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The Absolute Galois Group of a Pseudo Real Closed Field.

DAN HARAN (*) - MOSHE JARDEN (**)

Introduction.

The main problem in Galois theory is to describe the absolute Galois group $G(K)$ of a field K. The most interesting case, namely that of the field of rational numbers, is still very far from being accessible. Nevertheless, a few other interesting cases have been resolved. Among them there are the finite fields, with $\hat{\mathbb{Z}}$ as the absolute Galois groups, real closed fields R with $G(R) \cong \mathbb{Z}/2\mathbb{Z}$, the p-adic fields $\mathbb{Q}_p$, with a description of $G(\mathbb{Q}_p)$ by generators and relations (Jakovlev [12], Jansen-Winberg [13] and Winberg [25]) and the field $\mathbb{C}(t)$ with $G(\mathbb{C}(t))$ being free. Finally we mention PAC fields with projective groups as their absolute Galois groups. The last example motivates the present work, we therefore explain it in more detail.

Recall that a field K is said to be *pseudo algebraically closed* (PAC) if every absolutely irreducible variety defined over K has a K-rational point. On the other hand, a profinite group G is said to be *projective* if every finite embedding problem for G is solvable; in other words, given a diagram

(1)
$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow[\alpha]{} & A \end{array}$$

where α is an epimorphism of finite groups and φ a homomorphism, there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha\circ\gamma = \varphi$ (Gruenberg [9]).

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It is now well known that if K is a PAC field, then $G(K)$ is projective (Ax [1, p. 269]). Conversely, if G is a projective group, then there exists a PAC field K such that $G(K) \cong G$ (Lubotzky-v.d. Dries [20, p. 44]).

An attempt to enrich the structure of the PAC fields has led to the definition of PRC fields:

A field K is said to be *pseudo real closed*, if every absolutely irreducible variety V defined over K, which has a $\bar{K}$-rational point in every real closed field $\bar{K}$ containing K, has a K-rational point (Prestel [22]).

In particular, if K has no orderings, then K is a PAC field. The study of PRC fields has already attracted a lot of attention ([16], [17], Prestel [22], Ershov [7] and others). In order to extend these investigations it has become necessary to give a group theoretic characterization of the absolute Galois group of a PRC field, in other words, one has to find the «right» definition for «real projective group». Here is our suggestion: We consider the embedding problem (1) and call it *real* if for every involution $g \in G$ such that $\varphi(g) \neq 1$ there exists an involution $b \in B$ such that $\alpha(b) = \varphi(g)$. A profinite group G is said to be *real projective* if the subset $I(G)$ of all involutions of G is closed and for every finite real embedding problem (1) there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha\circ\gamma = \varphi$. We prove:

THEOREM. *If K is a* PRC *field, then $G(K)$ is real projective. Conversely, if G is a real projective group, then there exists a* PRC *field K such that $G(K) \cong G$.*

Unfortunately we have to go a long way in order to prove the Theorem. Nevertheless there is a bonus for the effort, namely the introduction of *Artin-Schreier structures*. In the same way that PRC fields generalize PAC fields, Artin-Schreier structures enrich Galois groups by taking into account the orderings. Indeed, to every Galois extension L/K with $\sqrt{-1} \in L$ we attach the *space of orderings* $X(L/K)$, consisting of all pairs $(L(\varepsilon), P)$ where ε is an involution of $\mathcal{G}(L/K)$, $L(\varepsilon)$ is its fixed field in L and P is an ordering of $L(\varepsilon)$. The corresponding Artin-Schreier structure is $\mathfrak{G}(L/K) = \langle \mathcal{G}(L/K), \mathcal{G}(L/K(\sqrt{-1})), X(L/K) \xrightarrow{d} \mathcal{G}(L/K)\rangle$, where $d(L(\varepsilon), P) = \varepsilon$. In particular the *absolute Artin-Schreier structure* of K is $\mathfrak{G}(K) = \mathfrak{G}(K_s/K)$.

In the category of Artin-Schreier structures there are projective objects (Section 7), the underlying groups of which are exactly the real projective groups (Proposition 7.7). If K is a PRC field, then $\mathfrak{G}(K)$ is a projective Artin-Schreier structure (Theorem 10.1). Conversely, for every Artin-Schreier structure $\mathfrak{G}$ there exists a PRC field K such that $\mathfrak{G}(K) \cong \mathfrak{G}$ (Theorem 10.2). This completes the main result, Theorem 10.4, mentioned above.

Notation.

$X(K)$ = the set of orderings of a field K.

K_s = the separable closure of a field K.

If L/K is a Galois extension, F is an extension of L and σ is an automorphism of F over L, then $L(\sigma) = \{x \in L: \sigma(x) = x\}$ is the fixed field of σ in L.

1. – Profinite topological transformation groups.

Sets of orderings of fields, profinite groups, etc. are projective limits of finite sets. The next Definition-Theorem characterizes these objects as topological spaces.

DEFINITION 1.1. *A topological space X is said to be a Boolean space, if it satisfies one of the following equivalent conditions:*

(i) *X is a totally disconnected compact Hausdorff space.*

(ii) *X is compact and every $x \in X$ has a basis of closed-open neighborurhoods, whose intersection is $\{x\}$.*

(iii) *X is an inverse limit of finite discrete spaces.*

(iv) *X is homeomorphic to a closed subset of $\{-1, 1\}^I$, for some set I.*

The conditions are indeed equivalent:

(ii) $\Rightarrow$ (ii): Hewitt and Ross [11, p. 12].

(ii) $\Rightarrow$ (iiii): Clearly X is Hausdorff. Since the required proof is a special case of a part of the proof of Proposition 1.5, we shall not bring it here.

(iii) $\Rightarrow$ (iv): Assume $X = \varinjlim_{j \in J} X_j$, with X_j finite. Then X is a closed subset of $\prod_{j \in J} X_j$: Also, we may assume that $X_j \subseteq \{\pm 1\}^{I_j}$, for some finite set I_j. Then X is closed in $\{\pm 1\}^I$, where I is the disjoint union of the sets I_j.

(iv) $\Rightarrow$ (i): Clear. //

Examples of Boolean spaces: profinite groups, sets of orderings of fields (Prestel [21, § 6]).

Throughout this paper we tacitly use the fact that a continuous map between compact Hausdorff spaces is closed; in particular, a continuous bijection is a homeomorphism.

LEMMA 1.2. *Let $p: X \to Y$ be a continuous closed and open map from a Boolean space X onto a topological space Y. Then Y is also a Boolean space.*

PROOF. It suffices to show that Y is Hausdorff, since the image of a compact set is compact, hence by 1.1 (i), Y is also a Boolean space. Thus our Lemma follows, e.g., by [6, Ch. 2, § 4, Theorem 4 and Theorem 5]. //

Let us consider the category of (*topological*) *transformation groups*, i.e. pairs (X, G), where X is a topological space and G is a topological group acting continuously on X (the action $X \times G \to X$ denoted henceforth by $(x, \sigma) \mapsto x^\sigma$) (cf. Bredon [2, Chapter 1]). A morphism in this category, say $(Y, H) \to (X, G)$, is a pair (f, φ) consisting of a continuous map $f\colon Y \to X$ and a continuous homomorphism $\varphi\colon H \to G$, such that

$$f(y)^{\varphi(h)} = f(y^h) \qquad \text{for every } y \in Y \text{ and } h \in H\,.$$

If $f(Y) = X$ and $\varphi(H) = G$, we call $[f, \varphi)$ en *epimorphism*.

A transformation group (X, G) is called *finite*, if both X and G are finite and discrete. A transformation group is *profinite*, it is an inverse limit of finite transformation groups.

Our first aim is to characterize the profinite transformation groups.

Let (X, G) be a transformation group. A *partition* $Y = \{V_1, \dots, V_n\}$ of X is a finite collection of disjoint non-empty closed-open subsets of X, such that $X = \dot{\bigcup}_{i=1}^{n} V_i$. We say that Y is a *G-partition*, if for every $\sigma \in G$ and every $1 \leqslant i \leqslant n$ there is a $1 \leqslant j \leqslant n$ such that $V_1^\sigma = V_j$.

For two partitions Y, Y' of X we write $Y' \geqslant Y$, if Y' is *finer* than Y, i.e., for every $V' \in Y'$ there is (a unique) $V \in Y$ such that $V' \subseteq V$. The family of partitions (resp. G-partitions) is thus partially ordered.

REMARK 1.3. Let Y be a finite collection of closed-open subsets of X. Then there is a partition Y' of X such that for every $V' \in Y'$ and $V \in Y$ either $V' \subseteq V$ or $V' \cap V = \emptyset$.

In particular, every two partitions of X have a common refinement.

If Y is a G-partition of X, there is an obvious way to consider (Y, G) as a transformation group and, furthermore, to define an empimorphism $(p_Y, \mathrm{id}_G)\colon (X, G) \to (Y, G)$ (i.e., by $p_Y(x) = V$ if $x \in V$). If $Y' \geqslant Y$ is another G-partition of X, there is an obvious epimorphism $(p_{Y'Y}, \mathrm{id}_G)\colon (Y', G) \to (Y, G)$, such that $(p_{Y'Y}, \mathrm{id}_G)\circ(p_{Y'}, \mathrm{id}_G) = (p_Y, \mathrm{id}_G)$.

LEMMA 1.4. *Let (X, G) be a transformation group. Assume that X is a Boolean space and G is a profinite group, and let Y be a partition of X.*

(i) *There exists an open normal subgroup N of G such that $V^\sigma = V$ for every $V \in Y$ and every $\sigma \in N$.*

(ii) *There exists a G-partition Y' of X finer than Y.*

PROOF. (i) It suffices to find for a closed-open subset V of X an open normal subgroup N of G such that $V^\sigma = V$ for every $\sigma \in N$. Let $x \in V$. By the continuity of the action $X \times G \to X$ there is a closed-open neighbourhood U_x of x and an open normal subgroup N_x of G such that $U_x^\sigma \subseteq V$ for every $\sigma \in N_x$. Since V is compact, there are $x_1, \dots, x_k \in V$ such that $V = \bigcup_{i=1}^{k} U_{x_i}$. Put $N = \bigcap_{i=1}^{k} N_{x_i}$; then N has the required property.

(ii) Assume $Y = \{V_1, \dots, V_n\}$ and choose N which satisfies (i). If $\sigma \equiv \sigma' \pmod N$, then $V_i^\sigma = V_i^{\sigma'}$ for every $\sigma, \sigma' \in G$ and $1 \leqslant i \leqslant n$. Let $\sigma_1, \dots, \sigma_n$ be representatives of G/N. For every function $\alpha\colon \{1, \dots, m\} \to \{1, \dots, n\}$ denote $V_\alpha = V_{\alpha(1)}^{\sigma_1} \cap \dots \cap V_{\alpha(m)}^{\sigma_m}$. It is easily checked that $Y' = \{V_\alpha\colon V_\alpha \neq \emptyset\}$ is a G-partition, finer than Y. //

PROPOSITION 1.5. *A transformation group (X, G) is profinite if and only if X is a Boolean space and G is a profinite group.*

PROOF. The necessity is obvious. To show the sufficiency, assume that X is a Boolean space and G is a profinite group. Let $\mathfrak{I}$ be the family of G-partitions of X. The maps $\{(p_Y, \mathrm{id}_G)\}_{Y \in \mathfrak{I}}$ define an enimorphism (p, id_G): $(X, G) \to \varprojlim_{Y \in \mathfrak{I}} (Y, G)$ (Ribes [23, Lemma 2.5]). But p is also injective: if $x, x' \in X$ are distinct, there is a closed-open set $U \subseteq X$, such that $x \in U$, $x' \notin U$; by Lemma 1.4 (ii) there exists a G-partition Y of X finer than $\{U, X - U\}$. Thus $p_Y(x) \neq p_Y(x')$, whence $p(x) \neq p(x')$. Therefore $(X, G) \cong \varprojlim_{Y \in \mathfrak{I}} (Y, G)$ (1), since both X and $\varprojlim_{Y \in \mathfrak{I}} Y$ are Hausdorff and compact spaces Thus we may assume that X is finite.

By the continuity of the action $X \times G \to X$, there is an open subgroup N_0 of G such that $x^\sigma = x$ for every $x \in X$ and $\sigma \in N_0$. Let $\mathcal{N}$ be the family of open normal subgroups of G contained in N_0. Then, clearly, $(X, G) = \varprojlim_{N \in \mathcal{N}} (X, G/N)$. //

As an application of the material accumulated in this Section we construct a quotient of a profinite transformation group.

(1) If $G = 1$, this part proves (ii) $\approx$ (iii) in Definition 1.1.

Let (X, G) be a profinite transformation group, and let N be a closed normal subgroup of G. Define an equivalence relation $\sim$ on X by: $x_1 \sim x_2$ if there is a $\sigma \in N$, such that $x_1^\sigma = x_2$, and let X/N be the quotient space. The quotient map $p: X \to X/N$ is open $\left(\text{if } U \subseteq X \text{ is open, then } p^{-1}(p(U)) = \bigcup_{\sigma \in N} U^\sigma\right)$ and closed $\left(\text{if } F \subseteq X \text{ is closed, then } p^{-1}(p(F)) = \bigcup_{\sigma \in N} F^\sigma \text{ is the image of the compact set } F \times N \text{ under the action } X \times G \to X, \text{ hence compact}\right)$. By Lemma 1.2, X/N is a Boolean space.

Let $\pi: G \to G/N$ be the natural epimorphism. It is easily verified that the action of G on X induces a continuous action of G/N on $X/N\left(p(x)^{\pi(\sigma)} = p(x^\sigma) \text{ for } x \in X \text{ and } \sigma \in G\right)$.

Thus we have shown:

CLAIM 1.6. $(X/N, G/N)$ is a profinite transformation group and (p, π): $(X, G) \to (X/N, G/N)$ is an epimorphism. Moreover, $p: X \to X/N$ is an open map.

2. – The space of orderings of a Galois extension.

Every Galois extension L/K is naturally accompanied by its Galois group $\mathcal{G}(L/K)$. Another natural structure associated with L/K is the set $X(L/K)$ of the maximal ordered subfields of L containing K. In this section we investigate this set and its relations to $\mathcal{G}(L/K)$. To ensure a good behaviour we assume that $\sqrt{-1} \in L$. It turns out that $X(L/K)$ is a Boolean space and $\mathcal{G}(L/K)$ acts on it. To attain full generality we do not require that K be of characteristic zero and formally real. Nevertheless, the interesting case arises when K can be ordered.

We begin by summing up some relevant facts from the Artin-Schreier theory. Recall that an *ordered field* is a pair (K, P), where K is a field and $P \subseteq K$, the *ordering*, satisfies $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = \emptyset$ and $P \cup -P = K^*$.

PROPOSITION 2.1. *Let L/K be a Galois extension such that $\sqrt{-1} \in L$.*

(i) *Let $\delta \in G(K) = \mathcal{G}(K_s/K)$ be an involution (i.e., $\delta^2 = 1$, $\delta \neq 1$). Then $K_s(\delta)$ is real closed, hence has a unique ordering.*

(ii) *Let P be an ordering of K, and let (L', Q) be a maximal ordered extension of (K, P) such that $L' \subseteq L$. Then there exists an involution $\varepsilon \in \mathcal{G}(L/K)$ such that $L' = L(\varepsilon)$.*

(iii) *Let P be an ordering of K and let $(L(\varepsilon_1), Q_1)$ and $(L(\varepsilon_2), Q_2)$ be two maximal ordered extension of (K, P) contained in L. Then there exists a unique $\sigma \in \mathcal{G}\left(L/K(\sqrt{-1})\right)$ such that $(L(\varepsilon_1), Q_1)^\sigma = (L(\varepsilon_2), Q_2)$; in particular $\varepsilon_1^\sigma = \varepsilon_2$.*

PROOF. (i) This follows from the fact that $[K_s: K_s(\delta)] = 2$ by Lang [18, Cor. 2 on p. 223 and Prop. 3 on p. 274].

(ii) There exists ([18, Theorem 1 on p. 274]) and involution $\delta \in G(L')$ such that Q extends to the real closed field $K_s(\delta)$. Let $\varepsilon = \mathrm{Res}_L\, \delta$; then $\varepsilon^2 = 1$. By the maximality of L', $L' = K_s(\delta) \cap L = L(\varepsilon)$.

(iii) By (ii), there are involutions $\delta_1, \delta_2 \in G(K)$ such that $(L(\varepsilon_i), Q_i) \subseteq (K_s(\delta_i), \bar{Q}_i)$, where $\bar{Q}_i$ is the (unique) ordering of $K_s(\delta_i)$, $i = 1, 2$. Thus $\mathrm{Res}_L\, \delta_1 = \varepsilon_1$, $\mathrm{Res}_L\, \delta_2 = \varepsilon_2$. By [18, Theorem 3 on p. 277] there is a unique K-isomorphism $(K_s(\delta_1), \bar{Q}_1) \to (K_s(\delta_2), \bar{Q}_2)$. Its restriction to $L(\varepsilon_1)$, $\bar{\sigma}$: $(L(\varepsilon_1), Q_1) \to (L(\varepsilon_2), Q_2)$, is a unique K-isomorphism between $(L(\varepsilon_1), Q_1)$ and $(L(\varepsilon_2), Q_2)$, by Prestel [21, p. 42]. Now $L \cong L(\varepsilon_1) \otimes_K K(\sqrt{-1})$, hence $\bar{\sigma}$ can be extended to a unique element $\sigma \in \mathcal{G}(L/K(\sqrt{-1}))$. //

Let L/K be a Galois extension, $\sqrt{-1} \in L$. An involution $\varepsilon \in \mathcal{G}(L/K)$ is *real*, if $L(\varepsilon)$ is a formally real field. The set $X(L/K)$ of the maximal ordered fields in L containing K is called *the space of orderings of L/K*. Proposition 2.1 implies that these fields are of the form $(L(\varepsilon), Q)$, where $\varepsilon \in \mathcal{G}(L/K)$ is a real involution. The map d: $X(L/K) \to \mathcal{G}(L/K)$, defined by $d(L(\varepsilon), Q) = \varepsilon$, is called the *forgetful map*.

If L_0/K is another Galois extension, such that $K(\sqrt{-1}) \subseteq L_0 \subseteq L$, then the *restriction map* Res: $X(L/K) \to X(L_0/K)$, given by $(L(\varepsilon), Q) \mapsto (L_0(\varepsilon), Q \cap L_0(\varepsilon))$, is surjective, by Zorn's Lemma. Note that the forgetful map commutes with the restriction of the spaces of orderings and the restriction of the Galois groups.

Consider the *Harrison topology* on $X(L/K)$ defined via the subbase $\{H_L(a): a \in L^*\}$, where $H_L(a) = \{(L(\varepsilon), Q) \mid a \in Q\}$. The sets $H_L(a)$ are closed-open. Indeed, let L_0/K be a finite Galois extension such that a, $\sqrt{-1} \in L_0 \subseteq L$, and pick up $b_1, \ldots, b_n \in L_0^*$ such that $K(b_1), \ldots, K(b_n)$ are all maximal formally real extensions of K in L_0 which do not contain a. Then clearly

$$X(L/K) - H_L(a) = H_L(-a) \,\dot\cup\, \dot\bigcup_{i=1}^{n} H_L(b_i) \,\dot\cup\, \dot\bigcup_{i=1}^{n} H_L(-b_i),$$

whence $H_L(a)$ is closed.

From this one may prove as an exercise that if L/K is finite, then $X(L/K)$ is a Boolean space (see Prestel [21, Theorem 6.5] for a similar proof). For an arbitrary Galois extension L/K such that $\sqrt{-1} \in L$ it may be verified that $X(L/K) = \varprojlim_{i \in I} X(L_i/K)$, where $\{L_i: i \in I\}$ is the family of finite Galois extensions of K contained in L and containing $\sqrt{-1}$. Thus $X(L/K)$ is a Boolean space.

The restriction map $\mathrm{Res}\colon X(L/K) \to X(L_0/K)$ defined above and the forgetful map $d\colon X(L/K) \to \mathcal{G}(L/K)$ are clearly continuous in the Harrison topology.

The group $\mathcal{G}(L/K)$ acts on $X(L/K)$ in an obvious (and continuous) way. By Proposition 2.1 (iiii) we have that

$$\left\{\sigma \in \mathcal{G}\big(L/K(\sqrt{-1})\big) \mid \big(L(\varepsilon), Q\big)^\sigma = \big(L(\varepsilon), Q\big)\right\} = 1\,,$$

for every $\big(L(\varepsilon), Q\big) \in X(L/K)$.

Finally note that $X\big(K(\sqrt{-1})/K\big) = X(K)$, the space of orderings of K (see [21, p. 88]).

3. – Artin-Schreier structures.

The discussion in Section 2 motivates (see Example 3.2 below) the following abstract definition.

DEFINITION 3.1. *An Artin-Schreier structure G is a system*

$$\mathfrak{G} = \langle G, G', X \xrightarrow{d} G\rangle\,, \tag{1}$$

where

(i) *(X, G) is a profinite topological transformation group (the case $X = \varphi$ is not excluded);*

(ii) *G' is an open subgroup of G of index $\leqslant 2$;*

(iii) *d is a continuous map such that $d(x)$ is an involution in G, $d(x) \notin G'$, $x^{d(x)} = x$ and $d(x^\sigma) = \big(d(x)\big)^\sigma$ for every $x \in X$ and $\sigma \in G$; and*

(iv) *we have for all $x \in X$: $\{\sigma \in G\colon x^\sigma = x\} = \{1, d(x)\}$.*

If a system $\mathfrak{G}$ satisfies only (i)-(iii), we call it a *weak Artin-Schreier structure.*

The Boolean space X is called *the space of orderings of G*; the map d is called *the forgetful map*; its image $d(X)$ is called *the set of real involutions.*

Note that (iv) is equivalent to the condition

(iv′) $\{\sigma \in G'\colon x^\sigma = x\} = \{1\}$ for all $x \in X$.

Also note that $G = G'$ implies $X = \varphi$.

EXAMPLE 3.2. If L/K is a Galois extension and $\sqrt{-1} \in L$, then

$$\mathfrak{G}(L/K) = \langle \mathcal{G}(L/K), \mathcal{G}\big(L/K(\sqrt{-1})\big), X(L/K) \xrightarrow{d} \mathcal{G}(L/K)\rangle$$

is, according to Section 2, an Artin-Schreier structure.

Let $I(L/K)$ be the set of real involutions in $\mathcal{G}(L/K)$, and let $i\colon I(L/K) \to \mathcal{G}(L/K)$ be the inclusion. Then

$$\langle \mathcal{G}(L/K), \mathcal{G}\big(L/K(\sqrt{-1})\big), I(L/K) \xrightarrow{d} \mathcal{G}(L/K)\rangle$$

is a weak Artin-Schreier structure.

If not explicitely stated otherwise, the underlying group, the underlying subgroup, the space of orderings and the forgetful map of an Artin-Schreier structure $\mathfrak{G}$ will be henceforth denoted by G, G', $X(\mathfrak{G})$ and d, respectively. Analogously for $\mathfrak{H}, \mathfrak{A}, \mathfrak{B}$, etc.

DEFINITION 3.3. *Let $\mathfrak{H}$, $\mathfrak{G}$ be (weak) Artin-Schreier structures. A morphism of (weak) Artin-Schreier structures $\varphi\colon \mathfrak{H} \to \mathfrak{G}$ is a pair of continuous maps (both denoted by abuse of notation by φ) $\varphi\colon H \to G$, $\varphi\colon X(\mathfrak{H}) \to X(\mathfrak{G})$ such that*

(i) *$d\big(\varphi(x)\big) = \varphi\big(d(x)\big)$ for every $x \in X(\mathfrak{H})$;*

(ii) *$(\varphi, \varphi)\colon \big(X(\mathfrak{H}), H\big) \to \big(X(\mathfrak{G}), G\big)$ is a morphism of profinite transformation groups, i.e., $\varphi(x^\sigma) = \varphi(x)^{\varphi(\sigma)}$ for all $x \in X(\mathfrak{H})$ and $\sigma \in H$;*

(iii) *$\varphi^{-1}(G') = H'$.*

A morphism $\varphi\colon \mathfrak{H} \to \mathfrak{G}$ is called an epimorphism if $\varphi(H) = G$ and $\varphi\big(X(\mathfrak{H})\big) = X(\mathfrak{G})$ (hence also $\varphi(H') = G'$).

An epimorphism of Aartin-Schreier structures $\varphi\colon \mathfrak{H} \to \mathfrak{G}$ is said to be a cover, if

(iv) *for all $x_1, x_2 \in X(\mathfrak{H})$ such that $\varphi(x_1) = \varphi(x_2)$ there exists a $\sigma \in G$ such that $x_1^\sigma = x_2$. (Then σ can be chosen to be an element of* Ker φ.)

Note that if (i) holds, then (ii) is equivalent to

(ii′) $\varphi(x^\tau) = \varphi(x)^{\tau(\varphi)}$ for all $x \in X(\mathfrak{H})$ and $\tau \in H'$.

Indeed, if $x \in X(\mathfrak{H})$ and $\sigma \in H - H'$, then there is a $\tau \in H'$ such that $\sigma = d(x)\tau$, since $(H\colon H') \leqslant 2$ and $d(x) \notin H'$. But $x^{d(x)} = x$ and $\varphi(x)^{d(\varphi(x))} = \varphi(x)$, hence $\varphi(x^\sigma) = \varphi(x^\tau) = \varphi(x)^{\varphi(\tau)} = \varphi(x)^{d(\varphi(x))\varphi(\tau)} = \varphi(x)^{\varphi(d(x))\varphi(\tau)} = \varphi(x)^{\varphi(\sigma)}$. Also observe that (iii) is equivalent to

(iii′) $\varphi'(H) \subseteq G'$ and $\varphi(H - H') \subseteq G - G'$,

in particular we have $\operatorname{Ker} \varphi \subseteq H'$.

Finally, let $\varphi: \mathfrak{H} \to \mathfrak{G}$ be a morphism of Artin-Schreier structures. Then the map spaces of orderings $\varphi: X(\mathfrak{H}) \to X(\mathfrak{G})$ induces a continuous map $\bar{\varphi}: X(\mathfrak{H})/H' \to X(\mathfrak{G})/G'$. Note that φ is a cover if and only if

(iv′) $\bar{\varphi}$ is a bijection, i.e., a homeomorphism, and $\varphi(H) = G$.

EXAMPKE 3.4. (*a*) Let $L_0 \subseteq L$ be two Galois extensions of K such that $\sqrt{-1} \in L_0$. Then the restriction map $\mathrm{Res}: \mathfrak{G}(L/K) \to \mathfrak{G}(L_0/K)$ is a cover (see Prop. 2.1 (iii)).

(*b*) The restriction map of the corresponding weak Artin-Achreier structures (Example 3.2) is an epimorphism, but need not satisfy condition (iv) of Definition 3.1. Indeed, there may exist two real involutions $\varepsilon, \varepsilon' \in \mathcal{G}(L/K)$ such that $\mathrm{Res}_{L_0} \varepsilon = \mathrm{Res}_{L_0} \varepsilon'$, but no ordering of $L_0(\varepsilon)$ extends both to $L(\varepsilon)$ and $L(\varepsilon')$. Thus ε and ε' are not conjugate.

(*c*) Let t be transcendental over $\mathbb{Q}$. Then the map $\mathrm{Res}: \mathfrak{G}\big(\mathbb{Q}(t, \sqrt{-1})/\mathbb{Q}(t)\big) \to \mathfrak{G}\big(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}\big)$ is an epimorphism of Artin-Schreier structures but not a cover.

Examples 3.4 (*a*) and (*c*) may be generalized as follows:

LEMMA 3.5. *Let L/K and F/E be Galois extensions such that $K \subseteq E$, $\sqrt{-1} \in L \subseteq F$. Then the restriction map* $\mathrm{Res}: \mathfrak{G}(F/E) \to \mathfrak{G}(L/K)$ *is a morphism of Artin-Schreier structures. It is an epimorphism if and only if E/K is a totally real extension, linearly disjoint from L/K. Here E/K is said to be totally real if every ordering of K extends to an ordering of E.*

PROOF. By Example 3.4 (*a*) we may assume that $F = LE$. The Lemma follows from v.d. Dries [4, Chapter II, Lemma 2.5]. //

4. – More about Artin-Schreier structures.

In this Section we develop come concepts and properties of (weak) Artin-Schreier structures needed later on.

First a few remarks:

4.1. Let $\mathfrak{G}$ be a (weak) Artin-Schreier structure and let $N \subseteq G'$ be a closed normal subgroup of G. Define

$$\mathfrak{G}/N = \langle G/N, G'/N, X(\mathfrak{G})/N \xrightarrow{\bar{d}} G/N \rangle$$

where $\big(X(\mathfrak{G})/N, G/N\big)$ is the quotient profinite transformation group (see 1.6) and $\bar{d}$ is the map induced by $d: X(\mathfrak{G}) \to G$.

We leave to the reader the straightforward check that $\mathfrak{G}/N$ is a weak Artin-Schreier structure and that if $\mathfrak{G}$ is an Artin-Schreier structure, then so is $\mathfrak{G}/N$. In the latter case the quotient maps $X(\mathfrak{G}) \to X(\mathfrak{G})/N$ and $G \to G/N$ define a cover. Moreover, every cover may be obtained this way.

4.2. An inverse limit of (weak) Artin-Schreier structures is a (weak) Artin-Schreier structure.

4.3. Let $\mathfrak{G}$ be a weak Artin-Schreier structure. Then $\mathfrak{G} = \varprojlim \mathfrak{G}/N$, where N runs through the family of open normal subgroups of G contained in G'.

In particular, if L/K is a Galois extension and $\sqrt{-1} \in L$, then $\mathfrak{G}(L/K) \cong \varprojlim_{i \in I} \mathfrak{G}(L_i/K)$, where $\{L_i\}_{i \in I}$ is the family of finite Galois extensions of K containing $\sqrt{-1}$ and contained in L.

LEMMA 4.4. *Every (weak) Artin-Schreier structure $\mathfrak{G}$ is an inverse limit of finite (weak) Artin-Schreier structures, which are epimorphic images of $\mathfrak{G}$.*

PROOF. By 4.3 we may assume that the group G is finite. Let $\mathfrak{T}$ be the family of G-partitions Y of $X(\mathfrak{G})$ which

(i) are finer than $\{d^{-1}(\varepsilon) : \varepsilon \in d(X(\mathfrak{G}))\}$, i.e., a map $d_Y : Y \to G$ may be defined by $d_Y(U) = d(x)$ for all $x \in U$ with $U \in Y$;

(ii) $U^\tau \cap U = \varphi$ for all $U \in Y$ and $\tau \in G' - \{1\}$, if $\mathfrak{G}$ is an Artin-Schreier structure.

Every $Y \in \mathfrak{T}$ defines a finite (weak) Artin-Schreier structure $\mathfrak{G}_Y = \langle G, G', Y \xrightarrow{d_Y} G \rangle$. Now to show that $\mathfrak{G} \cong \varprojlim G_Y$, we proceed exactly as in the first part of the proof of Prop. 1.5, but instead of using Lemma 1.4 (ii) we apply the following

CLAIM. Let Y' be a partition of $X = X(\mathfrak{G})$. Then there is a $Y \in \mathfrak{T}$ finer than Y'.

PROOF OF THE CLAIM. If G is a weak Artin-Schreier structure, this follows from Lemma 1.4 (ii). Assume, therefore, that $\mathfrak{G}$ is an Artin-Schreier structure. Let $x \in X$ and let $V \in Y'$ such that $x \in V$. There is a closed-open neighbourhood U_x of x such that $x^\tau \notin U_x$ for all $\tau \in G' - \{1\}$. We may assume that $U_x \subseteq V \cap d^{-1}(d(x))$ and $U_x^\tau \cap U_x = \emptyset$ for all $\tau \in G' - \{1\}$ $\Big($otherwise take $\big(d^{-1}(d(x)) \cap V \cap U_x\big) - \bigcup_{\tau \in G' - \{1\}} U_x^\tau$ instead of $U_x\Big)$. Since X is compact, finitely many of these neighbourhoods cover X. By remark 1.3 there is a partition Y_0 of X such that every $V \in Y_0$ is contained in U_x for some

$x \in X$, hence Y_0 satisfies (i) and (ii) above. Our claim therefore follows by Lemma 1.4 (ii). //

Finite weak Artin-Schreier structures appear naturally, but undesirably in the course of proofs in Section 7. Nevertheless we show in the next Lemma that such a structure $\mathfrak{A}$ is an epimorphic image of a minimal Artin-Schreier structure $\hat{\mathfrak{A}}$, which eventually replaces $\mathfrak{A}$ in the above mentioned proofs.

LEMMA 4.5. *Let $\mathfrak{A}$ be a finite weak Artin-Schreier structure. Then there exists a finite Artin-Schreier structure $\hat{\mathfrak{A}}$ and an epimorphism $p: \hat{\mathfrak{A}} \to \mathfrak{A}$, such that for every Artin-Schreier structure $\mathfrak{B}$ and for every (epi-)morphism of weak Artin-Schreier structures $\alpha: \mathfrak{B} \to \mathfrak{A}$ there exists an (epi-)morphism $\hat{\alpha}: \mathfrak{B} \to \hat{\mathfrak{A}}$ such that $p \circ \hat{\alpha} = \alpha$.*

PROOF. Let $x_1, \ldots, x_n$ be representatives of all the A-orbits in $X(\mathfrak{A})$, and denote $\varepsilon_i = d(x_i)$ (recall that $x_i^{\varepsilon_i} = x_i$), $i = 1, \ldots, n$. Let Z be the set of formal expressions Z_i^τ, where $1 \leq i \leq n$, $\tau \in A'$. The group A acts on each of the subsets $Z_i = \{z_i^\tau : \tau \in A'\}$ by $(z_i^\tau)^\sigma = z_i^{\tau\sigma}$ and $(z_i^\tau)^{\varepsilon_i \sigma} = z_i^{\tau^{\varepsilon_i}\sigma}$ for $\tau, \sigma \in A'$, (recall that $A = A' \,\dot\cup\, \varepsilon_i A'$), whence A acts on $Z = Z_1 \,\dot\cup \ldots \dot\cup\, Z_n$. The map $p: Z \to X(\mathfrak{A})$ defined by $p(z_i^\tau) = x_i^\tau$ is compatible with the action of A It is easily verified that $\hat{\mathfrak{A}} = \langle A, A', Z \xrightarrow{d\circ p} A \rangle$ is an Artin-Schreier structure, and p together with id_A define an epimorphism $p: \hat{\mathfrak{A}} \to \mathfrak{A}$.

Now let $\mathfrak{B}$ be an Artin-Schreier structure and $\alpha: \mathfrak{B} \to \mathfrak{A}$ a morphism We may assume that $\mathfrak{B}$ is finite, otherwise replace $\mathfrak{B}$ by a suitable epimorphic image, using Lemma 4.4. For every $1 \leq i \leq n$ let $\{y_{i1}, \ldots, y_{in_i}\}$ be a maximal subset of $\alpha^{-1}(x_i)$ such that $y_{i1}, \ldots, y_{in_i}$ represent distinct orbits in $X(\mathfrak{B})$. Then $X(\mathfrak{B}) = \{y_{ij}^\tau \mid 1 \leq j \leq n_i,\ 1 \leq i \leq n,\ \tau \in B'\}$. Define $\hat{\alpha}(y_{ij}) = z_i^{\alpha(\tau)}$. Then $\hat{\alpha}$ together with $\alpha: B \to A$ is the desired morphism $\hat{\alpha}: \hat{\mathfrak{B}} \to \hat{\mathfrak{A}}$. Moreover, $\alpha(X(\mathfrak{B})) = X(\mathfrak{A})$ if and only if $\hat{\alpha}(X(\mathfrak{B})) = \hat{X}(\mathfrak{A})$. //

Recall that if $\pi_i: Y_i \to Y$, $i = 1, 2$, are two continuous maps of topological spaces, we denote by $Y_1 \times_Y Y_2$ the (closed) subspace of $Y_1 \times Y_2$ consisting of pairs (y_1, y_2) such that $\pi_1(y_1) = \pi_2(y_2)$.

Let $\pi_1: \mathfrak{B}_1 \to \mathfrak{A}$ and $\pi_2: \mathfrak{B}_2 \to \mathfrak{A}$ be two morphisms of (weak) Artin-Schreier structures. Define

$$\mathfrak{B}_1 \times_{\mathfrak{A}} \mathfrak{B}_2 = \langle B_1 \times_A B_2, B_1' \times_{A'} B_2', X(\mathfrak{B}_1) \times_{X(\mathfrak{A})} X(\mathfrak{B}_2) \xrightarrow{d \times d} B_1 \times_A B_2 \rangle$$

and let $B_1 \times_A B_2$ act on $X(\mathfrak{B}_1) \times_{X(\mathfrak{A})} X(B_2)$ componentwise. It is an instructive exercise to check that the *fibred product* $\mathfrak{B}_1 \times_{\mathfrak{A}} \mathfrak{B}_2$ is a (weak) Artin-Schreier structure, and that the coordinate projections $p_i: \mathfrak{B}_1 \times_{\mathfrak{A}} \mathfrak{B}_2 \to \mathfrak{B}_i$, $i = 1, 2$, are morphism (cf. also Bredon [2, Chapter I, 6 (B)]).

To use fibred products we need the following characterization:

LEMMA 4.6.: *Consider a commutative diagram of (weak) Artin-Schreier structures*

$$(1)\qquad \begin{array}{ccc} \mathfrak{B} & \xrightarrow{p_2} & \mathfrak{B}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathfrak{B}_1 & \xrightarrow[\pi_1]{} & \mathfrak{A}. \end{array}$$

The following statements are equivalent:

(*a*) $\mathfrak{B}$ *is isomorphic to the fibred product* $\mathfrak{B}_1 \times_{\mathfrak{A}} \mathfrak{B}_2$, *i.e., there is an isomorphism* $\theta\colon \mathfrak{B} \to \mathfrak{B}_1 \times_{\mathfrak{A}} \mathfrak{B}_2$, *such that* $p_1 \circ \theta^{-1}$ *and* $p_2 \circ \theta^{-1}$ *are the coordinate projections.*

(*b*) $\mathfrak{B}$ *with* p_1, p_2 *is a pullback of the pair* (π_1, π_2), *i.e., given a weak Artin-Schreier structure* $\mathfrak{C}$ *with morphisms* $\psi_1\colon \mathfrak{C} \to \mathfrak{B}_1$ *and* $\psi_2\colon \mathfrak{C} \to \mathfrak{B}_2$ *such that* $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$, *there is a unique morphism* $\psi\colon \mathfrak{C} \to \mathfrak{B}$ *such that* $p_1 \circ \psi = \psi_1$ *and* $p_2 \circ \psi = \psi_2$.

(*c*) 1. *If* C *is a profinite group and* $\psi_1\colon C \to B_1$ *and* $\psi_2\colon C \to B_2$ *are continuous homomorphisms, then there exists a unique continuous homomorphism* $\psi\colon C \to B$ *such that* $p_1 \circ \psi = \psi_1$ *and* $p_2 \circ \psi = \psi_2$.

2. *If* X *is a topological space and* $\psi_1\colon X \to X(\mathfrak{B}_1)$ *and* $\psi_2\colon X \to X(\mathfrak{B}_2)$ *are continuous maps, there exists a unique continuous map* $\psi\colon X \to X(\mathfrak{B})$ *such that* $p_1 \circ \psi = \psi_1$ *and* $p_2 \circ \psi = \psi_2$.

(*d*) 1. *If* $b_1 \in B_1$, $b_2 \in B_2$ *and* $\pi_1(b_1) = \pi_2(b_2)$, *then there is a unique* $b \in B$ *such that* $p_1(b) = b_1$, $p_2(b) = b_2$ (*if* p_1 *and* p_2 *are surjective this is equivalent to* $\mathrm{Ker}\,(\pi_1 \circ p_1) = \mathrm{Ker}\,(p_1) \times \mathrm{Ker}\,(p_2)$); *and*:

2. *If* $x_1 \in X(\mathfrak{B}_1)$, $x_2 \in X(\mathfrak{B}_2)$ *and* $\pi_1(x_1) = \pi_2(x_2)$, *then there is a unique* $x \in X(\mathfrak{B})$ *such that* $p_1(x) = x_1$, $p_2(x) = x_2$.

PROOF. An analogue of [10, Lemma 1.1]. See also Bredon [2, Chapter I, 6 (B)]. //

We call a diagram (1) a *cartesian square*, if it satisfies one of the equivalent conditions of Lemma 4.6.

The following Lemma gives a very useful example of a cartesian square.

LEMMA 4.7. *Let* $p_1\colon \mathfrak{B} \to \mathfrak{B}_1$ *be an epimorphism of Artin-Schreier structures,* $K \subseteq B'$ *a closed normal subgroup of* B *such that* $K \cap \mathrm{Ker}\,(p_1) = 1$.

Let $p_2\colon \mathfrak{B} \to \mathfrak{B}/K$ *and* $\pi_1\colon \mathfrak{B}_1 \to \mathfrak{B}_1/p_1(K)$ *be the quotient maps. Then there exists a unique epimorphism* $\pi_2\colon \mathfrak{B}/K \to \mathfrak{B}_1/p_1(K)$ *such that*

$$(2)\qquad \begin{array}{ccc} \mathfrak{B} & \xrightarrow{p_2} & \mathfrak{B}/K \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathfrak{B}_1 & \xrightarrow[\pi_1]{} & \mathfrak{B}_1/p_1(K) \end{array}$$

commutes. Moreover, (2) *is a cartesian square.*

PROOF. The map π_2 is defined by the universal property of the quotient $\mathfrak{B}/K$. To show that (2) is a cartesian square we have to verify conditions 1. (which is trivial) and 2. of Lemma 4.6 (*d*).

Indeed, let $x_1 \in X(\mathfrak{B}_1)$ and $x_2 \in X(\mathfrak{B})/K$ with $\pi_1(x_1) = \pi_2(x_2)$. Then there exists an $x \in X(\mathfrak{B})$ such that $p_2(x) = x_2$. We have $\pi_1(x_1) = \pi_1(p_1(x))$, since (2) commutes, hence there exists a $\sigma \in K$ such that $p_1(x^\sigma) = p_1(x)^{p_1(\sigma)} = x_1$ Finally, the element $x^\sigma \in X(\mathfrak{B})$ satisfies also $p_2(x^\sigma) = p_2(x) = x_2$.

If an element $x' \in X(\mathfrak{B})$ also satisfies $p_i(x') = x_i = p_i(x)$, for $i = 1, 2$ then there is a $\tau \in K$ such that $x' = x^\tau$. Therefore $x_1 = p_1(x^\tau) = x_1^{p_1(\tau)}$, hence $p_1(\tau) = 1$. This implies $\tau = 1$, since $K \cap \mathrm{Ker}\,(p_1) = 1$; hence $x' = x$. //

5. - On PRCe fields.

A system $\mathcal{E} = (E, Q_1, \ldots, Q_e)$ consisting of a field E and e orderings $Q_1, \ldots, Q_e$ of E is called an *e-ordered field*. If E is PRC and $Q_1, \ldots, Q_e$ are all its distinct orderings, then $\mathcal{E}$ is said to be a PRCe *field*. An equivalent ([14, Lemmas 2.2 and 2.3] and Prestel [22, Theorems 2.1, 1.2 and Proposition 1.6]) definition is the following:

An e-ordered field $\mathcal{E} = (E, Q_1, \ldots, Q_e)$ is PRCe, if it satisfies;

(i) Let $f \in E[T_1, \ldots, T_r, X]$ be an absolutely irreducible polynomial, let $\boldsymbol{a}_0 \in E^r$ such that $f(\boldsymbol{a}_0, X)$ changes sign on E with respect to each of the Q'_ss, and let U_i be a Q_i-neighbourhood of $\boldsymbol{a}_0$ for $i = 1, \ldots, e$. Then there exists an $(\boldsymbol{a}, b) \in E^{r+1}$ such that $\boldsymbol{a} \in U_1 \cap \ldots \cap U_e$ and $f(\boldsymbol{a}, b) = 0$.

(ii) The orderings $Q_1, \ldots, Q_e$ induce distinct topologies on E.

Let K be a countable Hilbertian field and let $\mathcal{K} = (K, P_1, \ldots, P_e)$ be and e-ordered field, fixed for this Section. For integers $0 \leqslant e \leqslant m$ we denote by $\hat{D}_{e,m}$ the free product (in the category of profinite groups) of e copies of

$\mathbb{Z}/2\mathbb{Z}$ and $m-e$ copies of $\hat{\mathbb{Z}}$. Generalizing results of [16] and of Geyer [8] we show that there is an abundance of PRCe fields $\mathcal{E}$ that extend $\mathcal{K}$ such that E is algebraic over K and $G(E) \cong \hat{D}_{e,m}$.

To do this, fix involutions $\delta_1, \ldots, \delta_e \in G(K)$ such that the real closed fields $\bar{K}_i = K_s(\delta_i)$, $i = 1, \ldots, e$, induce $P_1, \ldots, P_e$ on K, respectively. For every $\sigma = (\sigma_1, \ldots, \sigma_m) \in G(K)^m$ let

$$K_\sigma = \bar{K}_1^{\sigma_1} \cap \ldots \cap \bar{K}_e^{\sigma_e} \cap K_s(\sigma_{e+1}) \cap \ldots \cap K_s(\sigma_m)$$

and denote by $P_{\sigma 1}, \ldots, P_{\sigma e}$ the orderings of K_σ induced by $\bar{K}_1^{\sigma_1}, \ldots, \bar{K}_e^{\sigma_e}$, respectively. Then $\mathcal{K}_\sigma = (K_\sigma, P_{\sigma 1}, \ldots, P_{\sigma e})$ extends $\mathcal{K}$ and $G(K_\sigma) = \langle \delta_1^{\sigma_1}, \ldots, \delta_e^{\sigma_e}, \sigma_{e+1}, \ldots, \sigma_m \rangle$.

LEMMA 5.1 (cf. [16, Lemma 6.4]): *Let $\mathcal{L} = (L, Q_1, \ldots, Q_e)$ be a finite extension of $\mathcal{K}$. Let $f \in L[T_1, \ldots, T_r, X]$ be an absolutely irreducible polynomial and let $0 \neq g \in L[T_1, \ldots, T_r]$. Suppose that there exists an $\mathbf{a}_0 \in L^r$ such that $f(\mathbf{a}_0, X)$ changes sign on L with respect to each of the Q_i's. Let U_i be a Q_i-neighbourhood of $\mathbf{a}_0$ in L^r. Then for almost all $\sigma \in G(K)^m$ for which $\mathcal{L} \subseteq \mathcal{K}_\sigma$ there exists an $(\mathbf{a}, b) \in K_\sigma^{r+1}$ such that $\mathbf{a} \in U_1 \cap \ldots \cap U_e$, $f(\mathbf{a}, b) = 0$ and $g(\mathbf{a}) \neq 0$.*

PROOF. Let $1 \leq i \leq e$, and let $\bar{L}_i$ be a real closure of L that induces Q_i; then there exists a $\tau_i \in G(K)$ such that $\bar{L} = \bar{K}_i^{\tau_i}$. If $\sigma_i \in G(K)$ is an additional element such that $\bar{K}_i^{\sigma_i}$ induces Q_i on L, then there exists a $\lambda \in G(L)$ such that $\bar{K}_i^{\sigma_i} = \bar{K}_i^{\tau_i \lambda}$, i.e., $\bar{L}_i^{\tau_i^{-1}\sigma_i\lambda^{-1}} = \bar{L}_i$. Thus $\tau_i^{-1}\sigma_i\lambda^{-1} \in G(\bar{L}_i)$, since $\bar{L}_i$ has no L-automorphisms besides the identity ([21, Cor. 3.11]), hence $\sigma_i \in \tau_i G(L)$. Conversely $\bar{K}_i^{\tau_i\lambda}$ induces the ordering Q_i on L for every $\lambda \in G(L)$. Put $\tau_{e+1} = \ldots = \tau_m = 1$, and $\tau = (\tau_1, \ldots, \tau_m)$; it follows that $\tau G(L)^m$ is the set of all m-tuples σ in $G(K)^m$ for which $\mathcal{L} \subseteq \mathcal{K}_\sigma$.

Withoug loss of generality we may assume that $f(\mathbf{a}, X)$ changes sign on L with respect to Q_i for every $\mathbf{a} \in U_i$ for $i = 1, \ldots, e$. By Lemma 8.4 of Geyer [8], and since L is Hilbertian, the set $H \cap U_1 \cap \ldots \cap U_e$ is not empty for every Hilbertian set H in L^r. Using the fact that f is absolutely irreducible, one can find a sequence $\mathbf{a}_1, \mathbf{a}_2, \ldots$ of elements in L^r, and a sequence $b_1, b_2, \ldots$ of elements in L_s such that:

a) $\mathbf{a}_j \in U_1 \cap \ldots \cap U_e$ and $f(\mathbf{a}_j, X)$ is an irreducible polynomial over L of degree $n = \deg_X f$ and changes sign on L with respect to Q_i for every $1 \leq i \leq e$ and every j;

b) $f(\mathbf{a}_j, b_j) = 0$ and $g(\mathbf{a}_j) \neq 0$ for every j;

c) denoting $L_j = L(b_j)$, we have that $L_1, L_2, \ldots$ is a linearly disjoint sequence of extensions of L of degree n (cf. the proof of Lemma 2.2 of [14]).

Condition a) implies that each of the Q_i's can be extended to an ordering Q_{ij} of L_j. Let $\mathfrak{L}_j = (L_j, Q_{1j}, \ldots, Q_{ej})$. As in the first paragraph of this proof there is a $\tau^{(j)} \in G(K)^m$ such that $\tau^{(j)}G(L_j)^m$ is the set of all m-tuples σ in $G(K)^m$ for which $\mathfrak{L}_j \subseteq \mathcal{K}_\sigma$. If $\sigma \in \tau^{(j)}G(L_j)^m$, then $(\boldsymbol{a}_j, b_j) \in K_\sigma^{r+1}$, hence σ has the required property. Thus it suffices to show that $\tau G(L)^m - \bigcup_j \tau^{(j)}G(L_j)^m$ is a zero set, or, equivalently, that $G(L)^m - \bigcup_j \tau^{-1}\tau^{(j)}G(L_j)^m$ is a zero set. Now observe that $\mathfrak{L} \subseteq \mathfrak{L}_j$ for every j, hence $\tau^{(j)}G(L_j)^m \subseteq \tau G(L)^m$; in particular, $\tau^{-1}\tau^{(j)} \in G(L)^m$. Hence our result follows by Lemma 6.3 of [16]. //

COROLLARY 5.2. *Almost all $\sigma \in G(K)^m$ have the following property: If $f \in K_\sigma[T_1, \ldots, T_r, X]$ is an absolutely irreducible polynomial for which there exists an $\boldsymbol{a}_0 \in K_\sigma^r$ such that $f(\boldsymbol{a}_0, X)$ changes sign on K_σ with respect to each of the porderings $P_{\sigma i}$, if U_i is a $P_{\sigma i}$-neighbourhood of $\boldsymbol{a}_0$, for $i = 1, \ldots, e$ and if $0 \neq g \in K_\sigma[T_1, \ldots, T_r]$, then there exists an $(\boldsymbol{a}, b) \in K_\sigma^{r+1}$ such that $\boldsymbol{a} \in U_1 \cap \ldots \cap U_e$, $f(\boldsymbol{a}, b) = 0$ and $g(\boldsymbol{a}) \neq 0$.*

PROOF. Use the countability of K and the fact that an intersection of countably many sets of measure 1 has also measure 1. Also observe, that if $f, U_1, \ldots, U_e$ are as above, there exists a finite extension L of K, over which they are defined. Compare the proof of Theorem 2, 5 of [14]. //

LEMMA 5.3. *The orderings $P_{\sigma 1}, \ldots, P_{\sigma e}$ induce distinct topologies on K- for almost all $\sigma \in G(K)^m$.*

PROOF. It suffices to prove that for every $1 \leq k < l \leq e$, every finite extension $\mathfrak{L}$ of $\mathcal{K}$, every $\delta_k, \delta_l \in L$ such that $0 <_k \delta_l <_k 1$, $0 <_l \delta_l <_l 1$ and for almost all $\sigma \in G(K)^m$ such that $\mathfrak{L} \subseteq \mathcal{K}_\sigma$ there exists a $b \in K_\sigma$ such that:

$$(1) \qquad 1 - \delta_k <_k b <_k 1 + \delta_k \quad \text{and} \quad 1 - \delta_l <_l -b <_l 1 + \delta_l\,.$$

With no loss let $k = 1$, $l = 2$.

Let $f(T, X) = X^2 - T$, $g(T) = 1$, $a_0 = 1$, $U_i = (1 - \delta_i/3, 1 + \delta_i/3)$, for $i = 1, 2$, $U_i = (0, 2)$, for $i = 3, \ldots, e$. With these data go over through the proof of Lemma 5.1 and note (at the instance of choosing Q_{ij}) that each of the Q_i's can be extended in exactly two ways to $L_j = L(b_j) = L(\sqrt{a_j})$. Assume, therefore, that we have chosen Q_{1j}, Q_{2j} such that $b_j >_1 0$, $b_j <_2 0$. Then b_j clearly satisfies (1), since $b_j^2 = a_j \in U_1 \cap U_2$, by the construction. Now continue in the proof of Lemma 5.1 and get the required result. //

An m-tuple $\sigma_1, \ldots, \sigma_e, \ldots, \sigma_m$ of elements of $\hat{D}_{e,m}$ is called a *basis*, if $\hat{D}_{e,m} = \langle \sigma_1, \ldots, \sigma_m \rangle$, and $\sigma_1^2 = \ldots = \sigma_e^2 = 1$.

By its definition, $\hat{D}_{e,m}$ has a basis $\sigma_1, \ldots, \sigma_m$ with the following *extension property*: if G is a profinite group, $\bar{\sigma}_1, \ldots, \bar{\sigma}_m \in G$, and $\bar{\sigma}_1^2 = \ldots = \bar{\sigma}_e^2 = 1$, then the map $\sigma_i \to \bar{\sigma}_i$, $i = 1, \ldots, m$, can be extended to a homomorphism $\hat{D}_{e,m} \to G$. In particular, if $\sigma_1', \ldots \sigma_m'$ is another basis of $\hat{D}_{e,m}$, then the map $\sigma_i \to \sigma_i'$ can be extended to an epimorphism $\hat{D}_{e,m} \to \hat{D}_{e,m}$. By [23, Cor. 7.7] this is an isomorphism. Therefore every basis of $\hat{D}_{e,m}$ has the extension property.

The following Lemma gives a useful characterization of $\hat{D}_{e,m}$.

LEMMA 5.4. *Let G be a profinite group generated by m elements, e of which are involutions. Then G is isomorphic to $\hat{D}_{e,m}$ if and only if every finite group generated by m elements, e of which are involutions, is a homomorphic image of G.*

PROOF. The isomorphism class of a finitely generated group is determined by its finite homomorphic images. Moreover, if $G = \langle \sigma_1, \ldots, \sigma_m \rangle$ is a finite group such that $\sigma_1^2 = \ldots \sigma_e^2 = 1$, one may easily construct another finite group $G' = \langle \sigma_1', \ldots, \sigma_m' \rangle$ such that $\sigma_1', \ldots, \sigma_e'$ are involutions, and G is a homomorphic image of G'. //

LEMMA 5.5. *For almost all $\sigma \in G(K)^m$, $\delta_1^{\sigma_1}, \ldots, \delta_e^{\sigma_e}, \sigma_{e+1}, \ldots, \sigma_m$ is a basis for $\hat{D}_{e,m}$.*

PROOF. By the preceding Lemma it suffices to show that if $G = \langle \tau_1, \ldots, \tau_m \rangle$ is a finite group and $\tau_1, \ldots, \tau_e$ are involutions, then G is a homomorhic image of $\langle \delta_1^{\sigma_1}, \ldots, \delta_e^{\sigma_e}, \sigma_{e+1}, \ldots, \sigma_m \rangle$, for almost all σ.

With no loss $m \geq 1$. Also, we may assume that G is of even order (otherwise $e = 0$, and, moreover, G may be replaced by the group $G \times \mathbb{Z}/2\mathbb{Z}$, which is also generated by m generators), say $|G| = 2n$.

Let $1 \leq i \leq 2$. The polynomial $g_0(X) = (X^2 + 1^2) \ldots (X^2 + n^2)$ has no roots in the real closed field $K_s(\delta_i)$. By Sturm,s Theorem (cf. [8, Lemma 8.2]), if $g \in K[X]$ is close enough to g_0, with respect to P_i, then g also has no roots in $K_s(\delta_i)$. Therefore by Lemma 8.4 of [8] we can construct a sequence of polynomials $g_1, g_2, \ldots$ in $K[X]$ that satisfy:

a) $\deg g_j = 2n$, and $\mathcal{G}(g_j, K)$, as the group of permutations on the roots of g_j, is the full symmetric group S_{2n};

b) g_j is close enough to g_0 with repsect to $P_1, \ldots, P_e$, in particular g_j has no roots in $K_s(\delta_1), \ldots, K_s(\delta_e)$;

c) denoting by L_j the splitting field of g_j over K we have that $L_1, L_2, \ldots$ is a linearly disjoint sequence of extensions of K of degree $(2m)!$.

Fix $j \geq 1$ and denote $\bar{\delta}_i = \mathrm{Res}_{L_j} \delta_i$ for $i = 1, \ldots, e$. Condition *b*) implies that g_j factors over each of the fields $K_s(\delta_1), \ldots., K_s(\delta_e)$ into a product of n irreducible quadratic factors. Hence the representations of $\delta_{1j}, \ldots, \delta_{ej}$ as permutations (in S_{2n}) of the roots of g_j are products of n disjoint 2-cycles.

We may embed G in $S_{2n} = \mathcal{G}(L_j/K)$ by letting it act on itself by multiplication from the right. In this representation all involutions of G, in in particular $\tau_1, \ldots, \tau_e$, are products of n disjoint 2-cycles.

It is a well known fact that all products of n disjoint 2-cycles in S_{2n} are conjugate to each other. Thus there are $\bar{\sigma}_1, \ldots, \bar{\sigma}_e \in \mathcal{G}(L_j/K)$ such that $\bar{\delta}_i^{\bar{\sigma}_i} = \tau_i$ for $i = 1, \ldots, e$. Choose $\sigma_1, \ldots, \sigma_m \in G(K)$ such that $\mathrm{Res}_{L_j} \sigma_i = \bar{\sigma}_i$ for $i = 1, \ldots, e$ and $\mathrm{Res}_{L_j} \sigma_i = \tau_i$ for $i = e+1, \ldots, m$. Then

$$G = \mathrm{Res}_{L_j} \langle \delta_1^{\sigma_1}, \ldots, \delta_e^{\sigma_e}, \sigma_{e+1}, \ldots, \sigma_m \rangle .$$

Condition c) implies (cf. Lemma 6.3 of [16]) that almost all $\sigma = (\sigma_1, \ldots, \sigma_m) \in G(K)^m$ have this property. //

REMARK. The case $m = e$ is Theorem 4.3 of Geyer [8]. The case $e = 1$ is proved by Mckenna. The case $e = 0$ is contained in Theorem 5.1 of [15].

The results proved in this section yield (cf., Lemma 2.3 and Lemma 2.6 of [16]).

PROPOSITION 5.6. *For almost all $\sigma \in G(K)^m$ the field $\mathcal{K}_\sigma$ is* PRC*e and $G(K_\sigma) \cong \hat{D}_{e,m}$.*

6. – The Artin-Schreier structures associated with $\hat{D}_{e,m}$.

There is an interesting group-theoretic corollary of Proposition 5.6:

PROPOSITION 6.1. *There are exactly e conjugacy classes of involutions in $\hat{D}_{e,m}$. If $\sigma_1, \ldots, \sigma_m$ is a basis for $\hat{D}_{e,m}$, then $\sigma_1, \ldots, \sigma_e$ represent these classes. Moreover, the subgroups $\langle\sigma_1\rangle, \ldots, \langle\sigma_e\rangle$ are their own normalizers in $\hat{D}_{e,m}$.*

PROOF. There exists an epimorphism $\varphi: \hat{D}_{e,m} \to (\mathbb{Z}/2\mathbb{Z})^e$ such that $\varphi(\sigma_1), \ldots, \varphi(\sigma_e)$ are distinct, hence not conjugate to each other. Thus $\sigma_1, \ldots, \sigma_e$ are not conjugate. On the other hand there exists a PRCe field M with $G(M) \cong \hat{D}_{e,m}$. Thus our assertion follows by Proposition 2.1. //

COROLLARY 6.2. *Let A be a finite group and let $I \subseteq A$ be a set of involutions closed under conjugation. Then there exists a finite group B and an epimorphism $\theta: B \to A$, which maps the involutions of B-*Ker *θ onto I.*

PROOF. Let $\bar{\sigma}_1, \ldots, \bar{\sigma}_e$ be representatives of the conjugacy classes of I and let $\bar{\sigma}_{e+1}, \ldots, \bar{\sigma}_m \in A$ such that $A = \langle \bar{\sigma}_1, \ldots, \bar{\sigma}_e, \ldots, \bar{\sigma}_m \rangle$. Let $\sigma_1, \ldots, \sigma_m$ be a basis for $\hat{D} = \hat{D}_{e,m}$. Define an epimorphism $\varphi: \hat{D}_{e,m} \to A$ by $\sigma_i \mapsto \bar{\sigma}_i$.

By Proposition 6.1 there are no involutions in the closed subset $S=\varphi^{-1}(A-(I\cup\{1\}))$ of $\hat{D}$, hence $S^2=\{\tau^2\mid\tau\in S\}$ is closed, and $1\notin S^2$. Thus there exists an open normal subgroup U of $\hat{D}$ such that $S^2\cap U=\emptyset$. With no loss $U\leqslant\operatorname{Ker}\varphi$. Let $B=\hat{D}/U$ and let $\theta\colon B\to A$ be the epimorphism induced by φ.

Now if $\sigma\in\hat{D}$ and σU is an involution in B, then $\sigma^2\in U$, hence $\sigma\notin S$. Therefore $\varphi(\sigma)\in I\cup\{1\}$, whence $\theta(\sigma U)\in I$ or $\theta(\sigma U)=1$. Conversely, $\theta(\sigma_i U)=\varphi(\sigma_i)=\bar{\sigma}_i$, and the Corollary follows. //

We now apply the results of this Section to Artin-Schreier structures. Let $\hat{D}=\hat{D}_{e,m}$, and denote by $I(\hat{D})$ the set of involutions in $\hat{D}$. Let

$$\mathfrak{D}'=\{D'<\hat{D}\mid D' \text{ is open in } \hat{D},\ (\hat{D}:D')\leqslant 2 \text{ and } D'\cap I(\hat{D})=\emptyset\}\,.$$

Note that $\mathfrak{D}'\neq\emptyset$. Indeed, if $\sigma_1,\ldots,\sigma_m$ is a basis of $\hat{D}$, then every map $\varphi_0\colon\{\sigma_1,\ldots,\sigma_m\}\to\{\pm 1\}$ such that $\varphi_0(\sigma_1)=\ldots=\varphi_0(\sigma_e)=-1$ extends to a unique homomorphism $\varphi\colon\hat{D}\to\{\pm 1\}$ and $\operatorname{Ker}\varphi\in\mathfrak{D}'$ Conversely, every $D'\in\mathfrak{D}'$ defines a map $\varphi_0\colon\{\sigma_1,\ldots,\sigma_m\}\to\{\pm 1\}$ by: $\varphi_0(\sigma_i)=1$ if and only if $\sigma_i\in D'$. The bijective correspondence $\varphi_0\leftrightarrow D'$ shows, in fact, that $\mathfrak{D}'$ has precisely 2^{m-e} elements.

PROPOSITION 6.3. *Let $D'\in\mathfrak{D}'$. Then*

(i) $\hat{\mathfrak{D}}=\langle\hat{D},D',I(\hat{D})\xrightarrow{\text{incl.}}\hat{D}\rangle$ *is an Artin-Schreier structure.*

(ii) *Let $\sigma_1,\ldots,\sigma_m$ be a basis of $\hat{D}=\hat{D}_{e,m}$. Let $\mathfrak{G}$ be a weak Artin-Schreier structure, let $x_1,\ldots,x_e\in X(\mathfrak{G})$ and $g_1,\ldots,g_m\in G$ such that $d(x_i)=g_i$, for $i=1,\ldots,e$. Assume that $g_i\in G'$ if and only if $\sigma_i\in D'$, for $i=1,\ldots,m$. Then there exist unique maps $\varphi_0\colon\hat{D}\to G$ and $\varphi_1\colon I(\hat{D})\to X(\mathfrak{G})$ such that $\varphi_0(\sigma_i)=g_i$, for $i=1,\ldots,m$ and $\varphi_1(\sigma_i)=x_i$, for $i=1,\ldots,e$, and $\varphi=(\varphi_0,\varphi_1)\colon\hat{D}\to\mathfrak{G}$ is a morphism of weak Artin-Schreier structures.*

Moreover, if $\mathfrak{G}$ is an Artin-Schreier structure, $x_1,\ldots,x_e$ are representatives of the distinct orbits in $X(\mathfrak{G})$ and $G=\langle g_1,\ldots,g_m\rangle$, then φ is a cover.

PROOF. (i) follows from Proposition 6.1.

(ii) The homomorphism φ_0 is uniquely defined by the extension property of $\hat{D}$. Define φ_1 by $\varphi_1(\sigma_i^\tau)=x_i^{\varphi_0(\tau)}$, for $i=1,\ldots,e$ and $\tau\in D'$. This is a well-defined continuous map, since the map $(\tau,\sigma)\mapsto\sigma^\tau$ from $D'\times\{\sigma_1,\ldots,\sigma_e\}$ into $I(\hat{D})$ is, by Proposition 6.1, a bijective continuous closed map, hence a homeomorphism and thus has a continuous inverse.

Finally note that $\varphi_0^{-1}(G')$ belongs to $\mathfrak{D}'$ and contains the same σ_i's as D', hence $\varphi_0^{-1}(G')=D'$. The map (φ_0,φ_1) defined above is clearly a morphism. //

7. – Projective Artin-Schreier structures.

In a complete analogy to the cateaory of profinite groups we introduce embedding problems and define the notion of projectivity for Artin-Schreier structures.

Let $\mathfrak{G}$ be a weak Artin-Schreier structure. A diagram

$$(1) \qquad \begin{array}{ccc} & & \mathfrak{G} \\ & & \downarrow{\scriptstyle \varphi} \\ \mathfrak{B} & \xrightarrow{\alpha} & \mathfrak{A} \end{array}$$

where φ is a morphism and α an epimorphism of weak Artin-Schreier structures, is called a *weak embedding problem for G*. If both $\mathfrak{B}$ and $\mathfrak{A}$ are Artin-Schreier structures, and α is a cover, we call (1) an *embedding problem*. The problem is said to be *finite* if both $\mathfrak{B}$ and $\mathfrak{A}$ are finite.

A morphism $\gamma\colon \mathfrak{G} \to \mathfrak{B}$ is called a *solution* of the problem, if $\alpha \circ \gamma = \varphi$.

DEFINITION 7.1. *An Artin-Schreier structure $\mathfrak{G}$ is projective, if every embedding problem for $\mathfrak{G}$ has a solution.*

EXAMPLE 7.2. Let $\hat{D} = \hat{D}_{e,m}$, let $I(\hat{D})$ be the set of involutions in $\hat{D}$ and D' an open subgroup of $\hat{D}$ of index $\leqslant 2$ which does not meet $I(\hat{D})$. Then $\hat{\mathfrak{D}} = \langle \hat{D}, D', I(\hat{D}) \overset{\text{incl.}}{\longrightarrow} \hat{D} \rangle$ is a projective Artin-Schreier structure. Indeed, let $\sigma_1, \ldots, \sigma_m$ be a basis of $\hat{D}$. Consider an embedding problem (1) with $\mathfrak{G} = \mathrm{D}$, and choose $x_1, \ldots, x_e \in X(\mathfrak{B})$ and $b_1, \ldots, b_m \in B$ such that $b_i = d(x_i)$ and $\alpha(x_i) = \varphi(\sigma_i)$, for $i = 1, \ldots, e$ and $\alpha(b_i) = \varphi(\sigma_i)$, for $i = 1, \ldots, m$. Note that for every $1 \leqslant i \leqslant m$ we have: $\sigma_i \in D' \Leftrightarrow \varphi(\sigma_i) \in A' \Leftrightarrow b_i \in B'$. By Proposition 6.3 the problem (1) has a solution.

As in the case of profinite groups (Gruenberg [9, Proposition 1]), we have the following test for the projectivity of Artin-Schreier structures:

LEMMA 7.3. *An Artin-Schreier structure $\mathfrak{G}$ is projective if and only if every finite embedding problem for $\mathfrak{G}$ has a solution.*

PROOF. Assume that every finite embedding problem for $\mathfrak{G}$ is solvable and let (1) be an embedding problem for $\mathfrak{G}$. Let K be the kernel of the empimorphism $\alpha\colon B \to A$. Then $K \leqslant B'$ and $\mathfrak{A} \cong \mathfrak{B}/K$, since $\alpha\colon \mathfrak{B} \to \mathfrak{A}$ is a cover. Thus, with no loss, α is the quotient map $\mathfrak{B} \to \mathfrak{B}/K$ (see 4.1).

We claim that (1) has a solution. The proof of this assertion is divided into two parts.

Part I. The kernel K is finite.

Assume that K is finite. Then $\{1\}$ is open in K, hence there is an open subgroup M in B such that $M \cap K = \{1\}$. By Lemma 4.4 there exists a finite Artin-Schreier structure $\mathfrak{B}_0$ and an epimorphism $p\colon \mathfrak{B} \to \mathfrak{B}_0$ such that $\operatorname{Ker} p \leqslant M$, whence $\operatorname{Ker}(p) \cap K = \{1\}$. Let $\mathfrak{A}_0 = \mathfrak{B}_0/p(K)$, and let $\alpha_0\colon \mathfrak{B}_0 \to \mathfrak{A}_0$ be the quotient map. By Lemma 4.7 there exists a cartesian diagram of epimorphisms of Artin-Schreier structures

(2)
$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\alpha} & \mathfrak{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{B}_0 & \xrightarrow[\alpha_0]{} & \mathfrak{A}_0 \end{array}$$

By assumption, there is a morphism $\gamma_0\colon \mathfrak{G} \to \mathfrak{B}_0$ such that $\alpha_0 \circ \gamma_0 = \pi \circ \varphi$. By 4.6 (*b*) there exists a morphism $\gamma\colon \mathfrak{G} \to \mathfrak{B}$ such that $\alpha \circ \gamma = \varphi$ (and $p \circ \gamma = \gamma_0$), i.e., γ is a solution of the embedding problem (1).

Part II. The general case.

Let Γ be the family of pairs (L, λ), where L is a closed normal subgroup of B contained in K and $\lambda\colon \mathfrak{G} \to \mathfrak{B}/L$ is a morphism such that

$$\begin{array}{ccc} & \overset{\lambda}{\swarrow} & \mathfrak{G} \\ & & \downarrow{\scriptstyle \varphi} \\ \mathfrak{B}/L & \xrightarrow[\alpha_L]{} & \mathfrak{B}/K \quad (=\mathfrak{A}) \end{array}$$

commutes (α_L is the cover induced by $L \leqslant K$). Partially order Γ by letting $(L', \lambda') \geqslant (L, \lambda)$ mean that $L' \leqslant L$ and

(3)
$$\begin{array}{ccccc} & & \overset{\lambda'}{\swarrow\!\!\!-\!\!\!-} & & \mathfrak{G} \\ & & & \overset{\lambda}{\swarrow} & \\ \mathfrak{B}/L' & & \longrightarrow & & \mathfrak{B}/L \end{array}$$

commutes. Then Γ is inductive and by Zorn's Lemma it has a maximal element (L, λ). If $L \neq 1$, there is an open normal subgroup N in B such that $L \not\leq N$; hence $L' = N \cap L$ is a proper open normal subgroup of L. By part I of this proof there is a morphism $\lambda' : \mathfrak{G} \to \mathfrak{B}/L'$ such that (3) commutes. Then $(L', \lambda') \in \Gamma$ and $(L', \lambda') > (L, \lambda)$, which is a contradition. Thus $L = 1$, as required. //

Projective Artin-Schreier structures have some interesting properties:

PROPOSITION 7.4. *Let $\mathfrak{G}$ be a projective Artin-Schreier structure. Then its forgetful map $d: X(\mathfrak{G}) \to G$ is injective and $d(X(\mathfrak{G}))$ is the set of all involutions in G.*

PROOF. Let $x_1, x_2 \in X(\mathfrak{G})$ such that $x_1 \neq x_2$. By Lemma 4.4 there is a finite Artin-Schreier structure $\mathfrak{A}$ and an epimorphism $\varphi: \mathfrak{G} \to \mathfrak{A}$ such that $\varphi(x_1) \neq \varphi(x_2)$. By Prop. 6.3 (ii) there exists an Artin-Schreier structure $\hat{\mathfrak{D}}$ and a cover $\alpha: \hat{\mathfrak{D}} \to \mathfrak{A}$ such that $d: X(\hat{\mathfrak{D}}) \to \hat{D}$ is injective and $d(X(\hat{\mathfrak{D}}))$ is the set of all involutions in $\hat{D}$. Now $\mathfrak{G}$ is projective, hence there is a morphism $\gamma: \mathfrak{G} \to \mathfrak{D}$ such that $\alpha \circ \gamma = \varphi$. The condition $\varphi(x_1) \neq \varphi(x_2)$ implies $\gamma(x_1) \neq \gamma(x_2)$, hence $d(\gamma(x_1)) \neq d(\gamma(x_2))$, i.e., $\gamma(d(x_1)) \neq \gamma(d(x_2))$, whence $d(x_1) \neq d(x_2)$.

Suppose that there is an involution $\varepsilon \in G$ and $\varepsilon \notin d(X(\mathfrak{G}))$. With no loss $\varphi(\varepsilon)$ is an involution in A and $\varphi(\varepsilon) \notin d(X(\mathfrak{A}))$, again, by 4.4. Then also $\gamma(\varepsilon)$ is an involution in $\hat{D}$ and $\gamma(\varepsilon) \notin d(X(\hat{\mathfrak{D}}))$, a contradiction. //

REMARK. *a*) We have the following corollary to Prop. 7.4:

Let $\mathfrak{G}$ be a projective Artin-Schreier structure and let L/K be a Galois extensin with $\sqrt{-1} \in L$. Assume that there exists an isomorphism $\varphi: G \to \mathcal{G}(L/K)$ such that $\varphi(G') = \mathcal{G}(L/K(\sqrt{-1}))$. Then $\mathfrak{G} \cong \mathfrak{G}(L/K)$ if and only if $L(\varepsilon)$ has a unique ordering for every involution $\varepsilon \in \mathcal{G}(L/K)$.

b) Observe that the absolute Artin-Schreier structure $\mathfrak{G}(K)$ of a field K satisfies the two assertions in the Proposition. Proposition 7.4 is therefore the first step in an effort to find for a projective Artin-Schreier structure $\mathfrak{G}$ a field K such that $\mathfrak{G} = \mathfrak{G}(K)$.

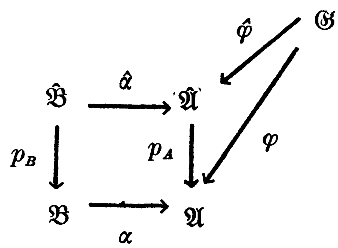
LEMMA 7.5. *Let $\mathfrak{G}$ be a weak Artin-Schreier structure. The following conditions are equivalent*:

(i) *$\mathfrak{G}$ is a projective Artin-Schreier structure*;

(ii) *every finite weak embedding problem for $\mathfrak{G}$ has a solution*;

(iii) *the forgetful map of* $\mathfrak{G}$ *is injective and every finite weak embedding problem* (1) *for* $\mathfrak{G}$, *in which the forgetful maps of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are inclusions, has a solution.*

Proof. (i) $\Rightarrow$ (ii). Let (1) be a finite weak embedding problem for $\mathfrak{G}$. Applying twice 4.5, we obtain Artin-Schreier structures $\hat{\mathfrak{A}}$, $\hat{\mathfrak{B}}$ a and a commutative diagram



in which $\hat{\alpha}$ is an epimorphism. If $X(\hat{\mathfrak{B}})$ is replaced by a minimal subset $X'(\hat{\mathfrak{B}})$ of $X(\hat{\mathfrak{B}})$, closed under the action of $\hat{B}$ and satisfying $\alpha(X'(\hat{\mathfrak{B}})) = X(\hat{\mathfrak{A}})$ then $\hat{\alpha}$ will be a cover (though p_B need not be surjective any more). By (i), there exists a $\hat{\gamma}\colon \mathfrak{G} \to \mathfrak{B}$ such that $\hat{\alpha}\circ\hat{\gamma} = \hat{\varphi}$. Clearly $p_B\circ\hat{\gamma}$ solves (1).

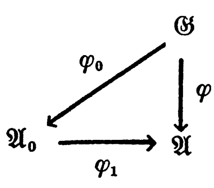
(ii) $\Rightarrow$ (i): We show that $\mathfrak{G}$ satisfies condition (iv′) of Definition 3.1; the rest follows from Lemma 7.3.

Let $x \in X(\mathfrak{G})$ and $\sigma \in G'$ such that $x^\sigma = x$, and assume that $\sigma \neq 1$. By Lemma 4.4 there is a finite weak Artin-Schreier structure $\mathfrak{A}$ and an epimorphism $\varphi\colon \mathfrak{G} \to \mathfrak{A}$ such that $\varphi(\sigma) \neq 1$. Let $p_A\colon \hat{\mathfrak{A}} \to \mathfrak{A}$ be an epimorphism which satisfies the conditions of Lemma 4.5. By (ii) there is a $\gamma\colon \mathfrak{G} \to \hat{\mathfrak{A}}$ such that $p_A\circ\gamma = \varphi$. Now $\gamma(x)^{\gamma(\sigma)} = \gamma(x)$ and $\gamma(\sigma) \in \hat{A}'$, hence $\gamma(\sigma) = 1$, since $\hat{\mathfrak{A}}$ is an Artin-Schreier structure. This implies $\varphi(\sigma) = p_A(\gamma(\sigma)) = 1$, a contraditction

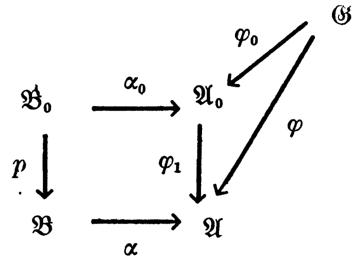
(i) and (ii) $\Rightarrow$ (iii): The forgetful map of $\mathfrak{G}$ is injective by Prop. 7.4.

(iii) $\Rightarrow$ (ii): It is easily seen that $\mathfrak{G}$ is an inverse limit of finite weak Artin-Schreier structures $\mathfrak{A}_0$ which are epimorphic images of $\mathfrak{G}$ and whose forgetful maps are inclusions. Thus there exists such an $\mathfrak{A}_0$ and a commutative diagram

(4)

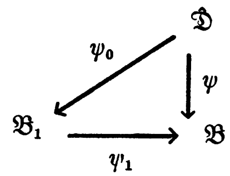


Let $\mathfrak{B}_0 = \mathfrak{B} \times_{\mathfrak{A}} \mathfrak{A}_0$. Then we have a commutative diagram



We may therefore assume that the forgetful map of $\mathfrak{A}$ is an inclusion and φ is an epimorphism.

By Prop. 6.3, there exists an epimorphism $\psi\colon \hat{\mathfrak{D}} \to \mathfrak{B}$ such that the forgetful map of $\hat{\mathfrak{D}}$ is an inclusion. Exactly as for φ in (4) there esists a finite weak Artin-Schreier structure $\mathfrak{B}_1$, whose forgetful map is an inclusion and a commutative diagram

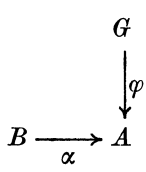


Replacing $\mathfrak{B} \xrightarrow{\alpha} \mathfrak{A}$ by $\mathfrak{B}_1 \xrightarrow{\alpha\circ\psi_1} \mathfrak{A}$ we may assume that the forgetful map of $\mathfrak{B}$ is an inclusion, and then apply (iii). //

Lemmas 7.4 and 7.5 indicate that the projectivity of Artin-Schreier structures might be expressed by pure group-theoretic terms. To this end we need some definitions.

Let G be a profinite group. A diagram

(5)



in which $\alpha\colon B \to A$ is an epimorphism of groups, and φ is a homomorphism, is called a *real embedding problem for* G, if for every involution $x \in G$ such that $\varphi(x) \neq 1$ there exists an involution $b \in B$ such that $\alpha(b) = \varphi(x)$.

A homomorphism $\gamma\colon G \to B$ is called a solution of (5), if $\alpha \circ \gamma = \varphi$.

A group G is said to be *real projective* if the set $I(G)$ of involutions in G is closed in G and every finite real embedding problem for G is solvable.

Note that if G is real projective, then G is a projective group if and only if there are no involutions in G.

REMARK 7.6. Let G be a profinite group and $I(G)$ the set of involutions in G. If $g \in G$ belongs to the closure of $I(G)$, then $g^2 = 1$, hence $g \in I(G)$ or $g = 1$. Therefore $I(G)$ is closed if and only if there exists an open normal subgroup U of G such that $U \cap I(G) = \emptyset$.

PROPOSITION 7.7. *Let G be a profinite group and $I(G)$ the set of involutions in G. Denote*

$$\mathcal{G}' = \{G' \lhd G \mid G' \text{ is open, } (G: G') \leqslant 2 \text{ and } G' \cap I(G) = \emptyset\}$$

Then the following two conditions are equivalent:

(i) *G is real projective;*

(ii) *$\mathcal{G}' \neq \emptyset$ and for every (or, equivalently, for some) $G' \in \mathcal{G}'$*

$$\mathfrak{G} = \langle G, G', I(G) \overset{\text{incl.}}{\to} G \rangle$$

is a projective Artin-Schreier structure.

PROOF. (i) $\Rightarrow$ (ii):

Part I. We show that $\mathcal{G}' \neq \emptyset$.

By remark 7.6, there is an open $U \lhd G$ such that $U \cap I(G) = \emptyset$. Let $\varphi: G \to G/U$ be the quotient map. Let $A_0 = \{\pm 1\} \times G/U, I = \{(-1, \varphi(\varepsilon)) \mid \varepsilon \in I(G)\}$ and let $\pi: A_0 \to G/U$ be the coordinate projection. By Cor. 6.2, there exists a finite group B_0 and an epimorphism $\alpha: B_0 \to A_0$ such that $\alpha(I(B_0)) = I$, where $I(B_0)$ is the set of involutions in $B_0 - \operatorname{Ker} \alpha$. By (i) there is a homomorphism $\gamma: G \to B_0$ such that $(\pi \circ \alpha) \circ \gamma = \varphi$. Let $A_0' = \{(1, a) \mid a \in G/U\}$ and let $G' = (\alpha \circ \gamma)^{-1}(A_0')$; then G' is an open subgroup of index 2 in G. Clearly $\gamma(I(G)) \subseteq I(B_0)$, hence $(\alpha \circ \gamma)(I(G)) \subseteq I$; on the other hand $(\alpha \circ \gamma)(G') \subseteq A_0'$. But $A' \cap I = \emptyset$, hence $G' \cap I(G) = \emptyset$, whence $G' \in \mathcal{G}'$.

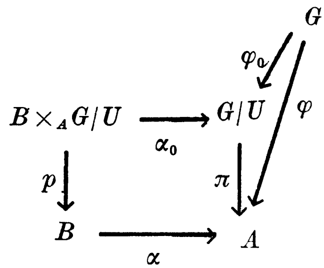
Part II. Projectivity.

Let $G' \in \mathcal{G}'$. We prove that the weak Artin-Schreier structure $\mathfrak{G} = \langle G, G', I(G) \overset{\text{incl.}}{\to} G \rangle$ is a projective Artin-Schreier structure. By Lemma 7.5, it suffices to show that every finite weak embedding problem (1), in which the forgetful maps of $\mathfrak{A}$ and $\mathfrak{B}$ are inclusions, has a solution.

The corresponding problem (5) for groups defined by (1) is real. Hence, by (i), there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha\circ\gamma = \varphi$. However, γ is not necessarily a morphism of weak Artin-Schreier structure, since it may happen that $\gamma(I(G)) \not\subseteq X(\mathfrak{B})$.

Nevertheless, by Corollary 6.2, there exists a finite group B_1 and an epimorphism $\theta\colon B_1 \to B$ such that $\theta(I(B_1)) = X(\mathfrak{B})$, where $I(B_1)$ is the set of all involutions in $B_1 - \operatorname{Ker}\theta$. We may replace $\mathfrak{B}$ by $\mathfrak{B}_1 = \langle B_1, \theta^{-1}(B'), I(B_1) \overset{\text{incl.}}{\to} B_1\rangle$, and thus assume that $X(\mathfrak{B})$ is the set of all involutions in $B - B'$. Then $\alpha(\gamma(\varepsilon)) = \varphi(\varepsilon) \in X(\mathfrak{A})$ for every $\varepsilon \in I(G)$, hence $\gamma(\varepsilon) \notin B'$, whence $\gamma(\varepsilon) \in X(\mathfrak{B})$. It follows that γ may be also regarded as a morphism $\gamma\colon \mathfrak{G} \to \mathfrak{B}$, and thus Problem (1) has a solution.

(ii) $\Rightarrow$ (i): By Remark 7.6, $I(G)$ is closed. Let (5) be a finite real embedding problem for G. There is an open $U \lhd G$ such that $U \cap I(G) = \emptyset$ and $U \leqslant G' \cap \operatorname{Ker}(\varphi)$. Let $\varphi_0\colon G \to G/U$ be the quotient map. Then $1 \notin \varphi_0(I(G))$. We obtain a commutative diagram of groups with a cartesian square



and it is easily seen that α_0 and φ_0 define a finite real embedding problem for G.

Thus with no loss $\varphi(I(G))$ is a set of involutions, φ is an epimorphism and $\operatorname{Ker}(\varphi) \leqslant G$. Let $A' = \varphi(G')$, $I(A) = \varphi(I(G))$, $B' = \alpha^{-1}(A')$ and $I(B) = \{\varepsilon \in B \mid \varepsilon^2 = 1,\ \alpha(\varepsilon) \in I(A)\}$. Then $\mathfrak{A} = \langle A, A', I(A) \overset{\text{incl.}}{\to} A\rangle$, $\mathfrak{B} = \langle B, B', I(B) \overset{\text{incl.}}{\to} B\rangle$ are weak Artin-Schreier structures, and $\alpha\colon \mathfrak{B} \to \mathfrak{A}$ and $\varphi\colon \mathfrak{G} \to \mathfrak{A}$ are epimorphisms. By (ii) and by Lemma 7.5, there is a $\gamma\colon \mathfrak{G} \to \mathfrak{B}$ such that $\alpha\circ\gamma = \varphi$. //

8. – Restrictions of orderings of fields.

We extend results of Elman, Lam and Wadsworth [5] and show that the restriction maps of orderings under finitely generated extensions have continuous sections. We also show that for every closed subset C of a space of orderings $X(K)$ of a field K there exists a regular extension E/K such that $\operatorname{Res}_{E/K}$ maps $X(E)$ injectively onto C.

LEMMA 8.1. *Let E/K be a finite extension. Then $\mathrm{Res}_{E/K}\colon X(E) \to X(K)$ is locally a homeomorphism, i.e., there is a closed-open covering $\{V_i\}_{i\in I}$ of $X(E)$ such that the restriction of $\mathrm{Res}_{E/K}$ to V_i is injective for every $i \in I$.*

PROOF. Let L/K be a finite Galois extension such that $E \subseteq L$ and $\sqrt{-1} \in L$. By the proof of Lemma 4.4, there exists a partition Y of $X(L/K)$ such that $\left\langle \mathcal{G}(L/K), \mathcal{G}\big(L/K(\sqrt{-1})\big), Y \xrightarrow{d_Y} \mathcal{G}(L/K)\right\rangle$ is an Artin-Schreier structure (with d_Y as in the proof of 4.4), in particular

(1) $\quad V^\sigma \cap C = \emptyset$ for every $\sigma \in \mathcal{G}\big(L/K(\sqrt{-1})\big) - \{1\}$ and $V \in Y$.

We have a commutative diagram

$$\begin{array}{ccc} X(L/E) & \xrightarrow{\ i\ } & X(L/K) \\ \Big\downarrow{\scriptstyle \mathrm{Res}_E} & & \Big\downarrow{\scriptstyle \mathrm{Res}_K} \\ X(E) & \xrightarrow[\mathrm{Res}_{E/K}]{} & X(K) \end{array}$$

in which i is the natural inclusion, and Res_E, Res_K are the obvious restriction maps. Note that $X(E) = X\big(E(\sqrt{-1})/E\big) = X(L/E)/\mathcal{G}\big(L/E(\sqrt{-1})\big)$, hence by 1.6, Res_E is an open map. Therefore $Y' = \big\{V' = \mathrm{Res}_E\,(i^{-1}(V)) \mid V \in Y\big\}$ is a closed open covering of $X(E)$. By (1) and by Proposition 2.1 (iii), $\mathrm{Res}_K\colon V \to X(K)$ is injective for every $V \in Y$; our diagram implies that $\mathrm{Res}_{E/K}\colon V' \to X(K)$ is also injective, for every $V' \in Y'$. //

PROPOSITION 8.2. *Let E/K be a finitely generated extension. Let H_E be a closed-open subset in $X(E)$ and denote $H_K = \mathrm{Res}_{E/K}\,(H_E)$. Then $\mathrm{Res}_{E/K}\colon H_E \to H_K$ has a continuous section.*

PROOF. There is a finite tower of simple extensions $K = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_n$. The set $H_i = \mathrm{Res}_{E/E_i}\,(H_E)$ is closed-open in $X(E_i)$, for every $0 \leqslant i \leqslant n$, by [5, Theorem 4.9]. Note that $H_0 = H_K$, $H_n = H_E$ and $\mathrm{Res}_{E_i/E_{i-1}}\,(H_i) = H_{i-1}$, for $i = 1, \ldots, n$. If we can find a section θ_i of $\mathrm{Res}_{E_i/E_{i-1}}\colon H_i \to H_{i-1}$, for $i = 1, \ldots, n$, then $\theta_n \circ \ldots \circ \theta_1$ is a section of $\mathrm{Res}_{E/K}\colon H_E \to H_K$. Thus we may assume that E/K is simple. Moreover, it is enough to find for every $P \in H_K$ a closed-open neighbourhood $V \subseteq H_K$ and a continuous map $\theta\colon V \to H_E$ such that $\theta(P')$ extends P', for every $P' \in V$.

Let $P \in H_K$: If E/K is finite our assertion follows easily from 8.1. Assume, therefore, that $E = K(t)$ is transcendental over K. With no loss $H_E = H_E(f_1, \ldots, f_m) = \{Q \in X(E) \mid f_1, \ldots, f_m \in Q\}$, where $f_1, \ldots, f_m \in K[t] - \{0\}$.

Suppose first that there is an $\alpha \in K$ such that $f_1(\alpha), \ldots, f_m(\alpha) \in P$. Let $V = H_K \cap H_K(f_1(\alpha), \ldots, f_m(\alpha))$, and define $\theta: X(K) \to X(E)$ in such a way that $\theta(P')$ is the unique extension of P' in which $u = 1/(t - \alpha)$ is infinitely large over K (cf. [18, p. 272]). Then θ is continuous: if $g_1(u), \ldots, g_k(u) \in K[u] - \{0\}$ have $a_1, \ldots, a_k \in K$ as their leading coefficients, respectively, then $\theta^{-1}\big(H_E(g_1(u), \ldots, g_k(u))\big) = H_K(a_1, \ldots, a_k)$. Moreover, $\theta(V) \subseteq H_E$. Hence the restriction of θ to V is the desired map.

If there is no $\alpha \in K$ such that $f_1(\alpha), \ldots, f_m(\alpha) \in P$, we can still find a finite extension (L, P_1) of (K, P) and an $\alpha \in L$ such that $f_1(\alpha), \ldots, f_m(\alpha) \in P_1$. Indeed, let $Q \in H_E$ such that $\mathrm{Res}_{E/K} Q = P$. If $(\bar{E}, \bar{Q})$ is a real closed field which extends (E, Q), then $f_1(t), \ldots, f_m(t) \in \bar{Q}$. By Tarski's principle ([21, Cor. 5.3]), there is an α in a real closure $(K, \bar{P})$ of (K, P) such that $f_1(\alpha), \ldots, f_m(\alpha) \in \bar{P}$. Let $L = K(\alpha)$ and $P_1 = \mathrm{Res}_{K/L} \bar{P}$.

Now let $F = L(t)$, $H_F = \mathrm{Res}_{F/E}^{-1}(H_E) = H_F(f_1, \ldots, f_m)$, $H_L = \mathrm{Res}_{F/L}(H_F)$. Then $H_K \supseteq \mathrm{Res}_{L/K}(H_L)$. By the finite extension case, there is a closed-open neighbourhood $V_1 \subseteq H_K$ of P and a map $\theta_1: V_1 \to H_L$ such that $\theta_1(P')$ extends P' for every $P' \in V_1$ and $\theta_1(P) = P_1$. By the previous case, there is a closed-open neighbourhood $V_2 \subseteq H_L$ of P_1 and a map $\theta_2: V_2 \to H_F$ such that $\theta_2(P_1')$ extends P_1' for every $P_1' \in V_2$. Now let $V = V_1 \cap \theta_1^{-1}(V_2)$ and let $\theta: V \to H_E$ be $\mathrm{Res}_{F/E} \circ \theta_2 \circ \theta_1$; then θ is the desired map. //

EXAMPLE 8.3. The preceding Lemma might lead one to a conjecture that if E/K is an arbitrary totally real extension, then $\mathrm{Res}_{E/K}: X(E) \to X(K)$ has a continuous section. However, this is false. Indeed, let X be the Boolean space $\{\pm 1/n \mid n \in \mathbb{N}\} \cup \{0\}$, with the topology inherited from the real line. Let $A_0 = \{x \in X \mid x \geqslant 0\}$ and $A_1 = \{x \in X \mid x \leqslant 0\}$. Craven [3, Theorem 5] has shown that there exists a field K with $X(K) \simeq X$. Identify $X(K)$ with X. Let t be transcendental over K and let

$$A_i' = \{Q \in X(K(t)) \mid \mathrm{Res}_{K(t)/K} Q \in A_i \text{ and } (-1)^i t \in Q\}\,, \quad i = 0, 1\,.$$

Then A' and A_i' are closed in $X(K(t))$. By [5, Theorem 4.18], there exists an extension E of $K(t)$ such that $\mathrm{Res}_{E/K(t)} X(E) = A_0' \cup A_1'$.

Assume that there is a continuous section θ of $\mathrm{Res}_{E/K}: X(E) \to X(K)$, and let $\theta' = \mathrm{Res}_{E/K(t)} \circ \theta$. Then $\theta'(1/n) \in H_k(t)$ and $\theta'(-1/n) \in H_K(-t)$, for every $n \in \mathbb{N}$. But θ' is continuous $\lim\limits_{n\to\infty} (1/n) = \lim\limits_{n\to\infty} (-1/n) = 0$, hence $\theta'(0) \in H_K(t) \cap H_K(-t) = \emptyset$, a contradiction.

LEMMA 8.4 (cf. [5, Theorem 4.18]). *Let K be a field and let C be a closed subset of $X(K)$. Then there exists a regular extension E of K, such that $\mathrm{Res}_{E/K} X(E) = C$, and a continuous section $\theta: C \to X(E)$ of $\mathrm{Res}_{E/K}$.*

PROOF. The proof consists of two parts.

Part I. Assume first that $C = X(K) - H$, where H is a basic closed-open subset of $X(K)$, i.e., $H = H_K(a_1, \ldots, a_m)$ for some $a_1, \ldots, a_m \in K^*$. With no loss $m > 1$. The *Pfister form* in 2^m variables

$$f = \sum_{\mathfrak{i} \in \{0,1\}^m} a_1^{i_1} a_2^{i_2} \ldots a_m^{i_m} X_{\mathfrak{i}}^2$$

is clearly an absolutely irreducible polynomial. This its function field E, i.e. the quotient field of $K[\mathbf{X}]/(f)$, is a finitely generated regular extension of K. A simple check shows that $\mathrm{Res}_{E/K} X(E) = C$; alternatively note that the form f is definite with respect to every $P \in H$ and indefinite with respect to every $P \in C$, and then use [5, Theorem 3.3]. The section θ exists by Lemma 8.2.

Part II. The general case.

There is a family $\{H_\lambda\}_{\lambda \in \Lambda}$ of basic closed-open sets in $X(K)$ such that $C = X(K) - \bigcup_{\lambda \in \Lambda} H_\lambda$. We may assume that Λ is well-ordered, i.e., Λ is the set of ordinals smaller than a fixed ordinal ω. For every ordinal $\mu \leqslant \omega$ let $C_\mu = X(K) - \bigcup_{\lambda < \mu} H_\lambda$. Thus $C_0 = X(K)$ and $C_\omega = C$. Furthermore, if $\lambda < \lambda' \leqslant \omega$, then $C_{\lambda'} \subseteq C_\lambda$ and we denote the inclusion map $C_{\lambda'} \to C_\lambda$ by $i_{\lambda',\lambda}$. Finally denote $E_0 = K$ and let θ_0 be the identity of $X(K)$.

Let $\mu \leqslant \omega$. Suppose, by transfinite induction, that we have constructed for every $\lambda < \mu$:

(i) a regular extension E_λ of K such that $\mathrm{Res}_{E_\lambda/K} X(E_\lambda) = C_\lambda$;

(ii) a continuous section $\theta_\lambda \colon C_\lambda \to X(E_\lambda)$ of $\mathrm{Res}_{E_\lambda/K'}$, such that for every $\lambda \leqslant \lambda' < \mu$ we have:

(iii) $E_\lambda \subseteq E_{\lambda'}$, and $\mathrm{Res}_{E_{\lambda'}/E_\lambda} \circ \theta_{\lambda'} = \theta_\lambda \circ_{\lambda',\lambda}$.

If the ordinal μ has no immediate predecessor, let $E_\mu = \bigcup_{\lambda < \mu} E_\lambda$. Then $X(E_\mu) = \varprojlim_{\lambda < \mu} X(E_\lambda)$, hence $\mathrm{Res}_{E_\mu/K} X(E_\mu) = \bigcap_{\lambda < \mu} C_\lambda = C_\mu$, and the maps $\{\theta_\lambda \circ i_{\mu,\lambda}\}_{\lambda < \mu}$ define a section $\theta_\mu \colon C_\mu \to X(E_\mu)$ of $\mathrm{Res}_{E_\mu/K}$ such that $\mathrm{Res}_{E_\mu/E_\lambda} \circ \theta_\lambda = \theta_\lambda \circ i_{\mu,\lambda}$, for every $\lambda < \mu$.

If μ has an immediate predecessor λ, let $C'_\mu = \mathrm{Res}_{E_\lambda/K^{-1}} C_\mu$. Note that $C_\mu = C_\lambda \cap (X(K) - H_\gamma)$ and $\mathrm{Res}_{E_\lambda/K} X(E_\lambda) = C_\lambda$, hence $C'_\mu = \mathrm{Res}_{E_\lambda/K}^{-1} (X(K) - H_\lambda)$, which is easily seen to be a complement of a basic closed-open subset of $X(E_\lambda)$. By Part I, there exists a regular extension E_μ of E_λ such that

$\mathrm{Res}_{E_\mu/E_\lambda} X(E_\mu) = C'_\mu$, and there is a section $\theta'_\mu\colon C'_\mu \to X(E_\mu)$ of $\mathrm{Res}_{E_\mu/E_\lambda}$. Let $\theta_\mu = \theta'_\mu \circ \theta_\lambda \circ i_{\mu,\lambda}$; then $\mathrm{Res}_{E_\mu/E_\lambda} \circ \theta_\mu = \theta_\lambda \circ i_{\mu,\lambda}$. Thus E_μ and θ_μ satisfy the induction hypthesis.

Let $E = E_\omega$ and $\theta = \theta_\omega$. Then $\mathrm{Res}_{E/K} X(K) = C_\omega = C$ and $\theta\colon C \to X(E)$ is a section of $\mathrm{Res}_{E/K}$. //

LEMMA 8.5. *Let K be a field and let C be a closed subset of $X(K)$. Then there exists a regular extension E/K such that $\mathrm{Res}_{E/k}$ maps $X(E)$ homeomorphically onto C.*

PROOF. By Lemma 8.4, we can construct a tower $K = E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$ of regular extensions and closed subsets $C_i \subseteq X(E_i)$ such that $C_0 = C$ and for every $i \geqslant 1$, $\mathrm{Res}_{E_i/E_{i-1}} X(E_i) = C_{i-1}$ and $\mathrm{Res}_{E_i/E_{i-1}}$ maps C_i homeomorphically onto C_{i-1} (i.e., $C_i = \theta_i(C_{i-1})$, for some section $\theta_i\colon X(E_{i-1}) \to X(E_i)$ of $\mathrm{Res}_{E_i/E_{i-1}}$). This is easily done by induction. Now let $E = \bigcup_{i=1}^{\infty} E_i$. Then $X(E) = \varprojlim C_i$, hence Res_{E/E_i} maps $X(E)$ homeomorphically onto C_i, for every $i \geqslant 0$. //

We apply the preceding results to PRC fields. Recall that a field K is PRC if and only if K is existentially closed (in the language of fields with parameters from K) is every regular totally real extension ([22, Section 1]). Let us call an extension E/K *exactly real* if $\mathrm{Res}_{E/K}\colon X_E \to X_K$ is a homeomorphism.

PROPOSITION 8.6: *A field K is* PRC *if and only if K is existentially closed in every regular exactly real extension.*

PROOF. The necessity is clear. To show the sufficiency it is enough to construct for every finitely generated regular totally real extension E/K a regular extension F/E such that F/K is exactly real. But this is now easy: by Lemma 8.2, there is a section $\theta\colon X(K) \to X(E)$ of $\mathrm{Res}_{E/K}$, and by 8.5, there exists a regular extension F/E such that $\mathrm{Res}_{F/E}$ maps $X(F)$ homeomorphically onto $\theta(X(K))$. Thus F/K is exactly real. //

We use this result to strengthen Theorem 1.1 of Prestel [22].

PROPOSITION 8.7. *Let K be a field and C a closed subset of $X(K)$. Then there exists a regular* PRC *extension E of K such that $\mathrm{Res}_{E/K}$ maps $X(E)$ homeomorphically onto C.*

PROOF. With no loss $C = X(K)$, otherwise, by Lemma 8.5, K may be replaced by a regular extension K' such that $\mathrm{Res}_{K'/K}$ maps $X(K')$ homeomorphically onto C. Denote by $\mathcal{M}$ the class of regular exactly real extension of K. Clearly, $\mathcal{M}$ is closed under unions of chains. Thus there exists an $E \in \mathcal{M}$ which is $\mathcal{M}$-existentially closed (v.d. Dries [4, p. 28]). To show that E is PRC, let F be a regular exactly real extension of E. Then $F \in \mathcal{M}$, hence every existential sentence with parameters from E which holds in F, also holds in E. //

9. – A transcendental construction.

A well known transcendental construction provides every profinite group G with a Galois extension F/E such that $G \cong \mathcal{G}(F/E)$. If G is the underlying group of an Artin-Schreier structure $\mathfrak{G}$, then $\mathfrak{G} \not\cong \mathfrak{G}(F/G)$, in general. Nevertheless, we show that the isomorphism of groups extends to a morphism $\mathfrak{G} \to \mathfrak{G}(F/E)$ of Artin-Schreier structures.

First we need some lemmas.

LEMMA 9.1. *Let* $\alpha\colon \mathfrak{B} \to \mathfrak{A}$ *be a cover of Artin-Schreier structures. Then the map* $\alpha\colon X(\mathfrak{B}) \to X(\mathfrak{A})$ *has a continuous section.*

PROOF. We follow the proof of Lemma 7.3. Let $K = \operatorname{Ker}\alpha$. With no loss $\mathfrak{A} = \mathfrak{B}/K$.

Part I. The kernel K is finite.

If K is finite, we obtain, as in 7.3, a cover $\alpha_0\colon \mathfrak{B}_0 \to \mathfrak{A}_0$ of finite Artin-Schreier structures $\mathfrak{A}_0$ and $\mathfrak{B}_0$ and a cartesian square

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\ \alpha\ } & \mathfrak{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{B}_0 & \xrightarrow[\ \alpha_0\ ]{} & \mathfrak{A}_0 \end{array}$$

of epimorphisms of Artin-Schreier structures. There exists a map $\gamma_0\colon X(\mathfrak{A}_0) \to X(\mathfrak{B}_0)$ such that $\alpha_0 \circ \gamma_0 = \mathrm{id}_{X(\mathfrak{A}_0)}$, since $X(\mathfrak{B}_0)$ is finite. Thus $\alpha_0 \circ (\gamma_0 \circ \pi) = \pi \circ \mathrm{id}_{X(\mathfrak{A})}$, whence by Lemma 4.6 (*c*), there exists a continuous map $\gamma\colon X(\mathfrak{A}) \to X(\mathfrak{B})$ such that $\alpha \circ \gamma = \mathrm{id}_{X(\mathfrak{A})}$ (and $p \circ \gamma = \gamma_0 \circ \pi$).

Part II. The general case.

Let Γ be the family of pairs (L, γ), where L is a closed normal subgroup of B contained in K and $\gamma\colon X(\mathfrak{A}) \to X(\mathfrak{B})/L$ is a continuous section of the quotient map $\alpha_L\colon X(\mathfrak{B})/L \to X(\mathfrak{B})/K$ $(= X(\mathfrak{A}))$. Partially order Γ by letting $(L', \gamma') \geqslant (L, \gamma)$ mean that $L' \leqslant L$ and

$$\begin{array}{ccc} & & X(A) \\ & \swarrow^{\gamma'} & \swarrow^{\gamma} \\ X(\mathfrak{B})/L' & \longrightarrow & X(\mathfrak{B})/L \end{array}$$

commutes. By Zorn's Lemma there exists a maximal element $(L, \gamma) \in \Gamma$. If $L \neq 1$, there is a proper open subgroup L' of L, normal in B. By Part. I, the map $X(\mathfrak{B})/L' \to X(\mathfrak{B})/L$ has a continuous section, say γ_1. Let $\gamma' = \gamma_1 \circ \gamma$. Then $(L', \gamma') \in \Gamma$ and $(L', \gamma') > (L, \gamma)$, a contradiction. Thus $L = 1$, as required. //

COROLLARY 9.2. *Let $\mathfrak{G}$ be an Artin-Schreier structure.*

(i) *There exists a closed complete system X of representatives of the G-orbits in $X(\mathfrak{G})$.*

(ii) *Let $X \subseteq X(\mathfrak{G})$ be a closed complete system of representatives of the G-orbits in $X(\mathfrak{G})$. Then the map $X \times G' \to X(\mathfrak{G})$, defined by $(x, \tau) \mapsto x^\tau$, is a homeomorphism.*

PROOF. (i) Let $\gamma\colon X(\mathfrak{G})/G' \to X(\mathfrak{G})$ be a continuous section of the quotient map $X(\mathfrak{G}) \to X(\mathfrak{G})/G'$. Put $X = \gamma(X(\mathfrak{G})/G')$. Then X is closed, since it is an image of a compact set. The required property of X follows from the fact that $G = G' \cup d(x)G'$ and $x^{d(x)} = x$ for every $x \in X(\mathfrak{G})$; hence the G'-orbit of X is the G-orbit of X.

(ii) The map $X \times G' \to X(\mathfrak{G})$ is clearly a continuous surjection. By condition (iv$'$) of Definition 3.1, it is injective. Finally the map is closed, since $X \times G'$ and $X(\mathfrak{G})$ are compact Hausdorff spaces. //

LEMMA 9.3. *Let $\mathfrak{B}, \mathfrak{A}$ be Artin-Schreier structures and let $X \subseteq X(\mathfrak{B})$ be a closed complete system of representatives of the G-orbits in $X(\mathfrak{B})$. Let $\theta_0\colon B \to A$ be a continuous homomorphism and $\theta_1'\colon X \to X(\mathfrak{A})$ a continuous map such that $\theta_0^{-1}(A') = B'$ and $d(\theta_1'(x)) = \theta_0(d(x))$, for every $x \in X$. Then θ_1' can be extended to a unique map $\theta_1\colon X(\mathfrak{B}) \to X(\mathfrak{A})$, such that the pair $\theta = (\theta_0, \theta_1)$ is a morphism of Artin-Schreier structures.*

Moreover, θ is a cover if and only if θ_0 is an epimorphism and $\theta_1'(X)$ is a complete system of representatives of the A-orbits in $X(\mathfrak{A})$.

PROOF. We define θ_1 by

$$\theta_1(x^\tau) = (\theta_1'(x))^{\theta_0(\tau)}, \quad \text{for } x \in X \text{ and } \tau \in G' .$$

This is a good definition, by Corollary 9.2 (ii). One can easily check that (θ_0, θ_1) satisfies the conditions (i) and (ii′) of Definition 3.3, hence θ_1 is a morphism. The uniqueness of θ_1 is obvious.

The last assertion of the Lemma follows from condition (iv′) of Definition 3.3. //

LEMMA 9.4. *Let L/K be a Galois extension such that $\sqrt{-1} \in L$. Let $\mathfrak{G}$ by an Artin-Schreier structure and $\pi\colon \mathfrak{G} \to \mathfrak{G}(L/K)$ a morphism such that:*

(i) *$\pi\colon G \to \mathcal{G}(L/K)$ is an epimorphism of groups;*

(ii) *for every real involution $\bar{\varepsilon}$ of $\mathcal{G}(L/K)$ there exists an involution $\delta \in G$ with $\pi(\delta) = \bar{\varepsilon}$.*

Then there exists a totally real regular extension E of K, a Galois extension F of E containing L and a commutative diagram

$$(1) \qquad \begin{array}{ccc} \mathfrak{G} & \xrightarrow{\quad\theta\quad} & \mathfrak{G}(F/E) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{Res}_L} & \\ & \mathfrak{G}(L/K) & \end{array}$$

in which θ is a morphism of Artin-Shreier structures such that $\theta\colon G \to \mathcal{G}(F/E)$ is an isomorphism of groups.

Moreover, E can be taken to be finitely generated over K, if G is a finite group.

REMARK. Conditions (1) and (ii) are satisfied, if $\pi\colon \mathfrak{G} \to \mathfrak{G}(L/K)$ is an epimorphism.

PROOF. We divide the proof into five parts.

Part I. The construction of F/E.

Denote by $\mathcal{N}$ the family of open normal subgroups of G contained in G', and by Λ the family of right cosets of groups in N in G. Let $T = \{t_{Ng} \mid Ng \in \Lambda\}$ be a set of algebraically independent elements over L. The group G acts on $F = L(T)$ in the following way:

$$z^g = z^{\pi(g)} \qquad \text{for } z \in L \text{ and } g \in G\,,$$
$$(t_{Ng'})^g = t_{Ng'g} \qquad \text{for } N \in \mathcal{N} \text{ and } g, g' \in G$$

as a group of automorphisms of F over K. Let E be the fixed field of G in F.

The action of G on F is faithful and clearly the stabilizer of every element of F is open in G. Therefore ([24, Theorem 1]), there exists an isomorphism $\theta\colon G \to \mathcal{G}(F/E)$ compatible with the action on F. In particular, the following diagram of groups commutes

$$
(2) \qquad \begin{array}{ccc} G & \xrightarrow{\ \theta\ } & \mathcal{G}(F/E) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{Res}_L} & \\ & \mathcal{G}(L/K) & \end{array}
$$

Note that $L \cap E = K$ and EL/L is regular extension, since F/L is transcendental. Hence E/K is regular. If G is finite, then T is finite, hence F/K is finitely generated; By [19, p. 64], E/K is also finitely generated.

Finally observe that, by (2)

$$\theta^{-1}\Big(\mathcal{G}\big(F/E(\sqrt{-1})\big)\Big) = \theta^{-1}\circ\mathrm{Res}_L^{-1}\Big(\mathcal{G}\big(L/K(\sqrt{-1})\big)\Big) = \pi^{-1}\Big(\mathcal{G}\big(L/K(\overline{-1})\big)\Big) = G'\,.$$

Part II. The map $\mathrm{Res}_L\colon X(F/E) \to X(L/K)$ is surjective.

Let $(L(\bar\varepsilon), P) \in X(L/K)$. By condition (ii), there exists an involution $\delta \in G$ with $\pi(\delta) = \bar\varepsilon$. Let $\varepsilon = \theta(\delta) \in \mathcal{G}(F/E)$; then $L(\bar\varepsilon) = L(\varepsilon)$. We show that the extension $F(\varepsilon)/L(\varepsilon)$ is purely transcendental, hence P can be extended to an ordering of $L(\varepsilon)$.

For every $Ng \in \Lambda$ denote

$$u_{\delta,Ng} = \begin{cases} t_{Ng} + t_{Ng\delta} & \text{for } g \in G' \\ \sqrt{-1}\,(t_{Ng} - t_{Ng\delta}) & \text{for } g \notin G'\,. \end{cases}$$

The elements of $U_\delta = \{u_{\delta,Ng} \mid Ng \in \Lambda\}$ are algebraically independent over $L(\varepsilon)$, since for every $Ng \in \Lambda$ the elements $u_{\delta,Ng}, u_{\delta,Ng\delta}$ are linear combinations of $t_{Ng}, t_{Ng\delta}$ with coefficients in L, and vice versa.

Clearly $L(\varepsilon)(U_\delta) \subseteq F(\varepsilon)$, but. $L(\varepsilon)\big(U_\delta, \sqrt{-1}\big) = L(\varepsilon)\big(\sqrt{-1}, T\big) = L(T) = F$, hence $[F : L(\varepsilon)(U_\delta)] = 2 = [F : F(\varepsilon)]$, which implies that $L(\varepsilon)(U_\delta) = F(\varepsilon)$.

Note that $\mathrm{Res}_L\colon \mathcal{G}(F/E) \to \mathcal{G}(L/K)$ is an epimorphism of groups, since (2) commutes. Therefore $\mathrm{Res}_L\colon \mathfrak{G}(F/E) \to \mathfrak{G}(L/K)$ is an epimorphism of Artin-Schreier structures. By Lemma 3.5, the extension E/K is totally real.

Part III. The definition of $\theta\colon X(\mathfrak{G}) \to X(F/E)$.

Let $x \in X(\mathfrak{G})$, $\delta = d(x)$ and $\varepsilon = \theta_0(\delta)$. Let P be the ordering of $L(\varepsilon)$ for which $\pi(x) = (L(\varepsilon), P)$. Recall that $F(\varepsilon) = L(\varepsilon)(U_\delta)$. We choose below an ordering Q of $F(\varepsilon)$ which extends P, and then define $\theta(x) = (F(\varepsilon), Q)$.

We may assume that Λ is totally ordered. This order defines the lexicographical order on the set of monomials in elements of U_δ: if $M = u_{\delta,\lambda_1}u_{\delta,\lambda_2} \dots u_{\delta,\lambda_m}$ and $M' = u_{\delta,\lambda'_1}u_{\delta,\lambda'_2}\dots u_{\delta,\lambda'_u}$, where $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_m$ and $\lambda'_1 \geqslant \lambda'_2 \geqslant \dots \geqslant \lambda'_m$ are elements of Λ, we define $M' > M$, if there exists an integer i such that $\lambda_1 = \lambda'_1, \dots, \lambda_{i-1} = \lambda'_{i-1}$, and $\lambda'_i > \lambda_i$ or $m \leqslant i < n$. Finally we define an orde ing Q on the ring of polynomials $L(\varepsilon)[U_\delta]$: we let a polynomial to be positive, if the coefficient of its largest monomial (which has a non-zero coefficient) is positive in P. This is easily seen to be an ordering of $L(\varepsilon)[U_\delta]$, and hence has a unique extension to an ordering Q of the quotient field $F(\varepsilon)$ (cf. [18, p. 272]). Clearly Q extends P. In fact, Q is the unique extension of P to $F(\varepsilon)$ in which every $u_{\delta,\lambda}$ is infinitely large with respect to the field $L(\varepsilon)(\{u_{\delta,\lambda'} \mid \lambda' < \lambda\})$.

Part IV. The map $\theta\colon X(\mathfrak{G}) \to X(F/E)$ is continuous.

Indeed, let $f_1, \dots, f_m \in F^*$ such that $(F(\varepsilon), Q) \in H_F(f_1, \dots, f_m)$, i.e, $f_1, \dots, f_m \in Q$. Then there are $\lambda_1 < \lambda_2 < \dots < \lambda_n \in \Lambda$ such that $f_1, \dots, f_m$ are rational functions in $u_{\delta,\lambda_1}, u_{\delta,\lambda_2}, \dots, u_{\delta,\lambda_m}$ with (non-zero) coefficients in $L(\varepsilon)$. With no loss they are polynomials: if $f_i = g/h$, replace f_i by gh. For every $1 \leqslant i \leqslant m$ let $a_i \in L(\varepsilon)$ be the coefficient of the largest monomial in f_i.

There are groups $N_1, \dots, N_n \in \mathcal{N}$ and $g_1, \dots, g_n \in G$ such that $\lambda_j = N_j g_j$ for $j = 1, \dots, n$. Let $N = N_1 \cap \dots \cap N_n$.

Consider the closed-open neighbourhood V of x consisting of $y \in X(\mathfrak{G})$ such that

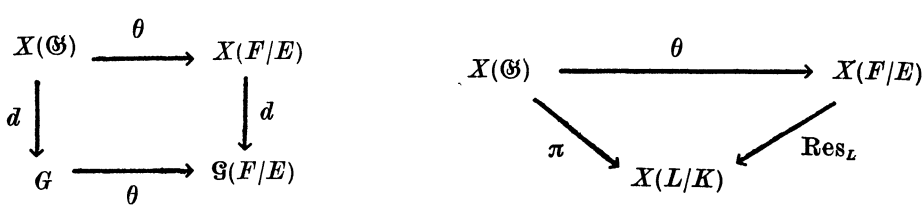
(i) $\pi(y) \in H_L(a_1, \dots, a_m)$ and

(ii) $d(y) \equiv \delta (\mathrm{mod}\ N)$.

We show that $\theta(V) \subseteq H_L(f_1, \dots, f_m)$, which proves the continuity of θ. Let $y \in V$, and let $(F(\varepsilon'), Q') = \theta(y)$, $(L(\varepsilon'), P') = \pi(y)$ and $\delta' = d(y)$. By (ii), we have that $N_j g_j \delta' = N_j g_j$ for $j = 1, \dots, n$, hence $u_{\delta,\lambda_j} = u_{\delta',\lambda_j}$, for $j = 1, \dots, n$. By (i), $a_1, \dots, a_m \in P'$. Hence, by the definition of Q' from Part III, $f_1, \dots, f_m \in Q'$. Thus $\theta(y) \in H_L(f_1, \dots, f_m)$.

Part V. End of the proof.

It follows directly from the definition of θ that the following two diagrams commute:



Unfortunately, it need not be true that $\theta(x)^{\theta(\sigma)} = \theta(x^\sigma)$ for all $x \in X(\mathfrak{G})$ and $\sigma \in G$, but we have a remedy. By cor. 9.2 (i) there exists a closed complete system X of representatives of the G-orbits in $X(\mathfrak{G})$. Denote by θ_1' and π_1' the restrictions of $\theta\colon X(\mathfrak{G}) \to X(F/E)$ and $\pi\colon X(\mathfrak{G}) \to X(L/K)$ to X, respectively; then $\mathrm{Res}_L \circ \theta_1' = \pi_1'$. Therefore, by Lemma 9.3, θ_1' can be extended to a map $X(\mathfrak{G}) \to X(F/E)$ which together with the group isomorphism $\theta\colon G \to \mathcal{G}(F/E)$ constitutes a morphism $\theta\colon \mathfrak{G} \to \mathfrak{G}(F/E)$. Moreover this morphism satisfies $\mathrm{Res}_L \circ \theta = \pi$, since both $\mathrm{Res}_L \circ \theta$ and π consist of an epimorphism of groups $\pi\colon G \to \mathcal{G}(L/K)$ and of a map $X(\mathfrak{G}) \to X(L/K)$ which extends π_1', hence they are equal by Lemma 9.3. //

10. – The main results.

In this Section we characterize the absolute Galois groups of PRC fields and the associated Artin-Schreier structures.

THEOREM 10.1. *Let K be a* PRC *field. Then*

(*a*) *$G(K)$ is a real projective group, and*

(*b*) *$\mathfrak{G}(K)$ is a projective Artin-Schreier structure.*

PROOF. Assertion (*b*) follows from (*a*) by Prop. 7.7. In order to prove (*a*), observe that there are no involutions in $G\big(K(\sqrt{-1})\big)$, hence the set $I(G(K))$ of involutions is closed in $G(K)$, by Remark 7.6.

Let L/K be a finite Galois extension and let $\alpha\colon B \to \mathcal{G}(L/K)$ be an epimorphism of finite groups such that for every $\delta \in I\big(G(K)\big)$ that satisfies $\mathrm{Res}_L\, \delta \neq 1$, there exists an involution $\varepsilon \in B$ for which $\alpha(\varepsilon) = \mathrm{Res}_L\, \delta$. We have to find a homomorphism $\gamma\colon G(K) \to B$ such that $\alpha \circ \gamma = \mathrm{Res}_L$. With no loss assume that $\sqrt{-1} \in L$, otherwise replace L by $L(\sqrt{-1})$ and B by $B \times_{\mathrm{G}(L/K)} \mathcal{G}\big(L(\sqrt{-1})/K\big)$.

Let $\mathfrak{B} = \Big\langle B, \alpha^{-1}\Big(\mathcal{G}\big(L/K(\sqrt{-1})\big)\Big), I(B) \overset{\text{incl.}}{\to} B\Big\rangle$, where $I(B) = \{\varepsilon \in B \mid \varepsilon^2 = 1,$ $\alpha(\varepsilon) \in I(L/K)\}$. Then α gives rise to a morphism of Artin-Schreier structures $\alpha\colon \mathfrak{B} \to \mathfrak{G}(L/K)$. By Lemma 9.4 there exists a finitely generated totally real regular extension E/K, a Galois extension F/E and an isomorphism of groups $\theta\colon B \to \mathcal{G}(F/E)$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad\theta\quad} & \mathcal{G}(F/E) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle \mathrm{Res}_{F/L}} & \\ & \mathcal{G}(L/K) & \end{array}$$

commutes. Thus with no loss we may assume that $B = \mathcal{G}(F/E)$ and $\alpha = \mathrm{Res}_{F/L}$.

Let x be a primitive element for F/E, let $f = \mathrm{irr}\,(x, E)$ and $d = \mathrm{discr}\,(f) \in E$. Let $R \subseteq E$ be an integrally closed domain finitely generated over K, which contains d^{-1} and the coefficients of f and such that E is its quotient field. By the definition of PRC fields ([22, Theorem 1.2]), there exists a K-homomorphism $\varphi\colon R \to K$. Let S be the integral closure of R in F (note that $L \subseteq S$) and extend φ to an L-homomorphism $\varphi\colon S \to K_s$. Denote by G_φ the decomposition group of φ in $\mathcal{G}(F/E)$ and let M be the splitting field of the polynomial $\varphi(f)$ over K. Then $L \subseteq M$ and $\varphi(f)$ has no multiple roots, since $\varphi(d) \neq 0$. By [18, Prop. 15 on p. 248], M/K is a Galois extension and φ induces an isomorphism $\varphi_*\colon G\colon \to \mathcal{G}(M/K)$ such that $(\varphi y)^{(\varphi_*\sigma)} = \varphi(y^\sigma)$ for every $\sigma \in G_\varphi$ and $y \in S$. The homomorphism $\varphi_*^{-1} \circ \mathrm{Res}_M\colon G(K) \to \mathcal{G}(F/E)$ solves our real embedding problem. //

By the way of converse we have:

THEOREM 10.2. *Let G be a projective Artin-Schreier structure. Let L/K be a Galois extension such that $\sqrt{-1} \in L$ and let $\pi\colon \mathfrak{G} \to \mathfrak{G}(L/K)$ be an epimorphism. Then there exists a* PRC *extension E of K and a commutativ- diagram*

(1)
$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\ \theta\ } & \mathfrak{G}(F) \\ & {}_{\pi}\searrow \quad \swarrow_{\mathrm{Res}_L} & \\ & \mathfrak{G}(L/K) & \end{array}$$

in which θ is an isomorphism.

PROOF. *Part* I. Epimorphisms of structures.

By Lemma 9.4, there exists a regular extension E_0 of K, a Galois extension F_0 of E_0 containing L and a commutative diagram

(2)
$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\ \theta_0\ } & \mathfrak{G}(F_0/E_0) \\ & {}_{\pi}\searrow \quad \swarrow_{\mathrm{Res}_L} & \\ & \mathfrak{G}(L/K) & \end{array}$$

in which $\theta_0\colon G \to \mathcal{G}(F_0/E_0)$ is an isomorphism of groups. If $\mathfrak{G}(F_0/E_0)$ is replaced in (2) by the Artin-Schreier structure

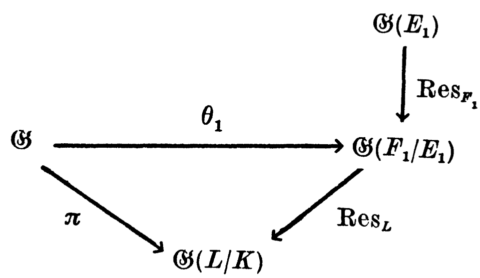
$$\mathfrak{G}'(F_0/E_0) = \left\langle \mathcal{G}(F_0/E_0), \mathcal{G}\big(F_0/E_0(\sqrt{-1})\big), C \xrightarrow{d} \mathcal{G}(F_0/E_0) \right\rangle ,$$

where $C = \theta_0\big(X(\mathfrak{G})\big) \subseteq X(F_0/E_0)$, then $\theta_0 \colon \mathfrak{G} \to \mathfrak{G}'(F_0/E_0)$ is an epimorphism.

Let $\bar{C} = \mathrm{Res}_{E_0} C \subseteq X(E)$. By Proposition 8.7, there exists a regular PRC extension E_1 of E_0 such that Res_{E_1/E_0} maps $X(E_1)$ homeomorphically onto $\bar{C}$. Let $F_1 = E_1 F_0$. The set C is closed under the action of $\mathcal{G}(F_0/E_0)$, hence, by Lemma 2.1, $C = \{x \in X(F_0/E_0) \mid \mathrm{Res}_{F_0} x \in \bar{C}\}$. Thus the map $\mathrm{Res}_{F_0} \colon X(F_1/E_1) \to C$ is well defined. By [4, Chapter II, Lemma 2.5] it is onto C. In fact, this map is also injective. Indeed, if $x, x' \in X(F_1/E_1)$ and $\mathrm{Res}_{F_0} x = \mathrm{Res}_{F_0} x'$, then $\mathrm{Res}_{E_1/E_0}(\mathrm{Res}_{E_1} x) = \mathrm{Res}_{E_1/E_0}(\mathrm{Res}_{E_1} x')$ hence $\mathrm{Res}_{E_1} x = \mathrm{Res}_{E_1} x'$. Thus there is a unique $\sigma \in \mathcal{G}\big(F_1/E_1(\sqrt{-1})\big)$ such that $x' = x^{\sigma}$. Let $\bar{\sigma} = \mathrm{Res}_{F_0} \sigma$; then $\mathrm{Res}_{F_0} x = \mathrm{Res}_{F_0} x' = (\mathrm{Res}_{F_0} x)^{\bar{\sigma}}$, whence $\bar{\sigma} = 1$. But $\mathrm{Res}_{F_0} \colon \mathcal{G}(F_1/E_1) \to \mathcal{G}(F_0/E_0)$ is an isomorphism, hence $\sigma = 1$, and therefore $x = x'$.

Thus $\mathrm{Res}_{F_0} \colon \mathfrak{G}(F_1/E_1) \to \mathfrak{G}'(F_0/E_0)$ is an isomorphism, and we obtain a commutative diagram

(3)



in which $\theta_1 = \mathrm{Res}_{F_0}^{-1} \circ \theta_0$ is an epimorphism such that $\theta_1 \colon G \to \mathcal{G}(F_1/E_1)$ is an isomorphism of groups.

Part II. The use of projectivity.

The forgetful map of $\mathfrak{G}$ is injective, by Prop. 7.4, hence $\theta_1 \colon X(\mathfrak{G}) \to X(F_1/E_1)$ is also injective. Therefore $\theta_1 \colon \mathfrak{G} \to \mathfrak{G}(F_1/E_1)$ is an isomorphism.

The restriction map $\mathrm{Res}_{F_1} \colon \mathfrak{G}(E_1) \to \mathfrak{G}(F_1/E_1)$ is a cover and $\mathfrak{G}$ is projective, hence there exists a morphism $\theta \colon \mathfrak{G} \to \mathfrak{G}(E_1)$ such that $\mathrm{Res}_F \circ \theta = \theta_1$. Let E be the fixed field of $\theta(G)$ in K_s. Then, clearly, $\mathfrak{G} \cong \langle \theta(G), \theta(G'), \theta\big(X(\mathfrak{G})\big) \xrightarrow{\text{incl.}} \theta(G) \rangle = \mathfrak{G}(E)$. Moreover, $\theta \colon \mathfrak{G} \to \mathfrak{G}(E)$ is an isomorphism which makes (1) commute. Finally, E is a PRC field ([22, Theorem 1.2]), which ends the proof. //

COROLLARY 10.3. *Let $\mathfrak{G}$ be an Artin-Schreier structure. Then there exists a Galois extension F/E such that $\sqrt{-1} \in F$ and $\mathfrak{G} \cong \mathfrak{G}(F/E)$.*

PROOF. We first show that there exists a field K and a cover $\pi: \mathfrak{G} \to \mathfrak{G}\big(K(\sqrt{-1})/K\big)$. If $X(\mathfrak{G}) = \emptyset$, this is trivial. If $X(\mathfrak{G}) \neq \emptyset$, there exists a field K such that $X(\mathfrak{G})/G' \cong X(K)$ (Craven [3, Theorem 5]). Thus $\mathfrak{G}/G' \cong \mathfrak{G}\big(K(\sqrt{-1})/K\big)$. Take π to be the composition of this isomorphism and the quotient map $\mathfrak{G} \to \mathfrak{G}/G'$.

Now denote $L = K(\sqrt{-1})$. By Part I of the proof of Theorem 10.2, there exists a commutative diagram (3), in which E_1 is a PRC field, $\theta_1: G \to \mathcal{G}(F_1/E_1)$ is an isomorphism of groups and $\theta_1: \mathfrak{G} \to \mathfrak{G}(F_1/E_1)$ is an epimorphism of Artin-Schreier structures.

We show that $\theta_1: X(\mathfrak{G}) \to X(F_1/E_1)$ is injective. If $x, x' \in X(\mathfrak{G})$ such that $\theta_1(x) = \theta_1(x')$, then $\pi(x) = \pi(x')$, hence there is a $\sigma \in G'$ such that $x' = x^\sigma$. Thus $\theta_1(x) = \theta_1(x') = \theta_1(x)^{\theta_1(\sigma)}$, and $\theta_1(\sigma) \in \mathcal{G}\big(F_1/E_1(\sqrt{-1})\big)$. By condition (iv′) of Definition 3.1, $\theta_1(\sigma) = 1$, hence $\sigma = 1$, whence $x = x'$. Thus $\mathfrak{G} \cong \mathfrak{G}(F_1/E_1)$. //

Combining Theorems 10.1, 10.2 with Prop. 7.7, we obtain the main result of this work:

THEOREM 10.4. *If K is a* PRC *field, then $G(K)$ is real projective.*

Conversely, if G is a real projective group, then there exists a PRC *field K such that $G \cong G(K)$.*

We use again the fact that algebraic extensions of PRC fields are PRC (Prestel [22, Theorem 3.1]):

COROLLARY 10.5. *A closed subgroup H of a real projective group G is real projective. Moreover, H is a projective group if and only if H contains no involutions of G. In particular, if σ is an element of finite order in G, then $\sigma^2 = 1$.*

11. – Concluding remarks.

The notions developed in this work and the results achieved open up new paths in the research of PRC fields. Results achieved for PAC fields may now be approached for PRC fields. For example, it has already been observed, using a simple logical principle, that the undecidability of the elementary theory of PAC fields implies the undecidability of the elementary theory of PRC fields (Ershov [7]). Therefore, the genuine question to be asked in this connection is about the undecidability of the theory of formally real PRC fields. We settle this question in a subsequent work by developing the appropriate analogue of Frattini covers. This may in turn help to prove the decidability of the theory of PRC fields with bounded corank. As a third topic in this list it should be of interest to set up real Frobenius fields and prove decidability results both model theoretically and by Galois stratification.

In the model theory of PAC fields an emphasis has been put upon algebraic models. To achieve analogous results for PRC fields one should complete Theorem 10.4:

PROBLEM. Given a real projective group G of rank $\leqslant \aleph_0$ and given a countable, formally real Hilbertian field K, does there exist a PRC field E, algebraic over K such that $G(E) \cong G$?

REFERENCES

[1] J. AX, *The elementary theory of finite fields*, Ann. of Math., **88** (1968), pp. 239-271.

[2] BREDON, *Introduction to compact transformation groups*, Academic Press, London, 1972.

[3] T. CRAVEN, *The Boolean space of orderings of a field*, Amer. Math. Soc. Transl., **209** (1975), pp. 225-235.

[4] L. VAN DEN DRIES, *Model theory of fields*, Ph. D. Thesis, Utrecht, 1978.

[5] R. ELMAN - T. Y. LAM - A. R. WADSWORTH, *Orderings under field extensions*, Journal für die reine und angewandte Math., **306** (1975), pp. 7-27.

[6] R. ENGELKING, *Outline of general topology*, North-Holland, Amsterdam, PWN, 1968.

[7] YU. L. ERSHOV, *Completely real extensions of fields*, Dokl. Akad. Nauk SSSR, **263** (1982), pp. 1047-1049.

[8] W.-D. GEYER, *Galois groups of intersections of local fields*, Israel J. Math., **30** (1978), pp. 383-396.

[9] K. GRUNBERG, *Projective profinite groups*, J. London Math. Soc., **42** (1967), pp. 155-165.

[10] D. HARAN - A. LUBOTZKY, *Embedding covers and the theory of Frobenius fields*, Israel J. Math., **41** (1982), pp. 181-202.

[11] E. HEWITT - A. K. ROSS, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin 1963.

[12] A. V. JAKOVLEV, *The Galois group of the algebraic closure of a local field*, Math. USSR-Izv. **2** (1968), pp. 1231-1269.

[13] U. JANNSEN - K. WINBERG, *Die Struktur der absoluten Galois gruppe $\mathfrak{p}$-adischer Zahlkörper*, Inventiones mathematicae, **70** (1982), pp. 71-98.

[14] M. JARDEN, *Elementary statements over large algebraic fields*, Amer. Math. Soc. Transl., **164** (1972), pp. 67-91.

[15] M. JARDEN, *Algebraic extensions of finite corank of Hilbertian fields*, Israel J. Math., **18** (1974), pp. 279-307.

[16] M. JARDEN, *The elementary theory of large e-fold ordered fields*, Acta Math. **149** (1982), pp. 239-240.
[17] M. JARDEN, *On the model companion of the theory of e-fold ordered fields*, Acta Math., **150** (1983), pp. 243-253.
[18] S. LANG, *Algebra*, Addison-Wesley, 1974.
[19] S. LANG, *Algebraic geometry*, Interscience Publishers, New York, 1964.
[20] A. LUBOTZKY - L. V. D. DRIES, *Subgroups of free profinite groups and large subfields of* $\tilde{\mathbb{Q}}$, Israel J. Math., **39** (1981), pp. 25-45.
[21] A. PRESTEL, *Lectures on formally real fields*, Monografias de Matemática 22, IMPA, Rio de Janeiro, 1975.
[22] A. PRESTEL, *Pseudo real closed fields*, in: *Set theory and model theory*, pp. 127-156, Lecture notes in Math., **782**, Springer, Berlin, 1981.
[23] L. RIBES, *Introduction to profinite groups and Galois cohomology*, Queen's University, Kingston, 1970.
[24] W. C. WATERHAUS, *Profinite groups are Galois groups*, Proc. Amer. Math. Soc., **42** (1974), pp. 639-640.
[25] K. WINBERG, *Der Eindeutigkeitssatz für Demuškinformationen*, Inventiones Mathematicae **70** (1982), pp. 99-113.

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