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Existence and regularity for semilinear parabolic evolution equations

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Existence and Regularity for Semilinear Parabolic Evolution Equations.

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Introduction.

In this paper we consider abstract semilinear parabolic systems

$$(1)\qquad \begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t)u &= F(t, u) \\ \mathcal{B}(t)u &= 0 \qquad t_0 < t \leqq T, \\ u(t_0) &= u_0, \end{aligned}$$

which can be interpreted as semilinear evolution equations of the form

$$(2)\qquad \begin{aligned} \dot{u} + A(t)u &= F(t, u), \qquad t_0 < t \leqq T, \\ u(t_0) &= u_0 \end{aligned}$$

in an appropriate Banach space X. Our main applications of the abstract results are to parabolic initial-boundary value problems

$$(3)\qquad \begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t)u &= f(t, x, u, \dots, D^{2m-1}u) && \text{in } \Omega \times (t_0, T], \\ \mathcal{B}(t)u &= 0 && \text{on } \partial\Omega \times (t_0, T], \\ u(0, t_0) &= u_0 && \text{on } \Omega, \end{aligned}$$

where $(\mathcal{A}(t), \mathcal{B}(t))$ denotes a (strongly coupled) very general parabolic system of order $2m$ (up to regularity conditions we assume essentially only

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the existence of L_p*-a-priori* estimates). In particular we do not assume that the boundary conditions are time-independent, that is, we do *not* assume that the domains $D(A(t))$ of $A(t)$ are constant in time. To cite one result, we shall show that problem (3) has for each sufficiently smooth initial value u_0 a unique solution $u(\cdot, t_0, u_0)$ on a maximal interval of existence J such that

$$u(\cdot, t_0, u_0) \in C^1(\dot{J}, C^{\mu}(\bar{\Omega}, \mathbb{R}^N)) \cap C(\dot{J}, C^{2m+\mu}(\bar{\Omega}, \mathbb{R}^N)) \cap C(J, C^{2m-1+\mu}(\bar{\Omega}, \mathbb{R}^N))$$

for some $\mu \in (0, 1)$, where $\dot{J} := J \setminus \{t_0\}$, provided $f \in C^2([0, T] \times \bar{\Omega} \times \mathbb{R}^M, \mathbb{R}^N)$ and Ω is either bounded, or Ω unbounded and f is independent of (t, x) and satisfies $f(0) = 0$. Here N is the number of components of u and $M = N \sum_{|\alpha| \leq 2m-1} 1$, where $\alpha \in \mathbb{N}^n$ and $\Omega \subset \mathbb{R}^n$. (For a more precise statement we refer to Corollary (15.7).) It should be noted—and this is an important point—that there are no compatibility conditions for f, that is, there are no other restrictions upon f besides the ones given above (for more general conditions we refer to Section 15).

Our approach is based on the « variations-of-constants formula »

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^{t} U(t, \tau)F(\tau, u(\tau))\,d\tau\,, \qquad t_0 \leq t \leq T\,, \tag{4}$$

where U is the evolution operator of the family $\{A(t) | 0 \leq t \leq T\}$, whose existence is guaranteed by general results of Kato and Tanabe [26] (where we use, in fact, a somewhat more restricted form of assumptions introduced by Yagi [51]). The standard approach for studying (4) in situations where the nonlinearity is « unbounded », that is, not defined on all of X, is by means of the theory of fractional powers (e.g. [21, 25, 37, 41]). This approach is, however, not well suited for the case where $D(A(t))$ varies with t and leads to serious difficulties. In fact, there seem to be no results in the literature in which the existence *and* regularity of solutions for (2), in the case of a time-dependent domain and an unbounded nonlinearity, have been shown.

In this paper we use a different approach. Namely we do not use fractional powers at all but study equations (2) and (4) in appropriate interpolation spaces. Of course, interpolation spaces have been used before by other authors in connection with evolution equations of type (2) (e.g. [15, 16, 17, 14, 40]. We refer to the end of this paper where a more detailed discussion of the relation of our results to earlier ones by other authors is given.). However, there are no results for nonlinear equations with time-dependent domains $D(A(t))$, and our approach is entirely different. In particular we shall show that—roughly speaking—the parabolic system (3) can be considered not only in L_p, as is well known, but also in W_p^s for $s > 0$. After having established this important fact we obtain the desired regularity almost automatically by means of embedding theorems.

In Chapter I we discuss integral-evolution equations in an abstract setting. These results can also be applied to other situations (e.g. in the case of nonlinear boundary conditions [6]). Besides of pure existence theorems we prove some results concerning the continuous dependence on initial values. These results are important for approximation arguments as well as for topological considerations (which are not given here).

Chapter II is the heart of the paper. Here we develop a theory of existence, regularity, and continuous dependence for abstract semilinear evolution equations in a general setting. The main results of this chapter are Theorem (8.7) and Corollary (8.8) and the «higher regularity» results of Theorem (9.6) and Theorem (10.3).

In Chapter III we apply the abstract results to parabolic equations. Here we consider the most general parabolic systems of even order and allow also unbounded domains. Our final results are contained in Section 15, though some intermediate theorems like Theorems (13.4) and (14.5) are of independent importance.

In this paper we do not consider the question of global existence. However the present paper is basic for this problem, which will be treated in a forthcoming publication. In this connection it will be important that we can work in the spaces W_p^s, where we can choose $s \in [0, 2m)$ and $p \in (1, \infty)$ essentially arbitrarily.

Throughout this work we use standard notation. We denote by $\mathbb{N}$ the nonnegative integers and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Moreover $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. All vector spaces are over $\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$ and we use complex quantities (for example in connection with spectral theory) it is always understood that we work with the natural complexification (of spaces and operators). Thus, by $\varrho(A)$, the resolvent set, and by $\sigma(A)$, the spectrum of a linear operator A, we mean always the resolvent set and the spectrum, respectively, of its complexification, if $\mathbb{K} = \mathbb{R}$.

X, Y, Z, with or without indices, denote always Banach spaces, and $\mathcal{L}(X, Y)$ is the Banach space of all continuous linear operators from X into Y. By $\mathcal{L}_s(X, Y)$ we mean the same vector space, but endowed with the topology of pointwise convergence. By $\mathbb{B}_X(a, r)$ we mean the open ball in X with center at a and radius r, and $\bar{\mathbb{B}}_X(a, r)$ is the corresponding closed ball. When no confusion seems possible we omit the index X. Moreover $\mathbb{B}_X$ is the open unit ball in X and $r\mathbb{B} := \mathbb{B}(0, r)$ for $r > 0$. Furthermore,

$\mathbb{B}^n := \mathbb{B}_{\mathbb{R}^n}$, $|\cdot|$ denotes the euclidian norm in $\mathbb{K}^n$ and $(\cdot|\cdot)$ the corresponding inner product. If M is a nonempty subset of X then we mean by $\mathbb{B}(M, \varepsilon)$ the ε-neighbourhood of M in X.

By T we denote a fixed but arbitrary positive number, and we let $\Delta_T := \{(t, s) \in \mathbb{R}^2 : 0 \leqq s \leqq t \leqq T\}$ and $\dot{\Delta}_T := \{(t, s) \in \mathbb{R}^2 : 0 \leqq s < t \leqq T\}$. The complement of a set M in a fixed superset is written as M^c, and $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geqq 0\}$.

If Ω is a measure space (more precisely: $(\Omega, \mathcal{A}, \mu)$ is a measure space), then $\mathcal{M}(\Omega, X)$ is the vector space of all strongly measurable functions $f: \Omega \to X$, which we simply call «measurable». For each $p \in [1, \infty]$ we let $\mathfrak{L}_p(\Omega, X)$ be the vector space of all measurable functions whose p-th power is integrable (with the usual modification if $p = \infty$), whereas $L_p(\Omega, X)$ is the usual Banach space obtained from $\mathfrak{L}_p(\Omega, X)$ by identifying functions differing only on sets of measure zero.

If Ω is a topological space then $B(\Omega, X)$, $BC(\Omega, X)$, $C(\Omega, X)$ denote the spaces of all bounded, all bounded and continuous, all continuous functions from Ω into X, respectively, the former two endowed with the usual supremum norm. The symbol $C(\Omega, M)$ is also used to denote the set of all continuous functions from Ω into M, where M is an arbitrary topological space (e.g. the locally convex space $\mathfrak{L}_s(X, Y)$). If M is a subset of Ω and the topology of M is stronger than the one of Ω we write $M \hookrightarrow \Omega$. If M and Ω are both linear spaces, this means also that M is a vector subspace of Ω.

If M is a metric space we denote by $C^{1-}(M, X)$ the set of all (locally) Lipschitz continuous maps from M into X. Here $f: M \to X$ is Lipschitz continuous if each point in M has a neighbourhood U in M such that $f|U$ is uniformly Lipschitz continuous. If M is a subset of some product space $E \times F$ then we write $f \in C^{0,1-}(M, X)$ if each point has a product neighbourhood $U \times V$ in M such that $f(\cdot, v): U \to X$ is continuous for each $v \in V$ and $f(u, \cdot): V \to X$ is uniformly Lipschitz continuous, also uniformly with respect to $u \in U$. Similar definitions apply to $C^{\alpha,\beta,\gamma}(M, X)$, where $\alpha, \beta, \gamma \in [0, 1) \cup \{1-\}$. If M is an open subset of some Banach space then $f \in C^{k-}(M, X)$ means that f is $(k-1)$-times continuously differentiable and the $(k-1)$-st derivative is Lipschitz continuous, provided of course, $k \geqq 2$.

If $B \in \mathfrak{L}(X, Y)$ and if $Z \hookrightarrow X$ then it is often convenient to denote the restriction of B to Z, that is, $B|Z \in \mathfrak{L}(Z, Y)$, again by B. Thus we can consider B as an element of $\mathfrak{L}(X, Y)$ as well as an element of $\mathfrak{L}(Z, Y)$, if no confusion seems possible.

Finally (2.5) means formula (5) in Section 2, etc.

CHAPTER I

INTEGRAL-EVOLUTION EQUATIONS

1. – Integral operators.

We begin with a simple technical

(1.1) LEMMA. *Suppose that* $a: \dot{\Delta}_T \to X$ *has the following properties*:

(i) $a(\cdot, s) \in C((s, T], X)$ *for a.a.* $s \in [0, T)$;

(ii) $a(t, \cdot) \in \mathcal{M}([0, t), X)$ *for each* $t \in (0, T]$;

(iii) *there exists a decreasing* $\hat{\alpha} \in \mathcal{L}_1((0, T], \mathbb{R})$ *such that*

$$\|\alpha(t, s)\| \leqq \hat{\alpha}(t-s) \qquad \forall (t, s) \in \dot{\Delta}_T .$$

Then

$$\left[t \mapsto \int_0^t a(t, \tau)\, d\tau\right] \in C([0, T], X) .$$

PROOF. For $0 \leqq \varepsilon < t$ let

$$a^{\varepsilon}(t) := \begin{cases} 0 & \text{for } 0 \leqq t \leqq \varepsilon , \\ \displaystyle\int_0^{t-\varepsilon} a(t, \tau)\, d\tau & \text{for } \varepsilon \leqq t \leqq T . \end{cases}$$

Then, due to (ii) and (iii), $a^{\varepsilon}: [0, T] \to X$ is well defined and

$$\|a^{\varepsilon}(t) - a^0(t)\| = \left\| \int_{(t-\varepsilon)\vee 0}^{t} a(t, \tau)\, d\tau \right\| \leqq \int_0^{\varepsilon} \hat{\alpha}(s)\, ds ,$$

which shows that $a^{\varepsilon} \to a^0$ as $\varepsilon \to 0$, uniformly with respect to $t \in [0, T]$. Hence it suffices to show that $a^{\varepsilon} \in C([0, T], X)$ for each $\varepsilon > 0$.

Let $\varepsilon \in (0, T)$ and $s \in [\varepsilon, T]$ be fixed. Then

$$a^{\varepsilon}(t) = \int_0^T \chi_{[0, t-\varepsilon]}(\tau)\, a(t, \tau)\, d\tau \qquad \forall t \in [\varepsilon, T] ,$$

where χ_M denotes the characteristic function of the set M. Observe that

$$\|\chi_{[0,t-\varepsilon]}(\tau)a(t,\tau)\| \leqq \hat{\alpha}(\varepsilon) \qquad \forall (t,\tau)\in \dot{\Delta}_T,\ t\geqq \varepsilon\,,$$

and that

$$\chi_{[0,t-\varepsilon]}(\tau)a(t,\tau) \to \chi_{[0,s-\varepsilon]}(\tau)a(s,\tau) \quad \text{for a.a. } \tau\in[0,T]$$

as $t\to s$ in $[\varepsilon, T]$. Hence Lebesgue's theorem implies $a^\varepsilon(t)\to a^\varepsilon(s)$ as $t\to s$ in $[\varepsilon, T]$. Thus $a^\varepsilon|[\varepsilon, T]$ is continuous and $a^\varepsilon(\varepsilon)=0$. Since $a^\varepsilon|[0,\varepsilon]=0$, it follows that $a^\varepsilon\in C([0,T],X)$. □

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite positive measure space. Then $B(\cdot)\colon \Omega\to\mathcal{L}(X,Y)$ is said to be *strongly measurable* if $B(\cdot)x\in\mathcal{M}(\Omega, Y)$ for each $x\in X$.

(1.2) LEMMA. *Let $B(\cdot)\colon\Omega\to\mathcal{L}(X,Y)$ be strongly measurable and let $u\in\mathcal{M}(\Omega,X)$. Then $B(\cdot)u(\cdot)\in\mathcal{M}(\Omega,Y)$.*

PROOF. There exists a sequence (u_j) of simple functions such that $u_j\to u$ μ-a.e. Hence $B(\cdot)u_j(\cdot)\to B(\cdot)u(\cdot)$ μ-a.e. Thus it suffices to prove the assertion if u is a simple function, $u:=\sum_{k=0}^{n} x_k\chi_{A_k}$. But this is obvious since $B(\cdot)x_k\chi_{A_k}\in\mathcal{M}(\Omega,Y)$ by the strong measurability of $B(\cdot)$. □

In the following we write

$$K\in\mathcal{K}(\hat{\alpha}, X, Y)$$

provided

(K i) $\hat{\alpha}\in\mathcal{L}_1((0,T),\mathbb{R}^+)$ and $\hat{\alpha}$ is decreasing;

(K ii) $K\colon\dot{\Delta}_T\to\mathcal{L}(X,Y)$;

(K iii) $\|K(t,s)\|_{\mathcal{L}(X,Y)}\leqq\hat{\alpha}(t-s)$, $(t,s)\in\dot{\Delta}_T$;

(K iv) $K(t,\cdot)\colon[0,t)\to\mathcal{L}(X,Y)$ is strongly measurable for each $t\in(0,T]$.

We write

$$K\in\mathcal{K}_s(\hat{\alpha},X,Y)$$

if $K\in C(\dot{\Delta}_T,\mathcal{L}_s(X,Y))$ and (Ki) and (Kiii) are satisfied. Since continuous functions (on intervals) are measurable it follows that

$$\mathcal{K}_s(\hat{\alpha},X,Y)\subset\mathcal{K}(\hat{\alpha},X,Y)\,. \tag{1}$$

In the special case that $\hat{\alpha}(t):=Mt^{-\alpha}$ for some $M\in\mathbb{R}^+$ and $\alpha\in[0,1)$ we let $\mathcal{K}(\alpha,M,X,Y):=\mathcal{K}(\hat{\alpha},X,Y)$, and a similar definition holds for $\mathcal{K}_s(\alpha,M,X,Y)$.

(1.3) PROPOSITION. *Suppose that* $K \in \mathcal{K}(\hat{\alpha}, X, Y)$ *and let*

$$\mathbf{K}_s u(t) := \int_s^t K(t, \tau) u(\tau) d\tau, \qquad s \leqq t \leqq T,$$

where $0 \leqq s < T$. *Then*

$$\mathbf{K}_s \in \mathcal{L}\big(L_\infty((s, T), X), B([s, T], Y)\big)$$

and

$$\|\mathbf{K}_s u(t)\|_Y \leqq \int_0^{t-s} \hat{\alpha}(\tau) d\tau \|u\|_{L_\infty((s,t),X)} \qquad \forall t \in [s, T].$$

If $K \in \mathcal{K}_s(\hat{\alpha}, X, Y)$ *then*

$$\mathbf{K}_s \in \mathcal{L}\big(L_\infty((s, T), X), C([s, T], Y)\big).$$

PROOF. It follows from (1) and Lemma (1.2) that $K(t, \cdot)u(\cdot) \in \mathcal{M}((0, t), Y)$ in each case. Since

$$\|K(t, \tau) u(\tau)\|_Y \leqq \hat{\alpha}(t - \tau) \|u\|_{L_\infty((s,t),X)}$$

for $s \leqq \tau < t \leqq T$, the first assertion is obvious. The second one is an easy consequence of Lemma (1.1). □

Our next result concerns the Hölder continuity of the function $\mathbf{K}_s u$.

(1.4) PROPOSITION. *Suppose that* $K \in \mathcal{K}(\alpha, M, X, Y)$ *and that there are constants* $\beta < 1$ *and* $N \in \mathbb{R}^+$ *such that*

$$\|K(t, \tau) - K(r, \tau)\|_{\mathcal{L}(X, Y)} \leqq N(t - r)^{1-\alpha}(r - \tau)^{-\beta} \tag{2}$$

for $0 \leqq \tau < r \leqq t \leqq T$. *Then* (*letting* $1 - 0 := 1-$)

$$\mathbf{K}_s \in \mathcal{L}\big(L_\infty((s, T), X), C^{1-\alpha}([s, T], Y)\big).$$

PROOF. (i) Let $s := 0$ and $\mathbf{K} := \mathbf{K}_0$, and let $\|\cdot\|_t$ denote the norm in $L_\infty((0, t), X)$.

If $0 \leqq r \leqq t/2$ and $t \leqq T$ it follows from Proposition (1.3) that

$$\|\mathbf{K}u(t) - \mathbf{K}u(r)\|_Y \leqq \frac{M}{1-\alpha}(t^{1-\alpha} + r^{1-\alpha})\|u\|_t \leqq \frac{3M}{1-\alpha}(t - r)^{1-\alpha}\|u\|_t.$$

If $0 < t/2 \leqq r \leqq t \leqq T$ then, by (2) and Proposition (1.3),

$$\|\mathbb{K}u(t) - \mathbb{K}u(r)\|_Y$$
$$\leqq \left\| \int_{2r-t}^{t} K(t, \tau) u(\tau)\, d\tau - \int_{2r-t}^{r} K(r, \tau) u(\tau)\, d\tau \right\|_Y + \int_0^{2r-t} \|(K(t, \tau) - K(r, \tau)) u(\tau)\|_Y d\tau$$
$$\leqq \left(\frac{3\, M}{1-\alpha} + \frac{NT^{1-\beta}}{1-\beta}\right)(t-r)^{1-\alpha}\|u\|_t .$$

This, together with Proposition (1.3), implies the assertion in this case.

(ii) If $0 < s < T$ the assertion follows from (i), from

$$\int_s^t K(t, \tau) u(\tau) d\tau = \int_0^{t-s} K(t - s + s, \tau + s) u(\tau + s) d\tau ,$$

and from the fact that the used estimates for $K(t, \tau)$ are translation invariant. $\square$

2. – Integral-evolution equations.

For completeness we include a proof of the following—in principle—well known

(2.1) PROPOSITION. *Let* $s \in [0, T)$ *and suppose that*

(i) $K \in \mathcal{K}_s(\hat{\alpha}, X, Y)$;

(ii) D *is open in* Y *and* $f \in C^{0,1-}([s, T] \times D, X)$;

(iii) $a \in C([s, T], Y)$ *and* $a(s) \in D$.

Then there exist δ *and a unique* $u \in C([s, s + \delta], D)$ *such that*

$$u(t) = a(t) + \int_s^t K(t, \tau) f(\tau, u(\tau)) d\tau , \qquad s \leqq t \leqq s + \delta .$$

PROOF. There exist $r > 0$, $\beta > 0$ and $\lambda \in \mathbb{R}^+$ such that

$$W := [s, s + \beta] \times \bar{\mathbb{B}}_Y(a(s), 2r) \subset [s, T] \times D , \tag{1}$$

$$\|f(t, x) - f(t, y)\|_X \leqq \lambda \|x - y\| \qquad \forall (t, x), (t, y) \in W, \tag{2}$$

and

$$M := \sup\{\|f(t, x)\|_X: (t, x) \in W\} < \infty .$$

Choose $\delta \in (0, \beta]$ such that

$$\|a(t) - a(s)\|_Y \leqq r \quad \text{for } s \leqq t \leqq s + \delta \tag{3}$$

and

$$\int_0^\delta \hat{\alpha}(\tau)\, d\tau \leqq \min\{1/2\lambda, r/M\}, \tag{4}$$

and let

$$Z := \{u \in C([s, s + \delta], Y): \|u(t) - a(t)\|_Y \leqq r \text{ for } s \leqq t \leqq s + \delta\}.$$

Then Z is a complete metric space (being a closed subset of the Banach space $C([s, s + \delta], Y)$), and, by (3),

$$u(t) \in \bar{\mathbf{B}}_Y(a(s), 2r) \quad \forall u \in Z,\ s \leqq t \leqq s + \delta . \tag{5}$$

Hence, by (1) and (iii),

$$f(\cdot, u(\cdot)) \in C([s, s + \delta], X) \subset L_\infty((s, s + \delta), X) \tag{6}$$

for $u \in Z$.

Let now

$$g(u)(t) := a(t) + \int_s^t K(t, \tau) f(\tau, u(\tau))\, d\tau, \qquad s \leqq t \leqq s + \delta .$$

Then it follows from (6), Proposition (1.3) and (4) that $g(Z) \subset Z$. Moreover, by (2), (4), (5) and Proposition (1.3),

$$\|g(u) - g(v)\|_Z \leqq \|u - v\|_Z/2 \qquad \forall u, v \in Z .$$

Now the assertion follows from Banach's fixed point theorem. □

By an *evolution system* (U, V) *in* (X, Y) (*of type* $\hat{\alpha}$) we mean a pair of functions possessing the following properties:

(ES i) $U: \Delta_T \to \mathfrak{L}(Y)$ and $V \in \mathcal{K}_s(\hat{\alpha}, X, Y)$.

(ES ii) $U(t, t) = id$ and $U(t, \tau)\, U(\tau, s) = U(t, s)$ for $0 \leqq s \leqq \tau \leqq t \leqq T$.

(ES iii) $U(t, \tau) V(\tau, s) = V(t, s)$ for $0 \leqq s < \tau \leqq t \leqq T$.

(ES iv) For each $t \in [0, T)$ there exists a closed linear subspace $Y(t)$ of Y such that

$$U(\cdot, t) \in C\big([t, T], \mathfrak{L}_s(Y(t), Y)\big)$$

and

$$U(t, s)(Y) \subset Y(t) \qquad \text{for } (t, s) \in \dot{\Delta}_T .$$

Clearly (ES iv) is satisfied with $Y(t) = Y$ if U is strongly continuous, that is, if $U \in C(\Delta_T, \mathfrak{L}_s(Y))$.

Throughout the remainder of this chapter we assume that:

(i) (U, V) *is an evolution system in* (X, Y).

(ii) D *is open in* Y *and* $f \in C^{0,1-}([0, T] \times D, X)$.

For each $(t_0, x_0) \in [0, T] \times D$ we consider the *integral-evolution equation*

$$(\mathfrak{J})_{(t_0,x_0)} \qquad u(t) = U(t, t_0)x_0 + \int_{t_0}^{t} V(t, \tau) f(\tau, u(\tau))\, d\tau , \qquad t_0 \leqq t \leqq T .$$

More precisely, $u\colon J_u \to D$ is said to be a *solution* of $(\mathfrak{J})_{(t_0,x_0)}$ (on J_u) if

(i) $J_u \subset [t_0, T]$ is an interval containing t_0 which is perfect (that is, contains more than one point) if $t_0 < T$;

(ii) $u \in C(J_u, D)$, and u satisfies $(\mathfrak{J})_{(t_0,x_0)}$ in Y for $t \in J_u$.

A solution u is *maximal* if there does not exist a solution of $(\mathfrak{J})_{(t_0,x_0)}$ which is a proper extension of u. In this case J_u is said to be a *maximal interval of existence.*

It is the purpose of the following considerations to show that $(\mathfrak{J})_{(t_0,x_0)}$ possesses a unique maximal solution for each $(t_0, x_0) \in [0, T] \times D(t_0)$, where

$$D(t_0) := D \cap Y(t_0) .$$

For this we need the following

(2.2) LEMMA. *Let* $0 \leqq t_0 < T$ *and* $a \in L_\infty([t_0, T], X)$. *Then*

$$U(t, t_1) \int_{t_0}^{t_1} V(t_1, \tau) a(\tau)\, d\tau \in C([t_1, T], Y)$$

for every $t_1 \in (t_0, T)$.

PROOF. Let $\varepsilon \in [0, t_1 - t_0)$ be arbitrary and let

$$\varphi_\varepsilon(t) := \int_{t_0}^{t_1-\varepsilon} V(t, \tau)a(\tau)d\tau , \qquad t_1 \leqq t \leqq T .$$

Then $\varphi_\varepsilon(t) \in Y$ by Proposition (1.3) and

$$\|\varphi_\varepsilon(t) - \varphi_0(t)\|_Y \leqq \|a\|_\infty \int_0^\varepsilon \hat{\alpha}(\tau)d\tau , \qquad t_1 \leqq t \leqq T ,$$

where $\|\cdot\|_\infty$ is the norm in $L_\infty((t_0, T), X)$. Hence

$$\varphi_\varepsilon(t) \to \varphi_0(t) = U(t, t_1) \int_{t_0}^{t_1} V(t_1, \tau)a(\tau)d\tau$$

as $\varepsilon \to 0$, uniformly with respect to $t \in [t_1, T]$. Thus, it suffices to show that $\varphi_\varepsilon \in C([t_1, T], Y)$ for $\varepsilon > 0$. But this follows from the fact that $\varphi_\varepsilon(t) = U(t, t_1)y$, where

$$y := U(t_1, t_1 - \varepsilon) \int_{t_0}^{t_1-\varepsilon} V(t_1 - \varepsilon, \tau)a(\tau)d\tau \in Y(t_1)$$

by (ES iv). $\square$

After these preparations we can prove the following *global existence and uniqueness*

(2.3) THEOREM. *For each $(t_0, x_0) \in [0, T] \times D(t_0)$ there exists a unique maximal solution*

$$u(\cdot, t_0, x_0) \in C(J(t_0, x_0), D)$$

of $(\mathfrak{J})_{(t_0,x_0)}$ and the maximal interval of existence $\mathfrak{J}(t_0, x_0)$ is right open in $[t_0, T]$.

PROOF. We can assume that $t_0 < T$. Then Proposition (2.1) implies the existence of a unique solution u on some compact perfect interval $[t_0, t_1] \subset [t_0, T]$. If $t_1 \neq T$ it follows from Lemma (2.2) that we can apply Proposition (2.1) again to find a unique solution v on $[t_1, t_2]$ of $(\mathfrak{J})_{(t_1,x_1)}$, where $x_1 := u(t_1)$ and $t_2 > t_1$. Let $w \in C([t_0, t_2], D)$ be defined by $w \,|\, [t_0, t_1] := u$ and $w \,|\, [t_1, t_2] := v$. Then it is an easy consequence of (ES ii) and (ES iii) that w is a solution of $(\mathfrak{J})_{(t_0,x_0)}$ on $[t_0, t_2]$. By Proposition (2.1) it is also the only solution on $[t_0, t_2]$.

Let now

$$J(t_0, x_0) := \cup \{[t_0, t] \subset [t_0, T]\colon (\mathfrak{J})_{(t_0,x_0)} \text{ has a solution on } [t_0, t]\}\,.$$

Then $J(t_0, x_0)$ is an interval in $[t_0, T]$, which contains t_0 and is right open in $[t_0, T]$, since otherwise an application of Proposition (2.1) and Lemma (2.2) to its right endpoint would give a contradiction. Clearly $J(t_0, x_0)$ is the maximal interval of existence of a solution $u(\cdot, t_0, x_0)$ of $(\mathfrak{J})_{(t_0,x_0)}$, which is defined in the obvious way. Moreover there is no other maximal solution of $(\mathfrak{J})_{(t_0,x_0)}$. $\square$

(2.4) COROLLARY. *$u(t, t_0, x_0) = u\big(t, s, u(s, t_0, x_0)\big)$ for all $s, t \in J(t_0, x_0)$ with $s \leq t$.*

It should be noted that in the above proofs we did not use the closedness of $Y(t)$ in Y.

3. – Global solutions.

In the following we let $(t_0, x_0) \in [0, T] \times D(t_0)$ and put

$$t^+(t_0, x_0) := \sup J(t_0, x_0)\,.$$

Then we study the behaviour of $u(\cdot, t_0, x_0)$ as $t \to t^+(t_0, x_0)$. The basic result is contained in the following

(3.1) THEOREM. *Suppose that*

$$t^+ := t^+(t_0, x_0) < T \quad \textit{and that} \quad f\big(\operatorname{graph}(u(\cdot, t_0, x_0))\big)$$

is bounded in X. Then $u(t, t_0, x_0) \to y \in D^c$ as $t \to t^+$.

PROOF. Let $u := u(\cdot, t_0, x_0)$ and $J := J(t_0, x_0)$. Then

$$v := f(\cdot, u(\cdot)) \in BC(J, X) \subset \mathfrak{L}_\infty(J, X)\,.$$

Consequently

$$\Big(t \mapsto w(t) := \int_{t_0}^{t} V(t, \tau) v(\tau)\, d\tau\Big) \in C\big([t_0, t^+], Y\big)$$

by Proposition (1.3). Hence $\bar{u} = U(\cdot, t_0)x_0 + w \in C\big([t_0, t^+], Y\big)$, since $x_0 \in Y(t_0)$.

Consequently $y := \bar{u}(t^+) = \lim_{t \to t^+} u(t)$ exists. If $y \in D$, then $\bar{u}$ is a solution of $(\mathfrak{J})_{(t_0, x_0)}$ on $[t_0, t^+]$ extending u, which contradicts the fact that $t^+ \notin J(t_0, x_0)$ if $t^+ < T$ by Theorem (2.3). □

A solution $u: J_u \to D$ of $(\mathfrak{J})_{(t_0, x_0)}$ is said to be *global* if $J_u = [t_0, T]$. Clearly every global solution is maximal.

(3.2) COROLLARY. *Suppose that $D = Y$ and $f(\text{graph}\,(u(\cdot, t_0, x_0)))$ is bounded in X. Then $u(\cdot, t_0, x_0)$ is a global solution.*

In order to obtain a simple sufficient criterion for the existence of global solution we need the following generalization of Gronwall's lemma.

(3.3) LEMMA. *Suppose that $0 \leqq t_0 < T$, that $b \in \mathfrak{L}_1([t_0, T], \mathbb{R}^+)$ is decreasing, and that $a, u \in C([t_0, T], \mathbb{R}^+)$. Moreover, suppose that*

$$u(t) \leqq a(t) + \int_{t_0}^{t} b(t-\tau) u(\tau)\, d\tau \qquad \forall t \in [t_0, T].$$

Then there exists a constant $\beta > 0$, depending only on b, such that

$$u(t) \leqq 2a^*(t) \exp[\beta(t-t_0)] \qquad \textit{for } t_0 \leqq t \leqq T,$$

where $a^*(t) := \max\{a(s): t_0 \leqq s \leqq t\}$.

PROOF. Choose $\varepsilon > 0$ such that $\int_{t_0}^{t} b(\tau)\, d\tau \leqq \frac{1}{2}$ and let $u(t) := 0$ for $t < t_0$. Then

$$\begin{aligned} u(s) &\leqq a(s) + \int_{t_0}^{s-\varepsilon} b(s-\tau) u(\tau)\, d\tau + \int_{s-\varepsilon}^{s} b(s-\tau) u(\tau)\, d\tau \\ &\leqq a^*(s) + b(\varepsilon) \int_{t_0}^{s-\varepsilon} u(\tau)\, d\tau + u^*(t) \int_{0}^{\varepsilon} b(\tau)\, d\tau \end{aligned}$$

for $t_0 \leqq s \leqq t \leqq T$. Hence

$$u^*(t) \leqq 2a^*(t) + \beta \int_{t_0}^{t} u^*(\tau)\, d\tau, \qquad t_0 \leqq t \leqq T,$$

where $\beta := 2b(\varepsilon)$. Now the assertion follows from the standard Gronwall's lemma. □

The principal idea for the above simple proof is due to Pazy (cf. [24, p. 33]).

In the important special case that $b(t) = \beta t^{-\alpha}$ for some $\beta > 0$ and $\alpha \in (0, 1)$ more precise estimates are given in [4, 25].

On the basis of Lemma (3.3) and Corollary (3.2) it is easy to prove the following important (and, in principle, familiar) criterion guaranteeing global solutions.

(3.4) PROPOSITION. *Suppose that $D = Y$ and that there exists a constant c such that*

$$\|f(t, x)\|_X \leqq c(1 + \|x\|_Y) \qquad \forall (t, x) \in \operatorname{graph}(u(\cdot, t_0, x_0)) . \tag{1}$$

Then $u(\cdot, t_0, x_0)$ is a global solution.

PROOF. The integral-evolution equation implies the estimate

$$\begin{aligned}\|u(t)\|_Y &\leqq \|U(t, t_0)x_0\|_Y + \int_{t_0}^{t} \hat{\alpha}(t-\tau)c(1 + \|u(\tau)\|_Y)\,d\tau \\ &\leqq c_1 + \int_{t_0}^{t} c\hat{\alpha}(t-\tau)\|u(\tau)\|_Y\,d\tau\end{aligned}$$

for $t_0 \leqq t < t^+$, where $u := u(\cdot, t_0, x_0)$ and

$$c_1 := \max\{\|U(t, t_0)x_0\|_Y : t_0 \leqq t \leqq T\} + c\int_0^{T-t_0} \hat{\alpha}(\tau)\,d\tau .$$

Hence, by Lemma (3.3), $\sup\{\|u(t)\|_Y : t_0 \leqq t < t^+\} < \infty$: Consequently $f(\operatorname{graph}(u))$ is bounded in X by (1), and Corollary (3.2) implies the assertion. $\square$

4. – Continuity properties.

In this section we study the continuity properties of the function $(t, x) \to u(t, t_0, x)$, where $t_0 \in [0, T)$ is fixed.

In the following we denote by $N \in \mathbb{R}^+$ a fixed constant such that

$$\|U(t, t_0)\|_{\mathcal{L}(Y(t_0), Y)} \leqq N \qquad \forall t \in [t_0, T] .$$

The existence of N is an easy consequence of the uniform boundedness principle since $Y(t_0)$ is a Banach space and $U(\cdot, t_0) \in C\big([t_0, T], \mathcal{L}_s(Y(t_0), Y)\big)$.

(4.1) LEMMA. *Let* $t_1 \in (t_0, T]$ *and* $B \subset D$ *and suppose that*

(i) *there exists a constant* L *such that*

$$\|f(t, x) - f(t, y)\|_Y \leqq L\|x - y\|_Y \qquad \forall (t, x), (t, y) \in [t_0, t_1] \times B\,;$$

(ii) $x_1, x_2 \in B \cap Y(t_0)$ *and* $u(t, t_0, x_j) \in B$ *for* $t_0 \leqq t \leqq t_1$ *and* $j = 1, 2$.

Then there exists a constant M, *depending only on* $L\hat{\alpha}$ *and* N, *such that*

$$\|u(t, t_0, x_1) - u(t, t_0, x_2)\|_Y \leqq M\|x_1 - x_2\|$$

for $t_0 \leqq t \leqq t_1$.

PROOF. The integral-evolution equation and the assumptions imply the estimate

$$\|u_1(t) - u_2(t)\|_Y \leqq \|U(t, t_0)(x_1 - x_2)\|_Y + \int_{t_0}^{t} L\hat{\alpha}(t - \tau)\|u_1(\tau) - u_2(\tau)\|_Y d\tau$$

for $t_0 \leqq t \leqq t_1$, where $u_j := u(\cdot, t_0, x_j)$, $j = 1, 2$. Hence the assertion follows from Lemma (3.3). $\square$

After these preparations we can now prove the following *global continuity*

(4.2) THEOREM. *Let*

$$\mathfrak{D}(t_0) := \{(t, x) \in [t_0, T] \times D(t_0) : t \in J(t_0, x)\}\,.$$

Then $\mathfrak{D}(t_0)$ *is open in* $[t_0, T] \times Y(t_0)$ *and* $u(\cdot, t_0, \cdot) \in C^{0,1-}(\mathfrak{D}(t_0), Y)$.

PROOF. Let $(\bar{t}, \bar{x}) \in \mathfrak{D}(t_0)$ be given, fix $t_1 \in (\bar{t}, t^+(t_0, \bar{x}))$, and let $\bar{u} := u(\cdot, t_0, \bar{x})$. Since $[t_0, t_1] \times \bar{u}([t_0, t_1])$ is compact in $\mathbb{R} \times D$ we can find constants $\varepsilon > 0$ and $L \in \mathbb{R}^+$ such that $t_1 + \varepsilon \leqq T$, $B := \mathbb{B}_Y(\bar{u}([t_0, t_1]), \varepsilon) \subset D$,

$$(1) \qquad \|f(t, x) - f(t, y)\|_X \leqq L\|x - y\|_Y \qquad \forall (t, x), (t, y) \in [t_0, t_1 + \varepsilon] \times B\,,$$

and such that

$$(2) \qquad \sup\{\|f(t, x)\|_X : (t, x) \in \mathbb{B}_{\mathbb{R} \times Y}(G, \varepsilon)\} < \infty\,,$$

where $G := \operatorname{graph}(\bar{u} \mid [t_0, t_1])$ (cf. [5, Satz 6.4]).

Let now $\varrho := \varepsilon/M$, where $M \geqq 1$ is a constant as in Lemma (4.1) belonging to L chosen above. Suppose that $x \in \mathbf{B}_{Y(t_0)}(\bar{x}, \varrho) \subset D(t_0)$ and that $t^+ := t^+(t_0, x) \leqq t_1$. If $(t, u(t)) \in \mathbf{B}_{\mathbf{R}\times Y}(G, \varepsilon) =: W$ for $t_0 \leqq t < t^+$, where $u := u(\cdot, t_0, x)$, then f is bounded on graph (u) by (2) and $t^+ = T$ by Theorem (3.1). Since $t_1 < T$, this is impossible. Hence there exists $t^* \in (t_0, t^+)$ such that $(t, u(t)) \in W$ for $t_0 \leqq t < t^*$ and $\|u(t^*) - \bar{u}(t^*)\|_Y = \varepsilon$. But then Lemma (4.1) implies the contradiction

$$\varepsilon = \|u(t^*) - \bar{u}(t^*)\|_Y \leqq M\|x - \bar{x}\|_Y < \varepsilon .$$

This shows that $t^+(t_0, x) > t_1$ and that

$$(3) \qquad (t, u(t, t_0, x)) \in W \quad \forall (t, x) \in [t_0, t_1] \times \mathbf{B}_{Y(t_0)}(\bar{x}, \varrho) .$$

Thus $[t_0, t_1] \times \mathbf{B}_{Y(t_0)}(\bar{x}, \varrho) \subset \mathfrak{D}(t_0)$ which proves that $\mathfrak{D}(t_0)$ is open in $[t_0, T] \times Y(t_0)$.

Since $W \subset [t_0, t_1 + \varepsilon] \times B$ it follows from (1), (3) and Lemma (4.1) that

$$\|u(t, t_0, x) - u(t, t_0, y)\|_Y \leqq M\|x - y\|$$

for all $t \in [t_0, t_1]$ and $x, y \in \mathbf{B}_{Y(t_0)}(\bar{x}, \varrho)$. Thus $u(\cdot, t_0, \cdot) \in C^{0,1-}(\mathfrak{D}(t_0), Y)$ since $u(\cdot, t_0, x) \in C(J(t_0, x), Y)$. $\square$

It is not difficult to extend this continuity result to families of integral-evolution equations

$$(4) \qquad u(t) = U_\lambda(t, t_0)v + \int_{t_0}^{t} V_\lambda(t, \tau) f_\lambda(\tau, u(\tau))\, d\tau ,$$

where λ varies in some topological space Λ. Given appropriate continuity hypotheses for the function $\lambda \mapsto (U_\lambda, V_\lambda, f_\lambda)$ it follows that the solution of (4) depends also continuously on λ.

5. – Autonomous integral-evolution equations.

Let M be a metric space and let $t^+ \colon M \to (0, \infty]$. Put

$$\mathfrak{D} := \bigcup_{x \in M} [0, t^+(x)) \times \{x\}$$

and suppose that $\varphi \colon \mathfrak{D} \to M$ is a map with the following properties:

(i) $\mathfrak{D}$ is open in $\mathbb{R}^+ \times M$;

(ii) $\varphi \in C(\mathfrak{D}, M)$;

(iii) $\varphi(0, \cdot) = \mathrm{id}_M$;

(iv) if $0 \leqq s < t^+(x)$ and $0 \leqq t < t^+(\varphi(s, x))$ then

$$0 \leqq s + t < t^+(x) \quad \text{and} \quad \varphi(t, \varphi(s, x)) = \varphi(s + t, x).$$

Then φ is said to be a (local) *semiflow on M*. If $t^+(x) = \infty$ for all $x \in M$ then φ is a *global* semiflow. For each x the set

$$\gamma^+(x) := \{\varphi(t, x) : 0 \leqq t < t^+(x)\}$$

is the (*positive*) *orbit through x* and $t^+(x)$ is the *positive exit time* of x (e.g. [5, 10]).

The following theorem shows that autonomous integral-evolution equations generate semiflows. Here an integral-evolution equation is said to be *autonomous* if U and V depend only on $t - \tau$ and if f is independent of t. More precisely we have the following

(5.1) THEOREM. *Suppose that*

(i) $\{U(t) : t \in \mathbb{R}^+\}$ *is a strongly continuous semigroup on* Y.

(ii) $V \in C(\Delta_T, \mathfrak{L}_s(X, Y))$ *and* $V(t + s) = U(t)V(s)$ *for* $t, s > 0$.

(iii) *For each* $T \in (0, \infty)$ *there exists a decreasing function* $\hat{\alpha}_T \in \mathfrak{L}_1((0, T], \mathbb{R}^+)$ *such that*

$$\|V(t)\|_{\mathfrak{L}(X,Y)} \leqq \hat{\alpha}_T(t) \quad \text{for } 0 < t \leqq T.$$

(iv) $D \subset Y$ *is open and* $f \in C^{1-}(D, X)$. *Then the autonomous integral-evolution equation*

$$u(t) = U(t)x + \int_0^t V(t - \tau) f(u(\tau))\, d\tau \tag{1}$$

has for each $x \in D$ *a unique maximal solution* $\varphi(\cdot, x) \in C\big([0, t^+(x)), D\big)$ *where,* $0 < t^+(x) \leqq \infty$. *Moreover*

$$\mathfrak{D} := \{(t, x) \in \mathbb{R}^+ \times D : 0 \leqq t < t^+(x)\}$$

is open in $\mathbb{R}^+ \times D$ *and* $\varphi : \mathfrak{D} \to D$ *is a semiflow on* D *such that* $\varphi \in C^{0,1-}(\mathfrak{D}, D)$.

PROOF. The assertion follows easily from Theorem (2.3), Corollary (2.4), Theorem (4.2), and from the «autonomous nature» of (1). □

CHAPTER II

ABSTRACT SEMILINEAR PARABOLIC EVOLUTION EQUATIONS

6. – Parabolic fundamental solutions.

We denote by $\mathrm{Isom}(X, Y)$ the set of all isomorphisms in $\mathcal{L}(X, Y)$. Then it is well known that $\mathrm{Isom}(X, Y)$ is open in $\mathcal{L}(X, Y)$ and

$$(B \mapsto B^{-1}) \in C^\infty\big(\mathrm{Isom}(X, Y), \mathcal{L}(Y, X)\big). \tag{1}$$

We consider now the following assumption:

(AP i) *X^1 is a Banach space such that $X^1 \hookrightarrow X$ and there exist $\mu_0 \in \mathbb{R}$, a Banach space Z, and a function $(\mathcal{A}, \mathcal{B})\colon [0, T] \to \mathcal{L}(X^1, X \times Z)$ such that*

$$\mathcal{T} := (\mu_0 + \mathcal{A}, \mathcal{B}) \in C\big([0, T], \mathrm{Isom}(X^1, X \times Z)\big).$$

Since $[0, T]$ is compact it follows from (AP i) and (1) that there are constants c_0, c_1 such that

$$\|x\|_1 \leqq c_0\Big(\big\|\big(\mu_0 + \mathcal{A}(t)\big)x\big\| + \|\mathcal{B}(t)x\|_Z\Big) \leqq c_1\|x\|_1 \tag{2}$$

for all $x \in X^1$ and all $t \in [0, T]$, where $\|\cdot\|$ denotes the norm in X and $\|\cdot\|_1$ is the norm in X^1.

For each $t \in [0, T]$ we let

$$X^1(t) := X^1_{\mathcal{B}(t)} := \ker\big(\mathcal{B}(t)\big).$$

Then $X^1(t)$ is a closed linear subspace of X^1 and $\big(\mu_0 + \mathcal{A}(t)\big) \,\big|\, X^1(t) \in \mathrm{Isom}\big(X^1(t), X\big)$. Since $X^1 \hookrightarrow X$ we can define an «unbounded» linear operator *in* X,

$$A(t)\colon D\big(A(t)\big) \subset X \to X,$$

by letting

$$D\big(A(t)\big) := X^1(t) \qquad \text{and} \qquad A(t)x := \mathcal{A}(t)x.$$

Then $A(t)$ is closed and $\mu_0 \in \varrho\big(-A(t)\big)$ for each $t \in [0, T]$.

Consider now the following additional assumption:

(AP ii) *$D(A(t))$ is dense in X and*

$$\varrho(-A(t)) \supset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqq \mu_0\}$$

for each $t \in [0, T]$. Moreover there exists a constant M_0 such that

$$\|(\lambda + A(t))^{-1}\|_{\mathfrak{L}(X)} \leqq \frac{M_0}{1+|\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqq \mu_0$ and all $t \in [0, T]$.

It is well known that this implies the existence of constants $\alpha \in (0, \pi/2)$ and c_2 such that

$$\varrho(-A(t)) \supset \mu_0 + S_{\alpha+\pi/2} \tag{3}$$

and

$$\|(\lambda + A(t))^{-1}\|_{\mathfrak{L}(X)} \leqq \frac{c_2}{1+|\lambda|} \tag{4}$$

for all $\lambda \in \mu_0 + S_{\alpha+\pi/2}$ and $t \in [0, T]$, where S_β denotes, for each $\beta \in (0, \pi)$, the closed sector

$$S_\beta := \{\lambda \in \mathbb{C}\colon |\arg\lambda| \leqq \beta\} \cup \{0\}$$

in the complex plane. Moreover $-A(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{\exp[-sA(t)]\colon s \geqq 0\}$ on X (that is: in $\mathfrak{L}(X)$), which is explicitly given by

$$\exp[-sA(t)] = \frac{1}{2\pi i}\int_\Gamma \exp[\lambda s](\lambda + A(t))^{-1} d\lambda\,, \tag{5}$$

where Γ is any smooth path in the interior of $\mu_0 + S_{\alpha+\pi/2}$ running from $\infty\exp[-i(\alpha+\pi/2)]$ to $\infty\exp[i(\alpha+\pi/2)]$ (e.g. [29, Theorem 13.2]).

Finally we impose the assumption

(AP iii) *$(\mu_0 + A(\cdot))^{-1} \in C^1([0, T], \mathfrak{L}(X))$ and there exist constants M_1 and $\varrho \in [0, 1)$ such that*

$$\|(\mu_0 + A(t))(\lambda + A(t))^{-1}[(\mu_0 + A(t))^{-1}]^{\cdot}\|_{\mathfrak{L}(X)} \leqq M_1|\lambda - \mu_0|^{-1+\varrho}$$

for all $\lambda \in \mu_0 + S_{\alpha+\pi/2}$ and all $t \in [0, T]$.

Here and often in the following we denote the derivative with respect to t by a dot.

It has been shown by Yagi [51] that (AP ii) and (AP iii) (together with the assumption that $A(t)$ is closed, which is automatically satisfied in our case) imply the existence of a parabolic fundamental solution U_{μ_0} for $\mu_0 + A(\cdot)$ in the sense defined below.

First we consider the evolution equation

$$\dot{u} + A(t)u = f(t)\,, \tag{E}$$

where $f\colon [0, T] \to X$. By a *solution* of (E) on $[t_0, T]$, $0 \leqq t_0 < T$, we mean a function $u \in C([t_0, T], X) \cap C^1((t_0, T], X)$ such that $u(t) \in D(A(t))$ and $\dot{u}(t) + A(t)u(t) = f(t)$ for $t_0 < t \leqq T$.

Then a function $U\colon \Delta_T \to \mathcal{L}(X)$ is said to be a *parabolic fundamental solution for* $A(\cdot)$ provided:

(PF i) $\quad U \in C(\Delta_T, \mathcal{L}_s(X))$.

(PF ii) $\quad U(t, t) = id$ and $U(t, \tau)U(\tau, s) = U(t, s)$ for $0 \leqq s \leqq \tau \leqq t \leqq T$.

(PF iii) $\quad R(U(t, s)) \subset D(A(t))$ for $(t, s) \in \dot{\Delta}_T$,

$$[(t, s) \mapsto A(t)U(t, s)A^{-1}(s)] \in C(\Delta_T, \mathcal{L}_s(X))\,,$$
$$[(t, s) \mapsto A(t)U(t, s)] \in C(\dot{\Delta}_T, \mathcal{L}_s(X))\,,$$

and there exists a constant c_3 such that

$$\|A(t)U(t, s)\| \leqq c_3(t - s)^{-1} \qquad \forall (t, s) \in \dot{\Delta}_T\,.$$

(PF iv) If $f \in C([0, T], X)$ and if u is a solution of (E) on $[t_0, T]$ then

$$u(t) = U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, \tau)f(\tau)d\tau\,, \qquad t_0 \leqq t \leqq T\,. \tag{6}$$

Conversely, if $f \in C^{\mu}([0, T], X)$ for some $\mu \in (0, 1)$ and if u satisfies (6) with any $u(t_0) \in X$ then u is a solution of (E) on $[t_0, T]$.

Observe that (PF iv) implies, in particular, that

$$U(\cdot, t_0) \in C^1((t_0, T], \mathcal{L}_s(X)) \tag{7}$$

and that

$$D_1 U(t, t_0) = -A(t) U(t, t_0), \qquad t_0 < t \leqq T, \tag{8}$$

where D_1 denotes the derivative with respect to the first variable. Moreover it implies also that there is at most one parabolic fundamental solution for $A(\cdot)$.

As mentioned above Yagi's results [51] imply the existence of a parabolic fundamental solution U_{μ_0} for $\mu_0 + A(\cdot)$. It is easily verified that $U(t, s) := \exp[\mu_0 t]\, U_{\mu_0}(t, s) \exp[-\mu_0 s]$, $(t, s) \in \Delta_T$, is a parabolic fundamental solution for $A(\cdot)$.

Throughout the remainder of this chapter we impose assumptions $(AP\,\mathrm{i})$-$(AP\,\mathrm{iii})$ *and we denote by* U *the parabolic fundamental solution for* $A(\cdot)$.

In the following we denote by c positive constants which may be different from formula to formula but are independent of the specific independent variables occurring in a particular place. Thus we use c in much the same way as the Landau symbol 0.

(6.1) LEMMA. (i) $U \in C(\dot{\Delta}_T, \mathcal{L}_s(X, X^1))$ *and*

$$\| U(t, s) \|_{\mathcal{L}(X, X^1)} \leqq c(t-s)^{-1} \qquad \forall (t, s) \in \dot{\Delta}_T .$$

(ii) $U(\cdot, s) \in C\big([s, T], \mathcal{L}_s(X^1(s), X^1)\big)$ *for every* $s \in [0, T]$.

PROOF. Without loss of generality we can assume that $\mu_0 = 0$.

(i) Let $x \in E$ and (t, s), $(\tau, \sigma) \in \dot{\Delta}_T$. Then

$$\begin{aligned}\mathcal{A}(t)\big(U(t, s)x - U(\tau, \sigma)x\big) \\ &= A(t)\, U(t, s)x - A(\tau)\, U(\tau, \sigma)x - [\mathcal{A}(t) - \mathcal{A}(\tau)]\, U(\tau, \sigma)x\end{aligned}$$

and, similarly,

$$\mathcal{B}(t)\big(U(t, s)x - U(\tau, \sigma)x\big) = -[\mathcal{B}(t) - \mathcal{B}(\tau)]\, U(\tau, \sigma)x$$

by $(PF\,\mathrm{iii})$ and $D(A(t)) = X^1_{\mathcal{B}(t)} = \ker(\mathcal{B}(t))$. Thus, by (2),

$$\begin{aligned}\| U(t, s)x - U(\tau, \sigma)x \|_1 &\leqq c_0\big(\| A(t)\, U(t, s)x - A(\tau)\, U(\tau, \sigma)x \| \\ &+ \| \mathcal{A}(t) - \mathcal{A}(\tau) \|_{\mathcal{L}(X^1, X)} \| U(\tau, \sigma)x \|_1 + \| \mathcal{B}(t) - \mathcal{B}(\tau) \|_{\mathcal{L}(X^1, Z)} \| U(\sigma, \tau)x \|_1\big) .\end{aligned} \tag{9}$$

Now $(AP\,\mathrm{i})$ and $(PF\,\mathrm{iii})$ imply $U \in C(\dot{\Delta}_T, \mathcal{L}_s(X, X^1))$. The estimate for the norm follows directly from the estimate in $(PF\,\mathrm{iii})$.

(ii) Letting $s=\sigma=\tau$ and $x\in X^1(s)$ in (9) we see that

$$\|U(t,s)x-x\|_1\leqq c_0\big(\|A(t)U(t,s)x-A(s)x\| \\ +\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(X^1,X)}\|x\|_1+\|\mathcal{B}(t)-\mathcal{B}(s)\|_{\mathcal{L}(X^1,Z)}\|x\|_1\big).$$

Thus, since $x=A^{-1}(s)y$ for some $y\in X$, the assertion follows from (i) and (*PF* iii). $\square$

We close this section by showing that $\mathcal{T}\in C^1([0,T],\mathcal{L}(X^1,X\times Z))$ implies the continuous differentiability of $A^{-1}(\cdot)$. In fact, we can prove somewhat more.

(6.2) LEMMA. *Suppose that* $\mathcal{T}\in C^{1+\nu}([0,T],\mathcal{L}(X^1,X\times Z))$ *for some* $\nu\in[0,1)$. *Then* $(\mu_0+A(\cdot))^{-1}\in C^{1+\nu}([0,T],\mathcal{L}(X,X^1))$.

PROOF. We can assume again that $\mu_0=0$. Let $(f,g)\in C^\nu([0,T],X\times Z)$ be given and put $h(\cdot):=\mathcal{T}^{-1}(\cdot)(f(\cdot),g(\cdot))$ so that

$$\mathcal{A}(t)h(t)=f(t)\,,\qquad \mathcal{B}(t)h(t)=g(t)\,,\qquad 0\leqq t\leqq T\,.$$

Then

$$\mathcal{A}(t)\big(h(t)-h(\tau)\big)=f(t)-f(\tau)+[\mathcal{A}(\tau)-\mathcal{A}(t)]h(\tau) \\ =f(t)-f(\tau)+[\mathcal{A}(\tau)-\mathcal{A}(t)]\mathcal{T}^{-1}(\tau)\big(f(\tau),g(\tau)\big)$$

and

$$\mathcal{B}(t)\big(h(t)-h(\tau)\big)=g(t)-g(\tau)+[\mathcal{B}(\tau)-\mathcal{B}(t)]h(\tau) \\ =g(t)-g(\tau)+[\mathcal{B}(\tau)-\mathcal{B}(t)]\mathcal{T}^{-1}(\tau)\big(f(\tau),g(\tau)\big)$$

for $0\leqq t,\ \tau\leqq T$. Hence we deduce from (2) that

$$h\in C^\nu([0,T],X^1)\,. \tag{10}$$

Let now $x\in X$ be given and put

$$u(\cdot):=\mathcal{T}^{-1}(\cdot)(x,0)\qquad\text{and}\qquad v(\cdot):=-\mathcal{T}^{-1}(\cdot)\big(\dot{\mathcal{A}}(\cdot)u(\cdot),\dot{\mathcal{B}}(\cdot)u(\cdot)\big)\,.$$

Then, by (1),

$$u(\cdot)=A^{-1}(\cdot)x\in C^1([0,T],X^1) \tag{11}$$

and, consequently, by applying (10),

$$v(\cdot) \in C^{\nu}([0, T], X^1). \tag{12}$$

Observe that

$$\mathcal{A}(t)[u(t+h) - u(t) - hv(t)]$$
$$= -[\mathcal{A}(t+h) - \mathcal{A}(t) - h\dot{\mathcal{A}}(t)]u(t+h) + h\dot{\mathcal{A}}(t)[u(t) - u(t+h)]$$

and

$$\mathcal{B}(t)[u(t+h) - u(t) - hv(t)]$$
$$= -[\mathcal{B}(t+h) - \mathcal{B}(t) - h\dot{\mathcal{B}}(t)]u(t+h) + h\dot{\mathcal{B}}(t)[u(t) - u(t+h)]$$

for $0 \leqq t,\ t+h \leqq T$. Hence we deduce from (2) and (11) that

$$\|u(t+h) - u(t) - hv(t)\|_1 = o(h)\|x\|$$

as $h \to 0$, which implies the assertion. $\square$

7. – Evolution equations in interpolation spaces.

In the following we use some basic facts from the theory of interpolation spaces for which we refer to [9, 45].

We suppose that for each $\Theta \in (0, 1)$ there is given an interpolation functor $\mathcal{F}_\Theta$ from the category $\mathcal{C}$ of compatible pairs of Banach spaces (in the sense of [9], that is, the category of interpolation couples in the sense of [45]) to the category of all Banach spaces possessing the following properties:

(*IPF i*) $\mathcal{F}_\Theta$ is an interpolation functor of exponent Θ, that is, there exists a constant c_Θ such that

$$\|T\|_{\mathcal{L}(\mathcal{F}_\Theta(A_0,A_1),\mathcal{F}_\Theta(B_0,B_1))} \leqq c_\Theta \|T\|^{1-\Theta}_{\mathcal{L}(A_0,B_0)} \|T\|^{\Theta}_{\mathcal{L}(A_1,B_1)}$$

whenever (A_0, A_1) and (B_0, B_1) are objects and $T\colon (A_0, A_1) \to (B_0, B_1)$ is a morphism in the category $\mathcal{C}$.

(*IPF* ii) $A_0 \cap A_1$ is dense in $\mathcal{F}_\Theta(A_0, A_1)$ for every object (A_0, A_1) in $\mathcal{C}$.

$(IPF\,\text{iii})$ There exists a constant $\bar{c}_\Theta$ such that

$$\|x\|_{\mathcal{F}_\Theta(A_0, A_1)} \leqq \bar{c}_\Theta \|x\|_{A_0}^{1-\Theta} \|x\|_{A_1}^{\Theta} \qquad \forall x \in A_0 \cap A_1 ,$$

where (A_0, A_1) is any object in $\mathcal{C}$.

Observe that the complex interpolation functors $[\ ,\ ]_\Theta$ and the real interpolation functors $(\ ,\ )_{\Theta,p}$, $0<\Theta<1$, $1 \leqq p < \infty$, satisfy $(IPF\,\text{i})$-$(IPF\,\text{iii})$.

In the following we let

$$X^\Theta := \mathcal{F}_\Theta(X, X^1) ,$$

and we denote by $\|\cdot\|_\Theta$ the norm in X^Θ. Since $X^1 \hookrightarrow X$ it follows that

$$X^1 \overset{d}{\hookrightarrow} X^\Theta \hookrightarrow X , \tag{1}$$

where the symbol d indicates that X^1 is dense in X^Θ. Similarly, letting

$$X^\Theta(t) := \mathcal{F}_\Theta\big(X, X^1(t)\big) , \qquad 0 \leqq t \leqq T ,$$

it follows that

$$X^1(t) \overset{d}{\hookrightarrow} X^\Theta(t) \hookrightarrow X , \qquad 0 \leqq t \leqq T . \tag{2}$$

Moreover, since $X^1(t)$ is a closed linear subspace of X^1, thus, in particular, $X^1(t) \hookrightarrow X^1$, it follows that $X^\Theta(t) \hookrightarrow X^\Theta$. However, in general it is not true that $X^\Theta(t)$ is a closed linear subspace of X^Θ (e.g. [32 I, Remark I.11.4]). For this reason F_Θ is said to be *admissible* if $X^\Theta(t)$ is a closed linear subspace of X^Θ for each $t \in [0, T]$.

After these preparations we can prove the following basic

(7.1) THEOREM. *Let F_Θ be admissible. Then (U, U) is an evolution system in (X, X^Θ) of type Θ, that is, of type $\mathfrak{a}$ with $\mathfrak{a}(t) := ct^{-\Theta}$.*

PROOF. Observe that $U \in C\big(\Delta_T, \mathcal{L}_s(X)\big)$, the compactness of Δ_T, and the uniform boundedness principle imply

$$\|U(t, s)\|_{\mathcal{L}(X)} \leqq c \qquad \forall (t, s) \in \Delta_T . \tag{3}$$

Since $U(t, s) \in \mathcal{L}(X) \cap \mathcal{L}(X, X^1)$ for $(t, s) \in \dot{\Delta}_T$, it is a morphism in $\mathcal{C}$. Hence $U(\dot{\Delta}_T) \subset \mathcal{L}(X, X^\Theta)$ and (3) and Lemma (6.1.i) imply

$$\|U(t, s)\|_{\mathcal{L}(X, X^\Theta)} \leqq c(t-s)^{-\Theta} \qquad \forall (t, s) \in \dot{\Delta}_T . \tag{4}$$

Since $U \in (\Delta_T, \mathfrak{L}_s(X, X^1))$ by Lemma (6.1.i) and since $X^1 \hookrightarrow X^\Theta$, it follows from (4) that $U \in \mathcal{K}_s(\Theta, c, X, X^\Theta)$. Since U obviously restricts to a map $\Delta_T \to \mathfrak{L}(X^\Theta)$, condition (*ES* i) is satisfied.

Conditions (*ES* ii) and (*ES* iii) are immediate consequences of (*PF* ii).

Let $s \in [0, T]$ be fixed. Then Lemma (6.1), the compactness of $[s, T]$, and the uniform boundedness principle imply

$$(5) \qquad \|U(t, s)\|_{\mathfrak{L}(X^1(s), X^1)} \leqq c \quad \forall t \in [s, T].$$

Hence, by (3) and (*IPF* i),

$$(6) \qquad \|U(t, s)\|_{(X^\Theta(s), X^\Theta)} \leqq c \quad \forall t \in [s, T].$$

Moreover (3), Lemma (6.1.ii) and (*IPF* iii) imply

$$U(\cdot, s)x \in C([s, T], X^\Theta) \quad \forall x \in X^1(s).$$

Hence, by (6) and the density of $X^1(s)$ in $X^\Theta(s)$,

$$U(\cdot, s) \in C([s, T], \mathfrak{L}_s(X^\Theta(s), X^\Theta)).$$

Since $U(t, s)(X) \subset D(A(t)) = X^1(t)$ for $(t, s) \in \dot{\Delta}_T$ by (*PF* iii) and since $X^1(t) \hookrightarrow X^\Theta(t) \hookrightarrow X^\Theta \hookrightarrow X$, we see that

$$U(t, s)(X^\Theta) \subset X^\Theta(t) \quad \forall (t, s) \in \dot{\Delta}_T.$$

Now (*ES* iv) follows from the admissibility of $\mathcal{F}_\Theta$. $\square$

Suppose now that, for some fixed $\Theta \in [0, 1)$:

($F\Theta$) *$\mathcal{F}_\Theta$ is admissible if $\Theta > 0$, that D is open in X^Θ, and that*

$$f \in C^{0,1-}([0, T] \times D, X).$$

Then, due to Theorem (7.1) we can consider the integral-evolution equation

$$(\mathfrak{J})_{(t_0, x_0)} \qquad u(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \tau) f(\tau, u(\tau))\, d\tau, \quad t_0 \leqq t \leqq T,$$

in X^Θ, where $(t_0, x_0) \in [0, T] \times D$. Every solution u of $(\mathfrak{J})_{(t_0, x_0)}$ is said to be a *mild X^Θ-solution* of the *semilinear initial value problem*

$$(SE)_{(t_0,x_0)} \qquad \begin{aligned} \dot u + A(t)u &= f(t,u), \quad t_0 < t \leqq T, \\ u(t_0) &= x_0 . \end{aligned}$$

Thus, by the results of Chapter I we have the following

(7.2) THEOREM. *Let assumption $(F\Theta)$ be satisfied. Then for each $(t_0, x_0) \in [0, T) \times D(t_0)$, where $D(t_0) := D \cap X^\Theta(t_0)$, there exists a unique maximal mild solution*

$$u(\cdot, t_0, x_0) \in C\big(J(t_0, x_0), D\big)$$

of $(SE)_{(t_0,x_0)}$, and the maximal interval of existence $J(t_0, x_0)$ is right open in $[t_0, T]$. Moreover,

$$\mathfrak{D}(t_0) := \{(t, x) \in [t_0, T] \times D(t_0) : t \in J(t_0, x_0)\}$$

is open in $[t_0, T] \times X^\Theta(t_0)$ and $u(\cdot, t_0, \cdot) \in C^{0,1-}\big(\mathfrak{D}(t_0), X^\Theta\big)$.

PROOF. This follows from the Theorems (7.1), (2.3) and (4.2). $\square$

A function $u: J \to X$ is said to be a *solution* (*on* J) of $(SE)_{(t_0,u_0)}$, where $0 \leqq t_0 < T$ and $u_0 \in D$, provided J is a perfect subinterval of $[t_0, T]$ containing t_0, $u \in C(J, X) \cap C^1\big(J \setminus \{t_0\}, X\big)$, $u(t) \in D\big(A(t)\big) \cap D$ and $\dot u(t) + A(t)u(t) = f\big(t, u(t)\big)$ for $t \in J \setminus \{t_0\}$, $u(t_0) = u_0$, and $f\big(\cdot, u(\cdot)\big) \in C(J, X)$. A solution u of $(SE)_{(t_0,x_0)}$ is *maximal* if there does not exist a solution of $(SE)_{(t_0,x_0)}$ which is a proper extension of u. In this case the domain of u is a *maximal interval of existence.*

(7.3) PROPOSITION. *Let $(F\Theta)$ be satisfied. Then every solution of $(SE)_{(t_0,x_0)}$ with $(t_0, x_0) \in [0, T) \times D(t_0)$ is a mild solution. Thus there is at most one maximal solution of $(SE)_{(t_0,x_0)}$ for $(t_0, x_0) \in [0, T) \times D(t_0)$.*

PROOF. The first assertion is an easy consequence of Theorem (7.1) and Proposition (1.3). The second assertion follows from the fact that the integral-evolution equation $(\mathfrak{J})_{(t_0,x_0)}$ has at most one solution on a given interval which is, for example, an immediate consequence of Lemma (3.3) and the fact that Lipschitz continuous maps are uniformly Lipschitz continuous on compact sets (cf. [5, Satz (6.4)] and the proof of [5, Satz 6.7]). $\square$

8. – Regularity properties.

It is the main purpose of this section to find conditions guaranteeing that mild solutions of $(SE)_{(t_0,u_0)}$ are in fact solutions.

Let $Y^1 \hookrightarrow Y^0$ and suppose that $B \in \mathfrak{L}(Y^0, Y^1)$. Then if follows from the decomposition

$$Y^1 \hookrightarrow Y^0 \xrightarrow{B} Y^1 \hookrightarrow Y^0$$

that

$$B \in \mathfrak{L}(Y^0) \cap \mathfrak{L}(Y^1) \cap \mathfrak{L}(Y^0, Y^1) \cap \mathfrak{L}(Y^1, Y^0) . \tag{1}$$

Suppose now that $0 < \eta < 1$ and put $Y^\eta := \mathcal{F}_\eta(Y^0, Y^1)$. The following lemma shows that $B \in \mathfrak{L}(Y^\eta, Y^\Theta)$ and gives an estimate for the norm of this operator.

(8.1) LEMMA. *Suppose that $Y^1 \hookrightarrow Y^0$ and that $B \in \mathfrak{L}(Y^0, Y^1)$. Then $B \in \mathfrak{L}(Y^\eta, Y^\Theta)$ and*

$$\|B\|_{\mathfrak{L}(Y^\eta, Y^\Theta)} \leqq c_\Theta c_\eta b_{00}^{(1-\eta)(1-\Theta)} b_{10}^{\eta(1-\Theta)} b_{01}^{(1-\eta)\Theta} b_{11}^{\eta\Theta} ,$$

where $\|B\|_{\mathfrak{L}(Y^j, Y^k)} \leqq b_{jk}$ for $j, k = 0, 1$ and where $0 \leqq \eta, \Theta \leqq 1$.

PROOF. Due to (1) we can apply $(IPF\,\mathrm{i})$ twice to obtain

$$\|B\|_{\mathfrak{L}(Y^\eta, Y^0)} \leqq c_\eta b_{00}^{1-\eta} b_{10}^\eta$$

and

$$\|B\|_{\mathfrak{L}(Y^\eta, Y^1)} \leqq c_\eta b_{01}^{1-\eta} b_{11}^\eta .$$

Now the assertion follows by applying again the same arguments in the obvious way. □

Recall that for every differentiable semigroup $\{\exp[sB]: s \geqq 0\}$ it is known that

$$R(\exp[sB]) \subset D(B^k)$$

and that $d^k/ds^k(\exp[sB]) = B^k \exp[sB]$ for $s > 0$ and $k \in \mathbb{N}$ (e.g. [37, Lemma 2.4.2]). Hence it follows from (6.5) that

$$A^k(t) \exp[-sA(t)] = \frac{(-1)^k}{2\pi i} \int_\Gamma \lambda^k \exp[\lambda s](\lambda + A(t))^{-1} d\lambda \tag{2}$$

for $s > 0$, $k \in \mathbf{N}$, and $0 \leqq t \leqq T$. Now (6.4) implies easily the existence of constants $\bar{c}_k$ and $\omega > \mu_0$ such that

$$(3)\quad \|A^k(t) \exp[-sA(t)]\|_{\mathfrak{L}(X)} \leqq \bar{c}_k s^{-k} \exp[\omega s], \qquad k \in \mathbf{N},\ s > 0,\ t \in [0, T].$$

On the basis of these estimates we obtain now easily the following lemma, where we let $X^0(t) := X$ for all $t \in [0, T]$.

(8.2) LEMMA. *Let $0 \leqq \eta, \Theta \leqq 1$. Then $A(t) \exp[-sA(t)] \in \mathfrak{L}(X^\eta(t), X^\Theta(t))$ and there is a constant $\omega > \mu_0$ such that*

$$\|A(t) \exp[-sA(t)]\|_{\mathfrak{L}(X^\eta(t), X^\Theta(t))} \leqq cs^{\eta-\Theta-1} \exp[\omega s]$$

for $s > 0$ and $0 \leqq t \leqq T$.

PROOF. From (3) (with $k = 2$) and from (6.2) we deduce that $A(t) \exp[-sA(t)] \in \mathfrak{L}(X, X^1(t))$ and (by increasing ω in (3))

$$(4)\qquad \|A(t) \exp[-sA(t)\|_{\mathfrak{L}(X, X^1(t))} \leqq cs^{-2} \exp[\omega s]$$

for $s > 0$ and $0 \leqq t \leqq T$. Since

$$A^2(t) \exp[-sA(t)]x = A(t) \exp[-sA(t)]A(t)x$$

for $x \in X^1(t)$ and $s > 0$ it follows again from (3) (with $k = 1$) and (6.2) that

$$(5)\qquad \|A(t) \exp[-sA(t)]\|_{\mathfrak{L}(X^1(t))} \leqq cs^{-1} \exp[\omega s]$$

for $s > 0$ and $0 \leqq t \leqq T$. Similarly we obtain

$$(6)\qquad \|A(t) \exp[-sA(t)]\|_{\mathfrak{L}(X^1(t), X)} \leqq c \exp[\omega s]$$

and, directly from (3),

$$(7)\qquad \|A(t) \exp[-sA(t)]\|_{\mathfrak{L}(X)} \leqq cs^{-1} \exp[\omega s]$$

for $s > 0$ and $0 \leqq t \leqq T$. Now the assertion follows from (4)-(7) and Lemma (8.1).

Since

$$(\lambda + A(t))^{-1} = [\mu_0 + A(t)]^{-1}[1 + (\lambda - \mu_0)(\mu_0 + A(t))^{-1}]^{-1}$$

it follows that $t \mapsto (\lambda + A(t))^{-1}$ is continuously differentiable and

$$(8)\qquad [(\lambda + A(t))^{-1}]^{\cdot} = (\mu_0 + A(t))(\lambda + A(t))^{-1}[(\mu_0 + A(t))^{-1}]^{\cdot}(\mu_0 + A(t))(\lambda + A(t))^{-1}$$

for $0 \leqq t \leqq T$ and $\lambda \in \mu_0 + S_{\alpha+\pi/2}$. Since $A(t)(\lambda + A(t))^{-1} = 1 - \lambda(\lambda + A(t))^{-1}$ we obtain from (8), (6.4) and (AP iii) that

$$(9)\qquad \|[(\lambda + A(t))^{-1}]^{\cdot}\|_{\mathfrak{L}(X)} \leqq c|\lambda - \mu_0|^{-1+\varrho}$$

for all $\lambda \in S_{\alpha+\pi/2}$ and $t \in [0, T]$. Hence it follows from [43, formulas (5.50)-(5.57)] and the fact that $U(t, s) = \exp[\mu_0 t]U_{\mu_0}(t, s)\exp[-\mu_0 s]$ that

$$(10)\qquad U(t, s) = \exp[-(t-s)A(t)] + \int_s^t \exp[-(t-\tau)A(t)]R(\tau, s)\,d\tau\,,$$

where $R \in C(\dot{\Delta}_T, \mathfrak{L}(X))$ and

$$(11)\qquad \|R(t, s)\|_{\mathfrak{L}(X)} \leqq c(t-s)^{-\varrho} \qquad \forall (t, s) \in \dot{\Delta}_T\,.$$

More precisely, letting

$$(12)\qquad R_0(t, s) := \frac{1}{2\pi i}\int_\Gamma \exp[\lambda(t-s)][(\lambda + A(t))^{-1}]^{\cdot}\,d\lambda\,,$$

it follows that $R_0 \in C(\dot{\Delta}_T, \mathfrak{L}(X))$ and that R is the unique solution of the Volterra equation

$$(13)\qquad R(t, s) - \int_s^t R_0(t, \tau)R(\tau, s)\,d\tau = R_0(t, s)\,, \qquad (t, s) \in \dot{\Delta}_T\,.$$

In particular $R(t, s)$ is given by the Neumann series

$$(14)\qquad R(t, s) = \sum_{k=0}^{\infty} R_k(t, s)\,,$$

where

$$(15)\qquad R_k(t, s) := \int_s^t R_0(t, \tau)R_{k-1}(\tau, s)\,d\tau$$

for $(t, s) \in \dot{\Delta}_T$ and $k \in \mathbb{N}$.

Indeed, in [43, pp. 131-136] (cf. also [26]) it is shown that $V(t, s)x$, where $V(t, s)$ denotes the right-hand side of (10), is, for each $x \in X$, the unique weak solution of the initial value problem $\dot{u} + A(t)u = 0$ in $s < t \leqq T$, $u(s) = x$. Since every solution is a weak solution, it follows from (6.8) that $U = V$.

We introduce now an additional assumption, namely:

$$(\dot{R})_\Theta \qquad \begin{aligned} &[(\lambda + A(t))^{-1}]^\bullet \in \mathfrak{L}(X^\Theta) \quad \textit{and} \quad \|[(\lambda + A(t))^{-1}]^\bullet\|_{\mathfrak{L}(X^\Theta)} \leqq c|\lambda - \mu_0|^{-1+\varrho+\Theta} \\ &\textit{for } \lambda \in \mu_0 + S_{\alpha+\pi/2} \textit{ and } 0 \leqq t \leqq T. \end{aligned}$$

Then we prove the following

(8.3) LEMMA. *Let $0 \leqq \Theta < 1 - \varrho$, let $(\dot{R})_\Theta$ be satisfied, and suppose that $\mathfrak{F}_\Theta$ is admissible if $\Theta > 0$. Moreover let $\eta := 1 - \varrho$ if $\varrho > 0$ and let $\eta \in (\Theta, 1)$ if $\varrho = 0$. Then*

$$\| U(t, \tau) - U(r, \tau) \|_{\mathfrak{L}(X, X^\Theta)} \leqq c(t - r)^{\eta - \Theta}(r - \tau)^{-\eta}$$

for $0 \leqq \tau < r < t \leqq T$.

PROOF. Observe that

$$U(t, \tau) - U(r, \tau) = \big(U(t, r) - id\big)\, U(r, \tau)\,. \tag{16}$$

Moreover, if $x \in X^1(r)$,

$$\begin{aligned} x = \exp[-(t-r)A(r)]x - \int_r^t (\partial/\partial s) \exp[-(s-r)A(r)]x\,ds \\ = \exp[-(t-r)A(r)]x + \int_r^t A(r) \exp[-(s-r)A(r)]x\,ds\,. \end{aligned}$$

Hence we deduce from (16) and (PF iii) that

$$\begin{aligned} U(t, \tau) - U(r, \tau) &= \big[U(t, r) - \exp[-(t-r)A(r)]\big]\, U(r, \tau) \\ -\int_r^t A(r) \exp[-(s-r)A(r)]\, U(r, \tau)\,d\tau &= \big[U(t, r) - \exp[-(t-r)A(t)]\big]\, U(r, \tau) \\ &+ \big[\exp[-(t-r)A(t)] - \exp[-(t-r)A(r)]\big]\, U(r, \tau) \\ &-\int_r^t A(r) \exp[-(s-r)A(r)]\, U(r, \tau)\,d\tau =: \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned} \tag{17}$$

Observe first that (3) (with $k=1$) and (6.2) imply (by increasing again ω)

$$\|\exp[-sA(t)]\|_{\mathfrak{L}(X,X^1)} \leqq cs^{-1}\exp[\omega s] \qquad \forall s>0,\ t\in[0,T].$$

Since

$$\|\exp[-sA(t)]\|_{\mathfrak{L}(X)} \leqq c\exp[\omega s] \qquad \forall s>0, t\in[0,T]$$

we obtain by interpolating

$$(18) \qquad \|\exp[-sA(t)]\|_{\mathfrak{L}(X,X^\Theta)} \leqq cs^{-\Theta}\exp[\omega s] \qquad \forall s>0, t\in[0,T].$$

Hence we deduce from (7.3), (10) and (11) that

$$(19) \qquad \|\mathrm{I}\|_{\mathfrak{L}(X,X^\Theta)} \leqq c\int_r^t \|\exp[-(t-s)A(t)]\|_{\mathfrak{L}(X,X^\Theta)}\|R(s,r)\|_{\mathfrak{L}(X)}ds$$
$$\leqq c\int_r^t (t-s)^{-\Theta}(s-r)^{-\varrho}ds = c(t-r)^{1-\varrho-\Theta} \leqq c(t-r)^{\eta-\Theta}.$$

From the representation (6.5) we obtain

$$\exp[-(t-r)A(t)] - \exp[-(t-r)A(r)]$$
$$= \frac{1}{2\pi i}\int_\Gamma \exp[(t-r)\lambda][(\lambda + A\ t))^{-1} - (\lambda + A(r))^{-1}]\,d\lambda$$
$$= \frac{1}{2\pi i}\int_\Gamma \exp[(t-r)\lambda]\int_r^t [(\lambda + A(\tau))^{-1}]^{\cdot}\,d\tau\,d\lambda.$$

Thus assumption $(\dot{R})_\Theta$ implies the estimate

$$\|\exp[-(t-r)A(t)] - \exp[-(t-r)A(r)]\|_{\mathfrak{L}(X^\Theta)} \leqq c(t-r)^{1-\varrho-\Theta}$$

for $0 \leqq r < t \leqq T$. Now it follows from (7.4) that

$$(20) \qquad \|\mathrm{II}\|_{\mathfrak{L}(X,X^\Theta)} \leqq c(t-r)^{1-\varrho-\Theta}(r-\tau)^{-\Theta} \leqq c(t-r)^{\eta-\Theta}(r-\tau)^{-\eta}.$$

Observe that for fixed $r\in(0,T)$ condition (PF iii) and (6.2) imply

$$\|U(r,\tau)\|_{\mathfrak{L}(X,X^1(r))} \leqq c(r-\tau)^{-1}, \quad 0 \leqq \tau < r,$$

where the constant c is independent of r. Thus, by (7.3) and interpolation, we obtain

$$\| U(r, \tau)\|_{\mathcal{L}(X, X^{\eta(r)})} \leqq c(r-\tau)^{-\eta}$$

for all $(r, \tau) \in \dot{\Delta}_T$. Consequently Lemma (8.2) gives

$$\begin{aligned}(21) \qquad \|\mathrm{III}\|_{\mathcal{L}(X, X^{\Theta})} &\leqq \int_r^t \|A(r)\exp[-(s-r)A(r)]\|_{\mathcal{L}(X^{\eta(r)}, X^{\Theta})} \| U(r, \tau)\|_{\mathcal{L}(X, X^{\eta(r)})} ds \\ &\leqq c\int_r^t (s-r)^{\eta-\Theta-1}(r-\tau)^{-\eta} ds = c(t-r)^{\eta-\Theta}(r-\tau)^{-\eta},\end{aligned}$$

where we used the fact that, due to the admissibility of $\mathcal{F}_\Theta$, $X^\Theta(t) \hookrightarrow X^\Theta$ for every $t \in [0, T]$. Now the assertion follows from (17) and (19)-(21). □

(8.4) COROLLARY. *Let $0 \leqq \Theta < 1-\varrho$, let $(\dot{R})_\Theta$ be satisfied, and assume that $\mathcal{F}_\Theta$ is admissible if $\Theta > 0$. Moreover let $\eta := 1-\varrho$ if $\varrho > 0$ and let $\eta \in (\Theta, 1)$ if $\varrho = 0$. Then*

$$\left(u \mapsto \int_{t_0}^{\bullet} U(\cdot, \tau)u(\tau)d\tau\right) \in \mathcal{L}\left(L_\infty((t_0, T), X), C^{\eta-\Theta}([t_0, T], X^\Theta)\right)$$

for every $t_0 \in [0, T)$.

PROOF. Let $\alpha := 1+\Theta-\eta < 1$ and observe that $\Theta < \alpha$. Hence it follows from Theorem (7.1) that $U \in \mathcal{K}_s(\alpha, c, X, X^\Theta)$, and Lemma (8.3) implies that condition (2) of Proposition (1.4) with $\beta := \eta$ and $Y := X^\Theta$ is satisfied. Thus Proposition (1.4) implies the assertion. □

Condition (*PF* iii), the compactness of Δ_T, and the uniform boundedness principle imply

$$\|A(t)U(t, s)A^{-1}(s)\|_{\mathcal{L}(X)} \leqq c \qquad \forall (t, s) \in \Delta_T .$$

Consequently, by (6.2),

$$(22) \qquad \| U(t, s)\|_{\mathcal{L}(X^1(s), X^1)} \leqq c \qquad \forall (t, s) \in \Delta_T .$$

Thus, by interpolation between (22) and the estimate of Lemma (6.1.i),

$$(23) \qquad \| U(t, s)\|_{\mathcal{L}(X^\Theta(s), X^1)} \leqq c(t-s)^{-1+\Theta} \qquad \forall (t, s) \in \dot{\Delta}_T .$$

Using this estimate we can now prove the Hölder continuity on $[t_0, T]$ of the solution of the homogeneous initial value problem

$$\dot{u} + A(t)u = 0 \quad \text{in } (t_0, T],$$
$$u(t_0) = x_0$$

provided $(t_0, x_0) \in [0, T) \times X^\Theta(t_0)$. Recall that by (*PF* iv) this solution is given by $U(\cdot, t_0)x_0$.

(8.5) PROPOSITION. *Suppose that* $(t_0, x_0) \in [0, T) \times X^\Theta(t_0)$, *where* $0 < \Theta < 1$. *Then*

$$U(\cdot, t_0)x_0 \in C^\Theta([t_0, T], X).$$

PROOF. Let $v := U(\cdot, t_0)x_0$. Then $\dot{v}(t) + A(t)v(t) = 0$ for $t_0 < t \leqq T$. Thus, by (6.2) and (23)

$$\|\dot{v}(t)\| = \|A(t)U(t, t_0)x_0\| \leqq c\|U(t, t_0)x_0\|_1$$
$$\leqq c\|U(t, t_0)\|_{\mathcal{L}(X^\Theta(t_0), X^1)}\|x_0\|_\Theta \leqq c(t - t_0)^{-1+\Theta}$$

for $t_0 < t \leqq T$. Consequently the relation

$$v(t) - v(t_0 + \varepsilon) = \int_{t_0+\varepsilon}^{t} \dot{v}(\tau)d\tau = \int_{t_0+\varepsilon}^{t} \dot{v}(\tau)(\tau - t_0)^{1-\Theta}(\tau - t_0)^{-1+\Theta}d\tau$$

implies the estimate

$$\|v(t) - v(t_0 + \varepsilon)\| \leqq c(t - t_0)^\Theta$$

for each $\varepsilon \in (0, t - t_0)$ and all $t \in (t_0, T]$. Hence, letting $\varepsilon \to 0$, we see that

$$\|v(t) - v(t_0)\| \leqq c(t - t_0)^\Theta \qquad \forall t \in (t_0, T].$$

Since $v \in C^1((t_0, T], X)$ the assertion follows. $\square$

Let $X^0 := X$ and suppose that $0 \leqq \alpha \leqq \beta \leqq \gamma \leqq 1$. Then X^β is said to be (X^α, X^γ)-*compatible* if $X^\gamma \hookrightarrow X^\beta \hookrightarrow X^\alpha$ and

$$\|x\|_\beta \leqq c\|x\|_\alpha^{(\gamma-\beta)/(\gamma-\alpha)}\|x\|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)} \qquad \forall x \in X^\gamma. \tag{24}$$

With this definition we obtain the

(8.6) COROLLARY. *Suppose that* $(t_0, x_0) \in [0, T) \times X^\Theta(t_0)$, $0 < \Theta < 1$, *that*

$\mathcal{F}_\Theta$ is admissible, that $0 \leqq \eta \leqq \Theta$, and that X^η is (X, X^Θ)-compatible. Then

$$U(\cdot, t_0)x_0 \in C^{\Theta-\eta}([t_0, T], X^\eta).$$

PROOF. Proposition (7.3) implies in particular that $v \in C([t_0, T], X^\Theta)$, where $v := U(\cdot, t_0)x_0$. Thus

$$\|v(t) - v(s)\|_\Theta \leqq c \qquad \forall t, s \in [t_0, T].$$

Since

$$\|v(t) - v(s)\| \leqq c|t-s|^\Theta \qquad \forall t, s \in [t_0, T]$$

by Proposition (8.5), the assertion follows from (24) and the (X, X^Θ)-compatibility of X^η. $\square$

After these preparations we can now prove the main result of this section.

(8.7) THEOREM. *Suppose that $0 < \xi < \Theta < 1 - \varrho$, that $\mathcal{F}_\Theta$ is admissible, that $(\dot{R})_\Theta$ is satisfied, and that X^ξ is (X, X^Θ)-compatible. Let D^ξ be open in X^ξ and let $D^\Theta := D^\xi \cap X^\Theta$, endowed with the X^Θ-topology. Assume that*

$$f \in C^\mu([0, T] \times D^\xi, X) \cap C^{0,1-}([0, T] \times D^\Theta, X) \tag{25}$$

for some $\mu \in (0, 1)$. Finally suppose that $u: J \to D^\Theta$ is a mild X^Θ-solution of

$$(SE)_{(t_0,x_0)} \qquad \begin{aligned} \dot{u} + A(t)u &= f(t, u), \qquad t_0 < t \leqq T, \\ u(t_0) &= x_0, \end{aligned}$$

where $(t_0, x_0) \in [0, T) \times D^\Theta(t_0)$.

Then u is a solution of $(SE)_{(t_0,x_0)}$ and

$$u \in C^1(\dot{J}, X) \cap C(\dot{J}, X^1) \cap C^{\Theta-\eta}(J, X^\eta)$$

for every $\eta \in [0, \Theta]$ such that X^η is (X, X^Θ)-compatible, where $\dot{J} := J \setminus \{t_0\}$.

PROOF. Let $j: X^\Theta \hookrightarrow X^\xi$ be the injection. Then $D^\Theta = j^{-1}(D^\xi)$, which shows that D^Θ is open in X^Θ.

Since $u \in C(J, X^\Theta)$ it follows that $f(\cdot, u(\cdot)) \in C(J, X)$. Hence, by Corollary (8.4),

$$u - U(\cdot, t_0)x_0 \in C^\nu(J, X^\Theta), \tag{26}$$

where $0<\nu<1-\varrho-\Theta$. By Corollary (8.6)

$$U(\cdot, t_0)x_0 \in C^{\Theta-\xi}([t_0, T], X^\xi). \tag{27}$$

Thus, since $X^\Theta \hookrightarrow X^\xi$, (26) and (27) imply

$$u \in C^\lambda(J, X^\xi),$$

where $\lambda := \min(\Theta-\xi, \nu)$. Hence it follows from (25) that

$$f(\cdot, u(\cdot)) \in C^\varkappa(J, X)$$

for some $\varkappa \in (0, 1)$. Now we deduce from (PF iv) that u is in fact a solution of $(SE)_{(t_0,x_0)}$. Hence $u \in C^1(\dot{J}, X)$.

It follows from Corollary (8.4) that

$$u - U(\cdot, t_0)x_0 \in C^\Theta([t_0, T], X).$$

Hence, due to Proposition (8.5),

$$u \in C^\Theta(J, X).$$

Now we deduce, similarly as in the proof of Corollary (8.6), that

$$u \in C^{\Theta-\eta}(J, X^\eta).$$

Since $u(t) \in D(A(t))$ for $t \in \dot{J}$ we see that

$$\mathcal{A}(t)\big(u(t)-u(\tau)\big) = [\mathcal{A}(\tau)-\mathcal{A}(t)]u(\tau) + f\big(t, u(t)\big) - f\big(\tau, u(\tau)\big) - [\dot{u}(t)-\dot{u}(\tau)]$$

and

$$\mathcal{B}(t)\big(u(t)-u(\tau)\big) = [\mathcal{B}(\tau)-\mathcal{B}(t)]u(\tau)$$

for $t, \tau \in \dot{J}$. Hence, by (6.2) and since we can assume that $\mu_0 = 0$,

$$\begin{aligned}\|u(t)-u(\tau)\|_1 \leqq c\Big(&\|\mathcal{A}(t)-\mathcal{A}(\tau)\|_{\mathcal{L}(X^1,X)}\|u(\tau)\|_1 + \|f\big(t, u(t)\big) - f\big(\tau, u(\tau)\big)\| \\ &+ \|\dot{u}(t)-\dot{u}(\tau)\| + \|\mathcal{B}(t)-\mathcal{B}(\tau)\|_{\mathcal{L}(X^1,Z)}\|u(\tau)\|_1\Big),\end{aligned}$$

which shows that $u \in C(\dot{J}, X^1)$. $\square$

(8.8) COROLLARY. *Let the hypotheses of Theorem* (8.7) *be satisfied. Then*

$(SE)_{(t_0,x_0)}$ has for each $(t_0, x_0) \in [0, T) \times D^{\Theta}(t_0)$ a unique maximal solution

$$u(\cdot, t_0, x_0) \in C(J(t_0, x_0), D^{\Theta}) .$$

The maximal interval of existence $J(t_0, x_0)$ is right open in $[t_0, T]$,

$$\mathfrak{D}^{\Theta}(t_0) := \{(t, x) \in [t_0, T] \times D^{\Theta}(t_0) : t \in J(t_0, x_0)\}$$

is open in $[t_0, T] \times X^{\Theta}(t_0)$, and

$$u(\cdot, t_0, \cdot) \in C^{0,1-}(\mathfrak{D}^{\Theta}(t_0), X^{\Theta}) .$$

Moreover

$$u(\cdot, t_0, x_0) \in C^1(\dot{J}(t_0, x_0), X) \cap C(\dot{J}(t_0, x_0), X^1) \cap C^{\Theta-\eta}(J(t_0, x_0), X^{\eta})$$

for every $\eta \in [0, \Theta]$ such that X^{η} is (X, X^{Θ})-compatible.

PROOF. This follows from Theorem (7.2), Proposition (7.3), and Theorem (8.7). $\square$

9. – Higher regularity.

In this section we study $U \big| X^{\eta}$, $0 < \eta < 1$. Since $X^{\eta} \hookrightarrow X$ we can expect that $U(\dot{\Delta}_T)(X^{\eta})$ is contained in some smaller space than X^1. If this is the case we shall show that $u(\cdot, t_0, x_0)$ has better regularity properties.

Let V and W be Banach spaces such that $V \hookrightarrow W$, and let $B: D(B) \subset W \to W$ be a linear operator in W. Then we define B_V, the *V-realization of B* (or the « part of B in V ») by

$$D(B_V) := \{w \in D(B) \cap V : Bw \in V\} \quad \text{and } B_V w := Bw ,$$

so that B_V is a linear operator in V:

$$B_V : D(B_V) \subset V \to V ,$$

the « maximal restriction of B to V ». Clearly, $D(B_V) = D(B) \cap V$ if $R(B) \subset V$. Moreover it is easily verified that B_V is closed in V if B is closed in W.

In the following we denote by $A_{\eta}(t)$ the X^{η}-realization of $A(t)$ where $t \in [0, T]$ and $0 \leqq \eta \leqq 1$. (Observe that $A_0(t) = A(t)$.) Since $D(A(t)) =$

$X^1(t) \subset X^1 \subset X^\eta$ it follows that

$$\varrho(-A_\eta(t)) \supset \mu_0 + S_{\alpha+\pi/2} \tag{1}$$

and that

$$(\lambda + A_\eta(t))^{-1} = (\lambda + A(t))^{-1} \,|\, X^\eta, \qquad \lambda \in \mu_0 + S_{\alpha+\pi/2}, \tag{2}$$

for $0 \leqq \eta \leqq 1$ and $0 \leqq t \leqq T$.

From

$$A(t)(\lambda + A(t))^{-1} = 1 - \lambda(\lambda + A(t))^{-1} \tag{3}$$

and (6.4) and (6.2) we see that

$$\|(\lambda + A(t))^{-1}\|_{\mathfrak{L}(X,X^1)} \leqq c \qquad \forall \lambda \in \mu_0 + S_{\alpha+\pi/2},\ 0 \leqq t \leqq T. \tag{4}$$

Thus, by interpolating between (4) and (6.4),

$$\|(\lambda + A(t))^{-1}\|_{\mathfrak{L}(X,X^\eta)} \leqq c|\lambda - \mu_0|^{-1+\eta} \qquad \forall \lambda \in \mu_0 + S_{\alpha+\pi/2},\ 0 \leqq t \leqq T.$$

Consequently, by (1) and (2),

$$\|(\lambda + A_\varkappa(t))^{-1}\|_{\mathfrak{L}(X^\varkappa,X^\eta)} \leqq c|\lambda - \mu_0|^{-1+\eta} \qquad \forall \lambda \in \mu_0 + S_{\alpha+\pi/2},\ 0 \leqq t \leqq T, \tag{5}$$

where $0 \leqq \varkappa,\ \eta \leqq 1$.

Let now $\varkappa \in [0, 1)$ be fixed. Then, by (2), (5) and (6.5),

$$\begin{aligned}\exp[-sA_\varkappa(t)] &:= \exp[-sA(t)] \,|\, X^\varkappa \\ &= \frac{1}{2\pi i}\int_\Gamma \exp[\lambda s](\lambda + A_\varkappa(t))^{-1}\, d\lambda \in \mathfrak{L}(X^\varkappa)\end{aligned} \tag{6}$$

for $s > 0$ and

$$\|\exp[-sA_\varkappa(t)]\|_{\mathfrak{L}(X^\varkappa)} \leqq cs^{-\varkappa}\exp[\omega s] \qquad \forall s > 0,\ t \in [0, T]. \tag{7}$$

Moreover, by (2), (3) and (5),

$$\|A_\varkappa(t)(\lambda + A_\varkappa(t))^{-1}\|_{\mathfrak{L}(X^\varkappa)} \leqq c|\lambda - \mu_0|^\varkappa \qquad \forall \lambda \in \mu_0 + S_{\alpha+\pi/2},\ t \in [0, T], \tag{8}$$

which, due to (6), implies

$$\|A_\varkappa(t)\exp[-sA_\varkappa(t)]\|_{\mathfrak{L}(X^\varkappa)} \leqq cs^{-1-\varkappa}\exp[\omega s] \qquad \forall s > 0,\ 0 \leqq t \leqq T. \tag{9}$$

From (5) and (6) we deduce also that

$$[s \mapsto \exp[-sA_\varkappa(t)]] \in C^\infty((0, \infty), \mathfrak{L}(X^\varkappa))$$

and that

$$\begin{aligned}(10)\qquad \frac{\partial}{\partial s}\exp[-sA_\varkappa(t)] &= \frac{1}{2\pi i}\int_\Gamma \exp[s\lambda]\lambda(\lambda + A_\varkappa(t))^{-1}d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma \exp[s\lambda]A_\varkappa(t)(\lambda + A_\varkappa(t))^{-1}d\lambda = -A_\varkappa(t)\exp[-sA_\varkappa(t)]\end{aligned}$$

for $s > 0$ and $0 \leqq t \leqq T$, where we used (3) and the fact that

$$\frac{1}{2\pi i}\int_\Gamma \exp[s\lambda](\lambda + A_\varkappa(t))^{-1}d\lambda = 0$$

by Cauchy's theorem.

(9.1) LEMMA. (i) $\exp[-sA_\varkappa(t)]x \to x$ *in* $X^\varkappa$ *as* $s \to 0$, *provided* $x \in D(A_\varkappa(t))$.

(ii) $R\left(\int_s^t \exp[-(t-\tau)A_\varkappa(t)]d\tau\right) \subset D(A_\varkappa(t))$ *and*

$$A_\varkappa(t)\int_s^t \exp[-(t-\tau)A_\varkappa(t)]d\tau = id_{X^\varkappa} - \exp[-(t-s)A_\varkappa(t)]$$

for all $(t, s) \in \dot{\Delta}_T$.

PROOF. (i) follows easily from (5) and (6) by standard arguments (e.g. [43, pp 66-67]).

(ii) From (10) we see (by assuming without loss of generality $\mu_0 = 0$) that

$$\frac{\partial}{\partial \tau}\exp[-(t-\tau)A_\varkappa(t)]A_\varkappa^{-1}(t) = \exp[-(t-\tau)A_\varkappa(t)] \qquad \forall (t, \tau) \in \dot{\Delta}_T.$$

Hence

$$\int_s^{t-\varepsilon} \exp[-(t-\tau)A_\varkappa(t)]d\tau = \exp[-\varepsilon A_\varkappa(t)]A_\varkappa^{-1}(t) - A_\varkappa^{-1}(t)\exp[-(t-s)A_\varkappa(t)]$$

for $0 < \varepsilon < t - s$ and $(t, s) \in \Delta_T$. Now the assertion follows from (7) and (i) by letting $\varepsilon \to 0$. $\square$

For completeness we included the simple proof of Lemma (9.1.ii) following [27, Lemma 3.1] (cf. also [39, Proposition 1.2]).

We impose now the following additional hypothesis:

$(HR)_\varkappa$ (i) $0 \leqq \varkappa < \frac{1}{4}$ *and* $\varrho = 0$.

(ii) *There exist Banach spaces* $X^{1+\varkappa} \hookrightarrow X^1$ *and* $Y^\varkappa \hookrightarrow Z$ *such that*

$$\mathcal{T} = (\mu_0 + \mathcal{A}, \mathcal{B}) \in C([0, T], \operatorname{Isom}(X^{1+\varkappa}, X^\varkappa \times Y^\varkappa)).$$

(iii) $(\mu_0 + A_\varkappa(\cdot))^{-1} \in C^{1+\mu}([0, T], \mathcal{L}(X^\varkappa))$ *for some* $\mu \in (3\varkappa, 1)$ *and*

$$\|[(\lambda + A_\varkappa(t))^{-1}]^\bullet\|_{\mathcal{L}(X^\varkappa)} \leqq c|\lambda - \mu_0|^{-1+\varkappa} \tag{11}$$

for all $\lambda \in \mu_0 + S_{\alpha+\pi/2}$ *and* $t \in [0, T]$.

Observe that (11) is just condition $(\dot{R})_\varkappa$ since $\varrho = 0$.

It follows from $(HR\,\mathrm{ii})_\varkappa$ that there exist constants $\bar{c}$ and $\hat{c}$ such that

$$\|x\|_{1+\varkappa} \leqq \bar{c}\big(\|(\mu_0 + \mathcal{A}(t))x\|_\varkappa + \|\mathcal{B}(t)x\|_{Y^\varkappa}\big) \leqq \hat{c}\|x\|_{1+\varkappa} \tag{12}$$

for all $x \in X^{1+\varkappa}$ and all $t \in [0, T]$, where $\|\cdot\|_{1+\varkappa}$ denotes the norm in $X^{1+\varkappa}$. Moreover, letting

$$X^{1+\varkappa}(t) := X^{1+\varkappa}_{\mathcal{B}(t)} := \ker\big(\mathcal{B}(t)|X^{1+\varkappa}\big),$$

it follows also that

$$A_\varkappa(t) = \mathcal{A}(t)|X^{1+\varkappa}(t) \qquad \forall t \in [0, T].$$

We obtain from (6), (8) and (12) easily (by arguments which are familiar by now) that

$$[(s, t) \mapsto \exp[-sA_\varkappa(t)]] \in C\big((0, \infty) \times [0, T], \mathcal{L}(X^\varkappa, X^{1+\varkappa})\big) \tag{13}$$

and, by (9) and (12), that, for some $\omega > \mu_0$,

$$\|\exp[-sA_\varkappa(t)]\|_{\mathcal{L}(X^\varkappa, X^{1+\varkappa})} \leqq cs^{-1-\varkappa}\exp[\omega s] \qquad \forall s > 0,\ t \in [0, T]. \tag{14}$$

Using these facts we prove the following

(9.2) LEMMA. *Suppose that* $0 \leqq t_0 < T$ *and* $g \in C^\nu([t_0, T], X^\varkappa)$ *for some*

$\nu \in (\varkappa, 1)$. *Then*

$$\left[t \mapsto \int_{t_0}^{t} \exp[-(t-\tau)A_\varkappa(t)]g(\tau)\,d\tau\right] \in C\big((t_0, T], X^{1+\varkappa}\big)\,.$$

PROOF. Let

$$\begin{aligned} f(t) :&= \int_{t_0}^{t} \exp[-(t-\tau)A_\varkappa(t)]g(\tau)\,d\tau \\ &= \int_{t_0}^{t} \exp[-(t-\tau)A_\varkappa(t)][g(\tau)-g(t)]\,d\tau + \int_{t_0}^{t} \exp[-(t-\tau)A_\varkappa(t)]g(t)\,d\tau \\ &=: f_0(t) + f_1(t) \end{aligned}$$

for $t_0 \leqq t \leqq T$, and let

$$a(t, \tau) := \exp[-(t-\tau)A_\varkappa(t)][g(\tau)-g(t)] \qquad \forall (t, \tau) \in \mathring{\Delta}_T\,.$$

Then $a(\cdot,\cdot) \in C(\mathring{\Delta}_T, X^{1+\varkappa})$ by (13) and

$$\|a(t, \tau)\|_{1+\varkappa} \leqq c(t-\tau)^{-1-\varkappa+\nu} \qquad \forall (t, \tau) \in \mathring{\Delta}_T$$

by (14). Hence it follows from Lemma (1.1) that $f_0 \in C\big([t_0, T], X^{1+\varkappa}\big)$.
From Lemma (9.1.ii) we deduce that $f_1(t) \in X^{1+\varkappa}(t) \hookrightarrow X^{1+\varkappa}$ and that

$$A_\varkappa(t)f_1(t) = g(t) - \exp[-(t-t_0)A_\varkappa(t)]g(t)$$

for $t_0 < t \leqq T$. Hence

$$\begin{aligned} \mathcal{A}(s)f_1(s) = \mathcal{A}(t)f_1(t) + g(s) - g(t) \\ + \exp[-(t-t_0)A_\varkappa(t)]g(t) - \exp[-(s-t_0)A_\varkappa(s)]g(s) \end{aligned}$$

and

$$\mathcal{B}(s)f_1(s) = 0 = \mathcal{B}(t)f_1(t)$$

for $t_0 < s,\ t \leqq T$. Thus it follows from (12)—where we can assume without loss of generality that $\mu_0 = 0$—that

$$\begin{aligned} \|f_1(s) - f_1(t)\|_{1+\varkappa} &\leqq c\Big(\|\mathcal{A}(s)\big(f_1(s)-f_1(t)\big)\|_\varkappa + \|\mathcal{B}(s)\big(f_1(s)-f_1(t)\big)\|_{Y^\varkappa}\Big) \\ &\leqq c\big(\|\mathcal{A}(s)-\mathcal{A}(t)\|_{\mathfrak{L}(X^{1+\varkappa}, X^\varkappa)}\|f_1(t)\|_{1+\varkappa} + \|g(s)-g(t)\|_\varkappa \\ &\quad + \|\exp[-(t-t_0)A_\varkappa(t)]g(t) - \exp[-(s-t_0)A_\varkappa(s)]g(s)\|_\varkappa \\ &\quad + \|\mathcal{B}(s)-\mathcal{B}(t)\|_{\mathfrak{L}(X^{1+\varkappa}, Y^\varkappa)}\|f_1(t)\|_{1+\varkappa}\big) \end{aligned}$$

for $t_0 < s, t \leqq T$. This shows that $f_1 \in C((t_0, T], X^{1+\varkappa})$ which proves the assertion. $\square$

Next we turn to the study of the function $R(\cdot)$ defined by (8.13). First we observe that, by (8.12), $(HR\,\mathrm{iii})_\varkappa$ and (2),

$$R_0(\cdot) \in C(\dot{\Delta}_T, \mathfrak{L}(X^\varkappa))$$

and that

$$\|R_0(t, s)\|_{\mathfrak{L}(X^\varkappa)} \leqq c(t-s)^{-\varkappa} \qquad \forall (t, s) \in \dot{\Delta}_T. \tag{15}$$

From this estimate, (6.14), and (6.15) we obtain easily that

$$R(\cdot) \in C(\dot{\Delta}_T, \mathfrak{L}(X^\varkappa))$$

and

$$\|R(t, s)\|_{\mathfrak{L}(X^\varkappa)} \leqq c(t-s)^{-\varkappa} \qquad \forall (t, s) \in \dot{\Delta}_T. \tag{16}$$

Now we can prove the

(9.3) LEMMA. *Suppose that $0 < \nu < \mu - 2\varkappa$. Then*

$$\|R(t, s) - R(\tau, s)\|_{\mathfrak{L}(X^\varkappa)} \leqq c\left(\frac{(t-s)^{\mu \wedge (1-\varkappa)}}{(t-s)(\tau-s)^{2\varkappa}} + \frac{(t-\tau)^\nu}{(\tau-s)^\varkappa}\right)$$

for $0 \leqq s < \tau < t \leqq T$.

PROOF. By using (8), (15) and (16) this follows by an obvious modification of the proof of [20, Corollary 7.2.4]. $\square$

(9.4) COROLLARY. *Suppose that $0 < \nu < \min\{\mu - 2\varkappa, 1 - 3\varkappa\}$. Then there exists a constant $\gamma < 1$ such that*

$$\|R(t, s) - R(\tau, s)\|_{\mathfrak{L}(X^\varkappa)} \leqq c(t-\tau)^\nu(\tau - s)^{-\gamma}$$

for $0 \leqq s < \tau < t \leqq T$.

After these preparations we can prove the following fundamental

(9.5) PROPOSITION. *Suppose that $0 \leqq t_0 < T$. Then*

$$U(\cdot, t_0) \in C((t_0, T], \mathfrak{L}_s(X^\varkappa, X^{1+\varkappa})).$$

Moreover, if $g \in C^{\nu}([t_0, T], X^{\varkappa})$ *for some* $\nu \in (\varkappa, 1)$ *then*

$$\left(t \to \int_{t_0}^{t} U(t,\tau) g(\tau) d\tau\right) \in C((t_0, T], X^{1+\varkappa}).$$

PROOF. It follows from (8.10) that

$$U(t, t_0)x = \exp[-(t-t_0)A(t)]x + \int_{t_0}^{t} \exp[-(t-\tau)A(t)] R(\tau, t_0) x \, d\tau \tag{17}$$

for $x \in X^{\varkappa}$ and $t_0 < t \leqq T$. By (13) the first term on the right-hand side of (17) belongs to $C((t_0, T], X^{1+\varkappa})$. As for the seond term, observe that it equals

$$h_0(t) + h_1(t) := \int_{t_0}^{t} \exp[-(t-\tau)A(t)][R(\tau, t_0)x - R(t, t_0)x] d\tau$$
$$+ \int_{t_0}^{t} \exp[-(t-\tau)A(t)] R(t, t_0) x \, d\tau .$$

Denoting by $a(t, \tau)$ the integrand of h_0 we see that $a(\cdot) \in C(\dot{\Delta}_T, X_{1+\varkappa})$ and that, by Corollary (9.4) and (14),

$$\|a(t,\tau)\|_{1+\varkappa} \leqq c(t-\tau)^{-1-\varkappa+\nu}(\tau - t_0)^{-\gamma},$$

where we can chose $\nu > \varkappa$. Let $0 < \varepsilon < T - t_0$ and define h_0^{ε} by $h_0^{\varepsilon}(t) := 0$ for $t_0 \leqq t \leqq t_0 + \varepsilon$ and by

$$h_0^{\varepsilon}(t) := \int_{t_0}^{t-\varepsilon} a(t,\tau) d\tau \qquad \text{for } t_0 + \varepsilon \leqq t \leqq T .$$

Then, similarly as in the proof of Lemma (1.1), we see that $h_0^{\varepsilon} \in C([t_0, T], X^{1+\varkappa})$ (since

$$\|\chi_{[t_0, t-\varepsilon)}(\tau) a(t,\tau)\|_{1+\varkappa} \leqq c\varepsilon^{-1-\varkappa+\nu}(\tau - t_0)^{-\gamma}$$

we can again apply Lebesgue's theorem). On the other hand

$$\|h_0(t) - h_0^{\varepsilon}(t)\|_{1+\varkappa} \leqq \int_{t-\varepsilon}^{t} \|a(t,\tau)\|_{1+\varkappa} d\tau \leqq c \int_{t-\varepsilon}^{t} (t-\tau)^{-1-\varkappa+\nu}(\tau - t_0)^{-\gamma} d\tau$$
$$\leqq c(t-t_0)^{\nu-\varkappa-\gamma} \int_{1-\varepsilon/(t-t_0)}^{1} (1-s)^{-1-\varkappa+\nu} s^{-\gamma} ds .$$

This shows that $h_0^\varepsilon \to h_0$ as $\varepsilon \to 0$, uniformly on every compact subinterval of $(t_0, T]$. Thus $h_0 \in C((t_0, T], X^{1+\varkappa})$.

From Lemma (9.1.ii) we deduce that $h_1(t) \in X^{1+\varkappa}(t)$ and that

$$A_\varkappa(t)h_1(t) = R(t, t_0)x - \exp[-(t-t_0)A(t)]R(t, t_0)x\,.$$

This implies, similarly as in the proof of Lemma (9.2), that $h_1 \in C((t_0, T], X^{1+\varkappa})$, which proves the first assertion.

As for the second assertion,

$$\int_{t_0}^{t} U(t, s)g(s)ds = \int_{t_0}^{t} \exp[-(t-s)A(t)]g(s)ds + \int_{t_0}^{t}\int_{s}^{t} \exp[-(t-\tau)A(t)]R(\tau, s)g(s)d\tau ds =: f_0(t) + f_1(t)$$

and $f_0 \in C((t_0, T], X^{1+\varkappa})$ by Lemma (9.2). Observe that

$$\|\exp[-(t-\tau)A_\varkappa(t)]R(\tau, s)g(s)\|_\varkappa \leqq c(t-\tau)^{-\varkappa}(\tau-s)^{-\varkappa}$$

for $t_0 \leqq s < \tau < t \leqq T$ by (7) and (16). Hence $f_1(t) \in X_\varkappa$ and we can apply Fubini's theorem (in $X^\varkappa$) to obtain

$$f_1(t) = \int_{t_0}^{t} \exp[-(t-\tau)A(t)]\int_{t_0}^{\tau} R(\tau, s)g(s)ds d\tau\,. \tag{18}$$

Since we can choose $\nu > \varkappa$ in Corollary (9.4) it follows from Proposition (1.4) that

$$\left(\tau \mapsto \int_{t_0}^{\tau} R(\tau, s)g(s)ds\right) \in C^\nu([t_0, T], X^\varkappa)\,.$$

Consequently we can apply Lemma (9.2) to (18) to obtain $f_1 \in C((t_0, T], X^{1+\varkappa})$. This proves the second assertion. $\square$

Now we are ready for the proof of the main result of this section.

(9.6) THEOREM. *Let the hypotheses of Theorem* (8.7) *and assumption* $(HR)_\varkappa$ *be satisfied. Assume also that* $\Theta > 2\varkappa$, *that* $X^\varkappa$ *is* (X, X^Θ)*-compatible, and that*

$$f \in C^{\nu,1-}([0, T] \times D^\Theta, X^\varkappa)$$

for some $\nu \in (\varkappa, 1)$. *Then*

$$u(\cdot, t_0, x_0) \in C^1(\dot{J}(t_0, x_0), X^\varkappa) \cap C(\dot{J}(t_0, x_0), X^{1+\varkappa})$$

for each $(t_0, x_0) \in [0, T) \times D^\Theta(t_0)$, *where* $\dot{J}(t_0, x_0) := J(t_0, x_0) \setminus \{t_0\}$.

PROOF. Let $(t_0, x_0) \in [0, T) \times D^\Theta(t_0)$ be fixed and put $u := u(\cdot, t_0, x_0)$ and $J := J(t_0, x_0)$. Then $u \in C^{\Theta-\varkappa}(J, X^\varkappa)$ by Corollary (8.8). Hence $f(\cdot, u(\cdot)) \in C^\nu(J, X^\varkappa)$ for an appropriate $\nu > \varkappa$. Now it follows from

$$u(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, \tau) f(\tau, u(\tau))\, d\tau, \qquad t \in J$$

and Proposition (9.5) that $u \in C(\dot{J}, X^{1+\varkappa})$. Thus, since $\mathcal{A}(\cdot) \in C([0, T], \mathcal{L}(X^{1+\varkappa}, X^\varkappa))$, we deduce that

$$w(\cdot) := -\mathcal{A}(\cdot)u(\cdot) + f(\cdot, u(\cdot)) \in C(\dot{J}, X^\varkappa). \tag{19}$$

Since u is a solution of $(SE)_{(t_0, x_0)}$ we see that $w(t) = \dot{u}(t)$ for $t \in \dot{J}$ in X. Hence

$$u(t) - u(s) = \int_s^t w(\tau)\, d\tau, \qquad s, t \in \dot{J}, \tag{20}$$

in X. Now it follows from (19) that (20) holds in $X^\varkappa$. Hence $u \in C^1(\dot{J}, X^\varkappa)$ and $\dot{u}(t) = w(t)$ for $t \in \dot{J}$ also in $X^\varkappa$. This proves the assertions. $\square$

10. – Strict solutions.

A solution u on J of $(SE)_{(t_0, x_0)}$ is said to be a *strict solution* if $u \in C^1(J, X) \cap C(J, X^1)$, that is, if u is also continuously differentiable at $t = t_0$. In this section we shall derive sufficient conditions for $u(\cdot, t_0, x_0)$ to be a strict solution.

In the following we let $0 < \eta < 1$ and suppose that $\mathcal{F}_\eta$ is admissible and that $X^\eta(t) = X^\eta(0)$ for all $t \in [0, T]$. Then $X^\eta(0)$ is a closed linear subspace of X^η and we can consider the $X^\eta(0)$-realization of $A_\eta(t)$, which we denote by $A_{\eta,\mathfrak{B}}(t)$ for $0 \leqq t \leqq T$. It is clear that $A_{\eta,\mathfrak{B}}(t)$ is also the $X^\eta(0)$-realization of $A(t)$ since

$$\begin{aligned} D(A_{\eta,\mathfrak{B}}(t)) &= \{x \in D(A_\eta(t)) \cap X^\eta(0) : A_\eta(t)x \in X^\eta(0)\} \\ &= \{x \in D(A(t)) \cap X^\eta \cap X^\eta(0) : A(t)x \in X^\eta(0)\} \\ &= \{x \in D(A(t)) : A(t)x \in X^\eta(0)\} \end{aligned} \tag{1}$$

due to the fact that $D(A(t)) = X^1(t) \hookrightarrow X^\eta(t) \hookrightarrow X^\eta$ and $X^\eta(t) = X^\eta(0)$. Moreover, by observing (9.1) and (9.2), it follows that

$$\varrho(-A_{\eta,\mathfrak{B}}(t)) \supset \mu_0 + S_{\alpha+\pi/2} \tag{2}$$

and

$$(\lambda + A_{\eta,\mathfrak{B}}(t))^{-1} = (\lambda + A_\eta(t))^{-1}|X^\eta(0) = (\lambda + A(t))^{-1}|X^\eta(0) \tag{3}$$

for $0 \leqq t \leqq T$.

(10.1) LEMMA. *$D(A_{\eta,\mathfrak{B}}(t))$ is dense in $X^\eta(0)$ and*

$$\|(\lambda + A_{\eta,\mathfrak{B}}(t))^{-1}\|_{\mathfrak{L}(X^\eta(0))} \leqq \frac{c}{1 + |\lambda|}$$

for all $\lambda \in \mu_0 + S_{\alpha+\pi/2}$ and $0 \leqq t \leqq T$.

PROOF. We know already that $-A(t)$ generates a strongly continuous analytic semigroup $\{\exp[-sA(t)]: s \geqq 0\}$ in $\mathfrak{L}(X)$ such that

$$\|A^k(t)\exp[-sA(t)]\|_{\mathfrak{L}(X)} \leqq cs^{-k}\exp[\omega s] \tag{4}$$

for $k = 0, 1$ and all $(s, t) \in (0, \infty) \times [0, T]$ (cf. (8.7)). Since strongly continuous semigroups commute with their generators it is easily verified that $\{\exp[-sA(t)]: s \geqq 0\}$ restricts to a strongly continuous semigroup in $\mathfrak{L}(X^1(t))$. Denoting by $-B(t)$ the infinitesimal generator of $\{\exp[-sA(t)]|X^1(t): s \geqq 0\}$ in $\mathfrak{L}(X^1(t))$ it is also easily verified that $B(t) \supset A(t)|D(A^2(t))$. We assume now without loss of generality that $\mu_0 = 0$.

Let $x \in X^1(t)$ and $\varepsilon > 0$ be given. Since $D(A(t))$ is dense in X there exists $y \in D(A^2(t))$ such that $\|A(t)x - A(t)y\| < \varepsilon$. Since $z \mapsto \|A(t)z\|$ is an equivalent norm on $X^1(t)$ by (6.2), it follows that $D(A^2(t))$ is dense in $X^1(t)$. Since $D(A^2(t))$ is also invariant under $\{\exp[-sA(t)]: s \geqq 0\}$ it is a core for $-B(t)$ (eg. [19, Theorem 1.9]). Thus $B(t)$ is the closure of $A(t) \mid D(A^2(t))$. However it is easily verified that the latter operator is closed in $X^1(t)$. Thus $B(t) = A(t)|D(A^2(t))$ which is precisely the X^1-realization of $A(t)$, denoted by $A_{1,\mathfrak{B}}(t)$.

Observe that, by (6.2), there exist positive constants $\bar{c}$ and $\hat{c}$ such that

$$\bar{c}\|G\|_{\mathfrak{L}(X^1(t))} \leqq \|A(t)GA^{-1}(t)\|_{\mathfrak{L}(X)} \leqq \hat{c}\|G\|_{\mathfrak{L}(X^1(t))}$$

for all $G \in \mathfrak{L}(X^1(t))$ and all $t \in [0, T]$. Hence we obtain from (4) and the above considerations that

$$\|A^k_{1,\mathfrak{B}}(t)\exp[-sA_{1,\mathfrak{B}}(t)\|_{\mathfrak{L}(X^1(t))} \leqq cs^{-k}\exp[\omega s] \tag{5}$$

for $k=0,1$ and $(s,t)\in(0,\infty)\times[0,T]$. Thus, by interpolating between (4) and (5), we see that

$$\|A^k(t)\exp[-sA(t)]\|_{\mathfrak{L}(X^\eta(t))}\leqq cs^{-k}\exp[\omega s] \tag{6}$$

for $k=0,1$, $s>0$, and $0\leqq t\leqq T$. Since

$$\big(s\mapsto\exp[-sA(t)]\big)\in C\big(\mathbb{R}^+,\mathfrak{L}_s(X)\big)\cap C\big(\mathbb{R}^+,\mathfrak{L}_s(X^1(t))\big)$$

it follows from (JPF iii) that $\{\exp[-sA(t)]: s\geqq 0\}$ restricts to a strongly continuous semigroup on $X^\eta(t)$. Let $-S(t)$ be the infinitesimal generator of this semigroup. Then it is a consequence of (JPF iii) that $S(t)\supset A_{1,\mathfrak{B}}(t)$. Since $D(A_{1,\mathfrak{B}}(t))$ is dense in $X^1(t)$ and $X^1(t)$ is dense in $X^\eta(t)$ by (JPF ii), it follows that $D(A_{1,\mathfrak{B}}(t))$ is dense in $X^\eta(t)$. Since $D(A_{1,\mathfrak{B}}(t))$ is also invariant under $\{\exp[-sA(t)]: s\geqq 0\}$ by the above considerations, it follows again from [19, Theorem 1.9] that $D(A_{1,\mathfrak{B}}(t))$ is a core for $S(t)$. Hence $S(t)$ is the closure in $X^\eta(t)$ of $A_{1,\mathfrak{B}}(t)$. Since $A_{\eta,\mathfrak{B}}(t)$ is closed in $X^\eta(t)$ and $A_{\eta,\mathfrak{B}}(t)\supset A_{1,\mathfrak{B}}(t)$ we see that $A_{\eta,\mathfrak{B}}(t)\supset S(t)$. Since $-S(t)$ is the infinitesimal generator of a strongly continuous semigroup in $\mathfrak{L}(X^\eta(t))$ there exists a number $\lambda>\mu_0$ such that $\lambda\in\varrho(-S(t))$. Since λ belongs also to $\varrho(-A_{\eta,\mathfrak{B}}(t))$ by (2) we see that $A_{\eta,\mathfrak{B}}(t)$ cannot be a proper extension of $S(t)$. Thus $A_{\eta,\mathfrak{B}}(t)=S(t)$, which implies in particular that $D(A_{\eta,\mathfrak{B}}(t))$ is dense in $X^\eta(t)=X^\eta(0)$. Moreover it follows from (6) that

$$\|A^k_{\eta,\mathfrak{B}}(t)\exp[-sA_{\eta,\mathfrak{B}}(t)]\|_{\mathfrak{L}(X^\eta(0))}\leqq cs^{-k}\exp[\omega s]$$

for $k=0,1$, $s>0$, and $0\leqq t\leqq T$. It is well known (e.g. [12, Proposition 1.1.11]) that this implies that the semigroup $\{\exp[-sA_{\eta,\mathfrak{B}}(t)]: s\geqq 0\}$ in $\mathfrak{L}(X^\eta(0))$ has a holomorphic extension to some sector S_β of $\mathbf{C}$, where $\beta\in(0,\pi/2)$ is independent of $t\in[0,T]$. Now the assertion follows by a well known characterization of infinitesimal generators of strongly continuous analytic semigroups (e.g. [29, Theorem 13.2]). □

We impose now the following hypothesis $(SS)_\eta$, where $0<\eta<1$.

$(SS)_\eta$ (i) *$\mathcal{F}_\eta$ is admissible and $X^\eta(t)=X^\eta(0)$ for $0\leqq t\leqq T$.*

(ii) *There exist Banach spaces $X^{1+\eta}\hookrightarrow X^1$ and $Z^\eta\hookrightarrow Z$ such that*

$$\mathcal{T}:=(\mu_0+\mathcal{A},\mathfrak{B})\in C\big([0,T],\mathrm{Isom}(X^{1+\eta},X^\eta\times Z^\eta)\big).$$

(iii) $(\mu_0 + A_{\eta,\mathfrak{B}}(\cdot))^{-1} \in C^1([0, T], \mathcal{L}(X^\eta(0)))$
and there exists a constant $\varrho_\eta \in [0, 1)$ *such that*

$$\|[\mu_0 + A_{\eta,\mathfrak{B}}(t)](\lambda + A_{\eta,\mathfrak{B}}(t))^{-1}[(\mu_0 + A_{\eta,\mathfrak{B}}(t))^{-1}]^{\bullet}\|_{\mathcal{L}(X^\eta(0))} \leqq c|\lambda - \mu_0|^{-1+\varrho_\eta}$$

for all $\lambda \in \mu_0 + S_{\alpha+\pi/2}$ *and all* $t \in [0, T]$.

Then we can prove the following proposition, where $\hat{\mathcal{F}}_\theta$ is any interpolation functor satisfying (*IPF* i)-(*IPF* iii).

(10.2) PROPOSITION. *Let* $0 < \theta < 1$. *Then*

$$U \in \mathcal{K}_s(\theta, c, X^\eta(0), \hat{\mathcal{F}}_\theta(X^\eta, X^{1+\eta})) .$$

Moreover

$$U(\cdot, t_0)x \in C([t_0, T], \hat{\mathcal{F}}_\theta(X^\eta, X^{1+\eta})) \tag{7}$$

for each $t_0 \in [0, T)$ *and* $x \in D(A_{\eta,\mathfrak{B}}(t_0))$.

PROOF. It follows from Lemma (10.1), (2) and $(SS)_\eta$ that the assumptions (*AP* ii) and (*AP* iii) are satisfied for $A_{\eta,\mathfrak{B}}(\cdot)$. Moreover, by $(SS\text{ ii})_\eta$, there are constants $\bar{c}$ and $\hat{c}$ such that

$$\|x\|_{1+\eta} \leqq \bar{c}\left(\|(\mu_0 + \mathcal{A}(t))x\|_\eta + \|\mathfrak{B}(t)x\|_{Z^\eta}\right) \leqq \hat{c}\|x\|_{1+\eta} \tag{8}$$

for all $x \in X^{1+\eta}$ and $t \in [0, T]$, where $\|\cdot\|_{1+\eta}$ is the norm in $X^{1+\eta}$. Hence it follows from Yagi's theorem [51] that there exists a parabolic fundamental solution $V: \Delta_T \to \mathcal{L}(X^\eta(0))$ for $A_{\eta,\mathfrak{B}}(\cdot)$. Due to (8) and the fact that

$$\mathcal{T} \in C([0, T], \mathcal{L}(X^{1+\eta}, X^\eta \times Z^\eta)) ,$$

one verifies that the proof of Lemma (6.1) applies literally to give $V \in C(\dot{\Delta}_T, \mathcal{L}_s(X^\eta(0), X^{1+\eta}))$ and

$$\|V(t, s)\|_{\mathcal{L}(X^\eta(0), X^{1+\eta})} \leqq c(t-s)^{-1} \qquad \forall (t, s) \in \dot{\Delta}_T ,$$

as well as

$$V(\cdot, s) \in C\left([s, T], \mathcal{L}_s(D(A_{\eta,\mathfrak{B}}(s)), X^{1+\eta})\right)$$

for each $s \in [0, T)$. Since also

$$\|V(t, s)\|_{\mathcal{L}(X^\eta(0))} \leqq c \qquad \forall (t, s) \in \Delta_T$$

we obtain the assertion with U replaced by V by interpolation.

Similarly as in (8.10) we see that

$$V(t, s) = \exp[-(t-s)A_{\eta,\mathfrak{B}}(t)] + \int_s^t \exp[-(t-\tau)A_{\eta,\mathfrak{B}}(t)]R_\eta(\tau, s)ds\,,$$

where R_η is the solution of the integral equation

$$R_\eta(t, s) - \int_s^t R_{0,\eta}(t, \tau)R_\eta(\tau, s)d\tau = R_{0,\eta}(t, s)\,,$$

and where

$$R_{0,\eta}(t, s) = \frac{1}{2\pi i}\int_\Gamma \exp[\lambda(t-s)][(\lambda + A_{\eta,\mathfrak{B}}(t))^{-1}]^{\cdot} d\lambda$$

(where, perhaps, the angle α has to be chosen smaller, but positive). Now it follows from (3) that $U(t, s) \supset V(t, s)$, which proves the assertion. $\square$

After these preparations we can prove our desired regularity result.

(10.3) THEOREM. *Let the hypotheses of Theorem* (8.7) *and assumption* $(SS)_\eta$ *for some* $\eta \in (0, \Theta)$ *be satisfied. Let* $0 < \theta < 1$, *put*

$$E^{1+\theta} := \hat{\mathfrak{F}}_\theta(X^\eta, X^{1+\eta})\,,$$

and suppose that there exists a Banach space E^θ *such that* $X^\eta(0) \hookrightarrow E^\theta \hookrightarrow X$ *and*

$$\mathcal{A}(\cdot) \in C([0, T], \mathfrak{L}(E^{1+\theta}, E^\theta))\,. \tag{9}$$

Finally, assume that

$$f \in C([0, T] \times D^\Theta, X^\eta)\,, \tag{10}$$

that

$$f(t, x) \in X^\eta(0) \qquad \forall x \in D(A(t)),\ t \in [0, T]\,, \tag{11}$$

that $0 \leqq t_0 < T$, *and that*

$$x_0 \in D(A(t_0)) \cap D^\Theta \quad \textit{and} \quad A(t_0)x_0 \in X^\eta(0)\,. \tag{12}$$

Then

$$u(\cdot, t_0, x_0) \in C^1(J(t_0, x_0), E^\theta) \cap C(J(t_0, x_0), E^{1+\theta})\,.$$

PROOF. Let $u := u(\cdot, t_0, x_0)$ and $J := J(t_0, x_0)$. By (12) and Corollary (8.8) we see that $u \in C(J, D^{\Theta})$. Hence, by (10), (11), the fact that $X^{\eta}(0)$ is a closed subspace of X^{η} and by $u(t) \in D(A(t))$ for all $t \in J$, it follows that

$$f(\cdot, u(\cdot)) \in C(J, X^{\eta}(0)) . \tag{13}$$

Thus, by Propositions (10.2) and (1.3),

$$\left(t \mapsto \int_{t_0}^{t} U(t, \tau) f(\tau, u(\tau)) d\tau\right) \in C(J, E^{1+\theta}) . \tag{14}$$

Since (12) means that $x_0 \in D(A_{\eta,\mathfrak{B}}(t_0))$, we obtain from Proposition (10.2) that

$$U(\cdot, t_0)x_0 \in C([t_0, T], E^{1+\theta}) . \tag{15}$$

Thus, since u satisfies the integral-evolution equation $(\mathfrak{J})_{(t_0, x_0)}$, (14) and (15) imply

$$u \in C(J, E^{1+\theta}) . \tag{16}$$

Since $X^{\eta}(0) \hookrightarrow E^{\theta}$ we obtain from (13), (9) and (16) that

$$v := -\mathcal{A}(\cdot)u(\cdot) + f(\cdot, u(\cdot)) \in C(J, E^{\theta}) . \tag{17}$$

Since u is a solution of $(SE)_{(t_0, x_0)}$ in X we know that $v(t) = \dot{u}(t)$ for $t > t_0$ in X. Hence

$$u(t) - u(s) = \int_{s}^{t} v(\tau) d\tau , \qquad s, t \in J \setminus \{t_0\} , \tag{18}$$

in X. By (17) this integral exists in E^{θ} for all $s, t \in J$. Thus it follows from (18) that $u \in C^1(J, E^{\theta})$ and $\dot{u} = v$ in E^{θ} on all of J. □

Conditions (11) and (12) are compatibility conditions which are easy to verify in concrete situations as we shall see in the next chapter. These compatibility conditions are not optimal, that is, they are sufficient for the existence of strict solutions (with values in $E^{1+\theta}$) but not necessary, in general. A necessary condition is obviously given by

$$x_0 \in D(A(t_0)) \quad \text{and} \quad -A(t_0)x_0 + f(t_0, x_0) \in E^{\theta} .$$

It is natural to expect that this condition is also necessary. There is some evidence for this conjecture, namely the compatibility conditions for parabolic equations (cf. [30, 42, 32 II, Chapter 4]) and the abstract results of Sinestrari [39] and Aquistapace and Terreni [7, 8] for linear evolution equations. The latter authors study equations of the form $\dot{u} + A(t)u = f(t)$ in a general Banach space E, where they impose upon $A(\cdot)$ the hypotheses of Sobolevskii [41] and Tanabe [44]—thus in particular that $D(A(t))$ is independent of $t \in [0, T]$—except that they do not require that $D(A(t))$ be dense in E. In [8] these authors give also a necessary and sufficient condition for the existence of strict solutions in the case where the hypotheses of Kato and Tanabe [26]—again except for the density of $D(A(t))$—are satisfied. However this condition is not explicit and it is not clear how to verify it in concrete situations. Since in [7, 8, 39] it is always assumed that

$$\|(\lambda + A(t))^{-1}\|_{\mathfrak{L}(E)} \leqq c/(1 + |\lambda|) \tag{19}$$

these results are not applicable to our situation. Moreover we shall see in the next chapter that (19) is not satisfied if E is a Sobolev space, the situation in which we are mainly interested.

Chapter III

SEMILINEAR PARABOLIC SYSTEMS

11. – Sobolev-Slobodeckii spaces.

Let Ω be a nonempty open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$. Then Ω is said *to be of class* C^l, $l \in \mathbb{N}^* \cup \{\infty\}$, if $\bar{\Omega}$ is an n-dimensional submanifold of $\mathbb{R}^n$ (with boundary $\partial\Omega$). Following F. E. Browder [11] we say that Ω is *uniformly regular of class* C^l if either $\Omega = \mathbb{R}^n$ or if the following condition (UR) is satisfied:

(UR) There exists a denumerable family (U_j, φ_j), $j = 1, 2, \ldots$ of coor dinate charts of $\bar{\Omega}$ with the following properties:

(i) Each φ_j is a C^l-diffeomorphism of U_j onto the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ mapping $U_j \cap \Omega$ onto the « upper half-ball » $\mathbb{B}^n \cap (\mathbb{R}^{n-1} \times (0, \infty))$ and $U_j \cap \partial\Omega$ onto the flat part $\mathbb{B}^n \cap (\mathbb{R}^{n-1} \times \{0\}) =: \Sigma^{n-1}$. Moreover there exists a constant M_0 such that

$$\sup_{U_j} |D^\alpha \varphi_j|, \quad \sup_{\mathbf{B}^n} |D^\alpha \varphi_j^{-1}| \leqq M_0$$

for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leqslant l$ and all $j = 1, 2, \ldots$.

(ii) The set $\bigcup_j \varphi_j^{-1}(\frac{1}{2}\mathbf{B}^n)$ contains an ε-neighbourhood of $\partial\Omega$ in $\bar{\Omega}$ for some $\varepsilon > 0$.

(iii) There exists $K_0 \in \mathbf{N}^*$ such that any $K_0 + 1$ distinct sets U_j have an empty intersection.

Here and in the following $|\cdot|$ denotes the euclidean norm.

Clearly every uniformly regular set of class C^l is a set of class C^l. Moreover, every open set of class C^l with a compact boundary or every open half-space is uniformly regular of class C^l.

In the remainder of this chapter Ω denotes throughout a nonempty open subset of $\mathbf{R}^n$, $n \in \mathbf{N}^*$, which is uniformly regular of class $C^{\hat{l}}$ for some $\hat{l} \in \mathbf{N}^*$.

It is not too difficult to see (cf. [11, Lemma 2]) that there exists a sequence (U_j, φ_j, π_j), $j \in \mathbf{N}$,—called a *regular localization system for Ω*—where $\{(U_j, \varphi_j): j \in \mathbf{N}\}$ is a $C^{\hat{l}}$-atlas for $\bar{\Omega}$ and $\{\pi_j: j \in \mathbf{N}\}$ a $C^{\hat{l}}$-partition of unity on $\bar{\Omega}$ subordinate to the open covering $\{U_j: j \in \mathbf{N}\}$ of $\bar{\Omega}$ in $\mathbf{R}^n$ such that:

(i) The open sets U_j are pairwise distinct.

(ii) The subset of those coordinate charts whose patches U_j intersect $\partial\Omega$ satisfies condition (UR).

(iii) There exists a constant $K \geqq K_0$ such that any intersection of at least $K + 1$ coordinate patches is empty.

(iv) There exists a constant $M \geqq M_0$ such that

$$\sup_{\mathbf{R}^n} |D^\alpha \pi_j| \leqq M$$

for all $j \in \mathbf{N}$ and $\alpha \in \mathbf{N}^n$ with $|\alpha| \leqq \hat{l}$.

In the remainder of this section E denotes a finite-dimensional Banach space and $W_p^m := W_p^m(\Omega, E)$, $m \in \mathbf{N}$, $1 \leqq p < \infty$, the Sobolev spaces consisting of all $u \in L_p := L_p(\Omega, E)$ whose distributional derivatives of order at most m belong to L_p, endowed with the norm

$$\|u\|_{m,p} := \|u\|_{m,p}^{\Omega} := \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{L_p}^p\Big)^{1/p}.$$

If $s \in \mathbb{R}^+ \setminus \mathbb{N}$ and $\sigma := s - [s]$, where $[s]$ is the integral part of s, we let

$$I_{\sigma,p}(u) := I^{\Omega}_{\sigma,p}(u) := \int\limits_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\sigma p}} \, d(x, y) .$$

Then the *Sobolev-Slobodeckii space* $W^s_p := W^s_p(\Omega, E)$ is the subspace of $W^{[s]}_p$ consisting of all elements u satisfying $I_{\sigma,p}(D^\alpha u) < \infty$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqq [s]$, endowed with the norm

$$\|u\|_{s,p} := \|u\|^{\Omega}_{s,p} := \Big(\|u\|^p_{[s],p} + \sum_{|\alpha| \leqq [s]} I_{\sigma,p}(D^\alpha u)\Big)^{1/p} .$$

It is well known that W^s_p is for each $s \in \mathbb{R}^+$ and $p \in [1, \infty)$ a Banach space.

The following technical lemma, which is closely related to [11, Lemma 3], is crucial.

(11.1) LEMMA. *Let $\{(U_j, \varphi_j, \pi_j) : j \in \mathbb{N}\}$ be a regular localization system for Ω and let*

$$|||u|||_{s,p} := \Big(\sum_j \|\pi_j u\|^p_{s,p}\Big)^{1/p} .$$

Then there exist positive constants c_0 and c_1-depending only on K, M, $\hat{l}$ and n, but not on $s \in \mathbb{R}^+$ and $p \in [1, \infty)$, such that

$$c_0 \|u\|_{s,p} \leqq |||u|||_{s,p} \leqq c_1 \|u\|_{s,p}$$

for all $u \in W^s_p$, all $s \in [0, \hat{l}]$, and all $p \in [1, \infty)$.

PROOF. Let $V_j := U_j \setminus \bigcup_{i=0}^{j-1} U_i$ and $\chi_j := \chi_{V_j}$ for $j \in \mathbb{N}$. Then the V_j are pairwise disjoint and, for each $i \in \mathbb{N}$, at most K of the functions $\chi_i | U_j$, $j \in \mathbb{N}$, are different from zero.

Let now $s \in \mathbb{N}$ and $|\alpha| \leqq s$. Then, by Hölder's inequality,

$$\int\limits_\Omega |D^\alpha u|^p dx = \sum_i \int\limits_\Omega \chi_i \Big|D^\alpha\Big(\sum_j \pi_j u\Big)\Big|^p dx \leqq K^{p-1} \sum_j \int\limits_\Omega |D^\alpha(\pi_j u)|^p dx ,$$

which shows that $\|u\|_{s,p} \leqq K |||u|||_{s,p}$. Similarly,

$$\begin{aligned}\sum_j \int\limits_\Omega |D^\alpha(\pi_j u)|^p dx &= \sum_j \sum_i \int\limits_\Omega \chi_i |D^\alpha(\pi_j u)|^p \, dx \\ &\leqslant c \sum_j \sum_i \sum_{|\beta| \leqq s} \int\limits_\Omega \chi_i \chi_{U_j} |D^\beta u|^p dx = Kc \sum_{|\beta| \leqq s} \int\limits_\Omega |D^\beta u|^p dx ,\end{aligned}$$

where c depends only on M, $\hat{l}$ and n. Hence $|||u|||_{s,p} \leqq c_1 \|u\|_{s,p}$, which proves the assertion in this case. A similar argument works if $s \notin \mathbf{N}$. $\square$

In the following we denote by $BUC^k := BUC^k(\Omega, E)$, $k \in \mathbf{N}$, the Banach space of all continuous functions $u\colon \Omega \to E$ whose derivatives of order at most k exist and are bounded and uniformly continuous on Ω, endowed with the norm

$$\|u\|_{C^k} := \|u\|_{C^k}^{\Omega} := \sum_{|\alpha| \leqq k} \sup_{\Omega} |D^\alpha u| .$$

Moreover, $BUC^k_\infty := BUC^k_\infty(\Omega, E)$ is the closed linear subspace of BUC^k consisting of all functions u such that $D^\alpha u$ « vanishes at infinity » for $|\alpha| \leqq k$. Here $v\colon \Omega \to E$ is said to vanish at infinity if for each $\varepsilon > 0$ there is a number $r > 0$ such that $|v(x)| < \varepsilon$ for all $x \in \Omega \setminus r\mathbf{B}^n$. If Ω is bounded then, as usual,

$$C^k(\bar{\Omega}, E) := BUC^k(\Omega, E) .$$

In this case $BUC^k_\infty = C^k(\bar{\Omega}, E)$.

Let $\mathfrak{R}_\Omega$ denote the restriction to Ω, that is, $\mathfrak{R}_\Omega u := u|\Omega$ for each $u\colon \mathbf{R}^n \to E$. Then it is well known that $\mathfrak{R}_\Omega \mathfrak{D}(\mathbf{R}^n, E)$ is dense in W^s_p, where $\mathfrak{D}(\mathbf{R}^n, E)$ is the space of all test functions on $\mathbf{R}^n$ with values in E. Consequently,

$$BUC^k_\infty \cap W^s_p \quad \textit{is dense in } W^s_p \textit{ if } k \geqq s . \tag{1}$$

A linear operator $\mathcal{E}\colon X \to \mathfrak{D}'(\Omega, E)$, where X is a subset of the space $\mathfrak{D}'(\Omega, E)$ of all E-valued distributions on Ω, is said to be an *extension operator* for Ω (and X) if $\mathfrak{R}_\Omega \mathcal{E} u = u$ for every $u \in X$.

(11.2) LEMMA. *There exists an extension operator $\mathcal{E}$ for Ω such that*

$$\mathcal{E} \in \mathcal{L}(W^s_p, W^s_p(\mathbf{R}^n, E)) \cap \mathcal{L}(BUC^k, BUC^k(\mathbf{R}^n, E)) \cap \mathcal{L}(BUC^k_\infty, BUC^k_\infty(\mathbf{R}^n, E))$$

for $0 \leqq s \leqq \hat{l}$, $1 \leqq p < \infty$, and $0 \leqq k \leqq \hat{l}$, $k \in \mathbf{N}$.

PROOF. Let $\{(U_j, \varphi_j, \pi_j)\colon j \in \mathbf{N}\}$ be a regular localization system for Ω and let $u \in BUC^{\hat{l}}_\infty \cap W^{\hat{l}}_p$. By means of local coordinates and a standard reflection procedure due to Hestenes (e.g. [1, Lemma 7.45] or [50, Satz 5.6]) we can extend each $\pi_j u$ with $U_j \cap \partial\Omega \neq \emptyset$ to an element $\tilde{u}_j \in BUC^{\hat{l}}_\infty(\mathbf{R}^n, E) \cap W^{\hat{l}}_p(\mathbf{R}^n, E)$ such that

$$\|\tilde{u}_j\|^{\mathbf{R}^n}_{s,p} \leqq c \|\pi_j u\|^{\Omega}_{s,p} \quad \text{and} \quad \|\tilde{u}_j\|^{\mathbf{R}^n}_{C^k} \leqq c \|\pi_j u\|^{\Omega}_{C^k} \tag{2}$$

for $0 \leqq s \leqq \hat{l}$, $1 \leqq p < \infty$, $0 \leqq k \leqq \hat{l}$ and $j \in \mathbb{N}$, and such that the map $u \mapsto \tilde{u}_j$ is linear (where c depends only on M, $\hat{l}$, and n).

Let now $\mathcal{E}u := \sum_j \pi_j \tilde{u}_j$: Then it follows from (2) that (by using obvious notations)

$$\|\pi_i \mathcal{E}u\|_{s,p}^{\mathbb{R}^n} \leqq c \sum_l \{\|\pi_l u\|_{s,p}^{\Omega} : U_j \cap U_l \neq \emptyset\} \tag{3}$$

and

$$\|\pi_j \mathcal{E}u\|_{C^k}^{\mathbb{R}^n} \leqq c \sum_l \{\|\pi_l u\|_{C^k}^{\Omega} : U_j \cap U_l \neq \emptyset\} \tag{4}$$

for $0 \leqq s \leqq \hat{l}$, $1 \leqq p < \infty$, $0 \leqq k \leqq \hat{l}$, and $j \in \mathbb{N}$. Hence, by (3) and (iii),

$$\Big[\sum_j \big(\|\pi_j \mathcal{E}u\|_{s,p}^{\mathbb{R}^n}\big)^p\Big]^{1/p} \leqq c|||u|||_{s,p}^{\Omega}. \tag{5}$$

As in the first part of the proof of Lemma (11.1) we see that

$$\|\mathcal{E}u\|_{s,p}^{\mathbb{R}^n} \leqq K\Big[\sum_j \big(\|\pi_j \mathcal{E}u\|_{s,p}^{\mathbb{R}^n}\big)^p\Big]^{1/p}.$$

Thus the first assertion follows from (5), Lemma (11.1) and (1).

It is obvious that

$$\|\mathcal{E}u\|_{C^k}^{\mathbb{R}^n} = \Big\|\sum_j \pi_j \mathcal{E}u\Big\|_{C^k}^{\mathbb{R}^n} \leqq K \sup_j \|\pi_j \mathcal{E}u\|_{C^k}^{\mathbb{R}^n}.$$

Now the last assertion follows from (4) since (4) is clearly true for every $u \in BUC_\infty^k$. The proof of the middle statement is now clear. □

As usual we denote by $H_p^s(\mathbb{R}^n, E)$ and $B_{p,q}^s(\mathbb{R}^n, E)$, $s \in \mathbb{R}$, $1 \leqq p$, $q \leqq \infty$, the Bessel potential (or generalized Lebesgue) spaces and the Besov spaces of E-valued distributions on $\mathbb{R}^n$, respectively (e.g. [9, Section 6.2]). Moreover,

$$H_p^s := H_p^s(\Omega, E) := \mathcal{R}_\Omega H_p^s(\mathbb{R}^n, E)$$

and

$$B_{p,q}^s := B_{p,q}^s(\Omega, E) := \mathcal{R}_\Omega B_{p,q}^s(\mathbb{R}^n, E),$$

where these spaces are endowed with the usual quotient space norms (e.g. [46, Definition 3.2.2.1]).

(11.3) THEOREM. *Let $s \in [0, \hat{l}]$ and $1 < p < \infty$: Then, up to equivalent norms,*

$$W_p^s = \begin{cases} H_p^s & \text{if } s \in \mathbb{N}, \\ B_{p,p}^s & \text{if } s \notin \mathbb{N}. \end{cases}$$

PROOF. This follows immediately from Lemma (11.2) and the fact that the assertion is true if $\Omega = \mathbb{R}^n$ ([45, Theorem 2.3.3 and Remark 2.5.1.4]). □

(11.4) COROLLARY. *Suppose that* $s \in (0, \hat{l})\setminus\mathbb{N}$ *and* $1 < p < \infty$. *Then*

$$u \mapsto \Big(\sum_{|\alpha|\leqslant[s]} \|D^\alpha u\|_{0,p}^p + \sum_{|\alpha|=[s]} I_{\sigma,p}(D^\alpha u)^{1/p}\Big)$$

is an equivalent norm for W_p^s.

PROOF. This follows from the corresponding renorming theorem for $B_{p,p}^s(\mathbb{R}^n, E)$ (cf. [45, Theorem 2.5.1 and Section 4.4.1]). □

As a further consequence of Lemma (11.2) we obtain the validity of the usual Sobolev-type imbedding theorems. In this connection we let

$$\|u\|_{C^s} := \|u\|_{C^s}^\Omega := \|u\|_{C^{[s]}} + \sum_{|\alpha|=[s]} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{s-[s]}}$$

for $s \in \mathbb{R}^+\setminus\mathbb{N}$. Then

$$BUC_\infty^s := BUC_\infty^s(\Omega, E) := \{u \in BUC_\infty^{[s]} \colon \|u\|_{C^s} < \infty\}$$

is a Banach space and

$$BUC_\infty^s = C^s(\bar{\Omega}, E) \quad \text{if } \Omega \text{ is bounded}.$$

(11.5) THEOREM. *Suppose that* $0 \leqq s,\ t \leqq \hat{l}$ *and* $1 < p,\ q < \infty$. *Then*

$$W_p^s \hookrightarrow W_q^t \textit{ if } 1/p \geqq 1/q \quad \textit{and} \quad s - n/p \geqq t - n/q,$$

and

$$W_p^s \hookrightarrow BUC_\infty^t \quad \textit{if } s - n/p > t,$$

where the equality sign is permitted if $t \notin \mathbb{N}$.

PROOF. This follows from Lemma (11.2) and the corresponding results for the case where $\Omega = \mathbb{R}^n$ (e.g. [45, Theorem 2.8.1]). □

Finally we obtain the following interpolation theorem which is of utmost importance for our purposes.

(11.6) THEOREM. *Let* $0 \leqq s_0,\ s_1 \leqq \hat{l},\ s_0 \neq s_1,\ 1 < p_0,\ p_1 < \infty$, *and* $0 < \Theta < 1$, *and define* $s \in [0, \hat{l}]$ *and* $p \in (1, \infty)$ *by*

$$s := (1-\Theta)s_0 + \Theta s_1 \quad \textit{and} \quad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1},$$

respectively. Then, up to equivalent norms,

$$(W^{s_0}_{p_0}, W^{s_1}_{p_0})_{\Theta,p} = W^s_p \quad \textit{if } s \notin \mathbb{N},$$

and

$$[W^{s_0}_{p_0}, W^{s_1}_{p_1}]_{\Theta} = W^s_p$$

if either $s_0, s_1, s \notin \mathbb{N}$ *or* $s_0, s_1, s \in \mathbb{N}$.

PROOF. Due to Theorem (11.3), Lemma (11.2) and [45, Theorem 1.2.4] it suffices to prove the assertion if $\Omega = \mathbb{R}^n$—In the latter case it is known that

$$(B^{s_0}_{p_0,p_0}, B^{s_1}_{p_1,p_1})_{\Theta,p} = B^s_{p,p}, \qquad [B^{s_0}_{p_0,p_0}, B^{s_1}_{p_1,p_1}]_{\Theta} = B^s_{p,p}$$

and that

$$(H^{s_0}_{p_0}, H^{s_1}_{p_1})_{\Theta,p} = B^s_{p,p}, \qquad [H^{s_0}_{p_0}, H^{p_1}_{s_1}]_{\Theta} = H^s_p$$

(e.g. [45, Theorem 2.4.1 and Remarks 2.4.2.2 (*b*) and (*d*)] or [9, Theorem 6.4.5]). Hence the desired result follows from Theorem (11.3). □

We turn now to the study of the trace operator on the boundary $\partial\Omega$. For this purpose we denote by Γ a nonempty open and closed subset of $\partial\Omega$ (that is, a nonempty union of components of $\partial\Omega$). Then we let

$$\|u\|^{\Gamma}_{s,p} := \Big[\sum_j{}^{\Gamma} \big(\|\pi_j u \circ \varphi_j^{-1}\|^{\Sigma^{n-1}}_{s,p}\big)^p\Big]^{1/p} \tag{6}$$

for $s \in [0, \hat{l}]$ and $1 \leq p < \infty$, where $\sum_j^{\Gamma}$ denotes the summation over all $j \in \mathbb{N}$ with $U_j \cap \Gamma \neq \emptyset$. Then we denote by $W^s_p(\Gamma, E)$ the vector subspace of $L_p(\Gamma, E)$ (where $L_p(\Gamma, E)$ is constructed by means of the usual hypersurface measure $d\sigma$ of Γ) consisting of all elements for which the norm (6) is finite. It is not difficult to see that $W^s_p(\Gamma, E)$ is a Banach space and that $|||\cdot|||^{\Gamma}_{s,p}$ is an equivalent norm for $W^s_p(\Gamma, E)$, where

$$|||u|||^{\Gamma}_{s,p} := \Big(\sum_i{}^{\Gamma} \sum_{|\beta|\leq s} \|D^\beta(\pi_j u)\circ\varphi_j\|^p_{L_p(\Gamma,E)}\Big)^{1/p}$$

if $s \in \mathbb{N}$, and where

$$|||u|||^{\Gamma}_{s,p} := \Bigg(|||u|||^p_{[s],p} + \sum_j^{\Gamma} \sum_{|\beta|\leq[s]} \int_\Gamma\int_\Gamma \frac{|D^\beta(\pi_j u)\circ\varphi_j(x) - D^\beta(\pi_j u)\circ\varphi_j(y)|^p}{|x-y|^{n-1+(s-[s])p}}\, d\sigma(x)\, d\sigma(y)\Bigg)^{1/p}$$

if $s \notin \mathbb{N}$. Using this norm and similar arguments as in the proof of Lemma (11.1) it follows that, up to equivalent norms, $W_p^0(\Gamma, E) = L_p(\Gamma, E)$, and that different regular localization systems for Ω induce equivalent norms on $W_p^s(\Gamma, E)$.

Let $u \in \mathcal{R}_\Omega \mathfrak{D}(\mathbb{R}^n, E)$ and let ν be the outer normal on $\partial\Omega$. Then we let

$$\vec{\gamma}_{\Gamma,k} u := \left(u|\Gamma, \frac{\partial u}{\partial \nu}\bigg|\Gamma, \ldots, \frac{\partial^k u}{\partial \nu^k}|\Gamma\right)$$

for $0 \leqq k \leqq \hat{l} - 1$, and we put $\gamma_\Gamma := \vec{\gamma}_{\Gamma,0}$. The following important trace theorem shows that $\vec{\gamma}_{\Gamma,k}$ can be extended as a bounded linear operator over W_p^s, which is again denoted by the same symbol. As usual we let $\mathring{W}_p^s$ be the closure of $\mathfrak{D}(\Omega, E)$ in W_p^s. Finally, following [45], a map $R \in \mathfrak{L}(X, Y)$ is said to be a *retraction* if it possesses a continuous right inverse $S \in \mathfrak{L}(Y, X)$.

(11.7) THEOREM. *Let* $p \in (1, \infty)$, $s \in [0, \hat{l}]$ *and* $k := [s - 1/p]$.

(i) *If* $s > 1/p$ *and* $s - 1/p \in \mathbb{N}$ *then* $\vec{\gamma}_{\Gamma,k}$ *is a retraction of* W_p^s *onto* $\prod_{j=0}^{k} W_p^{s-j-1/p}(\Gamma, E)$ *and* $\ker(\vec{\gamma}_{\Gamma,\cdot,k}) = \mathring{W}_p^s$.

(ii) *If* $0 \leqq s \leqq 1/p$ *then* $\mathring{W}_p^s = W_p^s$.

PROOF. This follows from the corresponding results for $\mathbb{R}^{n-1} \times (0, \infty)$ (e.g. [45, Theorem 2.9.3]) by means of « the method of local coordinates » (e.g. [46, Section 3.3.3]) on the basis of Lemma (11.1). □

Finally we shall need the following interpolation result.

(11.8) THEOREM. *Let* $1 < p < \infty$ *and* $0 < s_0$, $s_1 \leqq \hat{l}$ *with* $s_0 \neq s_1$ *and* s_0, $s_1 \in \mathbb{N}$. *Moreover, let* $0 < \Theta < 1$ *and* $s := (1 - \Theta)s_0 + \Theta s_1$. *Then*

$$\left(W_p^{s_0-1/p}(\Gamma, E), W_p^{s_1-1/p}(\Gamma, E)\right)_{\Theta,p} = W_p^{s-1/p}(\Gamma, E)$$

if $s, s - 1/p \notin \mathbb{N}$, *and*

$$[W_p^{s_0-1/p}(\Gamma, E), W_p^{s_1-1/p}(\Gamma, E)]_\Theta = W_p^{s-1/p}(\Gamma, E)$$

if $s \in \mathbb{N}$.

PROOF. This is a consequence of Theorems (11.6) and (11.7) and of [45, Theorem 1.2.4] (cf. also [9, Theorem 6.4.2]). □

It should be noted that this last result generalizes corresponding theorems of Lions and Magenes (cf. [31, p. 41]) to the case where Γ may be unbounded.

12. – A priori estimates for elliptic systems.

In the remainder of this chapter we let $m, N \in \mathbb{N}^*$ and put $E := \mathbb{K}^N$ and $\hat{l} := 2m + l$ for some $l \in \mathbb{N}$. Thus, in particular, $\Omega \subset \mathbb{R}^n$ is uniformly regular of class $2m + l$.

We denote by $\mathcal{A}$ a linear differential operator of order $2m$ of the form

$$\mathcal{A}u := (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha u \,,$$

acting on N-tuples of $\mathbb{K}$-valued functions, that is, on E-valued functions $u \colon \Omega \to E$. We assume throughout that

$$(1) \qquad a_\alpha \in BUC^l(\Omega, \mathcal{L}(E)) \,, \qquad |\alpha| \leq 2m \,.$$

For each $(\theta, x, \xi, t) \in \mathbb{R} \times \Omega \times \mathbb{R}^n \times \mathbb{R}$ we let

$$a_\theta(x, \xi, t) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha + \exp[i\theta] t^{2m} I_N \,,$$

where I_N denotes the N-dimensional identity matrix. We say that $\mathcal{A}$ satisfies the *α-root condition*, where $0 < \alpha < \pi/2$, if there exists a constant $c_0 > 0$ such that

$$(2) \qquad |\det a_\theta(x, \xi, t)| \geq c_0 \big(|\xi|^2 + t^2\big)^{mN}$$

for all $(\theta, x, \xi, t) \in [-\alpha - \pi/2, \alpha + \pi/2] \times \Omega \times \mathbb{R}^n \times \mathbb{R}$, and if the polynomial

$$\tau \mapsto \det a_\theta\big(x, \xi + \tau\nu(x), t\big)$$

has for each $(\theta, x, \xi, t) \in [-\alpha - \pi/2, \alpha + \pi/2] \times \partial\Omega \times \mathbb{R}^n \times \mathbb{R}$ with $(\xi, t) \neq (0, 0)$ and $\big(\xi | \nu(x)\big) = 0$ precisely mN roots $\tau_j^+(\theta, x, \xi, t)$, $j = 1, \ldots, mN$, with positive imaginary part.

Observe that (1) and (2) imply the existence of a further constant c_1 such that

$$c_0\big(|\xi|^2 + t^2\big)^{mN} \leq |\det a_\theta(x, \xi, t)| \leq c_1\big(|\xi|^2 + |t|^2\big)^{mN}$$

for all $(\theta, x, \xi, t) \in [-\alpha - \pi/2, \alpha + \pi/2] \times \Omega \times \mathbb{R}^n \times \mathbb{R}$. Thus the (system of) differential operator(s)

$$\mathcal{A}_\theta := \mathcal{A} + (-1)^m \exp[i\theta] D_t^{2m}$$

is for each $\theta \in [-\alpha - \pi/2, \alpha + \pi/2]$ uniformly elliptic on $\Omega \times \mathbb{R}$ (e.g. [3 II, p. 39], [36, Definition 6.1.2]).

Let Γ be a nonempty union of components of $\partial\Omega$ and let $\{(U_j, \varphi_j, \pi_j): j \in \mathbf{N}\}$ be a regular localization system for Ω. Then we denote by $BUC^k(\Gamma, X)$, $0 \leqq k \leqq 2m + l$, the vector spaces of all $v \in C^k(\Gamma, X)$ such that

$$(3) \qquad \|v\|_{C^k(\Gamma,X)} := \sup\{\|v\circ\varphi_j^{-1}\|_{C^k(\Sigma^{n-1},X)}: j\in\mathbf{N} \text{ with } U_j \cap \Gamma \neq \emptyset\} < \infty$$

and such that the family

$$\{D^\beta(v\circ\varphi_j^{-1}): \beta\in\mathbf{N}^{n-1}, |\beta|\leqq k, j\in\mathbf{N} \text{ with } U_j\cap\Gamma\neq\emptyset\}$$

is uniformly equicontinuous. It is easily verified that different regular localization systems for Ω induce equivalent norms (3) on $BUC^k(\Gamma, X)$.

We denote by $\mathcal{B}_\Gamma := (\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{mN})$ a *boundary operator on* Γ, that is,

$$\mathcal{B}_\Gamma^\varrho u := \sum_{|\alpha|\leqq m_{\varrho,\Gamma}} b_{\alpha,\Gamma}^\varrho D^\alpha u := \sum_{|\alpha|\leqq m_{\varrho,\Gamma}} b_{\alpha,\Gamma}^\varrho \gamma_\Gamma(D^\alpha u),$$

where $0 \leqq m_{\varrho,\Gamma} < 2m$ and

$$(4) \qquad b_{\alpha,\Gamma}^\varrho \in BUC^{2m+l-m_{\varrho,\Gamma}}(\Gamma, \mathcal{L}(\mathbf{K}^N, \mathbf{K}))$$

for $\alpha\in\mathbf{N}^n$ with $|\alpha|\leqq m_{\varrho,\Gamma}$ and $1\leqq\varrho\leqq mN$. We let

$$b_\Gamma^\varrho(x, \xi) := \sum_{|\alpha|=m_{\varrho,\Gamma}} b_{\alpha,\Gamma}^\varrho(x)\xi^\alpha$$

and denote by $b_\Gamma(x, \xi)$ the $(mN\times N)$-matrix with rows $b_\Gamma^\varrho(x, \xi)$.

Recall that the $(N\times N)$-matrix $\tilde{C}$ algebraically adjoint to the $(N\times N)$-matrix C is defined as follows: denote by C_{jk} the determinant of the $(N-1)\times(N-1)$-matrix which is obtained by deleting the j-th row and k-th column of C. Then $(-1)^{j+k}C_{jk}$ is the element of $\tilde{C}$ at the position (j, k). It is well known that $\tilde{C}C = C\tilde{C} = (\det C)I_N$. Hence if $N=1$ and $C\in\mathbf{K}^*$ we define $\tilde{C}$ consistently by $\tilde{C} := 1$.

The boundary operator $\mathcal{B}_\Gamma$ is said to satisfy the *α-complementing condition* with respect to $\mathcal{A}$ if, for every $(\theta, x, \xi, t)\in[-\alpha-\pi/2, \alpha+\pi/2]\times\partial\Omega\times\mathbf{R}^n\times\mathbf{R}$ with $(\xi, t)\neq(0, 0)$ and $(\xi|\nu(x)) = 0$, the rows of the matrix-valued function

$$\tau \mapsto b_\Gamma(x, \xi+\tau\nu(x))\tilde{a}_\theta(x, \xi+\tau\nu(x), t)$$

are linearly independent modulo $\prod_{j=1}^{mN}(\tau-\tau_j^+(\theta, x, \xi, t))$ (as polynomials in τ). Here $\tilde{a}_\theta(x, \eta, t)$ is the matrix algebraically adjoint to $a_\theta(x, \eta, t)$.

Finally we write

$$b_\Gamma(x, \xi + \tau\nu(x))\tilde{a}_\theta(x, \xi + \tau\nu(x), t) \equiv \sum_{j=0}^{mN-1} Q_j^\Gamma(\theta, x, \xi, t)\tau^j\Big(\operatorname{mod} \prod_{j=1}^{mN} (\tau - \tau^+(\theta, x, \xi, t))\Big)$$

and define the $(mN \times mN^2)$-matrix Q^Γ by

$$Q^\Gamma(\theta, x, \xi, t) := [Q_0(\theta, x, \xi, t), \ldots, Q_{mN-1}(\theta, x, \xi, t)] .$$

Then the *minor constant* δ_0^Γ *on* Γ is given by

$$\delta_0^\Gamma := \inf \max_\sigma |\det \hat{Q}_\sigma^\Gamma(\theta, x, \xi, t)| ,$$

where $\hat{Q}_\sigma^\Gamma$ denote the various $(mN \times mN)$-submatrices of Q^Γ and where the infimum is taken with respect to all $(\theta, x, \xi, t) \in [-\alpha - \pi/2, \alpha + \pi/2] \times \Gamma \times \mathbb{R}^n \times \mathbb{R}$ satisfying $(\xi|\nu(x)) = 0$ and $|(\xi, t)| = 1$. Observe that, due to the α-complementing condition, each one of the matrices $Q^\Gamma(\theta, x, \xi, t)$ has rank mN. Hence $\delta_0^\Gamma > 0$ *if* Γ *is compact.*

In the following we denote by $\mathbf{\Gamma}$ a finite set with the following properties:

(i) $\mathbf{\Gamma} = \emptyset$ if $\Omega = \mathbb{R}^n$.

(ii) If $\Omega \neq \mathbb{R}^n$ each $\Gamma \in \mathbf{\Gamma}$ is a nonempty union of components of $\partial\Omega$; the elements of $\mathbf{\Gamma}$ are pairwise disjoint and their union covers $\partial\Omega$.

Finally $(\mathcal{A}, \mathcal{B}, \Omega, \mathbf{\Gamma}, \alpha)$ is said to be an *α-regular elliptic boundary value problem* (BVP) of class C^l and order $2m$ provided:

(i) $0 < \alpha < \pi/2$.

(ii) $\mathcal{A}$ satisfies the α-root condition and the regularity assumption (1).

(iii) For each $\Gamma \in \mathbf{\Gamma}$ there is given a boundary operator $\mathcal{B}_\Gamma$ satisfying the regularity assumption (4) and the α-complementing condition with respect to $\mathcal{A}$ on Γ.

(iv) The minor constant $\delta_0 := \min\{\delta_0^\Gamma : \Gamma \in \mathbf{\Gamma}\}$ is positive.

Clearly, if $\Omega = \mathbb{R}^n$ there is no boundary operator and we write simply $(\mathcal{A}, \mathbb{R}^n, \alpha)$ in this case. In general $\mathcal{B} := \{\mathcal{B}_\Gamma : \Gamma \in \mathbf{\Gamma}\}$.

Throughout the remainder of this section we assume that $(\mathcal{A}, \mathcal{B}, \Omega, \mathbf{\Gamma}, \alpha)$ is an α-regular elliptic *BVP* of class C^l and order $2m$ for some α. Moreover, for each $s \in [2m, 2m + l]$ we let

$$\mathfrak{W}_p^{s-1/p} := \mathfrak{W}_p^{s-1/p}(\mathbf{\Gamma}, \mathbb{K}) := \prod_{\Gamma \in \mathbf{\Gamma}} \prod_{\varrho=1}^{mN} W_p^{s-1/p-m_{\varrho,\Gamma}}(\Gamma, \mathbb{K}^N) ,$$

where $\mathfrak{W}_p^{s-1/p} := \{0\}$ if $\Gamma = \emptyset$.

Observe that the boundary operator $\mathcal{B}$ is naturally extended to $\partial\Omega\times\mathbf{R}$ by letting $\mathcal{B}$ operate on the variable $x\in\partial\Omega$ and considering $t\in\mathbf{R}$ as a parameter. This fact will be used in the following without further mention. Moreover we let $\mathbf{\Gamma}\times\mathbf{R}:=\{\Gamma\times\mathbf{R}\colon \Gamma\in\mathbf{\Gamma}\}$.

After these preparations the following important a priori estimate is an easy consequence of the general results due to Agmon-Douglis-Nirenberg [3].

(12.1) THEOREM. *For each $p\in(1,\infty)$ there exists a constant c^* such that*

$$\|u\|_{2m+s}^{\Omega}\leqq c^*\Big(\|\mathcal{A}u\|_s^{\Omega}+\|\mathcal{B}u\|_{\mathfrak{W}_p^{2m+s-1/p}}+\|u\|_s^{\Omega}\Big) \tag{5}$$

for all $u\in W_p^{2m+s}(\Omega,\mathbf{C}^N)$ and such that

$$\|u\|_{2m+s}^{\Omega\times\mathbf{R}}\leqq c^*\Big(\|\mathcal{A}u\|_s^{\Omega\times\mathbf{R}}+\|\mathcal{B}u\|_{\mathfrak{W}_p^{2m+s-1/p}(\mathbf{\Gamma}\times\mathbf{R},\,\mathbf{C}^N)}+\|u\|_s^{\Omega\times\mathbf{R}}\Big) \tag{6}$$

for all $u\in W_p^{2m+s}(\Omega\times\mathbf{R},\mathbf{C}^N)$ and $s=0,1,\dots,l$. Moreover c^* depends only Ω, l, m, n, p, α, the moduli of continuity of the top order coefficients of $\mathcal{A}$, and on a bound for c_0^{-1}, δ_0^{-1}, the C^l-norms of the coefficients of $\mathcal{A}$ and the $C^{2m+l-m_{\varrho,\Gamma}}$-norms of the coefficients of $\mathcal{B}_\Gamma^\varrho$, $1\leqq\varrho\leqq mN$, $\Gamma\in\mathbf{\Gamma}$.

PROOF. It is easily seen that $\Omega\times\mathbf{R}$ is a uniformly regular open subset of $\mathbf{R}^{n+1}$ of class C^{2m+l}. Moreover our assumptions imply that the elliptic systems $(\mathcal{A},\mathcal{B},\Omega,\mathbf{\Gamma})$ and $(\mathcal{A}_\theta,\mathcal{B},\Omega\times\mathbf{R},\mathbf{\Gamma}\times\mathbf{R})$, $|\theta|\leqq\alpha+\pi/2$, satisfy the hypotheses of the basic paper [3 II] (cf. also [36, Chapter 6]). Hence the assertions follow from [3 II, Theorem 10.3 and 10.4] by means of regular localization systems for Ω and $\Omega\times\mathbf{R}$, respectively, and by taking into consideration Lemma (11.1) and Theorem (11.7). □

Using an idea due to Agmon [2] we can prove the following fundamental a priori estimate

(12.2) THEOREM. *For each $p\in(1,\infty)$ there exist positive constants c and λ_0, depending on the same quantities as c^*, such that*

$$\sum_{k=0}^{2m+s}|\lambda|^{1+(s-k)/2m}\|u\|_{k,p}^{\Omega}\leqq c\Big(\|(\lambda+\mathcal{A})u\|_{s,p}^{\Omega}+|\lambda|^{s/2m}\|(\lambda+\mathcal{A})u\|_{0,p}^{\Omega}$$
$$+\|\mathcal{B}u\|_{\mathfrak{W}_p^{2m+s-1/p}}+\sum_{\Gamma\in\mathbf{\Gamma}}\sum_{\varrho=1}^{mN}|\lambda|^{1+(s-m_{\varrho,\Gamma})/2m}\|\tilde{\mathcal{B}}_\Gamma^\varrho u\|_{0,p}^{\Omega}\Big),$$

for all $u\in W_p^{2m+s}(\Omega,\mathbf{C}^N)$, all $\lambda\in S_{\alpha+\pi/2}$ with $|\lambda|\geqq\lambda_0$, and $s=0,1,\dots,l$, *where $\tilde{\mathcal{B}}$ is an arbitrary extension of $\mathcal{B}$ over Ω possessing the same smoothness properties as $\mathcal{B}$. If $\mathcal{B}u = 0$ the last sum can be dropped.*

PROOF. Fix $\varphi \in \mathfrak{D}(\mathbf{R}, \mathbf{R})$ with $\operatorname{supp}(\varphi) \subset (-1, 1)$ and $\varphi|[-\frac{1}{2}, \frac{1}{2}] = 1$, let $r \geq 1$, $\theta \in [-\alpha - \pi/2, \alpha + \pi/2]$ and $u \in W_p^{2m+s}(\Omega, \mathbf{C})$, and put $v(x, t) := \varphi(t) \exp[irt]u(x)$. Then

$$\mathcal{A}_\theta v = \varphi \exp[irt][\mathcal{A}u + (-1)^m \exp[i\theta](ir)^{2m}u] + (-1)^m \exp[i\theta]u \exp[irt] \sum_{k=0}^{2m-1} \binom{2m}{k} (ir)^k D^{2m-k}\varphi .$$

Hence

$$(7) \qquad \|\mathcal{A}_\theta v\|_{s,p}^{\Omega\times\mathbf{R}} \leqq \|\varphi \exp[irt](\mathcal{A}u + r^{2m}\exp[i\theta]u)\|_{s,p}^{\Omega\times\mathbf{R}} + \sum_{k=0}^{2m-1} \binom{2m}{k} r^k \|\exp[irt]uD^{2m-k}\varphi\|_{s,p}^{\Omega\times\mathbf{R}}.$$

If $\varphi \in \mathfrak{D}(\mathbf{R}, \mathbf{C})$ satisfies $\operatorname{supp}(\varphi) \subset (-1, 1)$ then

$$\|\exp[irt]\varphi u\|_{s,p}^{\Omega\times\mathbf{R}} = \left(\sum_{j+|\alpha|\leqq s} \int_{-1}^{1} \int_{\Omega} |D^j(\exp[irt]\varphi) D^\alpha u|^p dx\, dt \right)^{1/p} \leqq c \sum_{j+|\alpha|\leqq s} r^j \|D^\alpha u\|_{0,p}^{\Omega} \leqq c \sum_{\sigma=0}^{s} r^{s-\sigma} \|u\|_{\sigma,p}^{\Omega} ,$$

where c depends only on φ, l and p. By applying these estimates to the right-hand side of (7) we see that

$$(8) \qquad \|\mathcal{A}_\theta v\|_{s,p}^{\Omega\times\mathbf{R}} \leqq c \sum_{0=\sigma}^{s} \left(r^{s-\sigma} \|\mathcal{A}u + r^{2m}\exp[i\theta]u\|_{\sigma,p}^{\Omega} + \sum_{k=0}^{2m-1} r^{k+s-\sigma} \|u\|_{\sigma,p}^{\Omega} \right).$$

By Theorem (11.6) and (*JPF* iii)

$$(9) \qquad \|u\|_{\sigma,p} \leqq c \|u\|_{s,p}^{\sigma/s} \|u\|_{0,p}^{1-\sigma/s}$$

for $0 < \sigma < s$ and $\sigma \in \mathbf{N}$. Hence, by applying (9) and Young's inequality $\xi\eta \leqq (1/q)\xi^q + (1/q')\xi^{q'}$ with $q := s/\sigma$ and $q' := q/(q-1) = s/(s-\sigma)$ to (8) we see that

$$(10) \qquad \|\mathcal{A}_\theta v\|_{s,p}^{\Omega\times\mathbf{R}} \leqq c(\|\mathcal{A}u + r^{2m}\exp[i\theta]u\|_{s,p}^{\Omega} + r^s \|\mathcal{A}u + r^{2m}\exp[i\theta]u\|_{0,p}^{\Omega} + r^{2m-1}\|u\|_{s,p}^{\Omega} + r^{2m-1+s}\|u\|_{0,p}^{\Omega}) .$$

Similarly Theorem (11.7) and the above arguments imply

$$\begin{aligned}\|\mathcal{B}_\Gamma^\varrho v\|_{2m+s-\mu-1/p,p}^{\Gamma\times\mathbf{R}} &\leq c\|\tilde{\mathcal{B}}_\Gamma^\varrho v\|_{2m+s-\mu,p}^{\Omega\times\mathbf{R}} \\ &\leq c\sum_{k=0}^{2m+s-\mu} r^{2m+s-\mu-k}\|\tilde{\mathcal{B}}_\Gamma^\varrho u\|_{k,p}^{\Omega} \\ &\leq c\left(\|\tilde{\mathcal{B}}_\Gamma^\varrho u\|_{2m+s-\mu,p}^{\Omega}+r^{2m+s-\mu}\|\tilde{\mathcal{B}}_\Gamma^\varrho u\|_{0,p}^{\Omega}\right),\end{aligned}$$

where $\mu := m_{\varrho,\Gamma}$.

On the other hand there exists a positive constant $\bar{c}$ such that

$$\|v\|_{2m+s,p}^{\Omega\times\mathbf{R}} \geq \left(\sum_{j+|\alpha|\leq 2m+s}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\Omega}|D^j(\exp[irt])D^\alpha u|^p\,dx\,dt\right)^{1/p} \geq \bar{c}\sum_{k=0}^{2m+s} r^{2m+s-k}\|u\|_{k,p}^{\Omega}.$$

Now we can apply (6) and observe that for sufficiently large r the third and the fourth term on the right-hand side of (10) can be absorbed by the left-hand side of the resulting inequality. Thus, letting finally $\lambda := r^{2m}\exp[i\theta]$, the assertion follows. $\square$

In the special case that Ω is bounded and $N = 1$, Theorem (12.2) has first been proven by Agmon [2, formula (2.11)] for $l = 0$ and $\mathcal{B}u = 0$. Agmon's result has been extended to arbitrary l (and for $p = 2$) in [32 II, Theorem IV.5.1]. The extension of Agmon's result to general $u \in W_p^{2m}(\Omega, \mathbf{C})$—that is, for Ω bounded and $l = 0$—is due to Tanabe [43, Lemma 3.8.1].

13. – Elliptic boundary value problems.

Let $(\mathcal{A}, \mathcal{B}, \Omega, \Gamma, \alpha)$ be an α-regular elliptic BVP of class C^l and order $2m$. Then it follows that

$$(1)\qquad (\mathcal{A}, \mathcal{B}) \in \mathfrak{L}(W_p^{2m+s}, W_p^s \times \mathfrak{W}_p^{2m+s-1/p})$$

for $1 < p < \infty$ and $s = 0, 1, 2, \ldots, l$. Moreover, it follows from Theorem (12.2) that $(\lambda + \mathcal{A}, \mathcal{B})$ is injective for every $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geq \lambda_0 =: \lambda_0(p)$.

In the following $(\mathcal{A}, \mathcal{B}, \Omega, \mathbf{\Gamma}, \alpha)$ is said to be a *strongly α-regular elliptic BVP of class C^l and order $2m$* if it is an α-regular elliptic BVP of class C^l and order $2m$ and if, for each $p \in (1, \infty)$, there exists $\lambda \geq \lambda_0(p)$ such that $(\lambda + \mathcal{A}, \mathcal{B})$ maps W_p^{2m} onto $L_p \times \mathfrak{W}_p^{2m-1/p}$.

(13.1) THEOREM. *Let $(\mathcal{A}, \mathcal{B}, \Omega, \Gamma, \alpha)$ be a strongly α-regular elliptic BVP of class C^l and order $2m$. Then there exists for each $p \in (1, \infty)$ a number $\lambda_0(p) \in \mathbb{R}^+$ such that*

$$(\lambda + \mathcal{A}, \mathcal{B}) \in \mathrm{Isom}(W_p^{2m+s}, W_p^s \times \mathfrak{W}_p^{2m+s-1/p})$$

for all $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0(p)$ and all $s \in [0, l]$ with $s - 1/p \notin \mathbb{N}$. For $\lambda_0(p)$ one can choose the corresponding number of Theorem (12.2).

PROOF. Let $p \in (1, \infty)$ be fixed. By assumption there exists $\mu \geqq \lambda_0$ such that $(\mu + \mathcal{A}, \mathcal{B})$ maps W_p^{2m} onto $L_p \times \mathfrak{W}_p^{2m-1/p}$. Since $(\mu + \mathcal{A}, \mathcal{B})$ is injective, the open mapping theorem implies that

$$(\mu + \mathcal{A}, \mathcal{B}) \in \mathrm{Isom}(W_p^{2m}, L_p \times \mathfrak{W}_p^{2m-1/p}).$$

Hence it follows in particular that $\mathcal{B}$ is a continuous surjection from W_p^{2m} onto $\mathfrak{W}_p^{2m-1/p}$ and that

$$\mu + A \in \mathrm{Isom}(W_{p,\mathcal{B}}^{2m}, L_p),$$

where $W_{p,\mathcal{B}}^{2m} := \{u \in W_p^{2m}: \mathcal{B}u = 0\}$ and $A := \mathcal{A}|W_{p,\mathcal{B}}^{2m}$. From Theorem (12.2) we deduce that $\|(\mu + A)^{-1}\|_{\mathcal{L}(L_p)} \leqq c/|\mu|$. This implies that the open disc in $\mathbb{C}$ with center μ and radius $|\mu|/c$ belongs to $\varrho(-A)$ (e.g. [52, Theorem VIII.2.1]). Since this argument can be applied to every $\lambda \in \varrho(-A) \cap S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0$ it follows that $\lambda + A \in \mathrm{Isom}(W_{p,\mathcal{B}}^{2m}, L_p)$ for every $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0$. Consequently $(\lambda + \mathcal{A}, \mathcal{B})$ is a bijection from W_p^{2m} onto $L_p \times \mathfrak{W}_p^{2m-1/p}$ for every $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0$. Now the assertion with $s = 0$ follows from the open mapping theorem.

Suppose now that $(f, g) \in W_p^l \times \mathfrak{W}_p^{2m+l-1/p}$ and let $u := (\lambda + \mathcal{A}, \mathcal{B})^{-1}(f, g) \in W_p^{2m}$, where $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0$ is fixed. Then we can localize the problem on the basis of Lemma (11.1) and apply [3 II, Theorems 10.3 and 10.4] to deduce that $u \in W_p^{2m+l}$. This shows that $(\lambda + \mathcal{A}, \mathcal{B})$ maps W_p^{2m+l} onto $W_p^l \times \mathfrak{W}_p^{2m+l-1/p}$ if $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0$: Since this map is also injective (by Theorem (12.2)) and continuous (by (11)), the assertion for $s := l$ follows again from the open mapping theorem.

The general case is now an easy consequence of Theorems (11.6) and (11.8). $\square$

In the following we let

$$W_{p,\mathcal{B}}^{2m+s} := \{u \in W_p^{2m+s}: \mathcal{B}u = 0\} \tag{2}$$

and we define the *W^s_p-realization of* $(\mathcal{A}, \mathcal{B}, \Omega, \mathbf{\Gamma}, \alpha)$,

$$A_{s,p}\colon D(A_{s,p}) \subset W^s_p \to W^s_p,$$

by

$$D(A_{s,p}) := W^{2m+s}_{p,\mathcal{B}} \quad \text{and} \quad A_{s,p}u := \mathcal{A}u$$

for $1 < p < \infty$ and $s \in [0, l]$. Thus $W^{2m+s}_{p,\mathcal{B}}$ is a closed linear subspace of W^{2m+s}_p and $A_{s,p}$ is an unbounded closed linear operator in W^s_p. Moreover it follows from Theorem (13.1) that

$$\varrho(-A_{s,p}) \supset \{\lambda \in S_{\alpha+\pi/2}\colon |\lambda| \geqq \lambda_0(p)\} \tag{3}$$

for every $p \in (1, \infty)$ and every $s \in [0, l]$ with $s - 1/p \notin \mathbb{N}$.

In the following we let $A_p := A_{0,p}$.

(13.2) LEMMA. *Suppose that* $1 < p < q < \infty$. *Then*

$$(\lambda + A_p)^{-1}|L_p \cap L_q = (\lambda + A_q)^{-1}|L_p \cap L_q$$

for every $\lambda \in S_{\alpha+\pi/2}$ *with* $|\lambda| \geqq \max\{\lambda_0(p), \lambda_0(q)\}$.

PROOF. Let $r := q$ if $n \leqq 2m$ and $r := \min\{q, np/(n-2m)\}$ otherwise. Then $W^{2m}_p \hookrightarrow L_r$ by Theorem (11.5).

Let $v \in L_p \cap L_q$ and let $u := (\lambda + A_p)^{-1}v$. Then $u \in W^{2m}_p \hookrightarrow L_r$. Hence $w := v - \lambda u \in L_r$ by Hölder's inequality. Choose now $\mu \in \varrho(-A_r)$. Then $u = (\mu + A_r)^{-1}(v + (\mu - \lambda)u) \in W^{2m}_r$. By repeating this argument a finite number of times we see that $u \in W^{2m}_p$. This implies the assertion. □

Generalizing (2) we define

$$W^s_{p,\mathcal{B}} := W^s_{p,\mathcal{B}}(\Omega, \mathbb{K}^N) := \{u \in W^s_p\colon \mathcal{B}^\varrho_\Gamma u = 0 \text{ for } m_{\varrho,\Gamma} < s - 1/p\} \tag{4}$$

for $1 < p < \infty$ and $0 \leqq s \leqq 2m$. Observe that $W^s_{p,\mathcal{B}}$ is a closed linear subspace of W^s_p and that

$$W^s_{p,\mathcal{B}} = W^s_p \quad \text{for } 0 \leqq s \leqq \hat{m}_{\partial\Omega} + 1/p, \tag{5}$$

where $\hat{m}_{\partial\Omega} := \min\{m_{\varrho,\Gamma} | 1 \leqq \varrho \leqq mN, \Gamma \in \mathbf{\Gamma}\}$.

The following important theorem is a consequence of results due to Grisvard [23, Theorem 7.5] and Seeley [38, Theorem 4.1]. The first author considered the real interpolation method for a bounded domain and $N = 1$, and the second author the complex interpolation method, also for a bounded domain but with an arbitrary $N \in \mathbb{N}^*$.

(13.3) THEOREM. *Suppose that* $1 < p < \infty$ *and* $0 < \Theta < 1$. *Moreover suppose that*

$$m_{\varrho,\Gamma} \neq 2m\Theta - 1/p \quad \text{for } \varrho = 1, \dots, mN,\ \Gamma \in \mathbf{\Gamma}. \tag{6}$$

Then

$$(L_p, W^{2m}_{p,\mathfrak{B}})_{\Theta,p} = W^{2m\Theta}_{p,\mathfrak{B}} \quad \text{if } 2m\Theta \notin \mathbb{N}$$

and

$$[L_p, W^{2m}_{p,\mathfrak{B}}]_\Theta = W^{2m\Theta}_{p,\mathfrak{B}} \quad \text{if } 2m\Theta \in \mathbb{N}.$$

PROOF. It follows from [3 II, Theorem 3.2] and from the considerations in [38, Section 3] that the boundary operators $\mathfrak{B}$ form a normal system in the sense of [38, Definition 3.1]. Now it is easily checked that, due to Lemmas (11.1) and (11.2), the proof of [38, Theorem 4.1] carries over to our situation. Hence

$$[L_p, H^{2m}_{p,\mathfrak{B}}]_\Theta = H^{2m\Theta}_{p,\mathfrak{B}}$$

if (6) is satisfied, where $H^s_{p,\mathfrak{B}}$ is defined in analogy to (4).

Since $\mathfrak{B}$ is a normal system in the sense of Seeley it is not difficult to see that the proof of [23, Theorem 7.5] carries over to the case $N > 1$ if $\bar{\Omega}$ is compact. Due to Lemmas (11.1) and (12.2) it is also not difficult to see that Grisvard's arguments work if Ω is unbounded. Thus

$$(L_p, W^{2m}_{p,\mathfrak{B}})_{\Theta,p} = B^{2m\Theta}_{p,p,\mathfrak{B}}$$

if (6) is satisfied, where the Besov space $B^s_{p,p,\mathfrak{B}}$ is again defined in analogy to (4). Now the assertion follows from Theorem (11.3). □

After these preparations we can now prove the main result of this section.

(13.4) THEOREM. *Suppose that* $(\mathcal{A}, \mathfrak{B}, \Omega, \mathbf{\Gamma}, \alpha)$ *is a strongly α-regular elliptic BVP of class* C^0 *and order* $2m$.

For each $p \in (1, \infty)$ *the operator*—A_p *generates a strongly continuous analytic semigroup* $\{\exp[-tA_p] : t \geqq 0\}$ *in* $\mathfrak{L}(L_p)$ *such that*

$$\exp[-tA_p]|L_p \cap L_q = \exp[-tA_q]|L_p \cap L_q$$

for all $t \geqq 0$ *and* $1 < p, q < \infty$. *Moreover, if* $0 < s \leqq 2m$ *and*

$$s - 1/p \neq m_{\varrho,\Gamma} \quad \text{for } \varrho = 1, \dots, mN,\ \Gamma \in \mathbf{\Gamma} \tag{7}$$

then $\{\exp[-tA_p]: t \geqq 0\}$ *restricts to a strongly continuous analytic semigroup on* $W^s_{p,\mathfrak{B}}$ *whose infinitesimal generator is the* $W^s_{p,\mathfrak{B}}$*-realization of* $-A_p$.

PROOF. Let $p \in (1, \infty)$ be fixed and choose $\mu > \lambda_0(p)$ such that

$$\varrho(-(\mu + A_p)) \supset S_{\alpha+\pi/2},$$

which is possible by Theorem (13.1). Then Theorem (12.2) implies the estimate

$$\|(\lambda + (\mu + A_p))^{-1}\|_{\mathfrak{L}(L_p)} \leqq \frac{c}{1+|\lambda|} \qquad \forall \lambda \in S_{\alpha+\pi/2}.$$

Since $\mathfrak{D}(\Omega, \mathbf{K}^N) \subset W^{2m}_{p,\mathfrak{B}} = D(A_p)$ it follows that $D(A_p)$ is dense in L_p. Hence $-A_p$ is the infinitesimal generator of a strongly continuous analytic semigroup which can be represented by

$$\exp[-tA_p] = \frac{1}{2\pi i}\int_\Gamma \exp[\lambda t](\lambda + A_p)^{-1} d\lambda \tag{8}$$

(cf. (6.5)). The same argument applies to any other $q \in (1, \infty)$. Since we can choose μ so large that (8) holds also for $\exp[-tA_q]$, we see from Lemma (13.2) that the first assertion is true.

For the second assertion let $X := L_p$, $X^1 := W^{2m}_p$ and $\mathcal{F}_\eta := (\cdot,\cdot)_{\eta,p}$ if $2m\eta \notin \mathbf{N}$ and $\mathcal{F}_\eta := [\cdot,\cdot]_\eta$ if $2m\eta \in \mathbf{N}$. Then $X^\eta = W^{2m\eta}_p$ by Theorem (11.6) and $X^\eta(0) = W^{2m\eta}_{p,\mathfrak{B}}$ if $s := 2m\eta$ satisfies (7) by Teorem (13.3). This shows that $\mathcal{F}_\eta$ is admissible if $\eta := s/2m$ and s satisfies (7). Now the second assertion is an easy consequence of Lemma (10.1) and the above arguments. □

In general it is not easy to verify that a given system $(\mathcal{A}, \mathfrak{B}, \Omega, \mathbf{\Gamma})$ satisfies the hypotheses guaranteeing that $(\mathcal{A}, \mathfrak{B}, \Omega, \mathbf{\Gamma}, \alpha)$ is a strongly α-regular elliptic *BVP* for some $\alpha \in (0, \pi/2)$. For this reason the following remarks are added.

The differential operator

$$\mathcal{A}u := (-1)^m \sum_{|\alpha| \leqq 2m} a_\alpha D^\alpha u$$

is said to be *uniformly strongly elliptic* if there exists a constant $c > 0$ such that

$$\operatorname{Re}\left(a(x, \xi)\eta \mid \eta\right) \geqq c|\xi|^{2m}|\eta|^2$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R}^n \times \mathbf{C}^N$. It is not too difficult to show that *a uniformly* *strongly elliptic differential operator satisfies the α-root condition for some* $\alpha \in (0, \pi/2)$. In fact, if β is the smallest number such that

$$\sigma\Big(\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha\Big) \subset S_\beta \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n \text{ with } |\xi| = 1,$$

then one can choose for α any positive number strictly smaller than $(\pi/2) - \beta$. In particular, if $\mathcal{A}$ has a hermitian principal part and is uniformly strongly elliptic then it satisfies the α-root condition for every $\alpha \in (0, \pi/2)$.

Suppose now that for each $\Gamma \in \mathbf{\Gamma}$ there are given m vector fields

$$\beta_j \in BUC^{2m+l-1}(\Gamma, \mathbb{R}^n), \qquad j = 1, \dots, m,$$

a positive constant c such that

$$\big(\beta_j(x) \,|\, \nu(x)\big) \geqq c \qquad \forall x \in \Gamma, \ j = 1, \dots, m,$$

and an integer $k \in \{0, \dots, m\}$ such that

$$\mathcal{B}_\Gamma u = \left\{ \frac{\partial^{k+j-1} u}{\partial \beta_j^{k+j-1}} \,\middle|\, j = 1, \dots, m \right\}. \tag{9}$$

Observe that in particular the *Dirichlet boundary operator* $\{u, \partial u/\partial \nu, \dots, (\partial^{m-1} u)/\partial \nu^m\}$ is of this form. Then it can be shown that $\mathcal{B}_\Gamma$ satisfies the α-complementing condition for some $\alpha \in (0, \pi/2)$ with respect to any $\mathcal{A}$ satisfying the α-root condition. Of course, it is also possible to add to each one of the differential operators making up $\mathcal{B}_\Gamma$ in (9) a lower order boundary operator.

Using these facts one can show that *there exists* $\alpha \in (0, \pi/2)$ *such that* $(\mathcal{A}, \mathcal{B}, \Omega, \mathbf{\Gamma}, \alpha)$ *is a strongly α-regular elliptic BVP provided $\mathcal{A}$ is uniformly strongly elliptic and each $\mathcal{B}_\Gamma$ is of the form* (9), *up to additional lower order boundary operators.*

Observe that this fact is well known if Ω is bounded and $N = 1$ and if the data are sufficiently smooth. If the data satisfy only the regularity assumptions specified above an approximation argument yields the desired result in this case (cf. the considerations in [43, Sections 3.7 and 3.8]). If Ω is unbounded and $N = 1$ then the assertion can be established on the basis of Lemmas (11.1) and (11.2) by appropriate duality arguments. Finally the case $N > 1$ can be reduced to the scalar case by an appropriate homotopy argument. Proofs for these facts as well as for different situations will be given elsewhere.

14. – Parabolic initial-boundary value problems.

We denote now for each $t \in [0, T]$ by $\mathcal{A}(t)$ a differential operator of the form

$$\mathcal{A}(t)u := (-1)^m \sum_{|\alpha| \leqq 2m} a_\alpha(\cdot, t) D^\alpha u$$

and by $\mathcal{B}(t)$ a boundary operator of the form

$$\mathcal{B}^\varrho_\Gamma(t)u := \sum_{|\alpha| \leqq m_{\varrho,\Gamma}} b^\varrho_{\alpha,\Gamma}(\cdot, t) D^\alpha u\,, \qquad \varrho = 1, \ldots, mN,\ \Gamma \in \mathbf{\Gamma}\,.$$

Then $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbf{\Gamma})$, $t \in [0, T]$, is said to be a *regular parabolic initial-boundary value problem* (*IBVP*) *of class* C^l *and order* $2m$ provided:

(i) There exists $\alpha \in (0, \pi/2)$ such that $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbf{\Gamma}, \alpha)$ is for each $t \in [0, T]$ a strongly α-regular elliptic *BVP* of class C^l and order $2m$, where $\mathbf{\Gamma}$ and the orders of the boundary operators $\mathcal{B}^\varrho_\Gamma(t)$ are independent of t.

(ii) $[t \mapsto a_\alpha(\cdot, t)] \in C^{2-}\big([0, T], BUC^l(\Omega, \mathcal{L}(\mathbf{K}^N))\big)$ for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leqq 2m$, and

$$[t \to b^\varrho_{\alpha,\Gamma}(\cdot, t)] \in C^{2-}\Big([0, T], BUC^{2m+l-m_{\varrho,\Gamma}}(\Gamma, \mathcal{L}(\mathbf{K}^N, \mathbf{K}))\Big)$$

for all $\alpha \in \mathbf{N}^u$ with $|\alpha| \leqq m_{\varrho,\Gamma}$ and for all $\varrho = 1, \ldots, mN$ and $\Gamma \in \mathbf{\Gamma}$.

Throughout the remainder of this section $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbf{\Gamma})$, $t \in [0, T]$, is supposed to be a regular parabolic *IBVP* of class C^l and order $2m$, where $l \in \mathbf{N}$. Of course, if $\Omega = \mathbf{R}^n$ it reduces to a pure initial value problem $(\mathcal{A}(t), \mathbf{R}^n)$, $t \in [0, T]$, since there are no boundary operators. Moreover by $\alpha \in (0, \pi/2)$ we denote always a fixed angle such that (i) is satisfied.

(14.1) LEMMA. *For each* $p \in (1, \infty)$ *there exists* $\lambda_0(p) \in \mathbf{R}^+$ *such that*

$$\big(\lambda + \mathcal{A}(\cdot), \mathcal{B}(\cdot)\big) \in C^{-2}([0, T], \mathrm{Isom}\,(W^{2m+s}_p, W^s_p \times \mathfrak{W}^{2m+s-1/p}_p))$$

for all $\lambda \in S_{\alpha+\pi/2}$ *with* $|\lambda| \geqq \lambda_0(p)$ *and all* $s \in [0, l]$ *with* $s - 1/p \notin \mathbf{N}$.

PROOF. This follows easily from Theorem (13.1), our regularity assumptions and the fact that the quantities determining the constant c^* in Theorem (12.1) can be chosen to be independent of $t \in [0, T]$. $\square$

(14.2) COROLLARY. *Let* $1 < p < \infty$. *Then*

$$[\lambda + A_{s,p}(\cdot)]^{-1} \in C^{2-}([0, T], \mathcal{L}(W_p^s, W_p^{2m+s}))$$

for all $\lambda \in S_{\alpha+\pi/2}$ *with* $|\lambda| \geqq \lambda_0(p)$ *and all* $s \in [0, l]$ *with* $s - 1/p \notin \mathbf{N}$.

PROOF. This follows from Lemmas (14.1) and (6.2). □

(14.3) LEMMA. *Let* $p \in (1, \infty)$ *and* $\mu \geqq \lambda_0(p)$ *be fixed. Then*

$$(1) \quad \|(\mu + A_{s,p}(t))(\lambda + A_{s,p}(t))^{-1}[(\mu + A_{s,p}(t))^{-1}]^{\bullet}\|_{\mathcal{L}(W_p^s)} \leqq c|\lambda - \mu|^{-1+s/2m}$$

and

$$(2) \qquad \|[(\lambda + A_{s,p}(t))^{-1}]^{\bullet}\|_{\mathcal{L}(W_p^s)} \leqq c|\lambda - \mu|^{-1+s/2m}$$

for all $\lambda \in S_{\alpha+\pi/2}$ *with* $|\lambda - \mu| \geqq \lambda_0(p)$ *and all* $s \in [0, l]$ *with* $s - 1/p \notin \mathbf{N}$.

PROOF. By replacing $A_{s,p}$ by $\mu + A_{s,p}$ and λ by $\lambda - \mu$ we can assume that $\mu = 0$.

Let $s \in [0, l] \cap \mathbf{N}$ and $v \in W_p^{2m+s}$ and put $u(\cdot) := A_{s,p}^{-1}(\cdot)v$. Then $u(\cdot) \in C^1([0, T], W_p^{2m+s})$ by Corollary (14.2) and—writing A for $A_{s,\cdot,p}$—

$$(\lambda + A(t))A(t)(\lambda + A(t))^{-1}\dot{u}(t) = (\lambda + \mathcal{A}(t))\big(1 - \lambda(\lambda + A(t))^{-1}\big)\dot{u}(t) = \mathcal{A}(t)\dot{u}(t)$$

and, similarly,

$$\mathcal{B}(t)A(t)(\lambda + A(t))^{-1}\dot{u}(t) = \mathcal{B}(t)\dot{u}(t)\,.$$

Since

$$(3) \qquad \mathcal{A}(t)u(t) = v \quad \text{and} \quad \mathcal{B}(t)u(t) = 0$$

we find that $\mathcal{A}(t)\dot{u}(t) = -\dot{\mathcal{A}}(t)u(t)$ and $\mathcal{B}(t)\dot{u}(t) = -\dot{\mathcal{B}}(t)u(t)$. Consequently, letting $C(t) := A(t)(\lambda + A(t))^{-1}[A^{-1}(t)]^{\bullet}$, we see that

$$(4) \qquad [\lambda + \mathcal{A}(t)]C(t)v = -\dot{\mathcal{A}}(t)u(t)$$

and

$$(5) \qquad \mathcal{B}(t)C(t)v = -\dot{\mathcal{B}}(t)u(t)$$

for $0 \leqq t \leqq T$. Hence from Theorem (12.2) we deduce that

$$(6) \quad \sum_{k=0}^{2m+s} |\lambda|^{1+(s-k)/2m}\|C(t)v\|_{k,p} \\ \leqq c\Big\{\|u(t)\|_{2m+s,p} + |\lambda|^{s/2m}\|u(t)\|_{2m,p} + \sum_{\Gamma\in\boldsymbol{\Gamma}}\sum_{\varrho=1}^{mN}|\lambda|^{1+(s-m_{\varrho,\Gamma})/2m}\|u(t)\|_{m_{\varrho,\Gamma},p}\Big\}\,.$$

By applying Theorem (12.2) once more, but to (3), we see that each one of the sumands on the right-hand side of (6) can be estimated by

$$c(\|v\|_{s,p} + |\lambda|^{s/2m}\|v\|_{0,p}) .$$

Hence

$$\|C(t)\|_{\mathfrak{L}(W_p^s)} \leqq c|\lambda|^{-1+s/2m}$$

for all $t \in [0, T]$ and $\lambda \in S_{\alpha+\pi/2}$ with $|\lambda| \geqq \lambda_0(p)$. The estimate (1) follows now from Theorem (11.6).

Let now $u(\cdot) := (\lambda + A(\cdot))^{-1}v$ and $C(t) := [(\lambda + A(t))^{-1}]^{\bullet}$. Since (4) and (5) hold again, the second assertion follows. □

(14.4) LEMMA. *Let $p \in (1, \infty)$ be fixed and put $X := L_p$, $X^1 := W_p^{2m}$, and $Z := \mathfrak{W}_p^{2m-1/p}$.*

(i) *Let $l \geqq 0$. Then assumptions (AP i)-(AP iii) are satisfied with $\varrho = 0$.*

(ii) *Suppose that $0 < s < 2m$ and*

$$m_{\varrho,\Gamma} \neq s - 1/p \quad \text{for } \varrho = 1, \dots, mN,\ \Gamma \in \mathbf{\Gamma} .$$

Then $(\cdot,\cdot)_{s/2m,p}$ is admissible if $s \notin \mathbf{N}$ and $[\cdot,\cdot]_{s/2m}$ is admissible if $s \in \mathbf{N}$.

(iii) *Suppose that $0 \leqq s \leqq l \leqq 2m$ and that $s - 1/p \notin \mathbf{N}$. Moreover let $X^{\Theta} := W_p^{2m\Theta}$ for $0 < \Theta < 1$. Then condition $(\mathring{R})_{s/2m}$ is satisfied with $\varrho = 0$.*

(iv) *Suppose that $0 \leqq s < m/2$ and that $l \geqq s$. Let $X^{1+\varkappa} := W_p^{2m(1+\varkappa)}$ and $Y^{\varkappa} := \mathfrak{W}_p^{2m(1+\varkappa)-1/p}$. Then condition $(HR)_{s/2m}$ is satisfied provided $s - 1/p \notin \mathbf{N}$.*

(v) *Suppose that $0 < s < 2m$, that $l \geqq s$, that $s - 1/p \notin \mathbf{N}$, and that $W^s_{p,\mathfrak{B}(t)} = W^s_{p,\mathfrak{B}(0)}$ for all $t \in [0, T]$. Then condition $(SS)_{s/2m}$ is satisfied with $\varrho_{s/2m} = s/2m$, $X^{1+s/2m} = W_p^{2m+s}$ and $Z^{s/2m} = \mathfrak{W}_p^{2m+s-1/p}$.*

(vi) *Suppose that $0 \leqq \sigma \leqq s \leqq 2m$. Then W_p^{σ} is (L_p, W_p^{σ})-compatible.*

PROOF. (i) follows easily from Lemma (14.1), Theorem (12.2) and Lemma (14.3).

(ii) is a consequence of Theorem (13.3).

(iii) follows from Corollary (14.2) and Lemma (14.3).

(iv) Observe that $\varkappa := s/2m < \frac{1}{4}$. Hence the assertion is a consequence of Lemma (14.1), Corollary (14.2) and Lemma (14.3).

(v) follows from (ii), from Lemmas (14.1) and (14.3), from Corollary (14.2), and from (10.1).

(vi) is a consequence of Theorem (11.6). $\square$

After these preparations we can now easily prove the following theorem in which we collect the most basic properties of the fundamental solution for $A_p(\cdot)$.

(14.5) THEOREM. (i) *Let* $1 < p < \infty$ *and* $l \geqq 0$. *Then there exists a unique parabolic fundamental solution* U_p *for* $A_p(\cdot)$ *and*

$$U_p|L_p \cap L_q = U_q|L_p \cap L_q \tag{7}$$

for each $q \in (1, \infty)$. *Moreover* (U_p, U_p) *is an evolution system in* (L_p, W_p^s) *of type* $s/2m$ *for each* $s \in [0, 2m)$ *with* $s - 1/p \notin \mathbb{N}$.

(ii) *Suppose that* $0 \leqq s < m/2$ *and* $l \geqq s$ *and that* $s - 1/p \notin \mathbb{N}$. *Then*

$$U_p(\cdot, t_0) \in C\big((t_0, T], \mathfrak{L}_s(W_p^s, W_p^{2m+s})\big)$$

and

$$\Big(t \to \int_{t_0}^{t} U_p(t, \tau) g(\tau) d\tau\Big) \in C\big((t_0, T], W_p^{2m+s}\big)$$

for every $g \in C^\nu([t_0, T], W_p^s)$ *with* $s/2m < \nu < 1$ *and for every* $t_0 \in [0, T)$.

(iii) *Suppose that* $0 < s < 2m$, *that* $l \geqq s$, *that* $s - 1/p \notin \mathbb{N}$, *and that*

$$W^s_{p,\mathfrak{B}(t)} = W^s_{p,\mathfrak{B}(0)} \quad \forall t \in [0, T].$$

Moreover let $A_{s,p,\mathfrak{B}}$ *be the* $W^s_{p,\mathfrak{B}(0)}$*-realization of* A_p. *Then* U_p *restricts to a parabolic fundamental solution on* $W^s_{p,\mathfrak{B}(0)}$ *for* $A_{s,p,\mathfrak{B}(0)}$.

PROOF. (i) Everything except (7) follows from Lemma (14.4.i, ii) and Theorem (7.1). The relation (7) is an easy consequence of Lemma (13.2), Theorem (13.4), and the representations (8.10), (8.12) and (8.13).

(ii) follows from Lemma (14.4.iv) and Proposition (9.5).

(iii) due to Lemma (14.4.v) this is a consequence of the proof of Proposition (10.2). $\square$

15. – Semilinear parabolic initial boundary value problems.

Throughout this section we suppose that $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbf{\Gamma})$, $t \in [0, T]$, is a regular parabolic *JBVP* of class C^l and order $2m$. Then we consider the semilinear parabolic *IBVP*

$$(P)_{(t_0,u_0)} \begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = F(t, u) & \text{in } \Omega \times (t_0, T], \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (t_0, T], \\ u(\cdot, t_0) = u_0 & \text{on } \Omega, \end{cases}$$

where $0 \leqq t_0 < T$ and F is an appropriate «nonlinearity». By an *L_p-solution, $1 < p < \infty$, of* $(P)_{(t_0,u_0)}$ we mean a solution of the semilinear evolution equation in L_p

$$(SE_p)_{(t_0,u_0)} \begin{cases} \dot{u} + A_p(t)u = F(t, u), \quad t_0 < t \leqq T, \\ u(t_0) = u_0, \end{cases}$$

provided $u_0 \in L_p$.

Suppose that D_p^τ is a subset of W_p^τ. Then we let $D_p^\sigma := D_p^\tau \cap W_p^\sigma$ and $D_{p,\mathcal{B}(t)}^\sigma := D_p^\tau \cap W_{p,\mathcal{B}(t)}^\sigma$ for $0 < \tau \leqq \sigma \leqq 2m$ and $0 \leqq t \leqq T$. Observe that D_p^σ is open in W_p^σ if D_p^τ is open in W_p^τ, due to $W_p^\sigma \hookrightarrow W_p^\tau$.

Our basic result concerning the solvability of problem $(P)_{(t_0,u_0)}$ is contained in the following

(15.1) THEOREM. (i) *Suppose that $1 < p < \infty$, that $0 \leqq s < \tau < \sigma < 2m$, that $2s < m$ and $2s < \sigma \leqq l$, and that $s, \sigma \notin \mathbb{N} + 1/p$. Moreover suppose that D_p^τ is open in W_p^τ and that*

$$F \in C^{\nu,1-}([0, T] \times D_p^\tau, W_p^s) \tag{1}$$

for some $\nu \in (s/2m, 1)$. Then problem $(P)_{(t_0,u_0)}$ has for each $(t_0, u_0) \in [0, T) \times D_{p,\mathcal{B}(t_0)}^\sigma$ a unique maximal L_p-solution $u(\cdot, t_0, u_0)$. The maximal interval of existence $J(t_0, u_0)$ is right open in $[t_0, T]$,

$$\mathfrak{D}_{p,\mathcal{B}(t_0)}^\sigma := \{(t, v) \in [t_0, T] \times D_{p,\mathcal{B}(t_0)}^\sigma : t \in J(t_0, v)\} \tag{2}$$

is open in $[t_0, T] \times W_{p,\mathcal{B}(t_0)}^\sigma$ and

$$u(\cdot, t_0, \cdot) \in C^{0,1-}(\mathfrak{D}_{p,\mathcal{B}(t_0)}^\sigma, W_p^\sigma). \tag{3}$$

Moreover

$$u(\cdot, t_0, u_0) \in C^1(\dot{J}, W_p^s) \cap C(\dot{J}, W_p^{2m+s}) \cap C^{(\sigma-\varrho)/2m}(J, W_p^\varrho) \tag{4}$$

for every $\varrho \in [0, \sigma]$, *where* $J := J(t_0, u_0)$ *and* $\dot{J} := J \setminus \{t_0\}$.

(ii) *Suppose in addition that*

$$W^r_{p,\mathcal{B}(t)} = W^r_{p,\mathcal{B}(0)} \quad \forall t \in [0, T], \tag{5}$$

where $0 < r < \sigma$ *and* $r \notin \mathbb{N} + 1/p$, *that*

$$F \in C\big([0, T] \times D_p^\sigma, W_p^r\big), \tag{6}$$

that

$$F(t, v) \in W^r_{p,\mathcal{B}(0} \qquad \forall v \in W^{2m}_{p,\mathcal{B}(t)}, \ t \in [0, T], \tag{7}$$

and that

$$u_0 \in D^{2m}_{p,\mathcal{B}(t_0)} \quad \textit{and} \quad \mathcal{A}(t_0) u_0 \in W^r_{p,\mathcal{B}(0}. \tag{8}$$

Then

$$u(\cdot, t_0, u_0) \in C^1\big(J(t_0, u_0), W^\lambda\big) \cap C\big(J(t_0, u_0), W_p^{2m+\lambda}\big) \tag{9}$$

for every $\lambda \in (0, r)$ *with* $\lambda \notin \mathbb{N} + 1/p$.

PROOF. (i) Since $W_p^\sigma \hookrightarrow W_p^\tau \hookrightarrow W_p^s \hookrightarrow L_p$ it follows from (1) that

$$F \in C^\nu\big([0, T] \times D_p^\tau, L_p\big) \cap C^{0,1-}\big([0, T] \times D_p^\sigma, L_p\big).$$

Hence the assertions up to (3) follow from Lemma (14.4.i-iii, vi) and Corollary (8.7). Moreover Corollary (8.7) and Lemma (14.4.i-iii, vi) imply also that $u(\cdot, t_0, u_0) \in C^{(\sigma-\varrho)/2m}\big(J(t_0, u_0), W_p^\varrho\big)$ for $0 \leqq \varrho \leqq \sigma$. The remaining part of (4) is a consequence of Lemma (14.4.iii.vi) and Theorem (9.6).

(ii) Let $\Theta := (\varrho - r)/2m$ for $r < \varrho < 2m + r$ and observe that, by Theorem (11.6),

$$(W_p^r, W_p^{2m+r})_{\Theta,p} = W_p^\sigma \quad \text{if } \varrho \notin \mathbb{N}.$$

Hence (9) follows from Lemmas (14.1) and (14.4.v) and Theorem (10.3). □

(15.2) COROLLARY. *Let the hypotheses of Theorem* (15.1) *be satisfied and suppose that* $\mathcal{A}$, $\mathcal{B}$ *and* F *are independent of* t. *Then* $\varphi := u(\cdot, 0, \cdot)$ *is a semi-*

flow on $D^\sigma_{p,\mathfrak{B}}$ such that $\varphi \in C^{0,1-}(\mathfrak{D}^\sigma_{p,\mathfrak{B}}, W^\sigma_p)$. Moreover, if Ω is bounded and if $\gamma^+(u_0)$ is a positive orbit through $u \in D^\sigma_{p,\mathfrak{B}}$ such that $\gamma^+(u_0)$ and $F(\gamma^+(u_0))$ are both bounded in L_p, then $\gamma^+(u_0)$ is relatively compact in W^σ_p.

PROOF. The first assertion follows from Theorems (5.1) and (13.4) and from the above proof. Recall that $\exp[-t(\lambda + A_p)] = \exp[-\lambda t] \exp[-tA_p]$ for $t \geqq 0$. Hence, by replacing A_p by $\lambda + A_p$ and F by $v \mapsto F(v) + \lambda v$ for a sufficiently large real number λ, we can assume without loss of generality that there exists a constant $\omega > 0$ such that

$$(10) \qquad \|\exp[-tA_p]\|_{\mathfrak{L}(L_p, W^\varrho_p)} \leqq ct^{-\varrho/2m} \exp[-\omega t]$$

and

$$(11) \qquad \|\exp[-tA_p]\|_{\mathfrak{L}(W^\varrho_{p,\mathfrak{B}})} \leqq c \exp[-\omega t]$$

for all $t > 0$ (cf. (8.18) and Theorem (13.4)), provided $0 < \varrho < 2m$ and $\varrho \notin \mathbb{N} + 1/p$.

Let $0 < t_1 < t^+(u_0)$ and observe that

$$(12) \qquad \varphi(t, u_0) = \exp[-(t-t_1)A_p] \exp[-t_1 A_p] u_0 + \int_0^t \exp[-(t-\tau)A_p] F(\varphi(\tau, u_0))\, d\tau$$

for $t_1 \leqq t < t^+(u_0)$. Since $\exp[-t_1 A_p] u_0 \in W^{2m}_{p,\mathfrak{B}} \hookrightarrow W^\varrho_{p,\mathfrak{B}}$, it follows from (10)-(12) and the boundedness of $F(\gamma^+(u_0))$ in L_p that

$$\|\varphi(t, u_0)\|_{\varrho,p} \leqq c \qquad \forall t \in [t_1, t^+(u_0)) .$$

Since we can choose $\varrho > \sigma$ and since, due to the boundedness of Ω, the space W^ϱ_p is compactly imbedded in W^σ_p, we see that $\{\varphi(t, u_0) : t_1 \leqq t < t^+(u_0)\}$ is relatively compact in W^σ_p. Now the assertion follows from the fact that $\{\varphi(t, u_0) : 0 \leqq t \leqq t_1\} = \varphi([0, t_1], u_0)$—being the continuous image of a compact set—is also compact in W^σ_p. $\square$

By a *classical solution* of $(P)_{(t_0,u_0)}$ on J we mean a function

$$u \in C(J \times \bar{\Omega}, \mathbb{K}^N) \cap C^{1,0}(\dot{J} \times \Omega, \mathbb{K}^N) \cap C^{0,2m}(\dot{J} \times \bar{\Omega}, \mathbb{K}^N),$$

where $\dot{J} := J \setminus \{t_0\}$, which satisfies $(P)_{(t_0,u_0)}$ pointwise. The following corollary shows that $(P)_{(t_0,u_0)}$ possesses a classical solution if $s > n/p$. In fact, much more is true.

(15.3) COROLLARY. (i) *Let the hypotheses of Theorem* (15.1) *be satisfied and suppose in addition that* $s > n/p$. *Then*

$$u(\cdot, t_0, u_0) \in C^1(\dot{J}, BUC^{\mu}_{\infty}) \cap C(\dot{J}, BUC^{2m+\mu}_{\infty})$$

for every $\mu \in (0, s - n/p)$, *or for* $\mu = s - n/p$ *if* $s - n/p \notin \mathbb{N}$.

(ii) *Suppose also that* $r > n/p$ *and that the conditions* (5)-(8) *are satisfied. Then*

$$u(\cdot, t_0, u_0) \in C^1(J, BUC^{\lambda}_{\infty}) \cap C(J, BUC^{2m+\lambda}_{\infty})$$

for every $\lambda \in (0, r - n/p)$, *or for* $\lambda = r - n/p$ *if* $r - n/p \notin \mathbb{N}$.

PROOF. This follows immediately from (4) and (9), respectively, and from Theorem (11.5). $\square$

In most applications F is a substitution operator of the form

$$F(t, u)(x) := f\big(t, x, u(x), Du(x), \dots, D^k u(x)\big) \tag{13}$$

for some $k \in \{0, 1, \dots, 2m - 1\}$. Hence it remains to find conditions for f guaranteeing that F satisfies (1) and (5)-(7), respectively.

In the following we fix $k \in \{0, 1, \dots, 2m - 1\}$ and let $M := N \sum_{|\alpha| \leqq k} 1$, where $\alpha \in \mathbb{N}^n$. Moreover we denote by G an open neighbourhood of zero in $\mathbb{K}^M$, we suppose that

$$f \in C\big([0, T] \times \bar{\Omega} \times G, \mathbb{K}^N\big),$$

and we define F by (13). Finally,

$$D^{\sigma}_p := \{u \in W^{\sigma}_p \,|\, (u, Du, \dots, D^k u)(\bar{\Omega}) \subset G\}.$$

In the following proposition we restrict ourselves to the most important case $n < p$ and $s < 1$. We leave it to the reader to prove analogous results in other situations. The differentiability requirements below concern the underlying real realization of f which is obtained by identifying $\mathbb{C}$ canonically with $\mathbb{R}^2$.

(15.4) PROPOSITION. *Suppose that* $n < p < \infty$, *that* $0 \leqq s < 1$, *and that* $s + k + n/p < \sigma \leqq 2m + l$.

(i) *Then* D^{σ}_p *is open in* W^{σ}_p.

(ii) *Suppose that*

$$[t \mapsto f(t,\cdot,0)] \in C([0,T], W_p^s)\,, \tag{14}$$

and that

$$f(t,x,\cdot) \in C^{1-}(G,\mathbf{K}^N)\,, \quad \textit{uniformly with respect to } (t,x) \in [0,T]\times\bar{\Omega}\,. \tag{15}$$

If $s > 0$ suppose in addition that there exists for each compact subset K of G a constant c_K such that

$$\|f_\xi(t,\cdot,\eta)\|_{0,\infty} + \|f_\xi(t,\cdot,\eta)\|_{s,p} \leqq c_K \tag{16}$$

for all $(t,\eta) \in [0,T]\times K$, and that

$$f_\xi(t,x,\cdot) \in C^{1-}(G,\mathbf{K}^N)\,, \quad \textit{uniformly with respect to } (t,x) \in [0,T]\times\bar{\Omega}\,. \tag{17}$$

where f_ξ is the derivative of f with respect to $\xi \in G$.

Then $F(t,\cdot) \in C^{1-}(D_p^\sigma, W_p^s)$, uniformly with respect to $t \in [0,T]$.

PROOF. (*a*) Suppose first that $k = 0$.

(i) Let $u_0 \in D_p^\sigma$ be fixed. Since $W_p^\sigma \hookrightarrow BUC_\infty$ by Theorem (11.5) it follows that $\overline{u_0(\Omega)}$ is compact and contained in G. Hence we can find a neighbourhood B of u_0 in $BUC_\infty \cap W_p^\sigma$ such that $\cup\,\{u(\bar{\Omega}) : u \in B\}$ has a compact closure which is contained in G. This implies the assertion.

(ii) Let $u_0 \in D_p^\sigma$ be fixed and let B be a convex neighbourhood of u_0 in D_p^σ such that $\cup\,\{u(\bar{\Omega}) : u \in B\}$ has a compact closure contained in G.

Since Lipschitz continuous maps are uniformly Lipschitz continuous on compact sets it is easily verified (see the proof of [5, Satz 6.4]) that, due to (15),

$$|f(t,x,u(x)) - f(t,x,v(x))| \leqq c|u(x) - v(x)|$$

for all $(t,x) \in [0,T]\times\bar{\Omega}$ and all $u, v \in B$. Hence

$$\|F(t,u) - F(t,v)\|_{0,p} \leqq c\|u - v\|_{0,p} \leqq c\|u - v\|_{s,p} \tag{18}$$

for all $t \in [0,T]$ and all $u, v \in B$. Thus it follows from (14) that $F(t,\cdot) \in C^{1-}(D_p^\sigma, L_p)$, uniformly with respect to $t \in [0,T]$.

Suppose now that $0 < s < 1$. Then

$$\begin{aligned} &F(t,u)(x) - F(t,v)(x) - [F(t,u)(y) - F(t,v)(y)] \\ &\quad = h(t,x)\big(u(x) - v(x) - [u(y) - v(y)]\big) + [h(t,x) - h(t,y)]\big(u(y) - v(y)\big)\,, \end{aligned} \tag{19}$$

where

$$h(t, x) := \int_0^1 f_\xi\big(t, x, v(x) + \tau(u(x) - v(x))\big)\, d\tau \,. \tag{20}$$

Since

$$f_\xi(t, x, \eta) - f_\xi(t, y, \zeta) = f_\xi(t, x, \eta) - f_\xi(t, x, \zeta) + f_\xi(t, x, \zeta) - f_\xi(t, y, \zeta)$$

it follows from (16), (17) and (19), (20) that

$$\begin{aligned}
&|F(t, u)(x) - F(t, v)(x) - F(t, u)(y) + F(t, v)(y)| \\
&\leqq c\Big\{|u(x) - v(x) - [u(y) - v(y)]| + |v(x) - v(y)||u(y) - v(y)| \\
&+ \int_0^1 \big|f_\xi\big(t, x, v(y) + \tau(u(y) - v(y))\big) - f_\xi\big(t, y, v(y) + \tau(u(y) - v(y))\big)\big| d\tau |u(y) - v(y)|\Big\}
\end{aligned}$$

for all $u, v \in B$. Since $\sigma > s$ and since

$$|u(y) - v(y)| \leqq \|u - v\|_C \leqq c\|u - v\|_{\sigma,p}$$

(16) and (18) imply

$$\|F(t, u) - F(t, v)\|_{s,p} \leqq c\|u - v\|_{\sigma,p} \qquad \forall u, v \in B,\ t \in [0, T]$$

(recall the definition of the norm in W_p^s). Now the assertion follows from (14).

(*b*) Let $k > 0$ and let $Hu := (u, Du, \ldots, D^k u)$. Then $H \in \mathfrak{L}(W_p^\sigma, W_p^{\sigma-k})$ and $F(t, u) = f(t, \cdot, Hu)$. Now the assertion follows from (*a*). $\square$

(15.5) Corollary. *Suppose in addition to the hypotheses of Proposition* (15.4.ii) *that* $F(\cdot, u) \in C^\nu([0, T], W_p^s)$ *for some* $\nu \in (0, 1)$. *Then*

$$F \in C^{\nu,1-}([0, T] \times D_p^\sigma, W_p^s) \,.$$

If $s = 0$ the above assumptions guaranteeing that $F \in C^{\nu,1-}([0, T] \times D_p^\sigma, L_p)$ are very simple and quite natural. If $s > 0$ we can give simpler conditions than the ones above if we consider a more specialized setting. This is the content of the following

(15.6) Proposition. *Suppose that* $n < p < \infty$, *that* $0 < s < 1$, *and that* $s + k + n/p < \sigma \leqq 2m + l$. *Moreover suppose that* either *f is independent* *of* (t, x) *and* $f(0) = 0$ or Ω *is bounded. Then* $F \in C^{1-}([0, T] \times D_p^\sigma, W_p^s)$ *provided* $f_t, f_\xi \in C^{0,\lambda,1-}([0, T] \times \bar{\Omega} \times G, \mathbb{K}_N)$ *for some* $\lambda \in (s, 1)$, *where* f_t *is the derivative with respect to* t.

PROOF. If f is independent of (t, x) the assertion is an obvious consequence of Corollary (15.5). Since Hölder continuous maps are uniformly Hölder continuous on compact sets (which follows similarly as for Lipschitz continuous maps) $f_\xi \in C^{0,\lambda,1-}([0, T] \times \bar{\Omega} \times G, \mathbb{K}^N)$ implies easily that (16) and (17) are true. Since

$$f(t, x, u(x)) - f(s, x, u(x)) - [f(t, y, u(y)) - f(s, y, u(y))]$$
$$= \int_s^t [f_t(\tau, x, u(x)) - f_t(\tau, y, u(y))]\, d\tau$$

it follows from $f_t \in C^{0,\lambda,1-}([0, T] \times \bar{\Omega} \times G, \mathbb{K}^N)$ that

$$|F(t, u)(x) - F(s, u)(x) - [F(t, u)(y) - F(s, u)(y)]|$$
$$\leqq c|s - t|\{|x - y|^\lambda + |u(x) - u(y)|\}\,.$$

This, together with the boundedness of Ω, implies easily that $F(\cdot, u) \in C^{1-}([0, T], W_p^s)$ for every $u \in D_p^\sigma$. Hence in particular (14) is satisfied. Now the assertion follows from Proposition (15.4). □

As an application we obtain the following corollary in which we do not give the most general assumptions upon f.

(15.7) COROLLARY. *Suppose that* $l = k + 1$ *and that* either Ω *is bounded* or f *is independent of* (t, x) *and* $f(0) = 0$. *Moreover suppose that* $f \in C^{2-}([0, T] \times \bar{\Omega} \times G, \mathbb{K}^N)$ *and that* $(t_0, u_0) \in [0, T) \times D^{2m}_{p,\mathcal{B}(t_0)}$ *for some* $p > 2n$. *Then the IBVP*

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t) u &= f(t, x, u, Du, \ldots, D^k u) && \text{in } \Omega \times (t_0, T]\,, \\ \mathcal{B}(t) u &= 0 && \text{on } \partial\Omega \times (t_0, T]\,, \\ u(\cdot, t_0) &= u_0 && \text{on } \Omega \end{aligned}$$

has a unique maximal L_p*-solution* $u(\cdot, t_0, u_0)$ *on* $J := J(t_0, u_0)$. *Moreover there exists a number* $\mu \in (0, 1)$ *such that*

$$(21) \qquad u(\cdot, t_0, u_0) \in C^1(\dot{J}, BUC^\mu_\infty) \cap C(\dot{J}, BUC^{2m+\mu}_\infty) \cap C(J, BUC^{k+\mu}_\infty)\,,$$

where $\dot{J} := J \setminus \{t_0\}$.

PROOF. Fix s, τ and σ such that $n/p < s < \frac{1}{2}$ and $s + k + n/p < \tau < \sigma < k + 1$ and such that $2s < \sigma$. Then D_p^σ is open W_p^σ by Proposition (15.4.i) and, due to Proposition (15.6), the assumptions of Theorem (15.1.i) are satisfied. Hence the assertion follows from Theorem (15.1.i), from Corollary (15.3), and from the fact that $W_p^\sigma \hookrightarrow BUC_\infty^{k+\mu}$ for $0 < \mu < \sigma - k - n/p$. $\square$

If Ω is bounded every classical solution is an L_p-solution for every $p \in (1, \infty)$. Thus Corollary (15.7) implies that there is a unique classical solution if Ω is bounded. It is well known that this is not the case, in general, if Ω is unbounded (e.g. [22, p. 31]).

We leave it to the reader to formulate and prove an analogous result for strict solutions based on the second parts of Theorem (15.1) and Corollary (15.3). However we add a few comments on the compatibility conditions (5) and (7).

Suppose that $\partial\Omega$ is a disjoint union $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$, where $\partial_0\Omega$ and $\partial_1\Omega$ are both unions of $\Gamma \in \mathbf{\Gamma}$. Moreover suppose that $\min\{m_{\varrho,\Gamma}: 1 \leqq \varrho \leqq mN\} \geqq 1$ for each $\Gamma \subset \partial_1\Omega$ and that $\mathcal{B}_\Gamma$ is the Dirichlet boundary operator for each $\Gamma \subset \partial_0\Omega$. Then it is clear that

$$W^r_{p,\mathcal{B}(t)} = \{u \in W^r_p : u|\partial_0\Omega = 0\} = W^r_{p,\mathcal{B}(0)} \quad \forall t \in [0, T]$$

provided $0 \leqq r \leqq 1 + 1/p$. Moreover

$$F(t, v) \in W^r_{p,\mathcal{B}(0)} \qquad \forall v \in W^{2m}_{p,\mathcal{B}(t)}, \qquad t \in [0, T],$$

where F is defined by (13) provided

$$f(t, x, 0, \eta) = 0 \qquad \forall (t, x) \in [0, T] \times \partial_0\Omega \tag{22}$$

and all $\eta \in \mathbb{R}^{M-M_0}$, where $M_0 := N \sum_{|\alpha| \leqq m-1} 1$ and η is a dummy variable for $(D^m u, \dots, D^k u)$, provided $k \geqq m$.

As already mentioned earlier there are no results known to the author guaranteeing the existence of classical solutions for semilinear parabolic equations (let alone systems!) with time-dependent boundary conditions. If the boundary conditions are time-independent, more precisely: in the case of Dirichlet boundary conditions, the existence of classical solutions to semilinear parabolic *IBVP*s (with $N = 1$) of arbitrary order has been studied by v. Wahl [47, 48, 49] and Kielhöfer [27, 28]. In [49] the existence and regularity of a solution without compatibility conditions for f is proven by working in Hölder spaces. However only a particular class of parabolic operators can be handled. In [47, 48] there are only regularity results odtained, whereas the existence of appropriate (weak) solutions is presupposeb.

Kielhöfer [27, 28] allows unbounded domains and obtains the existence of a classical solution. He also works in Hölder spaces. The papers by Kielhöfer and von Wahl depend heavily upon the fact that the boundary conditions are time-independent.

In a recent paper Mora [35] has studied semilinear autonomous parabolic equations in the spaces C^k and shown that they define semiflows. However he also has to impose compatibility conditions for f of the form (22).

A different approach, which is not based on the concept of a fundamental solution, has been initiated by DaPrato and Grisvard [15-17] and extensively exploited by DaPrato and his students (e.g. [7, 8, 14, 18, 33, 34, 39, 40]). In some sense their approach is related to our method though they are totally different. The above authors study abstract (mostly linear, but also quasi-linear) parabolic evolution equations in certain « continuous » interpolation spaces. In order to avoid compatibility conditions some effort is made to develop a theory which works when the domains $D(A(t))$ are not dense in the underlying Banach space [7, 8, 34, 39]. However, as mentioned earlier, they assume throughout the existence of an estimate of the form $\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} \leqq c(1 + |\lambda|)^{-1}$. This restricts the applicability of these results essentially to spaces of continuous functions or, in some cases, to subspaces of Hölder spaces, due to recent results of Campanato [13] and, of course, to L_p-spaces. Theorem (12.2) implies only an estimate of the form $\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} \leqq c(1 + |\lambda|)^{-1+\varrho}$ with $\varrho = s/2m$ if $X = W_p^s$. Hence the results of these authors are not applicable to these spaces. On the other hand Sobolev spaces are much better suited for many problems concerning semilinear equations—e.g. for a priori estimates leading to global solutions—than spaces of continuous functions or Hölder spaces, as will be shown in a forthcoming paper. Finally it should be noted that in the above mentioned papers there are also no results about nonlinear equations in the case that $D(A(t))$ is not constant in time.

Lastly it should be noted that, by a completely different method, which is based upon the classical results in [30], it is shown in [4] that L_p-solutions of semilinear second order parabolic equations ($N = 1$) with time-independent boundary conditions are classical solutions.

Note added in proof:

Professor Solonnikov informed the author that he has obtained in a joint work with Hačatrjan [53], existence theorems for semilinear parabolic systems by a completely different method. It is based upon the use of weighted norms and sharp a priori estimate for linear systems [54] under weaker compatibility conditions than in [30, 42].

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