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On the Comparative Theory of Primes.

JANOS PINTZ - SAVERIO SALERNO (*)

1. Knapowski and Turàn investigated in a series of papers ([3]) sign changes of the functions (in the case of $i = 1, 2, 4$)

$$(1.1) \quad \begin{cases} \Delta_1(x, q, l_1, l_2) = \sum_{p \leq x} \varepsilon(p, q, l_1, l_2) \\ \Delta_2(x, q, l_1, l_2) = \sum_{n \leq x} \varepsilon(n, q, l_1, l_2) \frac{A(n)}{\log n} \\ \Delta_3(x, q, l_1, l_2) = \sum_{p \leq x} \varepsilon(p, q, l_1, l_2) \log p \\ \Delta_4(x, q, l_1, l_2) = \sum_{n \leq x} \varepsilon(n, q, l_1, l_2) A(n) \end{cases}$$

where

$$(1.2) \quad A(n) = \begin{cases} \log p & \text{if } n = p^n \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon(n) = \varepsilon_1(n) - \varepsilon_2(n), \quad \varepsilon_i(n) = \begin{cases} 1 & \text{if } n \equiv l_i(q) \\ 0 & \text{otherwise} \end{cases}$$

and (what we shall always assume without mentioning)

$$(1.3) \quad (l_1, q) = (l_2, q) = 1, \quad l_1 \not\equiv l_2 \pmod{q}.$$

For a general modulus q Knapowski and Turàn needed always the so called Haselgrove condition (H), that the L -functions have no real non-trivial zeros or, in an explicit formulation, they assumed the existence of

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a number $A(q)$, $0 < A(q) < 1$, such that (for $s = \sigma + it$)

$$(1.4) \quad L(s, \chi, q) \neq 0 \quad \text{for} \quad |t| \leq A(q).$$

It is easy to see that in the case of the existence of a real zero, being on the right from all complex zeros of all $L(s, \chi, q)$ functions, the functions $\Delta_i(x)$ are of constant sign for suitable pairs l_1, l_2 for $x > x_0$.

Thus, at the present stage of analytic number theory, a hypothesis of type (1.4) is necessary in all investigations.

We note that (1.4) has been verified by Spira [6] for all $q < 25$. The other assumption, the so called finite Riemann-Piltz conjecture (FR-P),

$$(1.5) \quad L(s, \chi, q) \neq 0 \quad \text{for} \quad \sigma \neq \frac{1}{2}, \quad |t| \leq D = cq^{10},$$

used in many investigations of Knapowski and Turàn, is of more technical nature. The aim of Knapowski and Turàn was to achieve effective results, i.e. explicitly dependent only on $q, A(q)$ (and in given cases on D). In the case of $l_1 = 1$ or $l_2 = 1$ they were able to find, using only Haselgrove condition, infinitely many sign changes of $\Delta_i(x)$ for $1 \leq i \leq 4$. (Although they did not treat the case $i = 3$, we mention the corresponding results for $i = 3$ too, if it is possible to obtain it, using the method applied by them for the other cases). Also, they could give a lower estimation of the number $V_i(Y, q, l_1, l_2)$ of sign changes of $\Delta_i(x)$ in the interval $(2, Y)$ and an explicit upper bound for the first sign change (cf. parts I-III of [3]).

For general l_1 and l_2 , besides (H), they had to assume also the finite Riemann-Piltz conjecture and even this led to results only for $i = 2$ and 4 (with other numerical estimates, naturally than in the earlier mentioned case). In the case $i = 1$ and 3 they needed the additional assumption that l_1 and l_2 would be both quadratic residues or both non-residues, besides (FR-P) and (H), (cf. parts V and VI of [3]). In the case of $i = 4$ they succeeded in showing for general l_1 and l_2 the above mentioned results in a quantitatively much weaker but (as all the earlier mentioned results) also effective form, without supposing (FR-P), i.e. only assuming (H) (cf. part VII of [3]).

We note that the most important open problem of the comparative prime number theory is to assure infinitely many sign changes of $\Delta_i(x, q, l_1, l_2)$ for all pairs l_1, l_2 with (1.3). This problem seems to be hopeless at present, even supposing the (infinite) Riemann-Piltz conjecture, besides (H) naturally. (One has nearly the same difficulties for $\Delta_3(x)$, but this is not so important from an arithmetic point of view).

In this work (assuming (H) and (FR-P) in a slightly weaker form) we shall treat the case $i = 4$, for general l_1, l_2 , (and the case $i = 3$ if l_1 and l_2 have the same quadratic character) improving earlier results of Knapowski and Turàn, which we summarize now (in some cases with slight changes) as follows:

We shall use the notations $\exp_{r+1}(x) = \exp(\exp_r(x))$, $\exp_1(x) = \exp(x)$, $\log_{r+1} x = \log \log_r x$, $\log_1 x = \log x$.

THEOREM A. Assume (H) and (FR-P) (cf. (1.4)-(1,5)). Then for

$$(1.6) \quad Y > \max \left\{ \exp_2(cq^{2^0}), \exp_2\left(\frac{c}{A^3(q)}\right) \right\}$$

one has for $i = 2, 4$

$$(1.7) \quad \max_{Y^{1/2} \leq x \leq Y} \Delta_i(x, q, l_1, l_2) > \sqrt{Y} \exp\left(-44 \frac{\log Y \log_3 Y}{\log_2 Y}\right)$$

for all pairs l_1 and l_2 with (1.3), where, as always in the following, the generic symbol c replaces an explicitly calculable positive absolute constant, which might have different values at various appearances.

Since the opposite inequality clearly holds, by changing the role of l_1 and l_2 , one obtains the following

COROLLARY. On the above conditions, for $i = 2, 4$ one has for the number of sign changes of $\Delta_i(x)$ the lower estimate

$$(1.8) \quad V_i(Y, q, l_1, l_2) > \log_2 Y.$$

THEOREM B. Assume (H) (cf. (1.4)). Then for

$$(1.9) \quad Y > \max \left(\exp_2(q^c), \exp_2\left(\frac{c}{A^3(q)}\right) \right)$$

all functions $\Delta_4(x, q, l_1, l_2)$ with (1.3) change their sign in the interval

$$(1.10) \quad [\log^2 Y/4, Y].$$

For the proofs of Theorem A and B see parts V and VII of [3], respectively.

In the present work we shall show (roughly speaking) that in the case of $i = 4$ (1.7) can be improved to

$$(1.11) \quad \max_{x \in [Y \exp(-\frac{c(q)}{\log^3 Y}), Y]} \Delta_4(x) > \sqrt{Y} \exp(-c(q) \log^3 Y)$$

thereby obtaining

$$(1.12) \quad V_4(Y) > c(q) \log^{\dagger} Y.$$

Furthermore, we get for the first sign change of $\Delta_4(x)$ the upper bound

$$(1.13) \quad Y_0 = \exp\left(\frac{q^4}{A(q)}\right),$$

which improves (1.6). (In the above formulas we neglected some $\log q$, $\log(1/A(q))$ and $\log \log Y$ factors).

Due to the lower estimate (1.11) by the strong form of the prime number theorem of arithmetic progressions, we obtain the same results in the case of $i = 3$, if l_1 and l_2 are both quadratic residues or both quadratic non-residues. (In Knapowski-Turàn's proof this needs relatively many additional efforts, as can be seen from part VI of [3]).

Finally we remark that by ineffective methods (essentially due to Landau [5, § 197], Grosswald [2] and Anderson-Stark [1]) one obtains that if for a non-trivial zero $\rho_0 = \beta_0 + i\gamma_0$ of an $L(s, \chi, q)$, function one has $\gamma_0 \neq 0$, $\beta_0 \geq \frac{1}{2}$ and

$$(1.14) \quad a(\rho_0) = \sum_{\substack{\chi(\bmod q) \\ L(\rho_0, \chi) = 0}} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) m_{\chi}(\rho_0) \neq 0$$

(where $m_{\chi}(\rho_0)$ denotes the multiplicity of ρ_0 as a zero of $L(s, \chi)$), then for every l_1, l_2 with (1.3)

$$(1.15) \quad \overline{\lim}_{x \rightarrow \infty} \frac{\Delta_4(x)}{x^{\beta_0}} \geq \left| \frac{a(\rho_0)}{\rho_0} \right|, \quad \underline{\lim}_{x \rightarrow \infty} \frac{\Delta_4(x)}{\chi^{\beta_0}} \leq - \left| \frac{a(\rho_0)}{\rho_0} \right|.$$

This is valid for $\Delta_3(x)$ too, if l_1 and l_2 have the same quadratic character, or in case of $\beta_0 > \frac{1}{2}$ for every pair l_1, l_2 with (1.3). However, this theorem does not yield any localisation of sign changes or lower estimation of $V_i(Y, q, l_1, l_2)$.

We further note that the method of Knapowski and Turàn, (also in the present refined form) does not furnish better lower estimates than (essentially) \sqrt{Y} , even assuming the existence of a zero ρ_0 with $\beta_0 > \frac{1}{2}$, $\gamma_0 \neq 0$ and $a(\rho_0) \neq 0$.

2. In our results, we shall always assume that the Haselgrove condition (H) and the finite Riemann-Piltz conjecture (FR-P) hold, the second

one up to a level D with

$$(2.1) \quad D = c_0 q^2 \log^6 q$$

where c_0 is a sufficiently large positive absolute constant.

Here and in the sequel, we shall denote by c_i , $i = 0, 1, \dots$ explicitly calculable positive absolute constants; moreover, the generic symbol c , and the signs \gg , \ll , O , replace such constants; $\exp(x) = e^x$, $\log_2 Y = \log \log Y$.

We shall also assume without any further mention the trivial condition (1.3) on l_1, l_2 .

Our results are the following:

THEOREM 1. *Assume (H), (FR-P) and let be such that*

$$(2.2) \quad Y > Y_0 = \exp\left(c \frac{D^5}{A(q)} \log \frac{1}{A(q)}\right)$$

Then, there exists x with

$$(2.3) \quad x \in \left[Y \exp\left(-\frac{Cq}{\sqrt{A(q)}} (\log Y)^{\frac{1}{2}} (\log_2 Y)^{\frac{3}{2}}\right), Y \right]$$

such that

$$(2.4) \quad \Delta_4(x, q, l_1, l_2) > \sqrt{Y} \exp\left(-\frac{Cq}{\sqrt{A(q)}} (\log Y)^{\frac{1}{2}} (\log_2 Y)^{\frac{3}{2}}\right).$$

COROLLARY 1. *Under the assumptions of Theorem 1, for Y verifying (2.2) we have at least one sign change of $\Delta_4(x)$ for x belonging to the interval (2.3).*

Hence, we obtain

$$(2.5) \quad V_4(Y, q, l_1, l_2) \gg \frac{\sqrt{A(q)} \log^{\frac{1}{2}} Y}{q \log^{\frac{3}{2}} Y} \quad \text{for } Y > \exp\left(cD^5 A^{-1}(q) \log \frac{1}{A(q)}\right).$$

THEOREM 2. *Assume (H), (FR-P) and let*

$$(2.6) \quad Y > \exp\left(\frac{cD^2}{A(q)} \log^3 \frac{D}{A(q)}\right).$$

Then, there exists x with

$$(2.7) \quad x \in \left[Y \exp\left(-\frac{c \log^{\frac{3}{2}} Y \log_2 Y}{(DA(q))^{\frac{1}{2}}}\right), Y \right]$$

such that

$$(2.8) \quad \Delta_4(x) > \sqrt{x} \exp\left(-\frac{cq^{8/3} \log^4 q}{A^{\frac{1}{2}}(q)} (\log Y)^{\frac{1}{2}} (\log_2 Y)^2\right).$$

Using the fact that the error term given by the Prime Number Theorem for arithmetic progression (see for instance Prachar [5, pp. 297-298]) is smaller than the lower bounds (2.4), (2.7), we obtain

COROLLARY 2. *The statements of Theorem 1, Corollary 1 and Theorem 2 continue to be true without any change also for $\Delta_3(x)$, if l_1 and l_2 are both quadratic non-residues or if they are both quadratic residues (mod q).*

Actually one can assure (2.2)-(2.4) also for $i = 3$ if l_1 is a quadratic non-residue and l_2 a residue. However, in this case we cannot guarantee the opposite inequality, which follows in the earlier cases just by changing the role of l_1 and l_2 .

3. We introduce

$$(3.1) \quad F(s) = \sum_{n=1}^{\infty} \frac{\varepsilon(n)A(n)}{n}.$$

We have

$$(3.2) \quad F(s) = \frac{1}{\varphi(q)} \sum_{\chi(\bmod q)} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \frac{L'}{L}(s, \chi) = \frac{f_{-1}}{s} + f_0 + \dots$$

where (see [5], 7.4.21 and 7.4.46)

$$(3.3) \quad |f_{-1}| = \left| \frac{1}{\varphi(q)} \sum_{\substack{\chi(\bmod q) \\ \chi(-1)=1}} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \right| < 2,$$

$$(3.4) \quad f_0 = O(\log^2 q)$$

if there exists no Siegel-zero, as assured by (H).

Furthermore, let

$$(3.5) \quad a_e = \frac{1}{\varphi(q)} \sum_{\substack{\chi(\bmod q) \\ L(e, \chi)=0}} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) m(e) = \text{Res}_{s=e} F(s)$$

where $m_\chi(e)$ denotes the multiplicity of e as a zero of $L(s, \chi)$.

LEMMA 1. *For $\exp[-3\mu] < K < \mu/9$, $\mu \geq \log q$, we have*

$$(3.6) \quad \frac{1}{2\sqrt{\pi K}} \int_{\exp(\mu+3\sqrt{\mu K})}^{\exp(\mu-3\sqrt{\mu K})} \frac{A_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \\ = \sum_e \frac{a_e}{e} \exp(Ke^2 + \mu e) + \mu f_{-1} + f_0 + O(\exp(-\mu/4))$$

where the sum is performed on the non-trivial zeros ρ ($0 < \text{Re } \rho < 1$) of L -functions (mod q).

PROOF. We use the following integral formula

$$(3.7) \quad \frac{1}{2i\pi} \int_{(2)} \exp(Ks^2 + \alpha s) ds = \frac{1}{2\sqrt{\pi K}} \exp\left(-\frac{\alpha^2}{4K}\right), \quad K \in \mathbf{R}^+, \quad \alpha \in \mathbf{C}$$

and the fact that, by partial summation

$$(3.8) \quad F(s) = \sum_{n=1}^{\infty} \frac{\varepsilon(n)\Lambda(n)}{n^s} = s \int_1^{\infty} \frac{\Delta_4(x)}{x^{s+1}} dx.$$

Then, we have

$$(3.9) \quad \begin{aligned} \frac{1}{2\sqrt{\pi K}} \int_1^{\infty} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \\ = \frac{1}{2\pi i} \int_1^{\infty} \frac{\Delta_4(x)}{x} \int_{(2)} \exp[Ks^2 + (\mu - \log x)s] ds dx \\ = \frac{1}{2\pi i} \int_{(2)} \frac{F(s)}{s} \exp[Ks^2 + \mu s] ds. \end{aligned}$$

Since the following well-known estimate holds:

$$(3.10) \quad \frac{L'}{L}(s, \chi) \ll \log(q(|t| + 2)) \ll (\log q) \log(|t| + 2) \quad \text{for } s = -\frac{1}{2} + it$$

we obtain

$$(3.11) \quad \begin{aligned} R = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \frac{F(s)}{s} \exp(Ks^2 + \mu s) ds \ll (\log q) \exp\left[\frac{K}{4} - \frac{\mu}{2}\right] \\ \cdot \left\{ \int_0^{2+1/K} \frac{\log(|t| + 2)}{|t| + \frac{1}{2}} dt + \int_{2+1/K}^{\infty} \exp[-Kt^2] dt \right\} \\ \ll \mu \exp[-\mu/3] \left\{ \log^2\left(2 + \frac{1}{K}\right) + \frac{1}{\sqrt{K}} \exp\left[-K\left(2 + \frac{1}{K}\right)^2\right] \right\} \ll \exp[-\mu/4]. \end{aligned}$$

Now, by Cauchy's residues theorem, we get

$$(3.12) \quad \frac{1}{2\pi i} \int_{(2)} \frac{F(s)}{s} \exp[Ks^2 + \mu s] ds = \sum_{\rho} \frac{a_{\rho}}{\rho} \exp[K\rho^2 + \mu\rho] + \mu f_{-1} + f_0 + R$$

and, in view of (3.11),

$$(3.13) \quad \frac{1}{2\pi i} \int_{(2)} \frac{F(s)}{s} \exp(Ks^2 + \mu s) ds = \sum_{\rho} \frac{a_{\rho}}{\rho} \exp[K\rho^2 + \mu\rho] + \mu f_{-1} + f_0 + O(\exp(-\mu/4)).$$

Moreover, using the trivial estimate $\Delta_4(x) \ll x$, we obtain

$$(3.14) \quad \frac{1}{2\sqrt{\pi K}} \left| \int_{\exp(\mu+3\sqrt{\mu K})}^{\infty} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \right| \leq \left| \int_{\frac{3\sqrt{\mu}}{2}}^{\infty} \exp\left(-\left(y - \sqrt{K}\right)^2 + \mu + K\right) dy \right| \ll \exp\left(-\left(\frac{3\sqrt{\mu}}{2} - \sqrt{K}\right)^2 + \mu + K\right) \leq \exp(-\mu/4)$$

where we have introduced in the integral the variable $y = \log x - \mu/2\sqrt{K}$ and we have used $3\sqrt{\mu}/2 > \sqrt{K}$, $\mu > 9K$.

Since a similar estimate holds, completely trivially, also for $\int_1^{\exp(\mu-3\sqrt{\mu K})}$, we get

$$(3.15) \quad \frac{1}{2\sqrt{\pi K}} \int_1^{\infty} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx = \frac{1}{2\sqrt{\pi K}} \int_{\exp(\mu-3\sqrt{\mu K})}^{\exp(\mu+3\sqrt{\mu K})} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) \frac{\Delta_4(x)}{x} dx + O(\exp(-\mu/4)).$$

Now, our Lemma follows collecting together (3.9), (3.13) and (3.15). ■

Since the main part of (3.6) can be written as a powersum, our problem is reduced to give a good lower bound for it. This is accomplished by means of a «one-sided» powersum theorem of Knapowski and Turán (see Theorem 4.1 in part III of [3]). In order to obtain sharper estimates, we shall need this result in the following slightly modified form:

LEMMA 2. *Let $b_j, z_j \in \mathbb{C}$ for $j = 1, 2, \dots, n$, with*

$$(3.16) \quad 0 < \kappa \leq |\arg z_j| \leq \pi \quad \forall_j$$

$$(3.17) \quad |z_1| \geq |z_2| \geq \dots \geq |z_n|.$$

Then for any h with $1 < h < n$ and for any $m \geq 0$, there exists an integer ν with

$$(3.18) \quad \nu \in \left[m, m + n \left(3 + \frac{\pi}{\kappa} \right) \right]$$

such that

$$(3.19) \quad \operatorname{Re} \sum_{j=1}^n b_j z_j^\nu > \frac{E}{2n+1} \left(\frac{24e(m+n(5+\pi/\kappa))}{n} \right)^{-2n} |z|^\nu \left| \frac{z_h}{z_1} \right|^{n(5+\pi/\kappa)}$$

where

$$(3.20) \quad E = \min_{l \geq h} \left| \operatorname{Re} \sum_{j=1}^l b_j \right|.$$

PROOF. Following the lines the theorem of [3] quoted above, we obtain the following inequality, in the case $|z_1| = 1$,

$$(3.21) \quad \operatorname{Re} \sum_{j=1}^n b_j z_j^\nu > \frac{E}{2n+1} \left(\frac{|z_h| - \delta}{48} \right)^{2n} \delta^{m+n(3+\pi/\kappa)}$$

for a suitable ν verifying (3.18) and for every δ with

$$(3.22) \quad 0 < \delta < |z_h|$$

Then, (3.19) follows by choosing

$$(3.23) \quad \delta = |z_h| \left(1 - \frac{2n}{m+n(5+\pi/\kappa)} \right)$$

unlike to Knapowski-Turàn's choice $\delta = |z_h| - 2n/(m+n(3+\pi/\kappa))$. ■

According to (3.19), in our applications we shall need a non trivial lower estimate for E . This requires a modification of the coefficients in the power-sum, furnished by the following Lemma:

LEMMA 3. *There exists a prime $P \equiv l_1 \pmod{q}$ with*

$$(3.24) \quad \frac{D}{2} < P \log^2 P < D$$

such that, for $P_0 = P + \frac{1}{2}$ or $P_0 = P - \frac{1}{2}$ we have

$$(3.25) \quad \left| \sum_{\varrho} \frac{a_{\varrho}}{\varrho} \exp(K_0 \varrho^2 + i\mu_0 \varrho) \right| > \frac{\log P}{3}$$

where

$$(3.26) \quad K_0 = \frac{P_0^2 \log^2 P_0}{1}, \quad \mu_0 = \log P_0.$$

PROOF. Since the finite Riemann-Piltz conjecture is assumed to be true, the prime number formula of arithmetic progressions, truncated at D , assures the existence of a prime $P \equiv l_1 \pmod{q}$ verifying (3.24). Then, the function $\Delta_4(x)$ has a jump $\log P$ at the point P and we have

$$(3.27) \quad \Delta_4(P_2) - \Delta_4(P_1) = \log P$$

for $P_1 = P - \frac{1}{2}$, $P_2 = P + \frac{1}{2}$.

Now, we use Lemma 1 with $\mu_i = \log P_i$, $K_i = 1/P_i^2 \log^2 P_i$ for $i = 1, 2$. This choice implies

$$(3.28) \quad [\exp(\mu_i - 3\sqrt{\mu_i K_i}), \exp(\mu_i + 3\sqrt{\mu_i K_i})] \subset \left(P_i - \frac{1}{2}, P_i + \frac{1}{2}\right)$$

since P is large enough by (3.24) and (2.1).

Thus, $\Delta_4(x) = \Delta_4(P_i)$ in the above interval and, using Lemma 1, and setting $x = \exp(\mu + 2\sqrt{K}y)$ we obtain:

$$(3.29) \quad \begin{aligned} & \sum_{\varrho} \frac{a_{\varrho}}{\varrho} \exp(K_i \varrho^2 + \mu_i \varrho) + f_{-1} \mu_i + f_0 \\ &= \frac{\Delta_4(P_i)}{\sqrt{\pi}} \int_{\exp(\mu_i - 3\sqrt{\mu_i K_i})}^{\exp(\mu_i + 3\sqrt{\mu_i K_i})} \exp\left(-\frac{(\mu_i - \log x)^2}{4K_i}\right) \frac{dx}{2\sqrt{K_i}x} + O(\exp(-\mu/4)) \\ &= \frac{\Delta_4(P_i)}{\sqrt{\pi}} \int_{-3\sqrt{\mu_i/2}}^{3\sqrt{\mu_i/2}} \exp(-y^2) dy + O(\exp(-\mu/4)) = \Delta_4(P_i) + O(P^{-1/4}). \end{aligned}$$

Owing to $|f_{-1}| < 2$ and (3.27) we get from this

$$(3.30) \quad O\left(\frac{1}{P}\right) + \sum_{\varrho} \frac{a_{\varrho}}{\varrho} (\exp(K_2 \varrho^2 + \mu_2 \varrho) - \exp(K_1 \varrho^2 + \mu_1 \varrho)) = \log P + O(P^{-1/4})$$

which implies Lemma 3. ■

In view of the application of Lemma 2 to the power-sum appearing in (2.6), it is also necessary to assure the argument condition (3.16), and this is made by means of

LEMMA 4. Let a_j be real numbers for $j = 1, \dots, n$ with $a_j \neq 0$ and

$$(3.31) \quad \frac{1}{n} \sum_{j=1}^n |a_j|^{-1} \leq \eta.$$

Then, for every H there exists an y_0 with

$$(3.32) \quad y_0 \in [H, H + \eta]$$

such that, for any integer K and for every $j = 1, \dots, n$

$$(3.33) \quad |y_0 a_j - 2K\pi| \geq \frac{1}{4n}.$$

PROOF. For fixed j , (3.33) can be false for K in an interval of length at most $(\eta|a_j| + 1/2n)/2\pi$; for fixed K , leaving j fixed, this can happen for y in an interval of length at most $1/2n|a_j|$.

Thus, the total Lebesgue measure of y for which (3.33) is false for fixed j is majorised by

$$(3.34) \quad \frac{1}{2n|a_j|} \left(\frac{\eta|a_j|}{2} + \frac{3}{2} \right) = \frac{\eta}{4n} + \frac{3}{4} \frac{1}{n|a_j|}.$$

Summing over $j = 1, \dots, n$, we obtain our Lemma. ■

Now, we are in good position to apply Lemma 2.

We introduce the following position:

$$(3.35) \quad b_\varrho = \frac{a_\varrho}{\varrho} \exp(K_0 \varrho^2 + \mu_0 \varrho)$$

for a_ϱ given by (3.5) and K_0, μ_0 furnished by (3.26).

Furthermore, let

$$(3.36) \quad \lambda > 2D, \quad L > \frac{cq^2}{A(q)} \lambda \log^3 \lambda$$

Let B be a real number to be chosen later with

$$(3.37) \quad B \in \left[\frac{c \log \lambda}{A(q) \lambda}, \frac{2c \log \lambda}{A(q) \lambda} \right]$$

and let ν be an integer to be chosen later with

$$(3.38) \quad \nu \in \left[\frac{L - \mu_0}{B} - cq^2 \lambda^2 \log^2 L, \frac{L - \mu_0}{B} \right]$$

$$(3.39) \quad K = K_0 + \frac{B\nu}{\lambda^2} \quad \mu = \mu_0 + B\nu$$

$$(3.40) \quad z_\varrho = \exp \left[\frac{B\varrho^2}{\lambda^2} + B\varrho \right].$$

LEMMA 5. *With the positions (3.35) to (3.40), we have*

$$(3.41) \quad \operatorname{Re} \sum_{\varrho} \frac{a_{\varrho}}{\varrho} \exp(K\varrho^2 + \mu\varrho) > \exp \left\{ \frac{L}{2} - \frac{LD^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L \right\}$$

for suitable values of ν and B satisfying (3.38).

PROOF. By our definitions, we have

$$(3.42) \quad \sum_{\varrho} \frac{a_{\varrho}}{\varrho} \exp(K\varrho^2 + \mu\varrho) = \sum_{\varrho} b_{\varrho} \varrho^{\nu}$$

Moreover

$$(3.42) \quad \left| \sum_{|\varrho| \geq 2\lambda} \frac{a_{\varrho}}{\varrho} \exp(K\varrho^2 + \mu\varrho) \right| \ll \sum_{m=[2\lambda]}^{\infty} \exp(\mu + K(1 - m^2))(\log qm) \ll 1$$

and so we have only to consider

$$(3.43) \quad \sum_{|\varrho| < 2\lambda} b_{\varrho} \varrho^{\nu}$$

Here, the number n of terms is clearly

$$(3.44) \quad c_1 \varphi(q) \lambda \log \lambda < n < c_2 q \lambda \log \lambda$$

Now, we apply Lemma 4, setting $\varrho = \beta + i\gamma$,

$$(3.45) \quad a_j = \left(\frac{2\beta}{\lambda^2} + 1 \right) \gamma, \quad H = \frac{c \log \lambda}{A(q) \lambda}$$

Using Jensen's inequality, we have by $A(q) < 1$ and $\lambda > q$,

$$(3.46) \quad \sum_{j=1}^n \frac{1}{|a_j|} \ll \sum_{|\varrho| \leq 2\lambda} \frac{1}{|\gamma|} \ll \sum_{A(q) \leq |\gamma| \leq 1} \frac{1}{|\gamma|} + \sum_{\substack{|\varrho| \leq 2\lambda \\ |\gamma| > 1}} \frac{1}{|\gamma|} \ll \frac{\varphi(q) \log q}{A(q)} + \varphi(q) \log^2 \lambda;$$

so, in view of (3.44), condition (3.31) holds with

$$(3.47) \quad \eta = \frac{c \log \lambda}{A(q) \lambda}$$

Hence, Lemma 4 says that there exists a B in the interval (3.37) such that, for every $j = 1, \dots, n$

$$(3.48) \quad |Ba_j - 2K\pi| \geq \frac{1}{4n} = \frac{c}{q\lambda \log \lambda}$$

Since in our case $z_{\rho_j} = |z_{\rho_j}| \exp (i a_j B)$, this means that

$$(3.49) \quad |\text{Arg } z_{\rho}| > \frac{c}{\eta} = \frac{c}{q \lambda \log \lambda} = \kappa.$$

Now, we order the numbers z_{ρ} according to (3.17) and, in view of (FR-P), we choose h of Lemma 2 as the largest index corresponding to a zero ρ with $|\rho_0| \leq D$.

Since

$$(3.50) \quad \left| \sum_{|\rho| > D} \frac{a_{\rho}}{\rho} \exp (K_0 \rho^2 + \mu_0 \rho) \right| \ll \sum_{m=P \log^2 P}^{\infty} \exp \left[\frac{1-m^2}{P^2 \log^2 P} \right] P(\log m q) \ll 1$$

with constants implied by the \ll sign, independent of c_0 appearing in (2.1), we have by Lemma 3

$$(3.51) \quad \left| \sum_{\rho \in S} b_{\rho} \right| > \frac{\log P}{4}$$

for every set S containing all zeros ρ with $|\rho| \leq D$.

Finally, we have

$$(3.52) \quad |z_h| \geq \exp \left(\frac{B}{2} - \frac{B}{\lambda^2} D^2 \right)$$

because ρ_0 is on the critical line, and (by (3.40) and $\lambda > 2D$)

$$(3.52) \quad \left| \frac{z_h}{z_1} \right| \geq \exp \left(\frac{B}{2} - \frac{B}{\lambda^2} D^2 - B - \frac{B}{\lambda^2} \right) > \exp (-B).$$

Recalling (3.44), (3.49), (3.51), (3.52), (3.53) and choosing m as $m = (L - \mu_0)/B$ in view of (3.38), we obtain by Lemma 2 for suitable B and ν :

$$(3.54) \quad \text{Re} \sum_{|\rho| < 2\lambda} b_{\rho} z_{\rho}^{\nu} > \exp \left\{ \frac{B\nu}{2} - B\nu \frac{D^2}{\lambda^2} - c \frac{q^2}{A(q)} \lambda \log^3 \lambda - cq \lambda \log \lambda \log L \right\} \\ \gg \exp \left\{ \frac{L}{2} - \frac{D^2}{\lambda^2} L - \frac{cq^2}{A(q)} \lambda \log^3 L \right\},$$

from which our Lemma immediately follows. ■

4. We formulate the results of Section 3 as:

THEOREM 3. *Assume (H), (FR-P) and*

$$(4.1) \quad \lambda > 2D, \quad L > \max (cq^2 A^{-1}(q) \lambda \log^2 \lambda, \lambda^2).$$

Then there exist

$$(4.2) \quad \mu \in \left[L - \frac{cq^2}{A(q)} \lambda \log^3 \lambda, L \right]$$

$$(4.3) \quad K \in \left[\frac{L - \mu_0}{\lambda^2} - \frac{cq^2}{A(q)} \frac{\log^3 \lambda}{\lambda}, K_0 + \frac{L}{\lambda^2} \right]$$

such that

$$(4.4) \quad \frac{1}{2\sqrt{\pi K}} \int_{\exp(\mu - 3\sqrt{\mu K})}^{\exp(\mu + 3\sqrt{\mu K})} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx > \exp\left\{\frac{L}{2} - \frac{LD^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L\right\}.$$

PROOF. The theorem follows immediately from Lemmas 1 and 5. ■

PROOF OF THEOREM 2. We set, with the notations of Theorem 3

$$(4.5) \quad Y = \exp(L_1), \quad L = L_1 \left(1 - \frac{7}{2\lambda}\right).$$

Then we obtain, by easy calculations from (4.1)-(4.3) $\sqrt{\mu K} \leq 1, 1 L/\lambda$ and so

$$(4.6) \quad I \subseteq \left[Y \exp\left(-\frac{7L_1}{\lambda} - \frac{cq^2}{A(q)} \lambda \log^3 \lambda\right), Y \right].$$

We set also

$$(4.7) \quad A = \max_{x \in I} \frac{\Delta_4(x)}{\sqrt{x}}, \quad I = [\exp(\mu - 3\sqrt{\mu K}), \exp(\mu + 3\sqrt{\mu K})].$$

By Theorem 3, we have for suitable μ, K verifying (4.2), (4.3),

$$(4.8) \quad \frac{A}{2\sqrt{\pi K}} \int_{x \in I} \frac{1}{\sqrt{x}} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx > \int_{x \in I} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \cdot \frac{1}{2\sqrt{\pi K}} > \exp\left\{\frac{L}{2} - \frac{LD^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L\right\}.$$

Since, as it is easily verified,

$$(4.9) \quad \frac{1}{2\sqrt{\pi K}} \int_0^\infty \frac{1}{\sqrt{x}} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx = \exp\left(\frac{\mu}{2} + \frac{k}{4}\right)$$

and $\mu \ll L$, inequality (4.8) yields

$$(4.10) \quad A > \exp \left\{ -\frac{2L_1 D^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L_1 \right\}.$$

In order to optimise the lower bound (4.10), we choose

$$(4.11) \quad \lambda = cA^{\frac{1}{3}}(q) q^{\frac{1}{3}} \log^4 q \frac{L_1^{\frac{1}{3}}}{\log L_1}$$

which satisfies (4.1), in view of (2.6) and (4.5).

Thus, by (2.1), (4.5), (4.6), (4.10) we obtain

$$(4.12) \quad A > \exp \left(-\frac{cq^{8/3} \log q}{A^{\frac{1}{3}}(q)} (\log Y)^{\frac{1}{3}} (\log_2 Y)^2 \right)$$

and

$$(4.13) \quad I \subset \left[Y \exp \left(-\frac{c(\log Y)^{\frac{1}{3}} \log_2 Y}{(DA(q))^{\frac{1}{3}}} \right), Y \right]$$

which proves Theorem 2. ■

PROOF OF THEOREM 1. We set now also (4.5) and similarly we obtain (4.6)-(4.10) from (4.1)-(4.3).

In order to optimise localisation in (4.6) we choose

$$(4.14) \quad \lambda = \frac{cA^{\frac{1}{3}}(q)}{q} \frac{L^{\frac{1}{3}}}{(\log L_1)^{\frac{1}{3}}} > D^2,$$

in view of (2.1)-(2.2), and this proves also (4.1).

In such a way we obtain by (4.10)

$$(4.15) \quad I \subset \left[Y \exp \left(-\frac{cL_1}{\lambda} \right), Y \right]$$

and, for suitable x in I , by (4.10), (4.14) and (4.15)

$$(4.16) \quad \Delta_4(x) = A \sqrt{x} > \exp \left(-\frac{cL_1}{\lambda} \right) \sqrt{Y} \exp \left(-\frac{cL_1}{\lambda} \right),$$

which proves Theorem 1, due to the choice of λ in (4.14). ■

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