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Harmonic and Analytic Functions Admitting a Distribution Boundary Value.

EMIL J. STRAUBE (*)

0. – Introduction.

We consider harmonic functions defined in some smooth, bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. We say that f admits a distribution boundary value on $b\Omega$ if

$$\lim_{\varepsilon \to 0^+} \int f(x - \varepsilon n(x)) \varphi(x) \, d\sigma(x) \tag{1}$$

exists for all $\varphi \in \mathcal{D}(b\Omega)$; here $n(x)$ denotes the outward normal at $x \in b\Omega$. In section 1, we first characterize the harmonic functions which admit a distribution boundary value by several equivalent conditions: being an element of some Sobolev space $W^{-k}(\Omega)$, being polynomially bounded in $1/\operatorname{dist}(x, b\Omega)$, $(\operatorname{Re} f)^+$ and $(\operatorname{Im} f)^+$, the positive parts of $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively, being polynomially bounded near $b\Omega$, and the local existence of a harmonic primitive which is continuous up to the boundary. In view of this, we introduce the spaces

$$h^{-\infty}(\Omega) := \bigcup_{k \in \mathbb{N}} h^{-k}(\Omega)\,, \tag{2}$$

where

$$h^{-k}(\Omega) := \{h \in W^{-k}(\Omega) / \Delta h = 0\}\,. \tag{3}$$

$h^{-\infty}(\Omega)$ is proved with the inductive limit topology. The spaces $A^{-k}(\Omega)$ and $A^{-\infty}(\Omega)$ are defined analogously, with analytic functions instead of harmonic ones. We single out some properties of the topological vector spaces $h^{-\infty}(\Omega)$ (in particular the structure of bounded sets), all of which

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are related to the fact that $W^{-k}(\Omega)$ is compactly imbedded in $W^{-k-1}(\Omega)$, by Rellich's lemma. These properties will be applied in section 3. We show that, just as in the case of the harmonic Hardy spaces, the Poisson integral mediates between boundary values and the corresponding functions; denote by $P(x, y)$ the Poisson kernel of Ω. Then the map $\tau \mapsto P\tau$, with

$$P\tau(x) := \langle \tau, P(x, \cdot)\rangle \tag{4}$$

is an isomorphism (of TVS) of $\mathfrak{D}'(b\Omega)$ onto $h^{-\infty}(\Omega)$; its inverse is given by the map which assigns to each of $f \in h^{-\infty}(\Omega)$ its distribution boundary value. In (4), the duality is between $\mathfrak{D}'(b\Omega)$, and $\mathfrak{D}(b\Omega)$, of which $P(x, \cdot)$ is an element for all $x \in \Omega$. Finally, our characterization of harmonic functions with distribution boundary value allows for a very straightforward definition of a sesquilinear pairing on $h^{-\infty}(\Omega) \times C^{\infty}(\bar{\Omega})$, which extends the usual $\mathfrak{L}_2$-pairing:

$$\widetilde{\int_{\Omega}} f\bar{g}\,dV := \lim_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon} f\bar{g}\,dV, \tag{5}$$

where $\Omega_\varepsilon := \{x \in \Omega / \operatorname{dist}(x, b\Omega) > \varepsilon\}$. For this pairing, we prove the Sobolev inequality

$$\left|\widetilde{\int_{\Omega}} f\bar{g}\,dV\right| \leqslant C(k)\|f\|_{-k}\|g\|_k\,. \tag{6}$$

In section 2, we study the restriction of the map (4) to the space of CR-distributions on the boundary. We show that this restriction is an isomorphism onto $A^{-\infty}(\Omega)$, the subspace of analytic functions of $h^{-\infty}(\Omega)$. Furthermore, in this case, the Poisson extension given by (4) coincides with the Bockner-Martinelli extension. These results are, essentially, contained in § 6 of [23]. However, the approach of [23] is completely different from the point of view adopted here. We modify the arguments given in [10] for « weak solutions » (i.e. $\mathfrak{L}_1$-functions) and obtain a simple proof which is completely in the spirit of section 1.

In section 3, we apply the results and techniques of section 1 to the Szegö and Bergman (both harmonic and analytic) projections. We give a new, and we believe instructive, proof of a recent result of Boas, which characterizes regularity of the Szegö projection in terms of the Szegö kernel function ([9]). The results of section 1 also yield a continuous map $T: C^{\infty}(\bar{\Omega}) \to \bigcap_{k\in\mathbf{N}} W_0^k(\Omega)/L$ (L is a certain subspace), with the property that the Bergman projections Pg and Pg' agree for all g' in the equivalence class $Tg \in \bigcap_k W_0^k(\Omega)/L$. In particular, there exists $g' \in \bigcap_k W_0^k(\Omega)$ with $Pg' = Pg$ (see also [6]). However, the above continuous operator T, together with properties of the projection $\bigcap_k W_0^k(\Omega) \to \bigcap_k W_0^k(\Omega)/L$, allows for precise control of Sobolev norms. For example, if $g_n \to 0$ in $C^\infty(\bar{\Omega})$, g_n' can be chosen to converge to 0 in $\bigcap_k W_0^k(\Omega)$. The analogous result for the harmonic Bergman projection is also true.

Finally, we discuss an equivalence between global regularity of the Bergman projection P and duality between $A_{\mathrm{cl}}^{-\infty}(\Omega)$, the closure of the space of square integrable analytic functions in $A^{-\infty}(\Omega)$, and $A^\infty(\Omega)$. The duality is given by the pairing (5). An equivalence of this type was first shown in [8] under the assumption that Ω is pseudoconvex (then $A_{\mathrm{cl}}^{-\infty}(\Omega) = A^{-\infty}(\Omega)$). In [12] the condition $(R)_0^k$: P maps $W_0^k(\Omega)$ into $A^k(\Omega)$, is shown to be equivalent to a duality between $A^k(\Omega)$ and $A_{\mathrm{cl}}^{-k}(\Omega)$. In this connection the question arises whether $(R)_0^k$ and $(R)^k$: P maps $W^k(\Omega)$ into $A^k(\Omega)$, are equivalent. They are indeed: we construct continuous operators T^k: $W^k(\Omega) \to W_0^k(\Omega)$ with the property that

$$PT^k = P\,. \tag{7}$$

In fact, (7) holds for the harmonic Bergman projection, and therefore for the orthogonal projection onto an arbitrary subspace of $h^0(\Omega)$.

1. – Harmonic functions with boundary value.

Let Ω be a C^∞-smooth (i.e. having a C^∞-defining function), bounded domain in $\mathbb{R}^n$. By $\mathfrak{D}(b\Omega)$, we denote the space of C^∞-functions on the boundary of Ω, provided with its usual topology, which is the topology of locally uniform convergence of the pullbacks, as well as their derivatives, by each local coordinate system. Since $b\Omega$ is compact, this topology is metrizable: $\mathfrak{D}(b\Omega)$ becomes a Fréchet-space. $\mathfrak{D}'(b\Omega)$ denotes the strong dual of this space; we call it the space of distributions on $b\Omega$. We adopt the convention that integrable functions f on $b\Omega$ define distributions via

$$\langle f, \varphi\rangle := \int_{b\Omega} f(x)\varphi(x)\,d\sigma\,, \qquad \forall \varphi \in \mathfrak{D}(\Omega) \ (d\sigma = \text{surface element on } b\Omega)\,. \tag{1}$$

For a function f on Ω, we set

$$f_\varepsilon(x) := f\big(x - \varepsilon n(x)\big)\,, \tag{2}$$

where $n(x)$ is the unit outward normal to $b\Omega$ at x. So f_ε is defined on $b\Omega$.

Let now f be $\in C^\infty(\Omega)$. We say that f admits a distribution boundary value on $b\Omega$, if

$$\lim_{\varepsilon\to 0^+}\int_{b\Omega} f_\varepsilon(x)\varphi(x)\,d\sigma(x) \tag{3}$$

exists for all $\varphi\in\mathfrak{D}(b\Omega)$. In this case, the limit defines a distribution, f_0, and the convergence is not only in the weak sense, but in the strong dual topology on $\mathfrak{D}'(b\Omega)$, that is

$$\lim_{\varepsilon\to 0^+} f_\varepsilon = f_0 \quad \text{in } \mathfrak{D}'(b\Omega)\,. \tag{4}$$

This follows from standard distribution theory, see [19], chap. III, § 3. We wish to point out another useful fact: let $\tilde\varphi$ be *any* C^∞-extension of φ into Ω (C^∞ up to the boundary). Then

$$\lim_{\varepsilon\to 0^+}\int_{b\Omega} f_\varepsilon(x)\tilde\varphi_\varepsilon(x)\,d\sigma(x) = \lim_{\varepsilon\to 0^+}\langle f_\varepsilon, \tilde\varphi_\varepsilon\rangle = \langle f_0, \varphi\rangle\,, \tag{5}$$

since $\tilde\varphi_\varepsilon\to\varphi$ in $\mathfrak{D}(b\Omega)$; see again [19], chap. III, § 3.

Let us finally introduce the notation

$$\Omega_\varepsilon := \{x\in\Omega \mid \operatorname{dist}(x, b\Omega) =: d(x) > \varepsilon\}\,. \tag{6}$$

For ε small enough, Ω_ε is also a smooth domain. (5) also applies to the density of $d\sigma_\varepsilon$ with respect to $d\sigma$, thus the integrals (5) (or (3)) can be written as integrals over $b\Omega_\varepsilon$, without changing the definition of distribution boundary value.

Now we turn to *harmonic* functions admitting a distribution boundary value. Ω is still a smooth, bounded domain in $\mathbf{R}^n$. Then we have

THEOREM 1.1. *For a harmonic function f in Ω, the following properties are equivalent:*

i) *f admits a distribution boundary value on $b\Omega$*

ii) *f is in the Sobolev space $W^{-k}(\Omega)$, for some $k\in\mathbf{N}$*

iii) *there exist $C>0$, and $N\in\mathbf{N}$, such that*

$$|f(x)| \leqslant \frac{C}{d(x)^N}\,, \quad x\in\Omega \tag{7}$$

iv) (iii) *holds for* $(\operatorname{Re} f)^+$ *and* $(\operatorname{Im} f)^+$, *the positive parts of* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *respectively. Equivalently: a bound* (7) *holds* from above *for* $(\operatorname{Re} f)$ *and* $(\operatorname{Im} f)$.

v) *for all* $P \in b\Omega$ *there exists a neighborhood* $V(P)$ *of* P *in* $\mathbb{R}^n$, *a function* F *harmonic in* $V(P) \cap \Omega$, *and continuous in* $V(P) \cap \bar{\Omega}$, *constants* $a_1, \ldots, a_n$ *and an integer* N *such that*

$$f = \left(a_1 \frac{\partial}{\partial x_1} + \ldots + a_n \frac{\partial}{\partial x_n}\right)^N F \qquad \text{on } V(P) \cap \Omega \tag{8}$$

Here, $W^{-k}(\Omega)$ denotes the usual Sobolev spaces on Ω ([15]), $d(x) := \operatorname{dist}(x, b\Omega)$. Note that v) completely determines the local structure of the harmonic functions with distribution boundary value.

Proof. i $\to$ ii): By assumption $f_\varepsilon \to f_0$ in $\mathfrak{D}'(b\Omega)$. Let us restrict attention for a moment to a coordinate neighborhood U in $b\Omega$. The assumption implies that the pullbacks of the f_ε, by the coordinate mappings, converge in $\mathfrak{D}'(\tilde{u})$, where $\tilde{u}$ is the set corresponding to U in the coordinate space. For $K \subset\subset U$ compact, by the local structure of families of distributions depending continuously on a real parameter, there exists therefore a family of functions F_ε, continuous on U and depending continuously on ε for $\varepsilon \geqslant 0$, such that on K

$$f_\varepsilon(x(\xi)) = \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} F_\varepsilon(x(\xi)), x(\xi) \in K$$

Here, the ξ are local coordinates in U. For the local structure theorem just used, see [19]: theorème XXIII, chap. III., § 6, and the remark following the theorem. Now the $F_\varepsilon(x(\xi))$ define a continuous (up to U) function in a one sided neighborhood of U, and f is obtained by applying a differential operator with smooth coefficients, of order $|\alpha|$. Hence f is in $W^{-|\alpha|}$ « near U ». Since $b\Omega$ is compact, we can cover $b\Omega$ with finitely many coordinate neighborhoods of the above kind, and a partition of unity argument then gives the desired result.

ii) $\to$ iii): the equivalence of ii) and iii) has been fruitfully used by Bell. As it is so short, we reproduce his proof from [5], Lemma 2. Let $\chi \in \mathfrak{D}(\mathbb{R}^n)$ be a radially symmetric function supported in the unit ball, such that

$$\int \chi(\xi)\, dV(\xi) = 1\,. \tag{9}$$

Set $\chi_x(\xi) := \left(\dfrac{d(x)}{2}\right)^{-n} \chi\left(\dfrac{\xi - x}{d(x)/2}\right), \qquad \text{for } x \in \Omega\,.$

Then, by the mean value property of harmonic functions and by (9),

$$|f(x)| = \left| \int f(\xi)\, \chi_x(\xi)\, dV(\xi) \right| \leqslant \|f\|_{-k} \|\chi_x\|_k , \tag{10}$$

the last inequality being the Soblev inequality for the pairing of elements of $W^{-k}(\Omega)$ and $W_0^k(\Omega)$ respectively. Now

$$\|\chi_x\|_k \leqslant \frac{C}{d(x)^{+k+nh}} \tag{10a}$$

by inspection. (10) and (10a) imply the desired conclusion.

iii) $\to$ v): we assume without loss of generality that $P = 0 \in \mathbb{R}^n$ and that the interior normal at P coincides with the x_n-direction. Choose $V_1 \in \mathbb{R}^{n-1}$ so small that $b\Omega$ is locally given by

$$x_n = g(x_1, \dots, x_{n-1}) , \qquad (x_1, \dots, x_{n-1}) \in V_1 . \tag{11}$$

Boundedness by $C/d(x)^N$ implies boundedness by $C/[x_n - g(x_1, \dots, x_{n-1})]^N$, with possibly a different constant C (but the same N), for $(x_1, \dots, x_{n-1}) \in V_1$, $g(x_1, \dots, x_{n-1}) < x_n < (x_n)^0((x_n)^0$ suitably chosen). Choose $a < (x_n)^0$ such that the set $\{(x_1, \dots, x_{n-1}, a) | (x_1, \dots, x_{n-1}) \in V_1\}$ is relatively compact in Ω. For $(x_1, \dots, x_{n-1}) \in V_1$, $g(x_1, \dots, x_{n-1}) < x_n < (x_n)_0$, we define

$$F_1(x_1, \dots, x_n) := h(x_1, \dots, x_{n-1}) + \int_a^{x_n} f(x_1, \dots, x_{n-1}, s)\, ds . \tag{12}$$

The function h is to be determined in such a way that F_1 will be harmonic. This leads to the equation

$$\Delta_{n-1} h + \frac{\partial f}{\partial x_n} + \int_a^{x_n} \Delta_{n-1} f(x_1, \dots, x_{n-1}, s)\, ds = 0 , \tag{13}$$

where Δ_{n-1} is the Laplacian with respect to $(x_1, \dots, x_{n-1})$. The harmonicity of f yields

$$\Delta_{n-1} f(x_1, \dots, x_{n-1}, s) = -\frac{\partial^2}{\partial s^2} f(x_1, \dots, x_{n-1}, s) , \tag{14}$$

so that (13) becomes

$$\Delta_{n-1} h(x_1, \dots, x_{n-1}) + \frac{\partial f}{\partial x_n}(x_1, \dots, x_{n-1}, a) = 0 . \tag{15}$$

So F_1 will be harmonic if we choose h to be a solution of (15). This is possible if we take V_1 to be a ball (for simplicity). From (12) it is clear that F_1 satisfies an estimate of the form

$$|F_1(x_1, \ldots, x_n)| \leqslant \frac{C_1}{[x_n - g(x_1, \ldots, x_{n-1})]^{N-1}}, \tag{16}$$

and that

$$\frac{\partial F_1}{\partial x_n} = f. \tag{17}$$

Repeating this precedure (shrinking V_1 at each step), we obtain a function F_N which is estimated by

$$|F_N(x_1, \ldots, x_n)| \leqslant C_N \log [x_n - g(x_1, \ldots, x_{n-1})]. \tag{18}$$

Now log is integrable at zero; hence another repetition yields a bounded function, so that one more repetition then yields indeed a function F_{N+2} continuous up to $b\Omega$, by Lebesgue's dominated convergence theorem. Clearly,

$$\frac{\partial F_{N+2}}{\partial x_n^{N+2}} = f.$$

Taking $F := F_{N+2}$, and $V(P)$ a neighborhood of P in $\mathbb{R}^n$ which is small enough, v) is satisfied. The constants $a_1, \ldots, a_n$ are nothing but the components of the inward unit normal at P.

v) $\rightarrow$ i): Again by a partition of unity argument we conclude that $f \in W^{-k}(\Omega)$, for some $k \in \mathbb{N}$. This and the harmonicity of f imply that

$$f_\varepsilon \xrightarrow[\varepsilon \to 0+]{} f_0 \quad \text{in } W^{-k-\frac{1}{2}}(b\Omega). \tag{20}$$

$W^{-k-\frac{1}{2}}(b\Omega)$ again denotes the usual Sobolev spaces on $b\Omega$ (compare [15]). (20) then follows from [15], Theorems 6.5 and 8.1 of chapter 2. (20) implies a fortiori convergence in $\mathfrak{D}'(b\Omega)$.

The proof of theorem 1.1 will be complete when we show that iii) $\leftrightarrow$ iv). The implication iii) $\rightarrow$ iv) is trivial. The other direction is a consequence of a general method to obtain bounds for the negative part of a real-valued harmonic function in terms of bounds on the positive part, see Proposition 4.1 in section 4. ∎

REMARK 1. We point out for emphasis the following estimates, which are implicit in the above proof. Let $k \in \mathbb{N}$. There are constants C_1 and C_2, such that for f harmonic in Ω

$$C_1 \|f\|_{-k-1} \leqslant \sup_{x\in\Omega} |f(x)|d(x)^k \underset{\substack{\uparrow \\ \text{if } k > n/2\,.}}{\leqslant} C_2 \|f\|_{-k+n/2} \tag{21}$$

The second inequality is in ii) $\to$ iii), which was taken from [5], proof of Lemma 2. The first inequality follows from iii) $\to$ v): the local primitives of f, obtained after $k+1$ integrations, are bounded by const. times

$$\sup_{x\in\Omega} |f(x)|d(x)^k\,.$$

Essentially a partition of unity then yields the result. If one is willing to sacrifice some Sobolev regularity on the left side, the first inequality holds for non harmonic smooth functions as well (see [5], proof of Lemma 2).

The above proof shows that for a harmonic function f which admits a distribution boundary value on $b\Omega$, this boundary value is essentially the trace of f on $b\Omega$, as studied in [15] section 6.5 of chapter 2 (bear again Theorem 8.1 in mind). Furthermore there is a *local version* of Theorem 1.1 (with basically the same proof): instead of boundary values on all of $b\Omega$, one just considers boundary values in some neighborhood U of a point $P \in b\Omega$. Then the local versions of i)-v) «near» U are also equivalent.

From Theorem 1.1, we easily get the following

COROLLARY 1.2. *Let f be harmonic in Ω and assume that f admits a distribution boundary value. Let $a_\alpha(x) \in C^\infty(\bar{\Omega})$ for $|\alpha| \leqslant m$. Then the function*

$$g = \sum_{|\alpha|\leqslant m} a_\alpha(x) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$$

also admits a distribution boundary value on $b\Omega$.

Note that g is in general not harmonic.

PROOF. $(\partial^{|\alpha|}/\partial x^\alpha)f$ is harmonic. By Theorem 1.1, $f \in W^{-k}(\Omega)$ for some $k \in \mathbf{N}$, hence $(\partial^{|\alpha|}/\partial x^\alpha)f \in W^{-2-|\alpha|}(\Omega)$. Therefore

$$\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f\right)_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f\right)_0,$$

again by Theorem 2.1. By the same arguments as in the discussion of (3), (4) and (5) we conclude that $a_\alpha(\partial^{|\alpha|}/\partial x^\alpha)f$ admits a distribution boundary value for all α with $|\alpha| \leqslant m$, which gives the desired conclusion for g. ■

In section 2 we shall be interested in analytic functions with distribution boundary value. Of course, Theorem 1.1 applies. However, v) is obviously not the natural condition in this case. Also, it is clear that iv) must be required for the real part only. Thus:

THEOREM 1.3. *Let $\Omega \subset \mathbf{C}^n$ be a smooth, bounded domain, f analytic in Ω. Then* i), ii) *and* iii) *are each equivalent to each of the following*:

iv′) $\operatorname{Re} f$ *satisfies a bound* (7) *from above,*

v′) *for all $P \in b\Omega$ there exist a neighborhood $V(P)$ of P in $\mathbf{C}^n$, a function F, analytic in $V(P) \cap \Omega$ and continuous in $V(P) \cap \bar{\Omega}$, constants $a_1, \ldots, a_n$ and an integer N such that*

$$f = \left(a_1 \frac{\partial}{\partial z_1} + \ldots + a_n \frac{\partial}{\partial z_n}\right)^N F \quad \text{on } V(P) \cap \Omega . \tag{22}$$

PROOF. i), ii) and iii) are of course equivalent by Theorem 1.1. That in the analytic case iv′) is equivalent to iv) is seen by representing $\operatorname{Im} f$ as a line integral of certain first order derivatives of $\operatorname{Re} f$. That v′) implies i) needs a little adaption. Let $z_j = x_j + iy_j$, $j = 1, \ldots, n$. Without loss of generality we assume that the inward unit normal at P coincides with the y_n-direction, and that $P = 0 \in \mathbf{C}^n$. Then (12) becomes

$$F_1(z_1, \ldots, z_{n-1}, x_n + iy_n) := h(z_1, \ldots, z_{n-1}, x_n) + \int_a^{y_n} f(z_1, \ldots, z_{n-1}, x_n + is)\, ds . \tag{23}$$

The condition for analyticity of F_1 becomes:

$$\frac{\partial F_1}{\partial \bar{z}_j} = \frac{\partial h}{\partial \bar{z}_j} = 0 , \quad 1 \leqslant j \leqslant n - 1 \tag{24}$$

and

$$\begin{aligned} &\frac{\partial F_1}{\partial \bar{z}_n} \\ &= \frac{1}{2}\left(\frac{\partial h}{\partial x_n} + \int_a^{y_n} \frac{\partial f}{\partial x_n}(z_1, \ldots, z_{n-1}, x_n + is)\, ds + if(z_1, \ldots, z_{n-1}, x_n + iy_n)\right) = 0 . \end{aligned} \tag{25}$$

Using the fact that $\partial f/\partial x_n = -i(\partial f/\partial y_n)$, (25) becomes

$$\frac{\partial h}{\partial x_n}(z_1, \ldots, z_{n-1}, x_n) + if(z_1, \ldots, z_{n-1}, x_n + ia) = 0 . \tag{26}$$

If we take V_1 to be a «cube» $W \times (-\delta, \delta) \subset \mathbb{C}^{n-1} \times \mathbb{R}$, (26) is satisfied by

$$h(z_1, \ldots, z_{n-1}, x_n) := -i\int_{-\delta}^{x_n} f(z_1, \ldots, z_{n-1}, s + ia)\, ds\,. \tag{27}$$

Furthermore, h is analytic in $z_1, \ldots, z_{n-1}$, so that the equations (24) are also satisfied. Finally

$$f = \frac{\partial F_1}{\partial y_n} = i\,\frac{\partial F_1}{\partial z_n}$$

so that the proof is now completed analogous to the harmonic case. ∎

Let us point out that in the case of analytic functions, more can be said about the convergence of the f_ε in $\mathcal{D}'(b\Omega)$: the existence of the limit, as $\varepsilon \to 0^+$, in $\mathcal{D}'(b\Omega)$ entails the existence of a limit of the traces on 1-dimensional manifolds, as long as those manifolds «converge» to a 1-dimensional manifold on $b\Omega$ which is transversal to the complex tangent space of $b\Omega$, and convergence takes place in a space of C^∞-functions with values in the distributions on that transversal manifold. This is a special case of Theorem 4.1 in [20].

In view of Theorems 1.1 and 1.3, we introduce the following topological vector spaces:

$$W^{-\infty}(\Omega) := \bigcup_{k\in\mathbf{N}} W^{-k}(\Omega)\,,$$

provided with the inductive limit topology, *i.e.* the strongest locally convex topology such that all the injections are continuous. Furthermore

$$h^{-\infty}(\Omega) := \bigcup_{k\in\mathbf{N}} \big(W^{-k}(\Omega) \cap h(\Omega)\big)\,, \tag{28}$$

where $h(\Omega)$ denotes the set of harmonic functions in Ω. Then $W^{-k}(\Omega) \cap h(\Omega)$ is a closed subspace of $W^{-k}(\Omega)$, which we provide with the topology induced by $W^{-k}(\Omega)$. $h^{-\infty}(\Omega)$ then carries the inductive limit topology. Finally, we set

$$A^{-\infty}(\Omega) := \bigcup_{k\in\mathbf{N}} \big(W^{-k}(\Omega) \cap \mathcal{O}(\Omega)\big)\,, \tag{29}$$

where $\mathcal{O}(\Omega)$ denotes the set of analytic functions in Ω. The definition of all the topologies involved is analogous to the harmonic case. By Theorems 1.1 and 1.3 we know that $h^{-\infty}(\Omega)$ and $A^{-\infty}(\Omega)$ contain exactly those harmonic and analytic functions respectively, which admit a distribution boundary value on $b\Omega$. Later, we will need several properties of these spaces, all of which are essentially a consequence of the following observation: the embeddings

$$W^{-k}(\Omega) \to W^{-k-1}(\Omega) \tag{30}$$

are compact, by Rellich's lemma (see [15], Theorem 16.1 of chapter 1). Obviously, this property is then also enjoued by the defining sequences in (28) and (29). A first consequence of (30) is

LEMMA 1.4. *The inductive limit topologies on $h^{-\infty}(\Omega)$ and $A^{-\infty}(\Omega)$ coincide with the topology induced on these spaces by $W^{-\infty}(\Omega)$, when they are considered as subspaces of $W^{-\infty}(\Omega)$.*

The reader should think a moment to see that there is something to prove.

PROOF. It is clear that $h^{-\infty}(\Omega)$ and $A^{-\infty}(\Omega)$ are closed as subspace of $W^{-\infty}$. Since the embeddings (30) are compact, the lemma follows from [11], Theorem 7′. ∎

LEMMA 1.5. *$h^{-\infty}(\Omega)$ is a Montel space. A set is bounded in $h^{-\infty}(\Omega)$ if and only if it is contained in $h^{-k}(\Omega)$, for some $k \in \mathbb{N}$, and is bounded in $h^{-k}(\Omega)$. The analogous assertions hold for $A^{-\infty}(\Omega)$.*

PROOF. [11], Theorem 6′, where these properties, among others, are shown. ∎

As already mentioned, $h^{-\infty}(\Omega)$ contains exactly those harmonic functions which admit a distribution boundary value. In the next section, we shall need the result that, just as in the case of the more familar harmonic Hardy spaces, the Poisson integral mediates between boundary values and the corresponding functions (this is also of interest for its own sake). We first show that $P_\varepsilon(x, y) \to \delta_y(x)$ (Dirac distribution centered at $y \in b\Omega$), *not only in the familiar pointwise fashion, but in $C^\infty(b\Omega, \mathcal{D}'(b\Omega))$.*

PROPOSITION 1.6. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$ and $P(x, y) \in C^\infty(\Omega \times b\Omega)$ its Poisson kernel. Fix $\varphi \in \mathcal{D}(b\Omega)$. Then*

$$\lim_{\varepsilon \to 0^+} \int_{b\Omega} P_\varepsilon(x, y)\varphi(x)\,d\sigma(x) = \varphi(y)\,, \tag{31}$$

and the limit is attained in $\mathcal{D}(b\Omega)$.

REMARK 2. This is equivalent to saying that

$$\lim_{\varepsilon \to 0^+} P_\varepsilon(\cdot, y) = \delta_y(\cdot) \tag{32}$$

in $C^\infty(b\Omega, \mathfrak{D}'(b\Omega))$, the space of $\mathfrak{D}'(b\Omega)$-valued C^∞-functions on $b\Omega$, since $\mathfrak{D}'(b\Omega)$ is a Montel space.

PROOF. That $P(x, y) \in C^\infty(\Omega \times b\Omega)$ is well known. We first show that the left hand side of (31) converges in $\mathfrak{D}(b\Omega)$ to some function $\tilde{\varphi}$. If suffices to argue locally. Let $\xi = (\xi_1, \ldots, \xi_{n-1})$ be local coordinates in some coordinate neighborhood on $b\Omega$. We must show that all derivatives

$$\frac{\partial^{|\beta|}}{\partial \xi^\beta} \int_{b\Omega} P_\varepsilon(x, y(\xi)) \varphi(x) d\sigma = \int_{\Omega} \left(\frac{\partial^{|\beta|}}{\partial \xi^\beta} P \right)_\varepsilon (x, y(\xi)) \varphi(x) d\sigma \tag{33}$$

converge locally uniformly (in ξ). To this end, observe that $(\partial^{|\beta|}/\partial \xi^\beta) P_\varepsilon(x, y(\xi))$ is harmonic in x for fixed ξ. Also, recall for a moment the definition of the Poisson kernel: $P(x, y) = (\partial/\partial v) G(x, y)$, where the green's function $G(x, y)$ is given by

$$G(x, y) = \begin{cases} \dfrac{C_n}{|x-y|^{n-2}} \quad + h_x(y) & n > 2 \\ \frac{1}{2}\pi \log |x-y| & n = 2 \end{cases} \tag{34}$$

where $h_x(y)$ is the unique harmonic (in y) function such that $G(x, y)$ is zero for $y \in b\Omega$. (34) and standard elliptic theory of the Dirichlet problem, combined with Sobolev lemma arguments show that

$$\left| \frac{\partial^{|\beta|}}{\partial \xi^\beta} P(x, y(\xi)) \right| \leqslant \frac{C(n, \beta)}{d(x)^{2n+|\beta|}} , \tag{35}$$

for all ξ in some compact set K in coordinate space. The arguments in the proof of the implication iii) $\to$ v) and v) $\to$ i) show that there exist $k \in \mathbf{N}$ such that the set

$$\left\{ \frac{\partial^{|\beta|}}{\partial \xi^\beta} P(x, y(\xi)) \,\middle|\, \xi \in K \right\} \subset h^{-k}(\Omega) . \tag{36}$$

(Sobolev space with respect to x), and moreover, that this is a *bounded* set in $h^{-k}(\Omega)$. Then the sets

$$\left\{ \frac{\partial}{\partial x_j} \frac{\partial^{|\beta|}}{\partial \xi^\beta} P(x, y(\xi)) \,\middle|\, \xi \in K \right\}, \quad 1 \leqslant j \leqslant n \tag{37}$$

are bounded in $h^{-k-1}(\Omega)$. Thus

$$(38)\qquad \frac{d}{d\varepsilon}\int_{b\Omega}\left(\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}(x, y(\xi))\varphi(x)\,d\sigma = \int_{b\Omega}\left(\frac{\partial}{\partial\nu_x}\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}(x, y(\xi))\varphi(x)\,d\sigma$$
$$= \sum_{j=1}^{n}\int_{b\Omega}(a_j(x))_{\varepsilon}\left(\frac{\partial}{\partial x_j}\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}(x, y(\xi))\varphi(x)\,d\sigma\,,$$

with the $a_j(x)\in C^{\infty}(\bar{\Omega})$ being the j-th coefficient of the normal to $b\Omega_{\varepsilon}$ at $x\in b\Omega_{\varepsilon}$. So

$$(39)\qquad \frac{d}{d\varepsilon}\left\langle\left(\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}, \varphi\right\rangle = \sum_{j=1}^{n}\left\langle\left(\frac{\partial}{\partial x_j}\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}, (a_j)_{\varepsilon}\cdot\varphi\right\rangle.$$

Now the set of linear mappings

$$(40)\qquad T_{\varepsilon}\colon h^{-k-1}(\Omega)\to W^{-k-\frac{3}{2}}(b\Omega)\,,\qquad h\to h_{\varepsilon}$$

is a bounded set in $\mathfrak{L}\big(h^{-k-1}(\Omega), W^{-k-\frac{3}{2}}(b\Omega)\big)$, as $\varepsilon\to 0^{+}$, with the latter space being the usual Banach space of continuous linear mappings between two Hilbert spaces. This fact is proved in [15], chapter 2, (8.5), in the course of the proof of their Theorem 8.1, which has already been useful in the proof of our Theorem 1.1. Since $(a_j)_{\varepsilon}\to(a_j)_0$ in $\mathfrak{D}(b\Omega)$, the set $\{(a_j)_{\varepsilon}\varphi|\varepsilon>0\}$ is bounded in $\mathfrak{D}(b\Omega)$, hence in $W^{k+\frac{3}{2}}(b\Omega)$, which is the dual of $W^{-k-\frac{3}{2}}(b\Omega)$. Putting all these boundedness properties together, we obtain from (39):

$$(41)\qquad \left|\frac{d}{d\varepsilon}\left\langle\left(\frac{\partial^{|\beta|}}{\partial\xi^{\beta}}P\right)_{\varepsilon}, \varphi\right\rangle\right|\leqslant C(\varphi)\,,$$

for some C independent of ξ, as $\varepsilon\to 0^{+}$.

This implies uniform convergence for $\xi\in K$ of the integrals (33). We have shown so far that the left hand sides of (31) converge in $\mathfrak{D}(b\Omega)$ to some limit $\tilde{\varphi}$. That $\tilde{\varphi}=\varphi$ follows from the standard reproducing properties of the Poisson kernel: integrate both sides of (31) against $\psi(y)$ on $b\Omega$, for a $\psi\in\mathfrak{D}(b\Omega)$ to obtain that

$$(42)\qquad \int_{b\Omega}\tilde{\varphi}(y)\,\psi(y)\,d\sigma = \int_{b\Omega}\varphi(y)\,\psi(y)\,d\sigma\,.$$

As ψ was arbitrary, this implies $\tilde{\varphi}=\varphi$. This completes the proof of Proposition 1.6. ■

COROLLARY 1.7. *The map $\tau \mapsto P\tau$ with*

$$P\tau(x) := \langle P(x,\cdot), \tau\rangle \tag{43}$$

is an isomorphism of $W^{-k-\frac{1}{2}}(b\Omega)$ onto $h^{-k}(\Omega)$, for all $k \in \mathbb{N}$ (and, consequently, of $\mathfrak{D}'(b\Omega)$ onto $h^{-\infty}(\Omega)$). Its inverse is the map assigning to each $h \in h^{-k}(\Omega)$ its boundary value.

PROOF. For $x \in \Omega$, $P(x,\cdot) \in \mathfrak{D}(b\Omega)$, hence (43) is well defined. Obviously, $P\tau \in h(\Omega)$. We use again Theorems 6.7 and 8.1 of chapter 2 of [15] to conclude that the linear map which assigns for each $h \in h^{-k}(\Omega)$ its distribution boundary value on $b\Omega$ is an isomorphism of $h^{-k}(\Omega)$ onto $W^{-k-\frac{1}{2}}(b\Omega)$, for all $k \in \mathbb{N}$. Therefore, it remains to show that $P\tau$ has the distribution boundary value τ on $b\Omega$, $\forall \tau \in \mathfrak{D}'(b\Omega)$. This is animmediate con sequence of the foregoing proposition: for $\varphi \in \mathfrak{D}(b\Omega)$ we calculate

$$\begin{aligned}\int_{b\Omega} (P\tau)_\varepsilon(x)\varphi(x)\,d\sigma &= \int_{b\Omega} \langle \tau, P(x-\varepsilon n(x),\cdot)\rangle\varphi(x)\,d\sigma \\ &= \Big\langle \tau, \int_{b\Omega} P(x-\varepsilon n(x),\cdot)\varphi(x)\,d\sigma\Big\rangle \xrightarrow[\varepsilon\to 0^+]{} \langle\tau,\varphi\rangle;\end{aligned} \tag{44}$$

the last conclusion follows from Proposition 1.6. (44) says that the boundary value of $P\tau$ is τ, which we wanted to show. That (43) is an isomorphism from $\mathfrak{D}'(b\Omega)$ onto $h^{-\infty}(\Omega)$ follows from the fact that the two spaces are the inductive limits of $W^{-k-\frac{1}{2}}(b\Omega)$ and $h^{-k}(\Omega)$, respectively. For $h^{-\infty}(\Omega)$ this is so by definition. For $\mathfrak{D}'(b\Omega)$, the strong dual of $\mathfrak{D}(b\Omega)$, this follows for example by the observation that the embeddings $W^{k+\frac{3}{2}} \to W^{k+\frac{1}{2}}$, $k \in \mathbb{N}$ are compact (a partition of unity and Rellich's lemma in local coordinates) and by Theorem 11 in [11], since $\mathfrak{D}(b\Omega)$ is the projective limit of the spaces $W^{k+\frac{1}{2}}(b\Omega)$, $k \in \mathbb{N}$, and the dual of $W^s(b\Omega)$ is $W^{-s}(b\Omega)$ ([15]). ■

Corollary 1.7 says in particular that the boundary value on $b\Omega$ uniquely determines the function. In the case of *analytic* functions, much more can be said. It suffices to consider boundary values not only in an arbitrary small open set, but even on arbitrary small pieces of totally real *n-dimensional* submanifolds in $b\Omega$. These distribution boundary values then uniquely determine the function. This follows from [20], Theorems 4.1 and 2.1, (see also remark 2.4), and generalizes the corresponding result of Pinčuk for continuous boundary values ([17], [18]). Corollary 1.7 also answers in a general setting a question raised in [13] concerning the representation of harmonic functions by means of a generalized Poisson integral. Of course, for a space of harmonic functions (different from one of the $h^{-k}(\Omega)$, but still satisfying the conditions in Theorem 1.1), the problem arises to describe the resulting space of distributions on the boundary. We note that the space for which Korenblum does this, i.e. f real-valued, harmonic in the unit disc such that $f(0) = 0$ and $-\infty < f(z) \leqslant -C \log(1-|z|)$, is such a space: Prop. 4.1 ensures that the conditions of Theorem 1.1 are met. In section 2 we will describe the boundary values obtained from the functions in $A^{-\infty}(\Omega)$.

Theorem 1.1 now allows to define an « $\mathfrak{L}_2$-inner product » between f and g, whenever $f \in h^{-\infty}(\Omega)$ and $g \in C^\infty(\bar{\Omega})$ (in fact, as will be clear, $f \in h^{-k}(\Omega)$ and $g \in W^k(\Omega)$ will suffice), and to derive corresponding Sobolev inequalities. It was first realized by Bell ([4], [5]) that in the presence of harmonicity or analyticity special Sobolev inequalities hold. Here we present an approach which we believe provides new insight and which, in the case of harmonic functions, yields a sharper Sobolev estimate. Assume now that $f \in h^{-\infty}(\Omega)$, $g \in C^\infty(\bar{\Omega})$. We define

$$\tag{45} \widetilde{\int_{\Omega}} f\bar{g}\, dV := \int_{\Omega_\varepsilon} f\bar{g}\, dV + \int_0^\varepsilon \Big(\int_{b\Omega_{\varepsilon'}} f\bar{g}\, d\sigma \Big) d\varepsilon' = \lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} f\bar{g}\, dV .$$

Clearly, if $f \in \mathfrak{L}_2(\Omega)$, $\widetilde{\int} f\bar{g}\, dV = \int f\bar{g}\, dV$, provided the right hand side of (45) is well defined. However, since $f \in h^{-\infty}(\Omega)$, $g \in C^\infty(\bar{\Omega})$,

$$\tag{46} \lim_{\varepsilon \to 0^+} \int_{b\Omega_\varepsilon} f(x)\bar{g}(x)\, d\sigma_\varepsilon(x) = \langle \tau, \bar{g}|_{b\Omega} \rangle ,$$

where τ is the distribution boundary value of f, which we know to exist by Theorem 1.1 (we have once more used (5); note that $d\sigma_\varepsilon = \chi_\varepsilon\, d\sigma$). (46) immediately shows that everything in (45) is well defined, and that the last equality holds. We note that in general $f\bar{g}$ need not be integrable on Ω. For the pairing just defined, the following *Green's formula holds*:

LEMMA 1.8. *Let* $f \in h^{-\infty}(\Omega)$, $g \in C^\infty(\bar{\Omega})$. *Then*

$$\tag{47} \widetilde{\int_{\Omega}} f \Delta g\, dV = \Big\langle f, \frac{\partial g}{\partial \nu} \Big\rangle_{b\Omega} - \Big\langle \frac{\partial f}{\partial \nu}, g \Big\rangle_{b\Omega} .$$

The pairings on the right side of (47) *are in the duality between* $\mathfrak{D}'(b\Omega)$ *and* $\mathfrak{D}(b\Omega)$.

PROOF. Since $f \in h^{-\infty}(\Omega)$, both f and $\partial f/\partial \nu$ have distribution boundary values on $b\Omega$ (Corollary 1.2), so the right side of (47) is well defined. (47) now follows by integrating over Ω_ε, using the standard Green's formula for Ω_ε, and passing to the limit. ∎

PROPOSITION 1.9. *Let $f \in h^{-\infty}(\Omega)$, $g \in C^\infty(\bar{\Omega})$, $k \in \mathbb{N}$. Then*

$$\left|\tilde{\int_\Omega} f\bar{g}\, dV\right| \leqslant C(k)\|f\|_{-k}\|g\|_k , \tag{48}$$

where the norms are Sobolev norms.

PROOF. If $f \notin W^{-k}(\Omega)$, the right hand side of (48) is $+\infty$, so nothing is to prove. Now assume $f \in W^{-k}(\Omega) \cap h(\Omega) = h^{-k}(\Omega)$. Then, by Corollary 1.7, the boundary value of f is in $W^{-k-\frac{1}{2}}(b\Omega)$, and its Sobolev norm is bounded by const. times the $(-k)$-norm of f on Ω. Let h be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta h = \bar{g} & \text{on } \Omega \\ h = 0 & \text{on } b\Omega . \end{cases} \tag{49}$$

Then $h \in C^\infty(\bar{\Omega})$ and $\|h\|_{k+2} \leqslant C_1\|g\|_k$. Then, by Lemma 1.8,

$$\left|\tilde{\int_\Omega} f\bar{g}\, dV\right| = \left|\tilde{\int_\Omega} f \Delta h\, dV\right| = \left|\left\langle f, \frac{\partial h}{\partial \nu}\right\rangle_{b\Omega}\right| \leqslant \|f\|^{b\Omega}_{-k-\frac{1}{2}} \cdot \left\|\frac{\partial h}{\partial \nu}\right\|^{b\Omega}_{k+\frac{1}{2}} .$$

Now

$$\left\|\frac{\partial h}{\partial \nu}\right\|^{b\Omega}_{k+\frac{1}{2}} \leqslant C_2 \left\|\frac{\partial h}{\partial \nu}\right\|_{k+1} \leqslant C_3\|h\|_{k+2} \leqslant C_4\|g\|_k ,$$

the first inequality being part of the standard trace theorem in $W^{k+1}(\Omega)$ (note that $k+1 > \frac{1}{2}$). Therefore, since also $\|f\|^{b\Omega}_{-k-\frac{1}{2}} \leqslant C_5\|f\|_{-k}$,

$$\left|\tilde{\int} f\bar{g}\, dV\right| \leqslant C\|f\|_{-k}\|g\|_k . \qquad ∎$$

REMARK 3. It is easy to check that our definition (45) of the « integral » of $f \cdot \bar{g}$ for $f \in h^{-\infty}(\Omega)$, $g \in C^\infty(\bar{\Omega})$ coincides with that coming from the sesquilinear pairing introduced in [5]. For $f \in h^\infty(\Omega)$, equality is checked by inspection. Since for g fixed, both expressions are continuous on $h^{-\infty}(\Omega)$, the conclusion follows from the observation that h^∞ is dense in $h^{-\infty}$ ($\mathcal{D}(b\Omega)$ is dense in $\mathcal{D}'(b\Omega)$, and Corollary 1.7).

REMARK 4. Let $k \in \mathbf{N}$ be fixed. Since $C^\infty(\bar{\Omega})$ is dense in $W^k(\Omega)$, our sesquilinear pairing on $h^{-k}(\Omega) \times C^\infty(\bar{\Omega})$ is extended by continuity to $h^{-k}(\Omega) \times W^k(\Omega)$, by virtue of (48). For this extended pairing, the inequality (48) is of course again satisfied. We shall have the opportunity to make use of this extension in section 3, in the course of the proof of Theorem 3.3.

2. – Bochner-Hartogs extension.

The purpose of this section is to show that when boundary values of functions in $A^{-\infty}(\Omega)$ are considered, the resulting space on the boundary is *exactly* the space of CR-distributions (basically, distribution solutions of the tangential CR-equations); in other words, we want to generalize the Bochner-Hartogs phenomenon to CR-distributions. Since the conditions imposed on CR-distributions are *differential* conditions, hence *local* conditions, it is clear that we must require the complement of $\Omega \subset \mathbf{C}^n$ to be connected. Furthermore, these conditions are basically stated in terms of complex differentiability along complex tangent directions, which show up only for $n > 1$, and have therefore no analogue for $n = 1$. We thus assume $n > 1$ throughout this section. We point out, however, that the case with nonconnected complement, as well as the case $n = 1$, can be treated just as in the case of functions, namely by imposing *integral* (i.e. *global*) conditions on the distributions on the boundary analogous to those used in [10], chapter I. This will be apparent from the proof of Theorem 2.2 below. Finally, we remark that the results to be shown are, to a large extent, in § 6 of [23]. Here we give a different approach, which is natural in our context. (The notion of boundary value of section 1 works equally well for analytic functionals (hyperfunctions) on $b\Omega$; the basic trace theorems for this case are in [16]). Bochner-Hartogs extension of « weak solutions » of the tangential CR-equations, i.e. functions $f \in \mathfrak{L}_1(b\Omega)$ such that

$$\int_{b\Omega} f\, \bar{\partial}\omega = 0 \tag{1}$$

for all $\omega \in C^\infty_{n,n-2}(b\Omega)$, the space of smooth $(n, n-2)$-forms on $b\Omega$, is treated in [10]. It is easy to check that $f \in \mathfrak{L}_1(b\Omega)$ is a weak solution if and only if the corresponding distribution ((1) of section 1!) is a CR-distribution, to be defined next (compare the proof of Lemma 2.1).

Let $P \in b\Omega$. A vector X in $T_P(b\Omega)$ is called a complex tangent vector, $X \in T_P^{\mathbf{C}}(b\Omega)$, if iX is also a tangent vector, where the multiplication by i is the one induced by the ambient space $\mathbf{C}^n$. We will refer to vector fields X

such that $X(P)$ is a complex tangent vector for all P as complex tangent fields. This is not to be confused with the complexification of the tangent bundle, which is not used here. Then $T_P^{\mathbf{C}}(b\Omega)$ is a complex vector space of (complex) dimension $n-1$ (the orthogonal complement in $\mathbf{C}^n$ of $n(P)$, the normal to $b\Omega$ at P). Let $U(P)$ be a coordinate neighborhood of P, small enough such that there exist $n-1$ smooth complex vector fields $X_1, \ldots, X_{n-1}$ on $U(P)$ with the property that $X_1(Q), \ldots, X_{n-1}(Q)$ span $T_Q^{\mathbf{C}}$ for all $Q \in U$. Denote by $\sqrt{g}$ the density of the volume element of $b\Omega$ in the local coordinates $(\xi_1, \ldots, \xi_{2n-1})$ on $U(P)$. Then $\tau \in \mathfrak{D}'(b\Omega)$ is a *CR-distribution on* $U(P)$ if, in the local coordinates,

$$(2) \qquad (D_{X_k} + iD_{iX_k})\left(\frac{\tau}{\sqrt{g}}\right) = 0\,, \quad 1 \leqslant k \leqslant n-1$$

τ is a *CR-distribution on* $b\Omega$, if every $P \in b\Omega$ has a neighborhood $U(P)$ as described above, such that the restriction of τ to $U(P)$ is CR there. To make CR-distributions well-defined objects we must check that (2) is independent of the local coordinate system. One checks by inspection that

$$(3) \qquad \sqrt{g}\,(D_{X_k} + iD_{iX_k})\left(\frac{\tau}{\sqrt{g}}\right)$$

is coordinate invariant, i.e. defines an element of $\mathfrak{D}'(U(P))$. Since $\sqrt{g}$ is always different from zero, (2) is equivalent to saying that the distribution (3) equals zero, hence (2) is independent of the choice of coordinates. Let us also mention an invariant approach: the distribution defined by (3) is nothing but $(T_{X_k})'\,\tau$, where the prime denotes the adjoint, and T_{X_k} is the continuous operator on $\mathfrak{D}(U(P))$ defined by the equation

$$(4) \qquad (T_{X_k}\varphi)\cdot d\sigma = (D_{X_k} + iD_{iX_k})\varphi\, d\sigma\,.$$

The right hand side of (4) denotes the sum of the Lie-derivatives of the $(2n-1)$-form $\varphi\, d\sigma$ (note that iX_k is also a tangent field), which is again a $(2n-1)$-form and therefore is written uniquely as a function times $d\sigma$. That (3) and $(T_{X_k})'\tau$ are the same is seen for example by observing that

$$(5) \qquad D_X(\varphi\, d\sigma) = (D_X\varphi + \varphi \operatorname{div} X)\, d\sigma\,.$$

(5) and the analogous formula for D_{iX} combined with the standard formulas for the divergence in local coordinates then show that (3) indeed represents $(T_{X_k})'\tau$ in the local coordinates. Thus CR-distributions can be defined without having recourse to a local coordinate system by the require-

ment that

$$(6) \qquad (T_{X_k})'\tau = 0\,, \qquad 1 \leqslant k \leqslant n-1\,.$$

Finally, it is clear that our definitions are independent of the choice of the local basis fields X_k, $1 \leqslant k \leqslant n-1$.

Obviously (from (2)), a C^1-function is a classical CR-function if and only if the corresponding distribution (defined by (1) of section 1) is a CR-distribution. Also, boundary values of analytic functions are CR-distributions: they are CR on the manifolds Ω_ε (as restrictions of analytic functions) and (2) then follows by continuity as $\varepsilon \to 0^+$. To show that conversely, every CR-distribution on $b\Omega$ is the boundary value of an analytic function, we will need

LEMMA 2.1. *Let τ be CR on $b\Omega$, $\chi \in \mathfrak{D}(b\Omega)$ such that*

$$(7) \qquad \chi\, d\sigma = \bar{\partial}\omega$$

for some $\omega \in C^\infty_{(n,\,n-2)}(b\Omega)$. Then

$$(8) \qquad \langle \tau, \chi \rangle = 0\,.$$

PROOF. By a partition of unity, applied to ω, it suffices to prove (8) for an ω with compact support contained in an arbitrary small open neighborhood $U(P)$ of some $P \in b\Omega$. We assume $U(P)$ to be a coordinate neighborhood and small enough so that basis fields $X_1, \dots, X_{n-1}$ for the complex tangent space exist. Choose a sequence f_n of functions in $C^\infty(U(P))$ such that the associated distributions (via (1) of section 1) τ_{f_n} converge to τ in $\mathfrak{D}'(U(P))$. Then

$$(9) \qquad \langle \tau, \chi \rangle = \lim_{n\to\infty} \langle \tau_{f_n}, \chi \rangle = \lim_{n\to\infty} \int_{U(P)} f_n \chi\, d\sigma$$
$$= \lim_{n\to\infty} \int_{U(P)} f_n\, \bar{\partial}\omega = \lim_{n\to\infty} \left(\int_{U(P)} \bar{\partial}(f_n\omega) - \int_{U(P)} \bar{\partial} f_n \wedge \omega \right).$$

Now $\bar{\partial}(f_n\omega) = d(f_n\omega)$ on $b\Omega$, since ω is an $(n, n-2)$-form; therefore, the corresponding integral is zero by Stokes' theorem. Furthermore

$$(10) \qquad \int_{U(P)} \bar{\partial} f_n \wedge \omega = \int_{U(P)} \bar{\partial}_\tau f_n \wedge \omega$$

where $\bar{\partial}_\tau$ is the tangential part of $\bar{\partial}$, so that

$$\bar{\partial}_\tau f_n = \sum_{k=1}^{n-1} (D_{X_k} + iD_{iX_k}) f_n \cdot \alpha_k \tag{11}$$

for certain $(0, 1)$-forms α_k. Also, since $\tau_{f_n} \to \tau$ in $\mathfrak{D}'(U)$, we have that in the local coordinates ξ

$$f_n \to \frac{\tau}{\sqrt{g}} \quad \text{in } \mathfrak{D}'_\xi, \tag{12}$$

$\sqrt{g}$ again defined by $d\sigma = \sqrt{g}\, d\xi$. Putting (9), (10), (11), and (12) together, we obtain

$$\begin{aligned} \langle \tau, \chi \rangle &= -\lim_{n\to\infty} \sum_{k=1}^{n-1} \int_{U(P)} (D_{X_k} + iD_{iX_k}) f_n \cdot \alpha_k \wedge \omega \\ &= -\sum_{k=1}^{n-1} \left\langle (D_{X_k} + iD_{iX_k}) \left(\frac{\tau}{\sqrt{g}} \right), \varphi_k \right\rangle_\xi = 0 . \end{aligned} \tag{13}$$

Here $\varphi_k \in \mathfrak{D}_\xi$ is such that $\alpha_k \wedge \omega = \varphi_k\, d\xi$. Since τ is CR, (2) holds and yields the last equation in (13). ∎

We denote by $CR(b\Omega)$ the space of CR-distributions on $b\Omega$. Clearly, this is a closed subspace of $\mathfrak{D}'(b\Omega)$. We provide it with the topology induced by $\mathfrak{D}'(b\Omega)$. Set

$$CR^{-k-\frac{1}{2}}(b\Omega) := CR(b\Omega) \cap W^{-k-\frac{1}{2}}(b\Omega), \quad k \in \mathbb{N}, \tag{14}$$

provided with the topology induced by $W^{-k-\frac{1}{2}}(b\Omega)$. As in Lemma 1.4, we conclude that $CR(b\Omega)$ is the inductive limit of the sequence $CR^{-k-\frac{1}{2}}(b\Omega)$ (use that $\mathfrak{D}'(b\Omega)$ is the inductive limit of the sequence $W^{-k-\frac{1}{2}}$, $k \in \mathbb{N}$). By $P\tau$ we still denote the Poisson extension of $\tau \in \mathfrak{D}'(b\Omega)$ as introduced in (43) of section 1. We quickly recall the Bochner-Martinelli form. It is

$$\alpha(z, \zeta) := \frac{(n-1)!}{(2\pi i)^n} \sum_{n=1}^{n} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\zeta_k \wedge \lambda_k, \quad z \in \Omega,\ \zeta \in b\Omega \tag{15}$$

where

$$\lambda_k := \prod_{j \neq k} d\bar{\zeta}_j \wedge d\zeta_j . \tag{16}$$

We write α as

$$\alpha(z, \zeta) = \chi(z, \zeta)\, d\sigma(\zeta) , \tag{17}$$

thus defining $\chi(z,\cdot)\in\mathfrak{D}(b\Omega)$. For $\tau\in\mathfrak{D}'(b\Omega)$, we define the Bochner-Martinelli extension to be

$$BM\tau(z):=\langle\tau,\chi(z,\cdot)\rangle_\zeta\,. \tag{18}$$

Our result on Bochner-Hartogs extension for CR-distributions is

THEOREM 2.2. *Let $\Omega\subset\mathbb{C}^n$ $(n>1)$ be a smooth, bounded domain with connected complement. Then the map*

$$\tau\mapsto P\tau \tag{19}$$

is an isomorphism (of top. VS) of $CR^{-k-\frac12}(b\Omega)$ onto $A^{-k}(\Omega)$, $\forall k\in\mathbb{N}$ (and consequently of $CR(b\Omega)$ onto $A^{-\infty}(\Omega)$). The Poisson extension $P\tau$ concides in this case with the Bochner-Martinelli extension $BM\tau$.

PROOF. The proof is an adaption of the $\mathfrak{L}_1$-case as it is presented in [10]. The major new difficulty is to show that the boundary value of $P\tau$ is again τ, also in the distribution case. This we have already done (Corollary 1.7). In view of this corollary and the observation that boundary values of analytic functions are CR, which we made earlier in this section, we only need to prove that $\tau\in CR(b\Omega)$ implies $P\tau\in A^{-\infty}(\Omega)$ and that Poisson and Bochner-Martinelli extension agree. The former will be a consequence of the latter. The relationship between Poisson and Bochner-Martinelli kernel is as follows:

$$P(z,\zeta)\,d\sigma(\zeta)=\chi(z,\zeta)\,d\sigma(\zeta)-\beta(z,\zeta) \tag{20}$$

where

$$\beta(z,\zeta)=\frac{(n-2)!}{(2\pi i)^n}\sum_{k=1}^{n}\frac{\partial H(z,\zeta)}{\partial\zeta_k}\,d\zeta_k\wedge\lambda_k\,. \tag{21}$$

$H(z,\zeta)$ is the unique harmonic function (in ζ) with boundary values $1/|\zeta-z|^{2n-2}$. (20) follows by consideration of Green's function for Ω, compare [10], p. 615. In particular, since H is harmonic in ζ, the form β is $\bar\partial$-closed. Therefore, by Weinstock's approximation theorem ([22], Theorem 1), it can be approximated in $C^\infty_{n,n-1}(\bar\Omega)$ by a sequence β_k of forms belonging to $C^\infty_{n,n-1}(\mathbb{C})^n$ and $\bar\partial$-closed in all of $\mathbb{C}^n$. But then

$$\beta_k=\bar\partial\gamma_k \tag{22}$$

for some γ_k in $C^\infty_{n,n-2}(\mathbb{C}^n)$. (20), (22) and Lemma 2.1 now imply that $P\tau(z)$ and $BM\tau(z)$ agree, for all $z\in\Omega$, and our second assertion is proved. To prove the first, i.e. that $P\tau\in A^{-\infty}(\Omega)$, it suffices of course to show that $P\tau$

is analytic in Ω. We have, for j fixed,

$$\frac{\partial}{\partial \bar{z}_j} P\tau(z) = \frac{\partial}{\partial \bar{z}_j} BM\tau(z) = \left\langle \tau, \frac{\partial}{\partial \bar{z}_j} \chi(z,\cdot)_\zeta \right\rangle. \tag{23}$$

Differentiating (17) yields

$$\frac{\partial}{\partial \bar{z}_j} \alpha(z,\zeta) = \frac{\partial}{\partial \bar{z}_j} \chi(z,\zeta)\, d\sigma(\zeta). \tag{24}$$

Using the well known property of the Bochner-Martinelli form that

$$\frac{\partial}{\partial \bar{z}_j} \alpha(z,\zeta) = \bar{\partial}_\zeta \left(\frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\zeta_k \wedge d\zeta_j \wedge \lambda_{jk} \right), \tag{25}$$

with

$$\lambda_{jk} = \prod_{l \neq j,k} d\bar{\zeta}_l \wedge d\zeta_l, \tag{26}$$

we conclude again from Lemma 2.1 that

$$\left\langle \tau, \frac{\partial}{\partial \bar{z}_j} \chi(z,\cdot) \right\rangle_\zeta = 0. \tag{27}$$

(23) now shows that $P\tau$ is analytic, which completes the proof of the theorem. ∎

3. – Applications.

In this section, we give some applications of the preceding results and techniques to the Bergman and Szegö projections.

The *Bergman projection* P associated to a smooth bounded domain Ω in $\mathbf{C}^n$ is the orthogonal projection of $\mathfrak{L}_2(\Omega)$ onto $A^0(\Omega)$, the subspace of square integrable analytic functions. The Bergman kernel function associated to this projection is defined via

$$(Pg)(w) = \int_{b\Omega} K(w,z) g(z)\, dV(z) \tag{1}$$

for all $g \in \mathfrak{L}_2(\Omega)$. $K(w,z)$ is analytic in w, conjugate analytic in z and $K(w,z) = \overline{K(z,w)}$. These and other elementary properties of the projection and the kernel can be found in section 1.4 of [14]. The *Szegö projection* S associated to a bounded smooth domain Ω is the orthogonal projection of $\mathfrak{L}_2(b\Omega)$, onto $H_2(b\Omega)$, the $\mathfrak{L}_2(b\Omega)$-closure of the restrictions of $A^\infty(\Omega)$ functions to $b\Omega$. Each $f \in H_2(b\Omega)$ has a unique analytic extension to Ω, which we also denote by f. For $w \in \Omega$, $Sg(w)$ is given by

$$Sg(w) = \int_{b\Omega} S(w, z) g(z)\, d\sigma(z)\,, \qquad g \in \mathfrak{L}_2(b\Omega)\,. \tag{2}$$

The kernel $S(w, z)$ is the Szegö-kernel. For elementary properties, compare section 1.5 of [14].

The theorem which follows has been shown recently by Boas ([9]). The approach developed in section 1 leads to a short, yet natural proof.

Theorem 3.1 ([9]). *Let Ω be a smooth, bounded demain in $\mathbf{C}^n$. The following two conditions are equivalent*:

i) *The Szegö projection S maps $C^\infty(b\Omega)$ into $C^\infty(b\Omega)$.*

ii) *For every multi index α there are numbers $C > 0$ and $N \in \mathbf{N}$, such that*

$$\sup_{w\in\Omega} \left| \frac{\partial^{|\alpha|}}{\partial w^\alpha} S(w, z) \right| \leqslant \frac{C}{d(z)^N}\,, \tag{3}$$

where $d(z) := \operatorname{dist}(z, b\Omega)$.

Note that i) is equivalent to S mapping $C^\infty(b\Omega)$ continuously into $C^\infty(b\Omega)$, by the closed graph theorem.

PROOF. (3) is equivalent to $U_\alpha := \{(\partial^{|\alpha|}/\partial w^\alpha) S(w, \cdot) | w \in \Omega\}$ being bounded in $h^{-k}(\Omega)$, for some $k \in \mathbf{N}$, compare remark 1 of section 1. This in turn is equivalent to U_α being bounded in $h^{-\infty}(\Omega)$, by Lemma 1.5, hence to U_α being weakly bounded in $h^{-\infty}(\Omega)$ ($h^{-\infty}(\Omega)$ is Montel, or of course by Mackey's theorem). By Corollary 1.7 this holds if and only if the corresponding set of boundary values (which are now $\mathfrak{L}_2(b\Omega)$-functions) is weakly bounded in $\mathfrak{D}'(b\Omega)$. This, finally, is equivalent to:

$$\sup_{w\in\Omega} \left| \frac{\partial^{|\alpha|}}{\partial w^\alpha} S\varphi(w) \right| = \sup_{w\in\Omega} \left| \int_{b\Omega} \frac{\partial^{|\alpha|}}{\partial w^\alpha} S(w, z) \varphi(z)\, d\sigma(z) \right| \leqslant M(\varphi)\,, \tag{4}$$

for all $\varphi \in \mathfrak{D}''(b\Omega) = \mathfrak{D}(b\Omega) = C^\infty(b\Omega)$. But (4) (for all α!) clearly is equivalent to $S\varphi \in C^\infty(\bar{\Omega})$, which is the case if and only if the boundary value is in $C^\infty(b\Omega)$. ∎

REMARK 1. The corresponding result for the Bergman projection is also true and has been shown in [7]. Using on one hand that $h^{-\infty}(\Omega)'$ is a quotient of $\bigcap_{k\geq 0} W_0^k(\Omega)$, and on the other that each $g\in C^\infty(\bar{\Omega})$ defines a continuous linear functional on $h^{-\infty}(\Omega)$ via our extended $\mathfrak{L}_2$-pairing, one easily gets a proof of this result analogous to the one above.

It has been known since [2] (compare [1] for a modification) that for $g\in C^\infty(\bar{\Omega})$, there always exists $h\in C^\infty(\bar{\Omega})$ which vanishes to some prescribed finite order on $b\Omega$, and which has the same Bergman projection Ph as g. In [6] it is essentially shown that h can be chosen to vanish to infinite order on $b\Omega$. We recover this result by a different method which adds considerable precision to the statement.

THEOREM 3.2. *Let Ω be a smooth, bounded domain in $\mathbf{C}^n$. Fix $m\in\mathbf{N}$. There exist $C>0$ and $N\in\mathbf{N}$ such that for every $g\in C^\infty(\bar{\Omega})$ there exists $h\in\bigcap_{k\in\mathbf{N}} W_0^k(\Omega)$ with*

$$Ph = Pg \tag{5}$$

and

$$\|h\|_m \leqslant C\|g\|_N\,, \tag{6}$$

where the norms are Sobolev norms in $W_0^m(\Omega)$ and $W^N(\Omega)$ respectively. Furthermore, if $(g_n)_1^\infty$ is a sequence converging to 0 in $C^\infty(\bar{\Omega})$, there exists a sequence $(h_n)_1^\infty$ converging to zero in $C_0^\infty(\bar{\Omega}) = \bigcap_{k\in\mathbf{N}} W_0^k(\Omega)$, with $Ph_n = Pg_n$.

PROOF. $g\in C^\infty(\bar{\Omega})$ defines a continuous linear functional Tg on $h^{-\infty}(\Omega)$, via

$$\langle Tg, f\rangle := \tilde{\int_\Omega} fg\,dV\,,\qquad f\in h^{-\infty}(\Omega) \tag{7}$$

where $\tilde{\int}$ is the integral defined in (45), section 1. The map

$$C^\infty(\bar{\Omega}) \to h^{-\infty}(\Omega)'\,,\qquad g\mapsto Tg \tag{8}$$

is continuous if $h^{-\infty}(\Omega)'$ is provided with its weak topology, by Proposition 1.9. Since $h^{-\infty}(\Omega)$ is Montel (Lemma 1.5), so is $h^{-\infty}(\Omega)'$; therefore the weakly continuous map T is continuous. By Lemma 1.4, $h^{-\infty}(\Omega)$ can be considered a subspace (in the topological sense) of $W^{-\infty}(\Omega)$. Therefore, we have an isomorphism (of top. VS)

$$h^{-\infty}(\Omega)' \cong W^{-\infty}(\Omega)'/L \cong \bigcap_{k\in\mathbf{N}} W_0^k(\Omega)/L$$

by [11], Theorem 15. Here $L = \left\{h \in \bigcap_{k\in\mathbf{N}} W_0^k(\Omega) \middle| \int_\Omega fh\,dV = 0,\ \forall f \in h^{-\infty}(\Omega)\right\}$. Consider now T as a continuous operator

$$T\colon C^\infty(\bar{\Omega}) \to \bigcap_{k\in\mathbf{N}} W_0^k(\Omega)/L. \tag{9}$$

The topology on $\bigcap_{k\in\mathbf{N}} W_0^k(\Omega)/L$ is induced by the quotient semi norms ([21], chapter 7) corresponding to the Sobolev norms on $\bigcap_{k\in\mathbf{N}} W_0^k(\Omega)$. Therefore, there exist $N = N(m)$ and $C = C(m)$ such that

$$|||Tg|||_m \leqslant C\|g\|_N, \tag{10}$$

where $|||\ \ |||_m$ is the quotient semi norm corresponding to the m-th Sobolev norm. Now

$$|||Tg|||_m = \inf_{h+L=Tg} \{\|h\|_m\}; \tag{11}$$

so there exists h such that

$$h + L = Tg \tag{12}$$

and

$$\|h\|_m \leqslant 2C\|g\|_N. \tag{13}$$

Because of (12)

$$\int_\Omega fh\,dV = \int_\Omega f\tilde{g}\,dV, \quad \forall f \in h^{-\infty}(\Omega). \tag{14}$$

Setting $f := K(w, \cdot)$, the Bergman kernel function, (14) yields

$$Ph = Pg.$$

This and (13) prove the first part of the theorem. Assume now that $(g_n)_1^\infty$ is a sequence converging to zero in $C^\infty(\bar{\Omega})$. Then Tg_n also converges to zero in $\bigcap_{k\in\mathbf{N}} W_0^k(\Omega)/L$. Therefore, there exists a strictly increasing requence of integers $(N_j)_{j=1}^{\infty}$ such that

(15) $$|||Tg_n|||_j \leqslant \frac{1}{j}, \quad n \geqslant N_j .$$

Choose h_n such that

(16) $$h_n + L = Tg_n , \quad 1 \leqslant n < \infty$$

and

(17) $$\|h_n\|_j \leqslant 2|||Tg_n|||_j , \quad N_j \leqslant n < N_{j+1} .$$

Then, for $k \in \mathbf{N}$ fixed, (15) and (17) imply for all j_0:

(18) $$\|h_n\|_k \leqslant \|h_n\|_{k+j_0+j} \leqslant \frac{2}{k+j_0+j} \leqslant \frac{2}{k+j_0} , \quad n \geqslant N_{k+j_0} .$$

Clearly, (18) gives

(19) $$\|h_n\|_k \to 0 , \quad \text{as } n \to \infty .$$

Since k was arbitrary, this shows that $h_n \to 0$ in $\bigcap_{k \in \mathbf{N}} W_0^k(\Omega)$, and the proof is finished. ■

REMARK 2. Clearly, the above proof also works for the harmonic Bergman projection, that is, for the orthogonal projection of $\mathfrak{L}_2(\Omega)$ onto $h_0(\Omega)$. Therefore, Theorem 3.2 holds for this projection as well.

The method applied in [2] (and, in principle, in [1]) to find h vanishing to some prescribed order and having the same Bergman projection as a given g was to construct continuous operators

$$\varphi^k \colon W^{k+N(k)}(\Omega) \to W_0^k(\Omega) , \quad k \in \mathbf{N}$$

such that $P\varphi^k = P$. It is possible to construct operators from $W^k(\Omega)$ to $W_0^k(\Omega)$ (i.e. the same Sobolev index) with this property.

THEOREM 3.3. *Let Ω as before, $k \in \mathbf{N}$. There exist continuous linear operators*

$$T^k \colon W^k(\Omega) \to W_0^k(\Omega)$$

with

(20) $$PT^k = P .$$

PROOF. Let $g \in W^k(\Omega)$. Consider

$$f \mapsto \widetilde{\int_\Omega} f\bar{g}\,dV\,, \qquad f \in A^{-k}(\Omega) \tag{21}$$

where the pairing is the extension of (45), section 1, discussed in remark 4 of that section. Then (21) defines a continuous linear functional on $A^{-k}(\Omega)$, since

$$\left|\widetilde{\int_\Omega} f\bar{g}\,dV\right| \leqslant C(k)\|f\|_{-k}\|g\|_k\,, \tag{22}$$

compare again section 1. We thus have a continuous conjugate linear map (by (22))

$$\alpha^k\colon W^k(\Omega) \to A^{-k}(\Omega)'\,. \tag{23}$$

Let

$$P^k\colon W^{-k}(\Omega) \to A^{-k}(\Omega) \tag{24}$$

be the orthogonal Hilbert space projection, and let β^k be the canonical (conjugate) isomorphirm of $W^{-k}(\Omega)'$ onto $W_0^k(\Omega)$.

We set

$$T^k := \beta^k P^{k\prime}\alpha^k\,. \tag{25}$$

Then $T^k\colon W^k(\Omega) \to W_0^k(\Omega)$ is continuous, linear, and

$$\int_\Omega f\overline{T^k g}\,dV = \widetilde{\int_\Omega} f\bar{g}\,dV = \int_\Omega f\bar{g}\,dV \tag{26}$$

for all $f \in A_0(\Omega)$. This implies (20). ■

REMARK 3. Analogous arguments also work when P is replaced by the harmonic Bergman projection $Q\colon \mathfrak{L}_2(\Omega) \to h^0(\Omega)$. So the above proof yields operators T^k with $QT^k = Q$, $k \in \mathbb{N}$. It is then clear that (20) (with the same T^k!) holds for P the orthogonal projection onto an arbitrary subspace of $h^0(\Omega)$, since then $PQ = P$.

We give one last application to characterizing global regularity of the Bergman projection in terms of a duality between two spaces of analytic functions. P is said to be globally regular or to satisfy *condition R*, if P

takes $C^\infty(\bar{\Omega})$ into $C^\infty(\bar{\Omega})$. Note that then this mapping is automatically continuous, by the closed graph theorem. In [3], $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$ are exhibited as mutually dual via an extension of the usual $\mathfrak{L}_2$-pairing, constructed with the help of the operators mentioned before the statement of Theorem 3.3. Then, in [8] this duality is shown to be characteristic for global regularity of the Bergman projection, provided the domain Ω under consideration is (weakly) pseudoconvex. In [12], duality via a similar pairing between $A^k(\Omega)$ and $A^{-k}_{\mathrm{cl}}(\Omega)$, the closure of $A^0(\Omega)$ in $A^{-k}(\Omega)$, is shown to be equivalent to condition $(R)^k_0$: P maps $W^k_0(\Omega)$ into $A^k(\Omega)$ (no pseudoconvexity condition on the domain). We point out that Theorem 3.3 immediately implies that $(R)^k_0$ is equivalent to the seemingly stronger condition $(R)^k$: P maps $W^k(\Omega)$ into $A^k(\Omega)$. Using ideas from section 1, we will show that condition R is always equivalent to: $A^\infty(\Omega)$ and $A^{-\infty}_{\mathrm{cl}}(\Omega)$, the closure of $A^0(\Omega)$ in $A^{-\infty}(\Omega)$, are mutually dual to each other, via the pairing (45) of section 1. We note that this pairing coincides with the pairing used in the above cited cases, but has a simpler definition, thus rendering the various dualities even more natural. We need a few preparations. Let j be the following *conjugate* linear map

$$j\colon A^\infty(\Omega) \to A^{-\infty}_{\mathrm{cl}}(\Omega)'$$

(the prime denotes the strong dual), with

$$\langle j(f), h\rangle := \widetilde{\int_\Omega} \bar{f} h \, dV, \qquad (f, h) \in A^\infty(\Omega) \times A^{-\infty}_{\mathrm{cl}}(\Omega). \tag{27}$$

By Proposition 1.9, if $f_n \to 0$ in $A^\infty(\Omega)$, then $\widetilde{\int_\Omega} \bar{f} h\, dV \to 0$ uniformly as h ranges over a bounded set in $A^{-\infty}_{\mathrm{cl}}(\Omega)$, by the structure of bounded sets in $A^{-\infty}_{\mathrm{cl}}(\Omega)$ (analogous to Lemma 1.5). Thus j is continuous. j is injective: if $j(f) = 0$,

$$\langle j(f), f\rangle = \int_\Omega |f|^2 dV = 0, \tag{28}$$

whence $f = 0$. We denote by $\bar{j}'$ the adjoint of j, which is a linear map into the conjugate dual of $A^\infty(\Omega)$, composed with conjugation, so that $\bar{j}'$ is conjugate linear

$$\bar{j}'\colon A^{-\infty}_{\mathrm{cl}}(\Omega) \to A^\infty(\Omega)'. \tag{29}$$

Here, we have used that $A^{-\infty}_{\mathrm{cl}}(\Omega)$ is a Montel space ([11], Theorems 7′ and 6′)

and hence is reflexive. $\overline{j}'$ is given by

$$\langle \overline{j}'(h), f\rangle = \overline{\langle j(f), h\rangle} = \tilde{\int_{\Omega}} f\bar{h}\, dV \tag{30}$$

for $(h, f) \in A_{\mathrm{cl}}^{-\infty}(\Omega) \times A^{\infty}(\Omega)'$. Then we have

THEOREM 3.4. *Let Ω be a smooth, bounded domain in $\mathbf{C}^n$. Then the following conditions are equivalent*

i) *j is surjective*

ii) *Ω satisfies condition R*

iii) *j and $\overline{j}'$ are surjective*

iv) *j and $\overline{j}'$ are conjugate linear isomorphisms (of TVS).*

PROOF. i) $\to$ ii): Fix $g \in C^{\infty}(\bar{\Omega})$. Then

$$h \to \tilde{\int_{\Omega}} h\bar{g}\, dV \tag{31}$$

defines a continuous linear functional on $A^{-\infty}(\Omega)$ (Prop. 1.9), hence on $A_{\mathrm{cl}}^{-\infty}(\Omega)$. By i), there exists $f \in A^{\infty}(\Omega)$, such that

$$\tilde{\int_{\Omega}} h\bar{g}\, dV = \tilde{\int_{\Omega}} h\bar{f}\, dV\,, \qquad \forall h \in A_{\mathrm{cl}}^{-\infty}(\Omega)\,. \tag{32}$$

For $h \in A^0(\Omega) \subset A(\Omega)_{\mathrm{cl}}^{-\infty}$ we get

$$\int_{\Omega} h\overline{Pg}\, dV = \int_{\Omega} h\bar{g}\, dV = \int_{\Omega} h\bar{f}\, dV\,. \tag{33}$$

Since $h \in A^0(\Omega)$ was arbitrary, (33) implies $Pg = f \in A^{\infty}(\Omega) \subset C^{\infty}(\bar{\Omega})$, so that condition R is verified.

ii) $\to$ iii): Let $\tau \in A_{\mathrm{cl}}^{-\infty}(\Omega)'$. As in the proof of Theorem 3.2 we conclude that there is $g \in \bigcap_{k \in \mathbf{N}} W_0^k(\Omega)$ such that

$$\langle \tau, h\rangle = \int_{\Omega} h\bar{g}\, dV\,, \qquad \forall h \in A_{\mathrm{cl}}^{-\infty}\,. \tag{34}$$

Again, for $h \in A^0(\Omega)$, we have

$$\langle \tau, h\rangle = \int_\Omega h\bar{g}\,dV = \int_\Omega h\overline{Pg}\,dV\,. \tag{35}$$

By ii), $Pg \in A^\infty(\Omega)$. Thus all three expressions are continuous on $A_{\mathrm{cl}}^{-\infty}(\Omega)$, they therefore coincide on $A_{\mathrm{cl}}^{-\infty}(\Omega)$. This shows that $\tau = j(Pg)$. We now observe that $A_{\mathrm{cl}}^{-\infty}(\Omega)'$ is a Fréchet space ([11], Theorems 7′, 12, and 1′), and so is of course $A^\infty(\Omega)$. Since j is always injective, it follows from the surjectivity we have just shown and from the open mapping theorem that j is a conjugate linear isomorphism (of TVS). Hence this is true of the adjoint, so $\bar{j}'$ is in particular surjective.

iii) $\to$ iv): This is contained in the above argument.

iv) $\to$ i): trivial. ∎

It is worthwhile to note that everything used in the proof of Theorem 3.4 works just as well in the case of harmonic functions. In this case, the corresponding harmonic Bergman projection takes $C^\infty(\bar{\Omega})$ into $C^\infty(\bar{\Omega})$ ([5]). Furthermore, as already observed in remark 3 of section 1, $h^\infty(\Omega)$ is dense in $h^{-\infty}(\Omega)$ ($\mathfrak{D}(b\Omega)$ is dense in $\mathfrak{D}'(b\Omega)$, and Corollary 1.7), so that $h_{\mathrm{cl}}^{-\infty}(\Omega) = h^{-\infty}(\Omega)$. Therefore, the proof of ii)$\to$iii) above shows that *$h^{-\infty}(\Omega)$ and $h^\infty(\Omega)$ are mutually dual via the sesquilinear pairing which corresponds to* (27), *i.e.*

$$(h_1, h_2) \mapsto \widetilde{\int_\Omega} h_1\bar{h}_2\,dV\,, \qquad (h_1, h_2) \in h^{-\infty}(\Omega) \times h^\infty(\Omega)\,. \tag{36}$$

We have thus obtained a short proof of this duality result due to Bell ([5]). Consult again remark 3, section 1, to see that (36) and the pairing used in [5] coincide.

4. – Appendix.

We still need to show how the negative part of a real-valued harmonic function can be bounded in terms of bounds for the positive part. The proposition below looks like it should be well known, but we did not find a reference and therefore include a proof. We only treat the case of polynomial boundedness, it is clear from the proof what happens if one has bounds of a different nature for the positive part.

PROPOSITION 4.1. *Let Ω be a smooth, bounded domain in $\mathbf{R}^n$, f a real-valued harmonic function in Ω, such that its positive part, f^+, satisfies*

$$f^+(x) \leqslant \frac{C}{d(x)^N}, \tag{1}$$

for some $C > 0$ and $N \in \mathbf{N}$. Then the negative part of f satisfies

$$f^-(x) \leqslant \frac{C'}{d(x)^{N+n-1}}. \tag{2}$$

REMARK 1. The proposition says in particular: *for a real-valued harmonic function, polynomial boundedness from above implies polynomial boundedness from below.*

PROOF OF THE PROPOSITION. Fix $x_0 \in \Omega$. We may assume that $f(x_0) = 0$, since the estimates (1) and (2) are invariant under the addition of constants. For ε small enough, we have

$$0 = f(x_0) = \int_{b\Omega_\varepsilon} P_{\Omega_\varepsilon}(x_0, y) f(y)\, d\sigma_\varepsilon(y), \tag{3}$$

whence

$$\int_{b\Omega_\varepsilon} P_{\Omega_\varepsilon}(x_0, y) f^-(y)\, d\sigma_\varepsilon(y) = \int_{b\Omega_\varepsilon} P_{\Omega_\varepsilon}(x_0, y) f^+(y)\, d\sigma_\varepsilon(y) \leqslant \frac{C}{\varepsilon^N}. \tag{4}$$

The last inequality is obtained from the hypothesis (1) (of course we assume that ε is small, note that $d(y) = \varepsilon$ for $y \in b\Omega_\varepsilon$). Noting that for x_0 fixed, $P_{\Omega_\varepsilon}(x_0, \cdot)$ depends continuously on ε and the surface parameter, for small $\varepsilon \geqslant 0$, and that the Poisson kernel is always strictly positive, we see that there exists $\delta > 0$ such that

$$P_{\Omega_\varepsilon}(x_0, y) \geqslant \delta, \qquad 0 \leqslant \varepsilon \leqslant \varepsilon_0, \qquad y \in b\Omega_\varepsilon. \tag{5}$$

With this, we deduce from (4) that

$$\int_{b\Omega_\varepsilon} f^-(y)\, d\sigma_\varepsilon(y) \leqslant \frac{C}{\delta\varepsilon^N}. \tag{6}$$

Let now $x \in \Omega$. We assume that $d(x) < \varepsilon_0$. For x with $d(x) \geqslant \varepsilon_0$, the estimate (2) is trivial. Let $d(x) = \varepsilon$. Since f^- is subharmonic on Ω, we have

$$f^-(x) \leqslant \int_{b\Omega_{\varepsilon/2}} P_{\Omega_{\varepsilon/2}}(x, y) f^-(y)\, d\sigma_{\varepsilon/2}(y) \leqslant \frac{C_1}{\operatorname{dist}(x, b\Omega_{\varepsilon/2})^{n-1}} \int_{b\Omega_{\varepsilon/2}} f^-(y)\, d\sigma_{\varepsilon/2}(y)$$
$$\leqslant \frac{2^{n-1}\cdot C_1}{\varepsilon^{n-1}} \cdot \frac{C \cdot 2^N}{\delta \cdot \varepsilon^N} = \frac{C'}{\varepsilon^{N+n-1}}.$$

In the second inequality we have used that the constant in the polynomial estimate for the Poisson kernel can be chosen independently of ε, which follows by inspection of the proof ([14], Prop. 8.2.1). The last inequality follows from (6). ■

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