\int_{t_1}^{t_2}\int_{A(k,r_2,t,u_0)} 1\,dx\,dt\right\}^{1-1/p} \\
& + c^*\cdot(r_2-r_1)^{-1}\cdot M^3\cdot|g|^2_{L^{2p}(Q)}\cdot\left\{\int_{t_1}^{t_2}\int_{A(k,r_2,t,u_0)} 1\,dx\,dt\right\}^{1-1/(2p)} \\
& + \Phi\cdot c^*\cdot(r_2-r_1)^{-2}\cdot\int_{t_1}^{t_2}\int_{A(k,r_2 t,u_0)} \big(|u-u_0|^2-k\big)^2\,dx\,dt \\
& + (1-\Phi)\cdot c^*\cdot(r_2-r_1)^{-1}\cdot\int_{t_1}^{t_2}\int_{A(k,r_2,t,u_0)} \big(|u-u_0|^2-k\big)\cdot|\nabla|u-u_0|^2|\,dx\,dt\,,
\end{aligned}
$$

for all $k, t_1, t_2, \varepsilon_1, \varepsilon_2, r_1, r_2, \Phi\in\mathbf{R}^N$ satisfying

$$(2.6)\qquad \max\{0, t_0-R^2\} < t_1 < t_1+\varepsilon_1 < t_2-\varepsilon_2 < t_2 < t_0\,,$$

$$(2.7)\qquad 0 < r_1 < r_2 < R\,,$$

$$(2.8)\qquad u_0 \in K(z)\,,\quad \forall z\in\overline{Q_R(z_0)}\cap\bar{Q}\,,$$

$$(2.9)\qquad |u_0| \leqq \delta^*\cdot M\,,$$

$$(2.10)\qquad k \geqq k_0\,,$$

$$(2.11)\qquad \Phi = 1 \quad\text{or}\quad \Phi = 0\,,$$

where

$$A(k, r, t, u_0) = \{x\in B_r(x_0)\cap\Omega\colon |u(x,t)-u_0|^2 > k\}\,.$$

PROPOSITION 1. Strong Maximum Principle. *Choose an $\varepsilon > 0$ and suppose that*

$$(2.12)\qquad Q_R(z_0)\subset Q\,.$$

Then, one of the following two statements must be true.

(1) *There is a* $u_0 \in \mathbb{R}^N$ *satisfying*

$$|u_0| = M\,, \tag{2.13}$$

$$\{\operatorname{meas} Q_{R/4}(z_0)\}^{-1}\cdot \int_{Q_{R/4}(z_0)} |u-u_0|^2\, dz \leqq \varepsilon \cdot M^2\,. \tag{2.14}$$

(2) *There is a* $\delta > 0$ *depending only on* n, N, λ, p, δ^* *and* ε *such that*

$$\begin{aligned} |u| \leqq (1-\delta)\cdot M + \mu_K^{-1}\cdot M \cdot R + \delta^{-1}\cdot c_K \cdot R^{\mu_K} \\ + \delta^{-1}\cdot\big(|b^*|^{\frac12}_{L^p(Q)} + |g|_{L^{2p}(Q)}\big)\cdot R^{1-(n+2)/(2p)} + \delta^{-1}\cdot |b|_{L^p(Q)}\cdot R^{2-(n+2)/p}\,, \\ \text{in } Q_{\delta\cdot R}(z_0)\,. \end{aligned} \tag{2.15}$$

PROOF OF LEMMA 3. We set

$$\tau(t) = \begin{cases} \varepsilon_1^{-1}\cdot(t-t_1)\,, & \text{if } t \in [t_1, t_1+\varepsilon_1]\,, \\ 1\,, & \text{if } t\in[t_1+\varepsilon_1, t_2-\varepsilon_2]\,, \\ 1-\varepsilon_2^{-1}\cdot(t-t_2+\varepsilon_2)\,, & \text{if } t\in[t_2-\varepsilon_2, t_2]\,, \\ 0\,, & \text{if } t\notin[t_1,t_2]\,, \end{cases}$$

and choose a nonnegative function $\varrho \in C_c^\infty(B_{r_2}(x_0))$ satisfying

$$\begin{aligned} \varrho &= 1\,, && \text{in } B_{r_1}(x_0)\,, \\ |\nabla\varrho| &\leqq c\cdot(r_2-r_1)^{-1}\,, && \text{in } \mathbb{R}^n\,, \end{aligned}$$

for some constant c depending only on n. By (2.6), (2.8) and (2.10), we may insert

$$\begin{aligned} v &= u_0 \\ \psi(x,t) &= \varepsilon^{-1}\cdot\int_{t+\varepsilon}^{t}\Big\{\varepsilon^{-1}\cdot\int_s^{s+\varepsilon}|u(x,s')-u_0|^2\,ds' - k\Big\}_+\cdot\varrho^2(x)\cdot\tau(s)\,ds \end{aligned}$$

into (1.23), for sufficiently small $\varepsilon > 0$. We note that

$$\int_Q |u-v|^2\cdot\psi_t\,dz = \frac{1}{2}\cdot\iint_Q\Big\{\varepsilon^{-1}\cdot\int^{t+\varepsilon}|u(x,s)-u_0|^2\,ds - k\Big\}_+^2 \times\varrho^2(x)\cdot\tau_t(t)\,dx\,dt$$

and let ε tend to zero. Thus, we obtain that

$$-1/4\cdot\int_Q\{|u-u_0|^2-k\}_+^2\cdot\varrho^2\cdot\tau_t\,dz+\int_Q\sum_{\beta,\alpha=1}^n\gamma_{\alpha\beta}\cdot u_{x_\alpha}\cdot u_{x_\beta}\cdot\{|u-u_0|^2-k\}_+\cdot\varrho^2\cdot\tau$$

$$+\frac{1}{2}\cdot\gamma_{\alpha\beta}\cdot\frac{d}{dx_\alpha}\{|u-u_0|^2\}\cdot\frac{d}{dx_\alpha}\{(|u-u_0^2-k)_+\cdot\varrho^2\cdot\tau\}\,dz$$

$$\leqq\int_Q f\cdot(u-u_0)\cdot(|u-u_0|^2-k)_+\cdot\varrho^2\cdot\tau\,dz$$

$$+\int_Q g\cdot\nabla\{(u-u_0)\cdot(|u-u_0|^2-k)_+\cdot\varrho^2\}\cdot\tau\,dz\,.$$

This, (1.4), (1.5), (2.4), (2.9), the properties of ϱ and τ and Young's inequality imply that (2.5) is true.

Proof of Proposition 1. In order to prove Proposition 1, we suppose that

(2.16) $$\operatorname{meas}\{z\in Q_{R/2}(z_0):\ |u(z)|^2>(1-\varrho)\cdot M^2\}\geqq(1-\varrho)\cdot\operatorname{meas}Q_{R/2}(z_0)\,,$$

(2.17) $$R\leqq\mu_K\,,$$

(2.18) $$\varrho^2\cdot M\geqq c_K\cdot R^{\mu_K}+|b|_{L^{2p}(Q)}\cdot R^{2-(n+2)/p}+\left(|g|_{L^{2p}(Q)}+|b^*|^{\frac12}_{L^p(Q)}\right)\cdot R^{1-(n+2)/(2p)}\,,$$

and show that (2.13) and (2.14) are true, if $\varrho>0$ is « sufficiently small ». This proves Proposition 1, because, in the other cases, it is true, trivially or by Lemma 3 and 9.

In the following, c stands for a generic constant depending only on n, N, λ, p and δ^*. From (2.17), (2.18), Lemma 3 (with $\Phi=1$) and the easy fact that

$$|u|^2-(1-2\cdot\sigma)\cdot M^2\geqq\sigma\cdot M^2\,,\quad\text{if } |u|^2\geqq(1-\sigma)\cdot M^2\,,$$

one derives that

(2.19) $$\int_{t_0-R^2/4}^{t_0}\int_{A((1-\sigma)M^2,R/2,t,0)}|\nabla u|^2\,dx\,dt\leqq c\cdot\sigma\cdot M^2\cdot R^n\,,$$

for all $\sigma \in [\varrho, \frac{1}{2}]$. This, (2.16), (2.18) and Hölder's inequality imply that

$$(2.20)\quad \int\limits_{t_2-R^2/4}^{t_0} \int\limits_{A(M^2/2, R/2, t, 0)} |\nabla u|\, dx\, dt \leqq R^{(N+2)/2} \cdot \Big\{ \int\limits_{t_0-R^2/4}^{t_0} \int\limits_{A((1-\varrho)M^2, R/2, t, 0)} |\nabla u|^2\, dx\, dt \Big\}^{\frac{1}{2}}$$

$$+ \big(\text{meas}\,\{z \in Q_{R/2}(z_0): |u(z)|^2 \leqq (1-\varrho)\cdot M^2\}\big)^{\frac{1}{2}} \cdot \Big\{ \int\limits_{t_0-R^2/4}^{t_0} \int\limits_{A(M^2/2, R/2, t, 0)} |\nabla u|^2\, dx\, dt \Big\}^{\frac{1}{2}}$$
$$\leqq c \cdot \varrho^{\frac{1}{2}} \cdot M \cdot R^{N+1}.$$

It is easy to construct a function v satisfying

$$v = u, \qquad \text{if } |u|^2 \geqq \tfrac{3}{4} \cdot M^2,$$
$$v = 0, \qquad \text{if } |u|^2 \leqq \tfrac{1}{2} \cdot M^2,$$
$$\left.\begin{aligned} |v| &\leqq |u|, \\ |\nabla v| &\leqq c \cdot |\nabla u|, \end{aligned}\right\} \text{ in } Q.$$

We set

$$\overline{u}(t) = \big(\text{meas}\, B_{R/2}(x_0)\big)^{-1} \cdot \int\limits_{B_{R/2}(x_0)} v(x, t)\, dx.$$

With the aid of (2.20), the properties of v and Sobolev's embedding theorem, one obtains that

$$(2.21)\quad \int\limits_{Q_{R/2}(z_0)} |u(x, t) - \overline{u}(t)|\, dx\, dt \leqq \int\limits_{Q_{R/2}(z_0)} |u - v| + |v - \overline{u}|\, dz$$
$$\leqq 2 \cdot M \cdot \text{meas}\,\{z \in Q_{R/2}(z_0): |u(z)|^2 \leqq M^2/2\} + c \cdot R \cdot \int\limits_{Q_{R/2}(z_0)} |\nabla v|\, dz \leqq c \cdot \varrho^{\frac{1}{2}} \cdot M \cdot R^{n+2}.$$

An approximation of $u(t)$ by step-functions, (1.24), (1.25), (2.17), (2.18) and (2.21) show that there are a $\overline{u}_0 \in \mathbb{R}^N$, a $t_1 \in [t_0 - R^2/4, t_0 - R^2/8]$ and an $\varepsilon_1 \in [0, R^2/16]$ such that

$$(2.22)\qquad \overline{u}_0 \in K(z), \qquad \forall z \in \overline{Q_R(z_0)},$$

$$(2.23)\qquad \varepsilon_1^{-1} \cdot \int\limits_{t_1}^{t_1+\varepsilon_1} \int\limits_{B_{R/2}(x_0)} |u - \overline{u}_0|\, dz \leqq c \cdot \varrho^{\frac{1}{2}} \cdot M\ R^n.$$

We may suppose that

$$(2.24)\qquad \delta^* \in\,]0, 1/8[\,.$$

Lemma 3 (with $\Phi = 0$), (2.18), (2.20) and (2.22)-(2.24) can be combined to

$$\text{(2.25)} \qquad \operatorname*{ess\,sup}_{t_0 - R^2/16 \leq t \leq t_0} \int_{A((1-\delta^*)^2 M^2, R/4, t, \delta^* \bar{u}_0)} \{|u - \delta^* \cdot \bar{u}_0|^2 - (1-\delta^*)^2 \cdot M^2\}^2 \, dx \leq c \cdot \varrho^{\frac{1}{2}} \cdot M^4 \cdot R^N .$$

Now, we conclude the proof remarking that (2.13) anp (2.14) follow from (2.16) and (2.25), if $\varrho > 0$ is «sufficiently small». This can be easily seen when drawing the balls $B_M(0)$ and $B_{(1-\delta^*)M}(\delta^* \bar{u}_0)$

3. – Proof of the Hölder-estimates.

In this section, we consider a solution u of the variational inequality corresponding to (1.1) and K and we suppose that the hypotheses of Theorem 4 hold.

LEMMA 4. *Assume that*

$$\text{(3.1)} \qquad 0 < R_0 \leq \min\{\mu_R, R\} ,$$

and that

$$\text{(3.2)} \qquad u_0 \in K(z) ,$$

$$\text{(3.3)} \qquad |u(z) - u_0| \leq M_0 ,$$

$$\text{(3.4)} \qquad \big(u(z) - u_0\big) \cdot f(z, u, \nabla u) \leq a_0^* \cdot \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta}(z) \cdot u_{x_\alpha}(z) \cdot u_{x_\beta}(z) + b^*(z) ,$$

for all $z \in \overline{Q_{R_0}(z_0)} \cap \bar{Q}$, *some* $u_0 \in \mathbb{R}^N$, *some* $b^* \in L^p(Q)$, *some* $M_0 > 0$ *and some* $a_0^* \in \mathbb{R}$ *satisfying*

$$\text{(3.5)} \qquad a \cdot M_0 + a_0^* + \delta^* + a \cdot M_0 \cdot \delta^* \leq 2 ,$$

where $\delta^* > 0$. *Choose an* $\varepsilon > 0$. *Then, one of the following two statements must be true.*

(1) *There is a* $\delta > 0$ *and a* $u_1 \in \mathbb{R}^N$ *satisfying*

$$\text{(3.6)} \qquad u_1 \in K(z) , \qquad \forall z \in \overline{Q_{\delta \cdot R_0}(z_0)} \cap \bar{Q} ,$$

$$\text{(3.7)} \qquad |u_1 - u_0| \leq \delta^* \cdot M_0 ,$$

$$\text{(3.8)} \qquad |u - u_1| \leq (1 - \delta^* + \varepsilon) \cdot M_0 , \quad \text{in } Q_{\delta \cdot R_0}(z_0) \cap Q .$$

(2) *There are positive constants* δ *and* ϱ *such that*

$$|u-u_0| \leqq (1-2\cdot\delta)\cdot M_0 + \delta^{-1}\cdot R_0^{\varrho}\,, \qquad in\ Q_{\delta\cdot R_0}(z_0)\cap Q\,. \tag{3.9}$$

Moreover, the constants δ *and* ϱ *occurring in* (3.6), (3.8) *and* (3.9) *depend only on* n, N, λ, p, a, θ, c_D, μ_D, c_K, μ_K, δ^*, ε *and upper bounds for* M_0, $|b|_{L^p(Q)}$, $|b^*|_{L^p(Q)}$ *and* $|g|_{L^{2p}(Q)}$. *In the case that* $Q_{R_0}(z_0)\cap\big(\partial Q\setminus\Omega\times\{T\}\big)=\emptyset$, *the constants* δ *and* ϱ *are independent of* c_D, μ_D *and* θ.

PROOF OF THEOREM 4. By the assumptions of Theorem 4, (3.2)-(3.4) hold for $a_0^*=a$, $b^*=(2\cdot M+1)\cdot b$, $M_0=M$ and for all $z\in\overline{Q_R(z_0)}\cap\bar{Q}$. Moreover, there is a $\delta^*\in\,]0,\tfrac12]$ such that

$$a\cdot M_0 + a_0^* + \delta^* + 2\cdot a\cdot M_0\cdot\delta^* \leqq 2\,. \tag{3.10}$$

We choose an $\varepsilon\in\,]0,\delta^*/2]$ satisfying

$$\varepsilon\cdot\sum_{\nu=1}^{\infty}(1-\delta^*/2)^{\nu}\leqq\delta^*\,. \tag{3.11}$$

Now, let $\delta>0$ and $\varrho>0$ be such that the conclusion of Lemma 4 holds and that

$$\delta^{\varrho}\leqq\tfrac12\leqq 1-\delta\,. \tag{3.12}$$

We determine an R_0 satisfying (3.1) and

$$\delta^{-1}\cdot R_0^{\varrho}\leqq\delta\cdot M_0\,. \tag{3.13}$$

Applying Lemma 4 repeatedly, we obtain a_ν^*, M_ν, R_ν and u_ν such that

$$R_\nu=\delta^\nu\cdot R_0\,, \tag{3.14}$$

$$M_\nu=\begin{cases}(1-\delta^*+\varepsilon)\cdot M_{\nu-1}\,, & \text{if } u_\nu\neq u_{\nu-1}\,,\\ (1-\delta)\cdot M_{\nu-1}\,, & \text{if } u_\nu=u_{\nu-1}\,,\end{cases} \tag{3.15}$$

$$u_\nu=K(z)\,, \qquad \forall\ z\in\overline{Q_{R_\nu}(z_0)}\cap\bar{Q}\,, \tag{3.16}$$

$$|u-u_\nu|\leqq M_\nu\,, \qquad \text{in } Q_{R_\nu}(z_0)\cap Q\,, \tag{3.17}$$

$$a_\nu^*=\begin{cases}a_{\nu-1}^*+(\varepsilon+\delta^*)\cdot a\cdot M_{\nu-1}\,, & u_\nu\neq u_{\nu-1}\,,\\ a_{\nu-1}^*\,, & \text{if } u_\nu=u_{\nu-1}\,,\end{cases} \tag{3.18}$$

$$(u-u_\nu)\cdot f(\cdot,u,\nabla u)\leqq a^*\cdot\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\cdot u_{x_\alpha}\cdot u_{x_\beta}+b^*\,, \qquad \text{in } Q_{R_\nu}(z_0)\cap Q\,. \tag{3.19}$$

Moreover, by (3.10)-(3.13), (3.15) and (3.18),

$$a \cdot M + a_\nu^* + \delta^* + a \cdot M_\nu \cdot \delta^* \leqq 2 , \tag{3.20}$$

$$\delta^{-1} \cdot R_\nu^\varrho \leqq \delta \cdot M_\nu . \tag{3.21}$$

Therefore, we can apply Lemma 4 as often as we want, in order to obtain (3.14), (3.15) and (3.17), for all $\nu \in \mathbb{N}$. This obviously implies (1.11). Thus, we can conclude the proof remarking that the dependence on the constants is just as stated in Theorem 4.

PROOF OF LEMMA 4. In this proof, we suppose that

$$c_K \cdot R^0_{\mu_K} \leqq \varepsilon/8 \cdot M_0 , \tag{3.22}$$

because, in the other case, (3.9) is true, trivially. Moreover, we may assume that

$$\varepsilon \leqq \delta^* . \tag{3.23}$$

We have to distinguish between three cases.

1-st *case*. For some $\delta_1 \in]0, 1]$ to be determined later on,

$$t_0 \in [0, \delta_1 \cdot R_0^2] . \tag{3.24}$$

In this case, we set

$$\bar{u}_1 = u_0 + (\delta^* - \varepsilon/8) \cdot \big(u_D(x_0, 0) - u_0\big) .$$

By (1.24), (1.25), (3.1), (3.22) and (3.23), there is a $u_1 \in \mathbb{R}^N$ satisfying

$$|u_1 - \bar{u}_1| \leqq \varepsilon/8 \cdot M_0 ,$$

and (3.6) and (3.7), for all $z \in \overline{Q_{R_0}(z_0)} \cap \bar{Q}$. The assumption (1.14) implies that

$$|u_D - u_1| \leqq (1 - \delta^* + \varepsilon/4) \cdot M_0 + c_D \cdot R^0_{\mu_D} , \quad \text{on } \partial Q \cap Q_{R_0}(z_0) .$$

Hence, by (1.12), (1.13) and Lemma 3, the function

$$v(x, t) = \begin{cases} \big\{|u(x + x_0, t + t_0) - u_1|^2 - \{(1 - \delta^* + \varepsilon/4) \cdot M_0 + c_D \cdot R_0^{\mu_D}\}^2\big\}_+ , & \text{if } t \geqq -t_0 , \\ 0 , & \text{if } t \leqq -t_0 , \end{cases}$$

satisfies the hypotheses of Lemma 8. Moreover,

$$\int_{Q_{R_0}(z_0)} v^2\, dz \leqq \delta_1 \cdot M^4 \cdot \operatorname{meas} Q_{R_0}(z_0)\,.$$

Therefore, we can find a $\delta_1 > 0$ depending only on n, N, λ, p, a, c_D, μ_D, c_K, μ_K, δ^*, ε and upper bounds for M_0, $|b|_{L^p(Q)}$, $|b^*|_{L^p(Q)}$ and $|g|_{L^{2p}(Q)}$ such that

$$|u - u_1| \leqq (1 - \delta^* + \varepsilon/2)\cdot M_0 + \delta_1^{-1}\cdot R_0^{\delta_1}\,, \qquad \text{in } Q_{R_0/2}(z_0) \cap Q\,. \tag{3.25}$$

If $\varepsilon/2 \cdot M_0 \leqq \delta_1^{-1}\cdot R_0^{\delta_1}$, then (3.8) follows from (3.25). In the other case, (3.9) is true, trivially.

2-nd *case*. For the same δ_1 as above and for some $\delta_2 > 0$ to be determined later on,

$$t_0 \geqq \delta_1 \cdot R_0^2\,, \tag{3.26}$$

$$B_{\delta_2\cdot R_1}(x_0) \cap \partial\Omega \neq \emptyset\,, \tag{3.27}$$

where

$$R_1 = \begin{cases} R_0\,, & \text{if } t_0 \geqq R_0^2\,,\\ \delta_1\cdot R_0\,, & \text{if } t_0 \in [\delta_1\cdot R_0^2, R_0^2]\,. \end{cases}$$

In this case, we pick an $x_1 \in B_{\delta_2\cdot R_1}(x_0) \cap \partial\Omega$ and set

$$\bar{u}_2 = u_0 + (\delta^* - \varepsilon/8)\cdot\big(u_D(x_1, t_0) - u_0\big)\,.$$

Similarly as above, we find a $u_1 \in \mathbb{R}^N$ satisfying (3.6) and (3.7) and a $\delta_2 > 0$ depending only on n, N, λ, p, θ, a, c_D, μ_D, c_K, μ_K, δ^*, ε and upper bounds for M_0, $|b|_{L^p(Q)}$, $|b^*|_{L^p(Q)}$ and $|g|_{L^{2p}(Q)}$ such that

$$|u - u_1| \leqq (1 - \delta^* + \varepsilon/2)\cdot M_0 + \delta_2^{-1}\cdot R_0^{\delta_2}\,, \tag{3.28}$$
$$\text{in } Q_{2\delta_2 R_1}(x_1, t_0) \cap Q \supset Q_{\delta_2 R_1}(z_0) \cap Q\,.$$

The only difference is that one has to use Lemma 3 and 9 and (1.3), repeatedly, instead of Lemma 3 and 8. Just as in the 1-st case, (3.28) implies that (3.8) or (3.9) must be true.

3-rd *case*. For the δ_1, δ_2 and R_1 determined above, (3.26) holds and

$$B_{\delta_2\cdot R_1}(x_0) \cap \partial\Omega = \emptyset\,. \tag{3.29}$$

In this case, we set

$$R_2 = \begin{cases} \delta_2 \cdot R_1 , & \text{if } B_{R_1}(x_0) \cap \partial\Omega \neq \emptyset , \\ R_1 , & \text{if } B_{R_1}(x_0) \cap \partial\Omega = \emptyset . \end{cases}$$

We suppose that there is a $\bar{u}_1 \in \mathbb{R}^N$ and a $\sigma > 0$ such that

$$\int_{Q_{R_2/4}} |u - u_0 - \bar{u}_1|^2 \, dz \leqq \sigma \cdot M_0^2 \cdot R_2^{n+2} , \tag{3.30}$$

$$|u_0 - u_1| = M_0 , \tag{3.31}$$

$$\varepsilon/4 \cdot M_0 \geqq \sigma^{-1} \cdot R_2^{\sigma} , \tag{3.32}$$

and we show that there is a $u_1 \in \mathbb{R}^N$ satisfying (3.6)-(3.8), provided that $\sigma > 0$ is « sufficiently small ». This proves the conclusion of Lemma 4, in the third case, because, otherwise, one can apply the Strong Maximum Principle (Proposition 1) to $(u - u_0)$, in order to obtain (3.9). By (1.24), (1.25), (3.1), (3.22) and (3.23), there is a $u_1 \in \mathbb{R}^N$ satisfying (3.6) and (3.7), for all $z \in \overline{Q_{R_2}(z_0)}$, and

$$|u_1 - u_0 - (1 - \delta^* + \varepsilon/8) \cdot (\bar{u}_1 - u_0)| \leqq \varepsilon/8 \cdot M_0 . \tag{3.33}$$

From (3.30), (3.31) and (3.33), one derives that

$$\int_{Q_{R_2/4}} \{|u - u_1|^2 - (1 - \delta^* + \varepsilon/4)^2 \cdot M_0^2\}_+^2 \, dz \leqq \sigma \cdot M_0^4 \cdot R_0^{n+2} . \tag{3.34}$$

Now, we conclude the proof remarking that (3.9) follows from (3.6), (3.7), (3.32), (3.34) and Lemma 3 and 8, provided that $\varrho > 0$ is « sufficiently small ».

4. – Proof of the existence result.

In this section, we suppose that the assumptions of Theorem 5 hold and we prove its existence conclusion.

For each $e \in S^{n-1}$ and each $z \in \bar{Q}$, there is a uniquely determined $\psi_e(z) \in \mathbb{R}$ such that

$$\big(w - e \cdot \psi_e(z)\big) \cdot e \leqq 0 , \quad \forall w \in K(z) , \tag{4.1}$$

$$\big(w_0 - e \cdot \psi_e(z)\big) \cdot e = 0 , \quad \text{for some } w_0 \in K(z) . \tag{4.2}$$

Let $(e_\nu)_{\nu \in \mathbb{N}}$ be a sequence of vectors which is dense in S^{n-1}. For each $\nu \in \mathbb{N}$

and each $k \in \{1, 2, \ldots, \nu\}$, there is function $\psi_{\nu k} \in C^\infty(\bar{Q})$ such that

(4.3) $$\psi_{e_k}(z) \leqq \psi_{\nu k}(z) \leqq \psi_{e_k}(z) + \nu^{-1}, \qquad \forall z \in \bar{Q}.$$

We set

$$K_\nu(z) = \{w \in \mathbf{R}^N: (w - e_k \cdot \psi_{\nu k}(z)) \cdot e_k \leqq 0, \text{ for } k = 1, 2, \ldots, \nu\},$$

$$K_\nu \quad = \{v \in L^2(]0, T[; H^{1,2}(\Omega)): v(z) \in K_\nu(z), \ \forall z \in \bar{Q}\}.$$

We can choose the functions $\psi_{\nu k}$, the vectors e_ν and the constants c_K and μ_K in such a way that the regularity conditions (1.24) and (1.25) hold also for the sets K_ν. Moreover, there is a $\nu_0 \in \mathbf{N}$ and an $M_0 > 0$ such that

(4.4) $$|w - u_D(z)| \leqq M_0, \qquad \forall z \in \bar{Q}, \ \forall \nu \geqq \nu_0, \ \forall w \in K_\nu(z).$$

A convolution argument shows that, for each $\nu \in \mathbf{N}$, there are functions $\gamma^\nu_{\alpha\beta} \in C^\infty(\bar{Q})$ which satisfy (1.4) and (1.5), with the same constant λ as the coefficients $\gamma_{\alpha\beta}$, and

(4.5) $$\lim_{\nu \to \infty} \gamma^\nu_{\alpha\beta}(z) = \gamma_{\alpha\beta}(z),$$

for almost all $z \in Q$. For μ, $\varrho \in \mathbf{N}$, we set

$$f_\mu(z, \xi, \eta) = (1 + \mu^{-1} \cdot |\eta|^2)^{-1} \cdot f(z, \xi, \eta),$$

$$f_{\mu\varrho}(z, v) \quad = f_\mu(z, v, \nabla_\varrho v),$$

where v is an arbitrary function defined in Q and

$$\nabla_\varrho v(x, t) = \begin{cases} \varrho \cdot \{v(x + \varrho^{-1} \cdot \bar{x}_\alpha, t) - v(x, t)\}_{\alpha = 1, 2, \ldots, n}, & \text{if } \operatorname{dist}(x, \partial\Omega) > \varrho^{-1}, \\ 0, & \text{if } \operatorname{dist}(x, \partial\Omega) \leqq \varrho^{-1} \end{cases}$$

and where $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_R$ are the standard unit vectors of $\mathbf{R}^n$. Moreover, (g_μ) is a sequence of $C^\infty_c(Q)$-functions satisfying

(4.6) $$\lim_{\mu \to \infty} g_\mu = g,$$

strongly in the sense of $L^{2p}(Q)$. For the proof of Theorem 5, we need the following

LEMMA 5. *Pick μ, $\varrho \in \mathbf{N}$ and a $\nu \geqq \nu_0$. Then, there is a function $u \in K_\nu$*

$\cap\, C^0(\bar{Q})$ *such that*

$$u = u_D\,, \quad on\ \partial\Omega\setminus(\Omega\times\{T\})\,, \tag{4.7}$$

$$u_t \in L^2(]0, T[; H^{-1}(\Omega))\,, \tag{4.8}$$

$$\int_0^T \langle u_t, u - v\rangle\, dt + \int_Q \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta}\cdot u_{x_\alpha}\cdot(u-v)_{x_\beta}\, dz \leqq \int_Q f_{\mu\varrho}(z, u)\cdot(u-v) - \nabla g_\mu\cdot(u-v)\, dz\,, \tag{4.9}$$

for all $v \in K_\nu \cap C^0(\bar{Q})$ *satisfying*

$$v = u_D\,, \quad on\ \partial\Omega\times]0, T[\,. \tag{4.10}$$

Here and in the following, $\langle\ ,\ \rangle$ are the duality brackets between $H^{-1}(\Omega)$ and $H_0^{1,2}(\Omega)$.

Proof of Theorem 5. In the following, the solutions provided by Lemma 5 will be denoted by $u_{\mu,\varrho,\nu}$. It is easily checked that the functions $u_{\mu,\varrho,\nu}$ satisfy (1.23), for $\gamma_{\alpha\beta} = \gamma^\nu_{\alpha\beta}$, $f = f_{\mu\varrho}$, $g = g_\mu$, all $v \in K_\nu \cap H^{1,2}(Q)$ and all nonnegative $\psi \in C^\infty(\bar{Q})$ satisfying

$$(u_D - v)\cdot\psi \in H_0^{1,2}(\Omega)\,. \tag{4.11}$$

Therefore, we can use (4.4), Theorem 4 and Ascoli's theorem in order to obtain sequences $(\nu_{\mu\varrho})$ with the following properties. For each $\varrho \geqq 2$, $(\nu_{1\varrho})$ is a subsequence of $(\nu_{\varrho-1,\varrho-1})$ and, for each $\mu \geqq 2$ and $\varrho \geqq \mu$, $(\nu_{\mu\varrho})$ is a subsequence of $(\nu_{\mu-1,\varrho})$. Moreover, for each $\mu \in \mathbb{N}$ and each $\varrho \geqq \mu$,

$$\lim_{\nu_{\mu\varrho}\to\infty} u_{\mu,\varrho,\nu_{\mu\varrho}} = u_{\mu,\varrho}\,, \tag{4.12}$$

in the sense of $C^0(\bar{Q})$. Inserting $v = u_D$ into (4.9), we obtain a bound for

$$\int_Q |\nabla u_{\mu,\varrho,\nu}|^2\, dz$$

which is independent of $\nu \geqq \nu_0$. Hence, (4.12) holds also weakly in the sense of $L^2(]0, T[; H^{1,2}(\Omega))$. From (4.9), one derives that

$$\int_Q \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta}\cdot(u^1 - u^2)_{x_\alpha}\cdot(u^1 - u^2)_{x_\beta}\, dz \leqq \int_Q (f_{\mu_1,\varrho_1} - f_{\mu_2,\varrho_2} - \nabla g_{\mu_1} + \nabla g_{\mu_2})\cdot(u^1 - u^2)\, dz\,, \tag{4.13}$$

for

$$u^i = u_{\mu_i,\varrho_i} .$$

If a sequence (v^ν) of functions satisfies

$$\lim_{\nu\to\infty} v^\nu = v ,$$

weakly in the sense of $L^2(]0, T[; H^{1,2}(\Omega))$, then

$$\int_Q \sum_{\alpha,\beta=1}^n \gamma_{\alpha\beta}\cdot v_{x_\alpha}\cdot v_{x_\beta}\, dz \leqq \liminf_{\nu\in\mathbf{N}} \int_Q \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta}\cdot v^\nu_{x_\alpha}\cdot v^\nu_{x_\beta}\, dz .$$

This argument shows that $u_{\mu,\varrho}\in K$ solves (4.7) and (1.23), for $f = f_{\mu,\varrho}$, $g = g_\mu$, all $v\in K\cap H^{1,2}(Q)$ and all nonnegative $\psi\in C^\infty(\bar{Q})$ satisfying (4.11). Now, we can use the regularity result of Theorem 4 once again in order to obtain sequences (ϱ_μ) such that, for each $\mu\geqq 2$, (ϱ_μ) is a subsequence of $(\varrho_{\mu-1})$ and that

(4.14) $$\lim_{\varrho_\mu\to\infty} u_{\mu,\varrho_\mu} = u_\mu ,$$

in the sense of $C^0(\bar{Q})$. For

$$u^i = u_{\mu_i,\varrho_i} ,$$

an inequality similar to (4.13) is valid. Hence, (4.14) holds also strongly in the sense of $L^2(]0, T[; H^{1,2}(\Omega))$. This implies that

$$u^i = u_{\mu_i}$$

satisfy an inequality similar to (4.13), too. In addition to that, u_μ solves (4.7) and (1.23), for $f = f_{\mu'}$, $g = g_{\mu'}$, all $v\in K\cap H^{1,2}(Q)$ and all nonnegative $\psi\in C^\infty(\bar{Q})$ satisfying (4.11). Inserting $v = u_D$ and $\psi = 1$ into (1.23), we obtain a bound for

$$\int_Q |\nabla u_\mu|^2\, dz$$

which is independent of μ. Similarly as above, one sees that there is a subsequence $(u_{\mu'})$ of (u_μ) such that

$$\lim_{\mu'\to\infty} u_{\mu'} = u ,$$

in the sense of $C^0(\bar{Q})$ and strongly in the sense of $L^2(]0, T[; H^{1,2}(\Omega))$. Now, we conclude the proof remarking that this function u has all the properties stated in Theorem 5.

For the proof of Lemma 5, we need the following

LEMMA 6. *Pick a $\nu \geqq \nu_0$ and a function $h \in L^\infty(Q)$. Then, there is a function $u \in K_\nu \cap C^0(\bar{Q})$ such that* (4.7) *and* (4.8) *hold and that*

$$(4.15) \qquad \int_0^T \langle u_t, u - v\rangle \, dt + \int_Q \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta} \cdot u_{x_\alpha} \cdot (u - v)_{x_\alpha} \, dz \leqq \int_Q h \cdot (u - v) \, dz ,$$

for all $v \in K_\nu \cap C^0(\bar{Q})$ satisfying (4.10).

PROOF OF LEMMA 5. By Lemma 6, for each function $w \in C^0(\bar{Q})$, there is a function $u = Tw \in K_\nu \cap C^0(\bar{Q})$ such that (4.7) and (4.8) hold and that

$$(4.16) \qquad \int_0^T \langle u_t, u - v\rangle \, dt + \int_Q \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta} \cdot u_{x_\alpha} \cdot (u - v)_{x_\beta} \, dz \leqq \int_Q f_{\mu\varrho}(z, w) \cdot (u - v) - \nabla g_\mu \cdot (u - v) \, dz ,$$

for all $v \in K_\nu \cap C^0(\bar{Q})$ satisfying (4.10). It is easily checked that the function u is uniquely determined. To u, we can apply the Hölder-estimates of Theorem 4. Thus, the mapping T satisfies the hypotheses of the extension of the Brouwer fixed point theorem to Banach spaces (cf. [6; Corollary 10.2]) which implies the existence result of Lemma 5.

For the proof of Lemma 6, we pick a $\nu \geqq \nu_0$ and a function $h \in L^\infty(Q)$ and we introduce a further notation. Namely, for each $z \in \bar{Q}$ and each $u' \in \mathbb{R}^N$, there is a uniquely determined $P(z, u') \in K_\nu(z)$ such that

$$(4.17) \qquad |u' - P(z, u')| = \inf \{|u' - u''| : u'' \in K_\nu(z)\} .$$

Moreover, we need the following

LEMMA 7. *Pick an $\varepsilon > 0$. Then, there is a function $u \in C^0(\bar{Q}) \cap L^2(]0, T[; H^{1,2}(\Omega))$ such that* (4.7) *and* (4.8) *hold and that*

$$(4.18) \qquad u_t - \sum_{\alpha,\beta=1}^n \frac{d}{dx_\beta} (\gamma^\nu_{\alpha\beta} \cdot u_{x_\alpha}) + \varepsilon^{-1} \cdot (u - P(z, u)) = h ,$$

in the sense of $L^2(]0, T[; H^{-1}(\Omega))$.

PROOF OF LEMMA 6. There is a constant $c_\nu > 0$ such that, for each $z \in \bar{Q}$ and each $u' \in \mathbb{R}^N$, there is a $k \in \{1, 2, \dots, \nu\}$ satisfying

$$P(z, u') \in \{u'' \in K_\nu(z)\colon (u'' - \psi_{\nu k}(z)\cdot e_k \cdot e_k) = 0\}\,, \tag{4.19}$$

$$c_\nu \cdot |u'' - P(z, u')| \leqq (u' - \psi_{\nu k}(z)\cdot e_k)\cdot e_k \leqq |u' - P(z, u')|\,. \tag{4.20}$$

We pick an $\varepsilon \in]0, 1]$ and a $k \in \{1, 2, \dots, \nu\}$. By u_ε, we denote the solution provided by Lemma 7. As

$$(u_\varepsilon - P(z, u))\cdot e_k \geqq (u_\varepsilon - \psi_{\nu k}(z)\cdot e_k)\cdot e_k\,, \quad \text{in } Q\,, \tag{4.21}$$

the function

$$v(z) = u_\varepsilon(z)\cdot e_k - \psi_{\nu k}(z)$$

satisfies

$$v_t - \sum_{\alpha,\beta=1}^{n} \frac{d}{dx_\beta}\{\gamma^\nu_{\alpha\beta}\cdot v_{x_\alpha}\} + \varepsilon^{-1}\cdot v \leqq h + \sum_{\alpha,\beta=1}^{n} \frac{d}{dx_\beta}\{\gamma^\nu_{\alpha\beta}\cdot \psi_{\nu k, x_\alpha}\} - \psi_{\nu k, t}\,, \tag{4.22}$$

in the sense of $L^2(]0, T[; H^{-1}(\Omega))$, and

$$v \leqq 0\,, \quad \text{on } \partial\Omega \setminus (\Omega \times \{T\})\,. \tag{4.23}$$

Inserting $(v_+)^i$ as test functions into (4.22), for $i = 1, 2, \dots, n$, one obtains a bound for

$$\varepsilon^{-n-1}\cdot \int_Q (v_+)^{n+1}\, dz$$

which is independent of ε. This, (4.15) and (4.20) show that there is a constant c independent of ε such that

$$\varepsilon^{-n-1}\cdot \int_Q |u_\varepsilon - P(z, u_\varepsilon)|^{n+1}\, dz \leqq c\,. \tag{4.24}$$

From this and the equation (4.18), one easily derives that one can choose the constant c in such a way that

$$|u_\varepsilon|_{L^2(]0,T[;\,H^{1,2}(\Omega))} + |u_{\varepsilon,t}|_{L^2(]0,T[;\,H^{-1}(\Omega))} \leqq c\,. \tag{4.25}$$

Moreover, by (4.24) and Lemma 3 and 8, one can choose the constant c in such a way that

$$|u_\varepsilon|_{L^\infty(Q)} \leqq c\,. \tag{4.26}$$

Now, we can use the regularity result of Theorem 1 and Ascoli's theorem in order to obtain a sequence (ε_ν) tending to zero such that

$$\lim_{\nu\to\infty} u_{\varepsilon_\nu} = u \,, \tag{4.27}$$

in the sense of $C^0(\bar{Q})$ and weakly in the sense of $L^2(]0, T[; H^{1,2}(\Omega))$ and that

$$\lim_{\nu\to\infty} u_{\varepsilon_\nu,t} = u_t \,, \tag{4.28}$$

weakly in the sense of $L^2(]0, T[; H^{-1}(\Omega))$. The functions u_ε solve

$$\frac{1}{2}\cdot\int_Q |u_\varepsilon - v|^2(x, T) - |u_\varepsilon - v|^2(x, 0)\, dx + \int_\Omega \sum_{\alpha,\beta=1}^n \gamma^\nu_{\alpha\beta}\cdot u_{\varepsilon,x_\alpha}\cdot(u_\varepsilon - v)_{x_\beta}\, dz \leq \int_Q (h - v_t)\cdot(u_\varepsilon - v)\, dz \,, \tag{4.29}$$

for all $v \in K_\nu \cap H^{1,2}(\Omega)$ satisfying

$$v - u_D \in L^2(]0, T[; H_0^{1,2}(\Omega)) \,. \tag{4.30}$$

Hence, by (4.27) and the weak lower semicontinuity of quadratic integrals, the function u solves (4.29) and (4.15), for all $v \in K_\nu \cap H^{1,2}(Q)$ satisfying (4.30). Now, we can conclude the proof remarking that Lemma 6 follows from the above considerations and a simple approximation argument which has to be applied to the test functions v.

PROOF OF LEMMA 7. We pick a $v \in C^0(\bar{Q})$ and a $\sigma \in [0, 1]$. From the theory of linear parabolic equations (cf. [20; Theorem 40.1]), we know that there is a uniquely determined function $u = Tv \in L^2(]0, T[; H^{1,2}(\Omega))$ satisfying (4.7) and

$$(u - \sigma\cdot u_D) \in L^2(]0, T[; H_0^{1,2}(\Omega)) \,, \tag{4.31}$$

$$\lim_{t\to 0} u(\cdot, t) = u_D(\cdot, 0) \,, \tag{4.32}$$

in the sense of $L^2(\Omega)$ and

$$u_t - \sum_{\alpha,\beta=1}^n \frac{d}{dx_\beta}\{\gamma^\nu_{\alpha\beta}\cdot u_{x_\alpha}\} + \sigma\cdot\varepsilon^{-1}\cdot(v - P(z, v)) = \sigma\cdot h \,, \tag{4.33}$$

in the sense of $L^2(]0, T[; H^{-1}(\Omega))$. We choose a $U_D > 0$ satisfying

$$U_D \geqq |u_D|_{L^\infty(Q)} + \sup\{|w|: w \in K_\nu(z), z \in \bar{Q}\}, \tag{4.34}$$

and set

$$b(x, t) = \exp[\varkappa \cdot t] + U_D,$$

for $\varkappa > 0$. From (4.31)-(4.34), one derives that

$$\int_Q \sum_{\alpha,\beta=1}^n \gamma_{\alpha\beta}^\nu \cdot (u\cdot e - b)_{+,x_\alpha} \cdot (u\cdot e - b)_{+,x_\beta}\, dz \leqq \int_Q \{|h| - \sigma\cdot\varepsilon^{-1}\cdot(v - P(z, v))\cdot e - \varkappa\}\cdot(u\cdot e - b)_+\, dz, \tag{4.35}$$

for all $e \in S^{n-1}$. Choosing $\varkappa$ sufficiently large in (4.35), one obtains an $L^\infty(Q)$-bound for u in dependence on $|v|_{C^0(\bar{Q})}$. This and the Hölder-estimates of Theorem 1 imply that $T\colon [0, 1]\times C^0(\bar{Q}) \to C^0(\bar{Q})$ is a compact and continous mapping.

We note that

$$(w - P(z, w))\cdot e \geqq 0,$$

for all $z \in \bar{Q}$, all $e \in S^{n-1}$ and all $w \in \mathbb{R}^N$ satisfying

$$w\cdot e \geqq U_D.$$

Hence, one can use (4.35) in order to show that

$$|u|_{C^0(\bar{Q})} \leqq c,$$

for some constant c and all $\sigma \in [0, 1]$ and all $u \in C^0(\bar{Q})$ satisfying

$$u = T(\sigma, u).$$

Now, we can conclude the proof remarking that Lemma 7 follows from the Leray-Schauder fixed point theorem (cf. [6; Theorem 10.6]).

5. – Analytic tools.

In this section, we prove the refinements of the techniques due to De Giorgi [3], Ladyzhenskaya-Uraltseva [14] and Ladyzhenskaya-Solonnikov-Uraltseva [13]. For this, we consider a nonnegative function

$v \in L^2(]-R^2,\ 0[;\ H^{1,2}(B_R(0)))$ satisfying

$$(5.1)\qquad c^{*-1}\cdot \int_{t_1+\varepsilon_1}^{t_2-\varepsilon_2} \int_{A(k,r_1,t)} |\nabla v|^2\, dx\, dt + \varepsilon_2^{-1}\cdot \int_{t_2-\varepsilon_2}^{t_2} \int_{A(k,r_1,t)} (v-k)^2\, dx\, dt$$

$$\leqq \varepsilon_1^{-1}\cdot \int_{t_1}^{t_1+\varepsilon_1} \int_{A(k,r_2,t)} (v-k)^2\, dx\, dt + k^{*2}\cdot (r_2-r_1)^{-2}\cdot R^{(n+2)\cdot\alpha}\cdot \left\{\int_{t_1}^{t_2} \int_{A(k,r_2,t)} 1\, dx\, ds\right\}^{1-\alpha}$$

$$+ \Phi\cdot c^*\cdot (r_2-r_1)^{-2}\cdot \int_{t_1}^{t_2} \int_{A(k,r_2,t)} (v-k)^2\, dx\, dt$$

$$+ (1-\Phi)\cdot c^*\cdot (r_2-r_1)^{-1}\cdot \int_{t_1}^{t_2} \int_{A(k,r_2,t)} (v-k)\cdot |\nabla v|\, dx\, dt\,,$$

for some $c^* > 0$, some $k^* > 0$, some

$$(5.2)\qquad \alpha \in [0, 1/n[\,,$$

for all $k \geqq 0$ and all $t_1, t_2, \varepsilon_1, \varepsilon_2, r_1, r_2, \Phi \in R$ satisfying

$$-R^2 < t_1 < t_1 + \varepsilon_1 < t_2 - \varepsilon_2 < t_2 < 0\,,$$
$$0 < r_1 < r_2 < R\,,$$
$$\Phi = 0 \quad \text{or} \quad \Phi = 1\,,$$

where

$$A(k, r, t) = \{x \in B_r(0)\colon v(x, t) > k\}\,.$$

LEMMA 8. *There is a constant c depending only on n, α and c^* such that*

$$(5.3)\qquad v^2 \leqq c\cdot R^{-n-2}\cdot \int_{Q_R(0)} v^2\, dz + k^{*2}\,, \qquad \text{in } Q_{R/2}(0)\,.$$

LEMMA 9. *Suppose that*

$$(5.4)\qquad v \leqq M\,, \qquad \text{in } Q_R(0)\,,$$

$$(5.5)\qquad \operatorname{meas}\{z \in Q_{R/2}(0)\colon v(z) > 0\} \leqq (1-\delta)\cdot \operatorname{meas} Q_{R/2}(0)\,,$$

for some $M \geqq 0$ *and some* $\delta \in]0, \frac{1}{2}]$. *Then, there is an* $\varepsilon > 0$ *depending only on* n, α, c^* *and* δ *such that*

$$v \leqq (1-\varepsilon)\cdot M + \varepsilon^{-1}\cdot k^*\,, \quad \text{in } Q_{\varepsilon\cdot R}(0)\,. \tag{5.6}$$

PROOF OF LEMMA 8. A simple stretching argument shows that it is sufficient to prove Lemma 8, for one particular R. Therefore, we may suppose that

$$R = 1\,. \tag{5.7}$$

By g, we denote a generic constant depending only on n, α and c^*. We pick a $K > 0$ satisfying

$$K^2 \geqq \max\left\{k^{*2}, \int_{Q_1(0)} v^2\, dz\right\}, \tag{5.8}$$

and set

$$\begin{aligned} r_i &= \tfrac{1}{2} + 2^{-i-1}\,,\\ t_i &= -\,r_i^2\,,\\ k_i &= (1-2^{-i})\cdot K\,,\\ Q_i &= Q_{r_i}(0)\,,\\ A_i(t) &= A(k_i, r_i, t)\,. \end{aligned}$$

In order to prove Lemma 8, we verify the recursion formula

$$K^{-2}\cdot\int_{Q_{i+3}} (v-k_{i+3})^2\, dz \leqq g^i\cdot\left\{K^{-2}\cdot\int_{Q_i}(v-k_i)^2\, dz\right\}^{1-\alpha+1/n}. \tag{5.9}$$

By (5.2),

$$1-\alpha+1/n > 1\,.$$

Thus, an elementary calculation or Lemma 2.4.7 of [14] show that (5.9) implies (5.3), if K is « sufficiently large ».

For the proof of (5.9), we choose a nonnegative function $\psi \in C_c^\infty(B_{r_{i+2}}(0))$ satisfying

$$\begin{aligned} \psi &= 1\,, && \text{in } B_{r_{i+3}}(0)\,,\\ |\nabla\psi| &\leqq g\cdot 2^i\,, && \text{in } B_{r_{i+2}}(0)\,. \end{aligned}$$

Now, we use Sobolev's imbedding theorem and Hölder's and Young's inequality to obtain that

$$(5.10)\quad \int_{Q_{i+3}} (v - k_{i+3})^2\, dz$$

$$\leqq \quad (\operatorname{meas} Q_{i+2})^{1/n} \cdot \left\{ \int_{Q_{i+2}} \{(v - k_{i+2})^2 \cdot \psi^2\}^{n/(n-1)}\, dz \right\}^{(n-1)/n}$$

$$\leqq g \cdot (\operatorname{meas} Q_{i+2})^{1/n} \cdot \left\{ \int_{t_{i+2}}^{0} \left\{ \int_{A_{i+2}(t)} |\nabla(|v - k_{i+2}|^2 \cdot \psi^2)|\, dx \right\}^{n/(n-1)} dt \right\}^{(n-1)/n}$$

$$\leqq g \cdot (\operatorname{meas} Q_{i+2})^{1/n} \cdot \left\{ \int_{t_{i+2}}^{0} \left\{ \int_{A_{i+2}(t)} |v - k_{i+2}| \cdot \{|\nabla v| + 2^i \cdot |v - k_{i+2}|\}\, dx \right\}^2 dt \right\}^{\frac{1}{2}}$$

$$\leqq g \cdot (\operatorname{meas} Q_{i+2})^{1/n} \cdot \left\{ \int_{Q_{i+2}} |\nabla v|^2\, dz + \operatorname*{essup}_{t_{i+2} < t < 0} \int_{A_{i+2}(t)} (v - k_{i+2})^2\, dx \right\}$$

$$+ g \cdot (\operatorname{meas} Q_{i+2})^{1/n} \cdot 2^{2i} \cdot \int_{Q_{i+1}} (v - k_{i+2})^2\, dz\,.$$

From (5.1) (with $\Phi = 1$) and (5.8), one derives that

$$(5.11)\quad \int_{Q_{i+2}} |\nabla v|^2\, dz + \operatorname*{essup}_{t_{i+2} < t < 0} \int_{A_{i+2}(t)} |v - k_{i+2}|^2\, dz$$

$$\leqq g \cdot 2^i \cdot \int_{Q_{i+2}} |v - k_{i+1}|^2\, dz + g \cdot K^2 \cdot 2^{2i} \cdot \{\operatorname{meas} Q_{i+1}\}^{1-\alpha}$$

$$\leqq g \cdot 2^{4i} \cdot K^2 \cdot \left\{ K^{-2} \cdot \int_{Q_i} |v - k_i|^2\, dz \right\}^{1-\alpha}.$$

Here, we used the simple inequality

$$(5.12)\qquad \operatorname{meas} Q_{i+1} \leqq g \cdot 2^{2i} \cdot K^{-2} \cdot \int_{Q_i} |v - k_i|^2\, dz\,.$$

Now, we conclude the proof remarking that (5.10)-(5.12) imply the desired recursion formula (5.9).

Lemma 9 follows from Lemma 8 and a repeated application of the following

LEMMA 10. *Let v satisfy the hypotheses of Lemma 9. Then, there is an*

$\varepsilon > 0$ *depending only on* n, α, c^* *and* δ *such that* (5.6) *holds or that*

(5.13) $$\operatorname{meas}\{z \in Q_{R/2}(0)\colon\ 0 < v(z) < (1-\varepsilon)\cdot M\} \geqq \varepsilon \cdot \operatorname{meas} Q_{R/2}(0)\ .$$

PROOF OF LEMMA 10. We suppose that

(5.14) $$\operatorname{meas}\{z \in Q_{R/2}(0)\colon\ 0 < v(z) < (1-\varrho)\cdot M\} \leqq \varrho \cdot \operatorname{meas} Q_{R/2}(0)\ ,$$

(5.15) $$\varrho \cdot M \geqq k^*\ ,$$

and show that

(5.16) $$v \leqq (1-\delta/8)\cdot M\ , \qquad \text{in } Q_{\delta\cdot R/8}(0)\ ,$$

provided that $\varrho > 0$ is « sufficiently small » (in dependence on n, α, c^* and δ). This proves Lemma 10, because, in the other cases, it is true, trivially.

By g, we denote a generic constant depending only on n, α, c^* and δ. From (5.1) (with $\Phi = 1$) and (5.15), we derive that

$$\int_{-R^2/4}^{0} \int_{A((1-\sigma)M,\,R/2,t)} |\nabla v|^2\, dx\, dt \leqq g\cdot M^2\cdot\sigma\cdot R^{n+2}\ ,$$

for all $\sigma \in [0, 1]$. This, (5.14), (5.15) and Hölder's inequality imply that

(5.17) $$\int_{Q_{R/2}} |\nabla v|\, dz \leqq \{\operatorname{meas}\{t \in Q_{R/2}(0) : 0 < v(z) < (1-\varrho)\cdot M\}^{\frac12}\cdot\Big\{\int_{Q_{R/2}(0)} |\nabla v|^2\, dz\Big\}^{\frac12} + g\cdot R^{(n+2)/2}\cdot\Big\{\int_{-R^2/4} \int_{A((1-\varrho)M,\,R/2,\,t)} v^2\, dx\, dt\Big\}^{\frac12} \leqq g\cdot\varrho^{\frac12}\cdot M\cdot R^{n+1}.$$

We set

$$\bar v(t) = \{\operatorname{meas} Q_{R/2}(0)\}^{-1}\cdot \int_{Q_{R/2}(0)} v\, dz$$

and use (5.17) and Sobolev's imbedding theorem, in order to obtain that

(5.18) $$\int_{Q_{R/2}(0)} |v(x,t) - \bar v(t)|\, dx\, dt \leqq g\cdot R\cdot \int_{Q_{R/2}(0)} |\nabla v|\, dz \leqq g\cdot\varrho^{\frac12}\cdot M\cdot R^{n+2}\ .$$

Moreover, by (5.18) and the hypotheses of Lemma 9 and 10,

(5.19) $$\int_{Q_{R/2}(0)} |\bar v(t)|\, dx\, dt \leqq \int_{Q_{R/2}(0)} |v - \bar v| + |v|\, dz \leqq g\cdot\varrho^{\frac12}\cdot M\cdot R^{n+2} + (1-\delta)\cdot M\cdot \operatorname{meas} Q_{R/2}(0)\ .$$

An approximation of $\bar{v}(t)$ by step-functions, (5.18) and (5.19) show that there is a $t_0 \in [-R^2/2, -\delta \cdot R^2/4]$, an $\varepsilon_0 \in]0, -R^2/2]$ and a $v_0 \in \mathbf{R}^N$ such that

$$|v_0| \leqq (1-\delta/2)\cdot M\,,$$

$$\varepsilon_0^{-1}\cdot \int_{t_0-\varepsilon_0}^{t_0} \int_{B_{R/2}(0)} |v - v_0|\, dz \leqq g\cdot \varrho^{\frac{1}{2}}\cdot M \cdot R^n\,.$$

These inequalities imply that

$$(5.20)\quad \varepsilon_0^{-1}\cdot \int_{t_0-\varepsilon_0}^{t_0} \int_{A((1-\delta/2)M,\, R/2,\, t)} |v - (1-\delta/2)\cdot M|^2\, dz \leqq g\cdot \varrho^{\frac{1}{2}}\cdot M^2 \cdot R^n\,.$$

From (5.1) (with $\Phi = 0$), (5.15), (5.17) and (5.20), we derive that

$$\operatorname*{ess\,sup}_{-\delta R^2/4<t<0} \int_{A((1-\delta/2)M,\, R/4,\, t)} |v - (1-\delta/2)\cdot M|^2\, dx \leqq g\cdot \varrho^{\frac{1}{2}}\cdot M^2\cdot R^n\,.$$

In particular,

$$(5.21)\qquad \operatorname{meas}\{z\in Q_{\delta\cdot R/4}(0)\colon |v(z)| \geqq (1-\delta/4)\cdot M\} \leqq g\cdot \varrho^{\frac{1}{2}}\cdot \operatorname{meas} Q_{\delta\cdot R/4}(0)\,.$$

Now, we conclude the proof remarking that (5.21) and Lemma 8 imply (5.16), provided that $\varrho > 0$ is « sufficiently small ».

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