ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

TOM BARTON

Biholomorphic equivalence of bounded Reinhardt domains

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 13, nº 1 (1986), p. 1-13

<http://www.numdam.org/item?id=ASNSP_1986_4_13_1_1_0>

Biholomorphic Equivalence of Bounded Reinhardt Domains.

TOM BARTON (*)

Introduction.

A domain D in a complex Banach space E with basis $(e_n)_{n\in N}$ is said to be *Reinhardt* (w.r.t. (e_n)) if it contains the origin and is invariant under the transformations

$$\sum_n x_n e_n \to \sum_n \lambda_n x_n e_n \qquad \forall |\lambda_n| = 1 \ .$$

It is known that E contains a bounded Reinhardt domain precisely when (e_n) is unconditional. In this case, an appropriate diagonal linear isomorphism $T\colon E \to E$ *normalizes* D, i.e.,

$$e_n \in \partial\tilde{D} \ , \qquad \text{and } \lambda e_n \in \tilde{D} \Rightarrow |\lambda| < 1 \qquad \forall n \in \mathbf{N} \ ,$$

where $\tilde{D} = TD$. E may then be given an equivalent norm $\|\cdot\|$ for which $co(D)$ is the unit ball and (e_n) is 1-unconditional.

Let D and $\tilde{D}$ be bounded normalized Reinhardt domains in E and $\tilde{E}$ w.r.t. the 1-unconditional bases (e_n) and $(\tilde{e}_n)$. D and $\tilde{D}$ are said to be *biholomorphically equivalent* if there is a biholomorphic map $\psi\colon D \to \tilde{D}$. In [10], Sunada has shown that for finite dimensional E and $\tilde{E}$, D and $\tilde{D}$ are biholomorphically equivalent iff there is a surjective linear isomorphism $T\colon D \to \tilde{D}$ which is *basic*, i.e., there is a permutation σ so that $T(e_n) = \tilde{e}_{\sigma(n)}$ $\forall n$. We extend this result to infinitely many coordinates.

The methods used in [10] are Lie algebraic and peculiar to finite dimensions. We'll deduce the theorem instead by using the D skew-hermitian operators on E to examine the $\|\cdot\|$ isometric structure induced by D on

(*) Current address: Dept. of Math. Sciences, MSU, Memphis, TN 38152.

The results of this article are contained in the author's Ph. D. thesis, written at Kent State University.

Pervenuto alla Redazione il 4 Dicembre 1984 ed in forma definitiva il 28 ottobre 1985.

the coordinate subspaces of E. Specifically, we'll find a maximal partition $\mathscr{C}$ of $\mathbf{N}$ such that $E_c = [e_i \colon i \in c]$ is isometrically Hilbert space and $D_c = D \cap E_c$ is the Hilbert unit ball $\forall c \in \mathscr{C}$, and such that $x \in D$ is determined by the sequence $(\|x_c\|_2)_{c\in\mathscr{C}}$, where $\|\cdot\|_2$ is the Hilbertian norm on E_c and x_c is the canonical projection of x onto E_c.

For $D = B_E$ this structure was studied by Schneider and Turner [8] in finite dimensions and by Fleming and Jamison [4, 5] and Kalton and Wood [6] in infinite dimensions, where $\{E_c \colon c \in \mathscr{C}\}$ are referred to as the Hilbert components of E. Stachó, using different methods, re-discovered Hilbert components in [9]. The papers of Vigué [12] and Barton et al. [2] using Jordan theoretic techniques developed principally by Vigué [12] and Braun et al. [3], may be viewed in part as uncovering certain Hilbert components of E induced by arbitrary bounded Reinhardt domains admitting of a nonlinear biholomorphic automorphism.

In § 1 we'll use an elementary argument to establish the Hilbert components induced by a bounded normalized Reinhardt domain D. The argument is motivated in part by Auerbach [1], in which it is shown that a bounded group of linear transformations is a subgroup of a group of unitary transformations. We'll see in particular that the Hilbert components induced by D are generally proper subspaces of those induced by $co(D)$. In § 2 we'll prove Sunada's theorem. The argument here is standard; half of it is similar to that of [5]. The *normal form* of a bounded Reinhardt domain (cf. [2]) is described in § 3. This form shows the very special geometric structure required of a bounded Reihardt domain to support a nonlinear biholomorphic automorphism. A subset of the Hilbertian components induced by D can be computed from the parameters of the normal form, and these parameters furnish a set of biholomorphic invariants of D.

§1. We first recall some background (cf. [11]). Let D be a bounded domain containing the origin, and let $G_0(D)$ be the linear and continuous automorphisms of D. The infinitesimal transformations of $G_0(D)$ are a Banach Lie sub-algebra $\mathfrak{g}(D)^+$ of the linear operators on E, called the *D skew-hermitian* operators on E. $\mathfrak{g}(D)^+$ is closed under the Lie bracket

$$[f, g] = g \circ f - f \circ g , \quad f, g \in \mathfrak{g}(D)^+ .$$

Represent $f \in \mathfrak{g}(D)^+$ as a matrix $(f_{ij})_{i,j\in\mathbf{N}}$ where

$$f_{ij} = \langle f(e_i), e_j^* \rangle$$

and e_j^* is the coefficient functional associated with e_j. When D is Reinhardt,

$ia \in \mathfrak{g}(D)^+$ whenever a is a real diagonal matrix (a_j) since

$$\exp(ita)(x) = \sum_j \exp(ita_j)x_j e_j .$$

Let h be a linear operator on E such that $ih \in \mathfrak{g}(D)^+$. Then for any real diagonal matrices a and b,

$$[ib, [ia, ih]] \in \mathfrak{g}(D)^+ .$$

In particular, choosing $a_k = 1$ and $a_i = 0$ $\forall i \neq k$, and $b_l = 1$, $b_i = 0$ $\forall i \neq l$, for some $k \neq l$, we find that $i(h_{kl}e_{kl} + h_{lk}e_{lk}) \in \mathfrak{g}(D)^+$ where e_{ij} is the elementary matrix with a 1 in the $(i,j)^{th}$ place and zeros elsewhere.

Let D be a bounded normalized Reinhardt domain.

LEMMA 1: (compare with [9, Lemma 3.6] and [6, Proposition 4.2]). *Let $k \neq l$. Then the following are equivalent*:

(i) $\exists ih \in \mathfrak{g}(D)^+$ *such that* $h_{kl} \neq 0$.

(ii) $|x_k|^2 + |x_l|^2 = |y_k|^2 + |y_l|^2$ *and* $|x_i| = |y_i|$ $\forall i \neq k, l$ *imply* $x \in D$ *iff* $y \in D$.

PROOF. Suppose (i) holds. We may assume $ih = i(\alpha e_{kl} + \beta e_{lk}) \in \mathfrak{g}(D)^+$ for some $\alpha, \beta \in \mathbf{C}$ with $\alpha \neq 0$. Let $g = \exp(ith)$ for arbitrary $t \in \mathbf{R}$ and write $D_0 = D \cap [e_k, e_l]$. Then $g|_{D \cap [e_i:\, i \neq k,l]}$ is the identity and $g|_{D_0} \in G_0(D_0)$, i.e., we may assume that $D = D_0$ and $ih \in \mathfrak{g}(D_0)^+$. By calculating $\exp(ith)$, $t \in \mathbf{R}$, one easily finds that $\alpha\beta > 0$ and $(i/\gamma)h \in G_0(D_0)$, where $\gamma^2 = \alpha\beta$.

Since D_0 is normalized,

$$\frac{i}{\gamma} h(\lambda e_k) \in D_0 \Rightarrow |\lambda| < 1 , \qquad \text{and}$$

$$\lambda \frac{i\beta}{\gamma} e_l \in D_0 \Rightarrow \left| \frac{\lambda i \beta}{\gamma} \right| < 1 .$$

Since $e_k, e_l \in \partial D_0$, we may choose $|\lambda|$ as near to 1 as we please, and so conclude that $|i\beta/\gamma| = 1$. Similarly, $|i\alpha/\gamma| = 1$. It follows that

$$\bar{\alpha} = \beta .$$

The argument of [6, Proposition 4.2] can be easily adapted to complete the proof of (i) $\Rightarrow$ (ii).

Now assume (ii) holds. Let $|\alpha| = 1$, fix $x \in D_0$, and let

$$y = (x_k \cos t + i\alpha x_l \sin t) e_k + (x_l \cos t + i\bar{\alpha} x_k \sin t) e_l$$

for arbitrary $t \in \mathbb{R}$. Then $|y_k|^2 + |y_l|^2 = |x_k|^2 + |x_l|^2$, so $y \in D_0$: This shows that $i(\alpha e_{kl} + \bar{\alpha} e_{lk}) \in \mathfrak{g}(D_0)^+$. ■

REMARK. If $k \neq l$ satisfy (ii), the above arguments shows that any g in $G_0(B_{l_2^2})$ is naturally in $G_0(D_0)$, and $g \oplus id_{[e_i : i \neq k, l]} \in G_0(D)$. In particular, for each $x \in E$ there exists $g \in G_0(D)$ with

$$g(x) = \left(|x_k|^2 + |x_l|^2\right)^{\frac{1}{2}} e_k + \sum_{i \neq k,l} x_i e_i .$$

Define a relation on $\mathbb{N}$ by $k \sim l$ if $k = l$ or $\exists f \in \mathfrak{g}(D)^+$ with $f_{kl} \neq 0$. The proof of (i) $\Rightarrow$ (ii) shows that $f_{kl} = -\bar{f}_{lk}$, so $\sim$ is symmetric. If $i(\alpha e_{kl} + \bar{\alpha} e_{lk})$ and $i(\beta e_{lj} + \bar{\beta} e_{jl})$ are in $\mathfrak{g}(D)^+$, then so is their Lie bracket, which is just $i(\alpha\beta e_{kj} + \overline{\alpha\beta} e_{jk})$. So $\sim$ is an equivalence relation. Let $\mathscr{C}$ denote the induced equivalence classes. For $x \in E$, let x_c be the canonical projection of x onto $E_c = [e_i : i \in c]$, $c \in \mathscr{C}$. Write $D_c = D \cap E_c$, and for each $c \in \mathscr{C}$ choose a distinguished $i_c \in c$.

If, for some $c \in \mathscr{C}$ and $x \in E$, x_c has only finitely many nonzero coordinates, then the remark following Lemma 1 may be repetitively applied to find a $g \in G_0(D)$ such that

$$g(x) = \|x_c\|_2 e_{i_c} + \sum_{i \notin c} x_i e_i ,$$

where $\|x\|_2 := \left(\sum_i |x_i|^2\right)^{\frac{1}{2}}$.

Let $y^n = \sum_{i=1}^{n} y_i e_i$, and let $c \in \mathscr{C}$ such that $c \cap \{1, \ldots, n\} \neq \emptyset$. Then $\exists g_c \in G_0(D)$ such that

$$g_c(y^n) = \|y_c^n\|_2 e_{i_c} + \sum_{i \notin c} y_i^n e_i .$$

By composing the automorphisms g_c we get a $g_n \in G_0(D)$ such that

$$g_n(y^n) = \sum \|y_c^n\|_2 e_{i_c} ,$$

where the summation is over all $c \in \mathscr{C}$ with $c \cap \{1, \ldots, n\} \neq \emptyset$. Since $G_0(D)$ is naturally embedded in $G_0(co(D))$, every $g \in G_0(D)$ is a $\|\cdot\|$-isometry. Hence, if $m > n$, then

$$\|g_m(y^m) - g_n(y^n)\| = \|g_m(y^m - y^n)\| = \|y^m - y^n\| .$$

So $(\sum \|y_c^n\|_2 e_{i_c})_n$ converges in E. Also,

$$\|x - g_n^{-1}(\sum \|y_c^n\|_2 e_{i_c})\| = \|g_n x - \sum \|y_c^n\|_2 e_{i_c}\| = \|\sum_{i>n} x_i e_i\| ,$$

so $g_n^{-1}(\sum \|y_c^n\|_2 e_{i_c})$ and consequently $\sum \|y_c^n\|_2 e_{i_c}$ both converge to an element in D if and only if $x \in D$.

Should $x \in E_c$ for some $c \in \mathscr{C}$, then $\|y^n\| = \|g_n y^n\| = \|y^n\|_2$ $\forall n$. Hence $\|x\| = \|x\|_2$. Summarizing the above discussion, we've shown (compare with [4, Lemma 4.2]):

LEMMA 2. *Let $x \in E$. Then $\|x_c\| = \|x_c\|_2$ $\forall c \in \mathscr{C}$. Furthermore,*

$$x \in D \quad \textit{iff} \ \sum_{\mathscr{C}} \|x_c\|_2 e_{i_c} \in D .$$

As a consequence we obtain

LEMMA 3. $D_c \cong B_{l_2}$ $\forall c \in \mathscr{C}$.

PROOF. By the normalization of D and Lemma 2 we may consider D_c to be a subset of B_{l_2}. Since D_c contains a relatively open neighborhood of the origin, $\exists 0 < \zeta < 1$ such that $|t| < \zeta$ implies $te_{i_c} \in D_c$. By Lemma 2, $\zeta B_{l_2} \subseteq D_c$. Hence, the $\|\cdot\|$ and $\|\cdot\|_2$ topologies on E_c coincide. Thus, D is an open connected subset of B_{l_2}. Since each of the sets

$$D_c \cap \{x \in B_{l_2} \colon \|x\|_2 < t\} , \quad \text{and}$$
$$D_c \cap \{x \in B_{l_2} \colon t < \|x\|_2 < 1\} , \quad 0 < t < 1$$

is open and nonempty, their union cannot exhaust D_c. Therefore, $\forall 0 < t < l$ $\exists x \in D_c$ with $\|x\|_2 = t$. Another application of Lemma 2 completes the proof. ■

It's now a small step to

PROPOSITION 4. $\bigoplus_{c \in \mathscr{C}} G_0(D_c) = \bigoplus_{c \in \mathscr{C}} G_0(B_{l_2}) \subseteq G_0(D)$.

PROOF. Let $g_c \in G_0(D_c)$ $\forall c \in \mathscr{C}$ and let $g = \bigoplus_{\mathscr{C}} g_c$. Let $x \in E$. By Lemma 3,

$\|g_c(x_c)\|_2 = \|x_c\|_2$, so by Lemma 2

$$\begin{aligned} x \in D \quad & \text{iff } \sum \|x_c\|_2 e_{i_c} \in D \\ & \text{iff } \sum \|g_c(x_c)\|_2 e_{i_c} \in D \\ & \text{iff } \sum \|(g(x))_c\|_2 e_{i_c} \in D \\ & \text{iff } g(x) \in D\,. \end{aligned}$$

$\therefore\ g \in G_0(D)\,.$ ■

We conclude this section by noting that the embedding of $G_0(D)$ into $G_0(co(D))$ induces an embedding of $\mathfrak{g}(D)^+$ into $\mathfrak{g}(co(D))^+$, and so $\mathscr{C}$ refines the equivalence classes induced by $co(D)$. The refinement is strict in general, as is apparent from the example

$$D = B_{l_2^2} \setminus \{x \in B_{l_2^2} \colon |x_1| = |a_1|,\, |x_2| = |a_2|\}\,,$$

where $0 < |a_1|^2 + |a_2|^2 < 1$. Choosing $|a_1| \neq |a_2|$ ensures, in particular, that the map $g \in G_0(B_{l_2^2})$ given by $g(e_1) = e_2$ and $g(e_2) = e_1$ is not in $G_0(D)$.

§ 2. In this section we prove our main result.

THEOREM. *Let D and $\tilde{D}$ be bounded normalized Reinhardt domains in E and $\tilde{E}$ with respect to the bases (e_n) and $(\tilde{e}_n)$. Then D is biholomorphically equivalent to $\tilde{D}$ if and only if there is a surjective linear isomorphism $S\colon E \to \tilde{E}$ taking D onto $\tilde{D}$ such that $S(\tilde{e}_n) = \tilde{e}_{\sigma(n)}$ for some bijection $\sigma\colon \mathbf{N} \to \mathbf{N}$.*

PROOF. Sufficiency is trivial.

Suppose that D is biholomorphically equivalent to $\tilde{D}$ and let $\psi\colon D \to \tilde{D}$ be a biholomorphic mapping. We first show that there is a surjective linear isomorphism $T\colon E \to \tilde{E}$ taking D onto $\tilde{D}$. The argument is standard. For $x \in D$ denote the orbit of x under $G(D)$, the biholomorphic automorphisms of D, by

$$G(D)\cdot x = \{g(x)\colon g \in G(D)\}\,.$$

If $f \in G(D)$, then $\psi f \psi^{-1} \in G(\tilde{D})$, so

$$\psi f(x) = \psi f \psi^{-1}(\psi(x)) \in G(\tilde{D})\cdot\psi(x) \qquad \forall f \in G(D)\,.$$

Hence,

$$\psi(G(D)\cdot x) \subseteq G(\tilde{D})\cdot\psi(x)\,.$$

Likewise,

$$\psi^{-1}(G(\tilde{D})\cdot\psi(x)) \subseteq G(D)\cdot x\,.$$

Thus,

$$\psi(G(D)\cdot x) = G(\tilde{D})\cdot\psi(x)\,.$$

Kaup and Upmeier [7] have shown that

$$G(D)\cdot 0 = \{x\in D\colon G(D)\cdot x \text{ is a closed complex submanifold of } D\}\,.$$

Since ψ preserves these properties, $G(\tilde{D})\cdot\psi(0)$ is a closed complex submanifold of $\tilde{D}$. Hence, $\psi(0)\in G(\tilde{D})\cdot 0$, and so $\exists g\in G(\tilde{D})$ with $g\psi(0) = 0$. By H. Cartan's theorem, $g\psi$ is linear. $T = g\psi$ is the desired mapping. Observe that by Proposition 4 and Lemma 1

$$ih\in \mathfrak{g}(D)^+ \quad \text{if and only if } h = \oplus_{\mathscr{C}}\, h_c\,,$$

where $ih_c\in\mathfrak{g}(D_c)^+$ for all $c\in\mathscr{C}$.

List the elements of $\mathscr{C} = \{c_1, c_2, \ldots\}$, and $\tilde{\mathscr{C}} = \{\tilde{c}_1, \tilde{c}_2, \ldots\}$, where each set is finite or infinite according to the cardinality of $\mathscr{C}$ and $\tilde{\mathscr{C}}$. Write

$$T = (T_{i,j})_{i,j} \qquad \text{where } T_{i,j}\colon E_{c_j}\to \tilde{E}_{\tilde{c}_i}$$

and

$$T^{-1} = (S_{ij})_{i,j} \qquad \text{where } S_{ij}\colon \tilde{E}_{\tilde{c}_j}\to E_{c_i}\,.$$

Fix k and $ih\in\mathfrak{g}(D_{c_k})^+$. Since $T\circ\exp\big(\mathfrak{g}(D)^+\big)\circ T^{-1} = \exp\big(T\circ\mathfrak{g}(D)^+\circ T^{-1}\big)$ it follows that $T\circ\mathfrak{g}(D)^+\circ T^{-1}\subseteq\mathfrak{g}(\tilde{D})^+$. Hence,

$$T\circ(ih\oplus 0)\circ T^{-1} = (T_{i,k}\circ ih\circ S_{k,j})_{i,j} = \oplus_j\, i\tilde{h}_{\tilde{c}_j}$$

for some $i\tilde{h}_{\tilde{c}_j}\in\mathfrak{g}(\tilde{D}_{\tilde{c}_j})^+$, where 0 is the zero map on $E_{\mathrm{N}-c_k}$. Consequently

$$T_{i,k}\circ ih\circ S_{k,j} = 0 \qquad \text{for all } i\neq j \text{ and } ih\in\mathfrak{g}(D_{c_k})^+. \tag{2.1}$$

Since T is invertible there is a j with $S_{k,j}\neq 0$. Because $D_{c_k}\simeq B_{z_2}$ every $g\in G_0(D_{c_k})$ may be written $g = ih_1 - h_2$, where $ih_1, ih_2\in\mathfrak{g}(D_{c_k})^+$. Hence (2.1) implies $T_{i,k} = 0$ for all $i\neq j$. Since T is invertible, $T_{j,k}\neq 0$. Thus for each k there is a unique j with $T_{j,k}\neq 0$. This defines a map

$$\tau\colon \mathscr{C}\to\tilde{\mathscr{C}}$$

by

$$\tau(c_k) = \tilde{c}_{j(c)}\,.$$

T's invertibility ensures that τ is a bijection. Consequently,

$$T|_{E_c} E_b \to \tilde{E}_{\tau(c)} .$$

Hence, there is an appropriate choice of $\tilde{g}_{\tilde{c}} \in G_0(\tilde{D}_{\tilde{c}})$ so that $\big(\bigoplus_{\tilde{\mathscr{C}}} \tilde{g}_{\tilde{c}}\big) \circ T$ is the desired mapping. ■

§3. We first recall some notation and background (see [2] and [12]). Suppose that D supports a nonlinear biholomorphic automorphism, so that $G(D)\cdot 0 \supsetneqq \{0\}$. Then $\exists I \subseteq \mathbf{N}$ with $E_I = [e_i : i \in I] = [G(D)\cdot 0]$, a partition $\mathscr{P}$ of I with $E_p = [e_i : i \in p] \cong l_2$ and $D \cap E_I \cong B_{(\bigoplus_{\mathscr{P}} E_p)_{c_0}}$, and nonnegative constants $r_{p,j}$ with $\sup_{j \in \mathbf{N}\setminus I} \sum_{p \in \mathscr{P}} r_{p,j} < \infty$ so that

$$D = \left\{ \sum_{\mathscr{P}} x_p + \sum_J x_j e_j : \left(\|x_p\|_2\right)_{\mathscr{P}} \in B_{c_0},\ \sum_{i \in J} \frac{x_j e_j}{\varphi_j(x_{\mathscr{P}})} \in D_1 \right\},$$

where $J = \mathbf{N}\setminus I$, $x_{\mathscr{P}}$ denotes $\sum_{p \in \mathscr{P}} x_p$, $D_1 = D \cap E_J$, and

$$\varphi_j(x_{\mathscr{P}}) = \prod_{p \in \mathscr{P}} \left(1 - \|x_p\|_2^2\right)^{r_{p,j}} .$$

We'll abbreviate these notations by writing

$$D = B_1[+]D_1 ,$$

where $B_1 = B_{(\bigoplus_{\mathscr{P}} E_p)_{c_0}}$: For each $k \in J$, define

$$S_k = \{ j \in J : r_{p,j} = r_{p,k}\ \ \forall p \in \mathscr{P} \} .$$

Then $S_k \cap S_l \neq \emptyset \Rightarrow S_k = S_l$ and $\bigcup_k S_k = J$. Thus, the distinct members $\mathscr{S}$ of $\{S_k : k \in J\}$ form a partition of J. For $j, k \in s \in \mathscr{S}$, we write $r_{p,s}$ for the common value of $r_{p,j}$ and $r_{p,k}$, and we write φ_s for the function

$$\varphi_s(x_{\mathscr{P}}) = \prod_{p \in \mathscr{P}} \left(1 - \|x_p\|_2^2\right)^{r_{p,s}} , \qquad s \in \mathscr{S}.$$

With these notations,

$$x \in D \Leftrightarrow x_{\mathscr{P}} \in B_1 \quad \text{and} \ \sum_{s \in \mathscr{S}} \frac{x_s}{\varphi_s(x_{\mathscr{P}})} \in D_1 ,$$

where x_s is the canonical projection of x onto $E_s = [e_i\colon i \in s]$. Observe that, since $\varphi_s(x_{\mathscr{P}})$ depends only on $(\|x_p\|_2)_{p\in\mathscr{P}}$,

$$\bigoplus_{p\in\mathscr{P}} G_0(D \cap E_p) \oplus G_0(D_1) \subseteq G_0(D)\,.$$

Associated with D is its triple product $\{\,,\,,\}\colon E\times E_I \times E \to E$. We'll not study the triple product here, but simply use several properties it possesses, namely that it is symmetric bilinear in the outer two variables, conjugate linear in the middle variable, and satisfies

$$\{x\xi x\} = 0 \qquad \forall \xi \in E_I\,, \qquad x \in E_I$$

$$\{x\xi y\} = -\,(x|\xi)\sum_{\mathscr{S}} r_{p,s} y_s \qquad \forall \xi, x \in E_p\,, \qquad \forall y \in E_J$$

$$\{x\xi y\} = -\,\tfrac{1}{2}\sum_{\mathscr{S}}\left[(x_p|\xi_p)y_p + (y_p|\xi_p)x\right] \qquad \forall \xi, x, y \in E_I\,,$$

where $(\cdot|\cdot)$ is the inner product on E_p. In [3] and [12] it's been shown that $ih \in \mathfrak{g}(D)^+$ implies

$$h(\{x\xi y\}) = -\,\{x, h(\xi), y\} + \{h(x), \xi, y\} + \{x, \xi, h(y)\} \quad \forall \xi \in E_I\,, \quad \forall x, y \in E\,.$$

It may occur that $G(D_1)\cdot 0 \supsetneqq \{0\}$, in which case D_1 may be decomposed

$$D_1 = B_2[+]D_2\,.$$

Assume this process continues at least n times, so

$$\begin{aligned} D = D_0 &= B_1[+]D_1 \\ &= B_1[+](B_2[+]D_2) \\ &= \ldots \\ &= B_1[+]\Big(B_2[+]\big(\ldots[+](B_n[+]D_n)\ldots\big)\Big)\,. \end{aligned}$$

Write $\mathscr{P}_k, \mathscr{S}_k, \varphi_s^k$, etc., for the quantities and objects associated with D_{k-1}, $1 \leqslant k \leqslant n$. Define

$$\mathscr{C}_1 = \mathscr{P}_1\,,$$

$$\mathscr{C}_k = \{s_1 \cap \ldots \cap s_{k-1} \cap p_k\colon s_i \in \mathscr{S}_i, p_k \in \mathscr{P}_k\}\,, \qquad 1 < k < n\,.$$

THEOREM. $\bigcup_{k=1}^{n} \mathscr{C}_k \subseteq \mathscr{C}$, *where* $\mathscr{C}$ *determines the Hilbertian components induced by* D.

The theorem is established by the following two lemmas. Note that if $c \in \mathscr{C}_k$, then $\exists p \in \mathscr{P}_k$ with $c \subseteq p$, so $D_c = D \cap E_c \cong B_{l_2}$.

LEMMA 6. *If* $ih \in \mathfrak{g}(D)^+$, *then* $h(E_c) \subseteq E_c$ $\forall c \in \bigcup_{k=1}^{n} \mathscr{C}_k$.

PROOF. Choose $p \in \mathscr{P}_1$ and $i \in p$. Then

$$\begin{aligned} -h(e_i) &= h(\{e_i, e_i, e_i\}) \\ &= -\{e_i, h(e_i), e_i\} + 2\{e_i, e_i, h(e_i)\} \\ &= \big(e_i|(h(e_i))_p\big)e_i - (e_i|e_i)(h(e_i))_p - \big((h(e_i))_p|e_i\big)e_i \\ &\in E_p\,. \end{aligned}$$

So $h(E_p) \subseteq E_p$ $\forall p \in \mathscr{P}_1$.

Now choose $s_0 \in \mathscr{S}_1$, and let p and i be as above. Then for $y \in E_{s_0}$,

$$\begin{aligned} r_{p,s_0}h(y) &= -h(\{e_i, e_i, y\}) \\ &= \{e_i, h(e_i), y\} - \{h(e_i), e_i, y\} - \{e_i, e_i, h(y)\} \\ &= -(e_i|h(e_i))r_{p,s_0}y + (h(e_i)|e_i)r_{p,s_0}y + (e_i|e_i)\sum_{s\in\mathscr{S}_1} r_{p,s}(h(y))_s\,. \end{aligned}$$

Since $(e_i|e_i) = 1$, this implies that

$$r_{p,s_0}(h(y))_s = r_{p,s}(h(y))_s \qquad \forall p \in \mathscr{P}_1\,,$$

for all $s \neq s_0$ for which $(h(y))_s \neq 0$. It follows from the definition of $\mathscr{S}_1$ that $(h(y))_s = 0$ $\forall s \neq s_0$. Thus $h(E_s) \subseteq E_s$ $\forall s \in \mathscr{S}_1$. In particular, h decomposes so that $ih|_{E_{J_1}} \in \mathfrak{g}(D_1)^+$. Hence, the above argument can be repeated for $D_1, \ldots, D_{n-1}$ to yield

$$\begin{aligned} h(E_p) &\subseteq E_p \qquad \forall p \in \bigcup_{k=1}^{n} \mathscr{P}_k \\ h(E_s) &\subseteq E_s \qquad \forall s \in \bigcup_{k=1}^{n} \mathscr{S}_k\,. \end{aligned}$$

Taking intersections completes the proof. ■

In view of Lemma 1, Lemma 6 establishes that for each $c \in \bigcup_{k=1}^{n} \mathscr{C}_k$, there is $c' \in \mathscr{C}$ with $c' \subseteq c$. The reverse inclusion follows from

LEMMA 7. *Let $c_0 \in \mathscr{C}_k$ for some $1 \leqslant k \leqslant n$. Suppose $y_i = x_i\ \forall i \notin c_0$ and that $\|y_{c_0}\|_2 = \|x_{c_0}\|_2$. Then $x \in D \Leftrightarrow y \in D$.*

PROOF. It suffices to show that if $g_0 \in G_0(D_{c_0}) = G_0(B_{l_2})$, then $g = g_0 \oplus id_{E_{\mathbf{N}\setminus c_0}} \in G_0(D)$. Write $c_0 = s_1 \cap \ldots \cap s_{k-1} \cap p_k$ for some $s_i \in \mathscr{S}_i$ and $p_k \in \mathscr{P}_k$. Fix $1 \leqslant i \leqslant k-1$ and $s \in \mathscr{S}_i$. If $s \neq s_i$, then $g|_{E_s} = id$. If $s = s_i$, then $g|_{E_s} = g_0 \oplus id_{E_{s\setminus c_0}}$, so $g(E_s) \subseteq E_s$. Hence,

$$g(E_s) \subseteq E_s \qquad \forall s \in \bigcup_{i=1}^{k-1} \mathscr{S}_i\,.$$

It follows that

$$\begin{aligned} g \in G_0(D) &\Leftrightarrow \forall x \in D\,, \quad g(x) \in D \\ &\Leftrightarrow \forall x \in D\,, \quad x_{\mathscr{P}_1} \in B_1 \quad \text{and} \sum_{\mathscr{S}_1} \frac{(g(x))_s}{\varphi_s^1(x_{\mathscr{P}_1})} \in D_1 \\ &\Leftrightarrow \forall x \in D\,, \quad x_{\mathscr{P}_1} \in B_1 \quad \text{and } g\Big(\sum_{\mathscr{S}_1} \frac{x_s}{\varphi_s^1(x_{\mathscr{P}_1})}\Big) \in D_1\,, \end{aligned}$$

which is implied by $g|_{D_1} \in G_0(D_1)$. Repeating this argument $k-1$ times we see that

$$g|_{D_{k-1}} \in G_0(D_{k-1}) \Rightarrow g|_{D_{k-2}} \in G_0(D_{k-2}) \Rightarrow \ldots \Rightarrow g \in G_0(D)\,.$$

Since $g|_{D_{k-1}} \in \bigoplus_{p \in \mathscr{P}_k} G_0(D \cap E_p) \oplus G_0(D_k) \subseteq G_0(D_{k-1})$ the proof is complete. ∎

We conclude this section with the following:

PROPOSITION 8. *Let D and $\tilde{D}$ be biholomorphically equivalent bounded normalized Reinhardt domains in E and $\tilde{E}$. Then the matrices $(r_{p,s})_{\mathscr{P},\mathscr{S}}$ and $(\tilde{r}_{p,s})_{\tilde{\mathscr{P}},\tilde{\mathscr{S}}}$ agree up to a permutation.*

PROOF. In light of the Theorem of § 2, we may assume there is a basic isomorphism $T\colon D \to \tilde{D}$ and a bijection $\tau\colon \mathscr{C} \to \tilde{\mathscr{C}}$ such that $T(E_c) = \tilde{E}_{\tau(c)}$ $\forall c \in \mathscr{C}$. By [12, Theorem 2.1], $T|_{B_1} = \tilde{B}_1$ and $T|_{D_1} = \tilde{D}_1$, and if $\xi \in E_I$, $x \in E$, then

$$T(\{x\xi x\}_D) = \{T(x), T(\xi), T(x)\}_{\tilde{D}}$$

where all quantities with a « ~ » above them refer to $\tilde{D}$. In particular,

$\tau|_{\mathscr{P}}: \mathscr{P} \to \tilde{\mathscr{P}}$. Fix $p \in \mathscr{P}$, $s \in \mathscr{S}$, $i \in p$ and $x \in E_s$. Then

$$\begin{aligned} T(e_i) + 2r_{p,s}T(x) &= -\, T(\{(x+e_i), e_i, (x+e_i)\}_D) \\ &= -\, \{T(x+e_i), T(e_i), T(x+e_i)\}_{\tilde{D}} \\ &= \big(T(e_i)|T(e_i)\big)T(e_i) + 2\big(T(e_i)|T(e_i)\big)\sum_{\tilde{s}\in\tilde{\mathscr{S}}} \tilde{r}_{\tau(p),\tilde{s}}\big(T(x)\big)_{\tilde{s}} . \end{aligned}$$

Since $\big(T(e_i)|T(e_i)\big) = \big(e_i|e_i\big) = 1$, we have

$$r_{p,s}T(x) = \sum_{\tilde{s}\in\tilde{\mathscr{S}}} \tilde{r}_{\tau(p),\tilde{s}}\big(T(x)\big)_{\tilde{s}} \qquad \forall p \in \mathscr{P} .$$

Using the definition of $\tilde{\mathscr{S}}$, it follows that there is a unique $\tilde{s}$, depending on s, with $\big(T(x)\big)_{\tilde{s}} \neq 0$, and that $r_{p,s} = \tilde{r}_{\tau(p),\tilde{s}}$ $\forall p \in \mathscr{P}$. The map $s \to \tilde{s}(s)$ determines a bijection $\sigma: \mathscr{S} \to \tilde{\mathscr{S}}$ since T is invertible. Evidently,

$$(r_{p,s})_{\mathscr{P},\mathscr{S}} = \big(\tilde{r}_{\tau(p),\sigma(s)}\big)_{\mathscr{P},\mathscr{S}} . \qquad \blacksquare$$

REMARK. If D can be decomposed n times, iteration of the above argument shows that $\tilde{D}$ can be decomposed n times, and that $(r^k_{p,s})_{\mathscr{P}_k,\mathscr{S}_k}$ and $(\tilde{r}^k_{p,s})_{\tilde{\mathscr{P}}_k,\tilde{\mathscr{S}}_k}$ agree up to a permutation $\forall 1 \leqslant k \leqslant n$.

BIBLIOGRAPHY

[1] H. AUERBACH, *Sur les groupes lineaires*, Studia Math., **4** (1933), pp. 113-125; Studia Math., **4** (1933), pp. 158-166, and Studia Math., **5** (1934), pp. 43-49.

[2] T. J. BARTON - S. DINEEN - R. TIMONEY, *Bounded Reinhardt domains in Banach spaces*, Compositio Math., to appear.

[3] R. BRAUN - W. KAUP - H. UPMEIER, *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, Manuscripta Math., **25** (1978). pp. 97-133.

[4] R. J. FLEMING - J. E. JAMISON, *Hermitian and adjoint abelian operators on certain Banach spaces*, Pacific J. Math., **52** (1974), pp. 67-84.

[5] —, *Isometries on certain Banach spaces*, J. London Math. Soc. (2), **9** (1975), pp. 121-127.

[6] N. J. KALTON - G. V. WOOD, *Orthonormal systems on Banach spaces and their applications*, Math. Proc. Cambridge Philos. Soc., **79** (1976), pp. 493-510.

[7] W. KAUP - H. UPMEIER, *Banach spaces with biholomorphically equivalent unit balls are isomorphic*, Proc. Amer. Math. Soc., **58** (1976), pp. 129-133.

[8] H. SCHNEIDER - E. L. TURNER, *Matrices hermitian for an absolute norm*, Linear and Multilinear Algebra, **1** (1973), pp. 9-31.

[9] L. L. STACHÓ, *A projection principle concerning biholomorphic automorphisms*, Acta Sci. Math., **44** (1982), pp. 99-124.

[10] T. SUNADA, *Holomorphic equivalence problem for bounded Reinhardt domains*, Math. Ann **235** (1978), pp. 111-128.

[11] J. P. VIGUÉ, *Le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe. Application aux domaines borné symétriques.* Ann. Sci. Ècole Norm. Sup., **9** (1976), pp. 203-282.

[12] —, *Automorphismes analytiques d'un domaine de Reinhardt borné d'un espace de Banach a base*, Ann. Inst. Fourier, **34** (1984), pp. 67-87.

Department of Mathematics
Kent State University
Kent, Ohio 44242