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A Free Boundary Problem for Quasi-Linear Elliptic Equations (*).

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0. – Introduction.

Consider the problem of minimizing the functional

$$J(u) = \int_{\Omega} \big(F(|\nabla u|^2) + \lambda^2 I_{\{u>0\}}\big)\, dx \qquad (\lambda > 0) \tag{0.1}$$

in the class of functions u satisfying $u = u^0$ on a part S of $\partial\Omega$, $u^0 \geqslant 0$. Here Ω is a domain in R^n and $F(t)$ is a convex function of t for $t > 0$, $F(0) = 0$, $F'(0) > 0$. The special case $F(t) = t$ was studied by Alt and Caffarelli [1] who proved Lipschitz continuity and nondegeneracy of a minimum u. They also studied the free boundary $\Gamma = \partial\{u > 0\} \cap \Omega$ and proved the analyticity of Γ if $n = 2$; further, if $n \geqslant 3$, $x^0 \in \Gamma$ and Γ satisfies the « flatness condition » at x^0, then Γ is analytic in a neighborhood of x^0.

The results of [1] were used by Alt, Caffarelli and Friedman [2-4] in their study of jet flows of inviscid, irrotational and incompressible fluid.

In this paper we shall extend all the results of [1] to the functional (0.1). In particular we establish Lipschitz continuity and nondegeneracy of a minimum, and analyticity of the free boundary (if $n \geqslant 3$, the flatness condition is assumed, as before).

The results of this paper extend with obvious changes to the more general

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functional

$$J(u) = \int_{\Omega} [F(x, |\nabla u(x)|^2) + \lambda^2(x) I_{\{u>0\}}(x)]\, dx \tag{0.2}$$

where $\lambda(x) > 0$, $\lambda \in C^{\alpha}(\Omega)$.

In a future publication we shall apply the results of this paper to the study of jets and cavities of compressible fluids.

1. – The minimization problem; basic properties.

Let $F(t)$ be a function in $C^{2,1}[0, \infty)$, satisfying:

$$\begin{aligned} &F(0) = 0\,, \qquad c_0 \leqslant F'(t) \leqslant C_0\,, \\ &0 \leqslant F''(t) \leqslant \frac{C_0}{1+t} \qquad (c_0, C_0 \text{ positive constants})\,. \end{aligned} \tag{1.1}$$

Let Ω be a domain in R^n, not necessarily bounded. For any $\lambda > 0$, consider the functional

$$J(v) = \int_{\Omega} \big(F(|\nabla v|^2) + \lambda^2 I_{\{u>0\}}\big)\, dx$$

over the class of admissible functions

$$K = \{v \in L^1_{\text{loc}}(\Omega), \nabla v \in L^2(\Omega) \text{ and } v = u^0 \text{ on } S\}$$

where S is a given subset of $\partial\Omega$ and u^0 is a given function. We assume that locally $\partial\Omega$ is a Lipschitz graph, that S is measurable with $H^{n-1}(S) > 0$, and that

$$u^0 \geqslant 0\,, \quad u^0 \in L^1_{\text{loc}}(\Omega)\,, \quad \nabla u_0 \in L^2(\Omega)\,.$$

Set

$$f(p) = F(|p|^2)\,.$$

From (1.1) we find that $f(p)$ is convex; moreover,

$$\beta|\xi|^2 \leqslant \sum \frac{\partial^2 f(p)}{\partial p_i \partial p_j} \xi_i \xi_0 \leqslant \beta^{-1}|\xi|^2 \qquad \text{for all } \xi \in R^n, \tag{1.2}$$

consequently

$$\begin{cases} \beta|p|^2 \leqslant f_p(p)\cdot p\,, \\ \beta|p|^2 \leqslant f(p) \leqslant \beta^{-1}|p|^2 \end{cases} \tag{1.3}$$

for some small constant $\beta > 0$, where $f_p = \nabla_p f$.

Consider the problem: find u such that

$$J(u) = \min_{v\in K} J(v)\,, \qquad u \in K\,. \tag{1.4}$$

This means that we have to deal with the differential operator

$$\mathscr{L}v = \nabla\cdot\big(f_p(\nabla v)\big)$$

THEOREM 1.1. *If $J(u^0) < \infty$ then there exists a solution to problem* (1.4).

PROOF. The proof is the same as in [1], Theorem 1.3. Let $\{u_k\}$ be a minimizing sequence. Then, by (1.3),

$$\int_\Omega |\nabla u_k|^2 \leqslant \frac{1}{\beta} J(u_k)\,.$$

It follows that $u_k - u^0$ are bounded in $L^2(\Omega \cap B_R)$ for any large R. Therefore there is a $u \in K$ such that, for a subsequence,

$$\begin{aligned} \nabla u_k &\to \nabla u \qquad \text{weakly in } L^2_{\text{loc}}(\Omega)\,,\\ u_k &\to u \qquad \text{a.e. in } \Omega\,. \end{aligned}$$

The pointwise convergence implies

$$\int_{\Omega\cap B_R} I_{\{u>0\}} \leqslant \liminf_{k\to\infty} \int_{\Omega\cap B_R} I_{\{u_k>0\}}\,,$$

and since f is convex we have (see, for instance [6; pp. 232-3])

$$\int_{\Omega\cap B_R} f\big(|\nabla u|\big) \leqslant \liminf_{k\to\infty} \int_{\Omega\cap B_R} f\big(|\nabla u_k|\big)\,;$$

thus

$$J(u) \leqslant \liminf_{k\to\infty} J(u_k)\,.$$

DEFINITION 1.2. $u \in K$ is called a *local minimum* of J if, for some $\varepsilon_0 > 0$, $J(u) \leqslant J(v)$ for any $v \in K$ with

$$\big\|\nabla(v-u)\big\|_{L^2(\Omega)} + \big\|I_{\{v>0\}} - I_{\{u>0\}}\big\|_{L^1(\Omega)} < \varepsilon_0\,.$$

LEMMA 1.3. *If u is a local minimum then u is $\mathscr{L}$-subsolution, that is,*

$$\int_\Omega \nabla\zeta\cdot f_p(\nabla u)\leqslant 0 \qquad \textit{for all } \zeta\in C_0^\infty(\Omega)\,,\quad \zeta\geqslant 0\,. \tag{1.5}$$

PROOF. For any $\varepsilon > 0$,

$$\begin{aligned}0\leqslant\frac{1}{\varepsilon}\big(J(u-\varepsilon\zeta)-J(u)\big)&\leqslant\frac{1}{\varepsilon}\int_\Omega\big(f(\nabla u-\varepsilon\nabla\zeta)-f(\nabla u)\big)\\&\leqslant-\int_\Omega\nabla\zeta\cdot f_p(\nabla u-\varepsilon\nabla\zeta)\,,\end{aligned}$$

since by convexity

$$f(p)-f(q)\leqslant(p-q)\cdot f_p(p)\,. \tag{1.6}$$

Now take $\varepsilon\to 0$.

LEMMA 1.4. *If w is a local $\mathscr{L}$-solution in B_R, that is, $w\in H^{1,2}(B_R)$ and*

$$\int\nabla\zeta\cdot f_p(\nabla w)=0 \qquad \textit{for all } \zeta\in C_0^\infty(B_R)\,, \tag{1.7}$$

and if $w\geqslant u$ on ∂B_R, then $w\geqslant u$ in B_R. If $w=u$ on ∂B_R then w is uniquely determined.

PROOF. Taking $\zeta=(u-w)^+$ in (1.5), (1.7) and comparing, we get, using the convexity (1.2) of f:

$$\begin{aligned}0&\geqslant\int_{B_R}\big(f_p(\nabla u)-f_p(\nabla w)\big)\cdot\nabla(u-w)^+\\&=\int_{B_R\cap\{u>w\}}\big(f_p(\nabla u)-f_p(\nabla w)\big)\cdot\nabla(\nabla u-\nabla w)\\&\geqslant\beta\int_{B_R}|\nabla(u-w)^+|^2;\end{aligned}$$

hence $u-w\leqslant 0$. If w_1 is another solution of (1.7) then the above proof gives $w_1-w\leqslant 0$. Similarly $w-w_1\leqslant 0$, so that $w=w_1$:

LEMMA 1.5. *If u is a local minimum then*

$$0\leqslant u\leqslant\sup_\Omega u^0\,. \tag{1.8}$$

PROOF. Setting $M = \sup_{\Omega} u^0$, we have, for small $\varepsilon > 0$,

$$
\begin{aligned}
0 &\leqslant \frac{1}{\varepsilon}\Big(J\big(u + \varepsilon \min(M-u, 0)\big) - J(u)\Big) \\
&= \frac{1}{\varepsilon}\int_{\Omega}\Big(f\big(\nabla u + \varepsilon\nabla \min(M-u, 0)\big) - f(\nabla u)\Big) \\
&\leqslant \int_{\Omega} \nabla \min(M-u, 0)\cdot f_p\big(\nabla u + \varepsilon \nabla \min(M-u, 0)\big), \qquad \text{by (1.6)}.
\end{aligned}
$$

Taking $\varepsilon \to 0$ we get

$$
\begin{aligned}
0 &\leqslant \int_{\Omega} \nabla \min(M-u, 0)\cdot f_p(\nabla u) \\
&= -\int_{\{u>M\}} \nabla u\cdot f_p(\nabla u) \leqslant -\beta \int_{\{u>M\}} |\nabla u|^2 \\
&= -\beta\int_{\Omega} |\nabla \min(M-u, 0)|^2
\end{aligned}
$$

by (1.3), which yields $u \leqslant M$ a.e. in Ω.

Similarly, to prove that $u \geqslant 0$ we begin with (ε positive and small)

$$
\begin{aligned}
0 &\leqslant \frac{1}{\varepsilon}\Big(J\big(u - \varepsilon \min(u, 0)\big) - J(u)\Big) \\
&= \frac{1}{\varepsilon}\int_{\Omega}\Big(f\Big(\nabla\big(u - \varepsilon \min(u, 0)\big)\Big) - f(\nabla u)\Big) \\
&\leqslant -\int_{\Omega} \nabla \min(u, 0)\cdot f_p\Big(\nabla\big(u - \varepsilon \min(u, 0)\big)\Big).
\end{aligned}
$$

Taking $\varepsilon \to 0$ we get

$$
\begin{aligned}
0 &\geqslant \int_{\Omega} \nabla \min(u, 0)\cdot f_p(\nabla u) \\
&= \int_{\{u<0\}} \nabla u\cdot f_p(\nabla u) \geqslant \beta \int_{\{u<0\}} |\nabla u|^2 = \beta\int_{\Omega} |\nabla \min(u, 0)|^2,
\end{aligned}
$$

from which we deduce that $u \geqslant 0$.

In § 2 it will be shown that, for any local minimum u, the set $\{u > 0\}$ is open. Let us use this fact already in the next statements of this section.

LEMMA 1.6. *If u is a local minimum then u is $\mathscr{L}$-solution in $\{u > 0\}$.*

The proof is the same as that of Lemma 1.3, taking ζ any function in $C_0^\infty(\{u > 0\})$.

LEMMA 1.7. *If G is an open set, $u \in H^{1,2}(G)$ and $\mathscr{L}u = 0$ in G, then $u \in C^{2+\alpha}$* for any $0 < \alpha < 1$.

PROOF. We can take G to be a ball. The equation $\mathscr{L}u = 0$ has the form

$$\partial_j(f_{p_j}(\nabla u)) = 0 \qquad \left(\partial_j = \frac{\partial}{\partial x_j}\right)$$

Applying ∂_k and setting $w_k = \partial_k u$ we get, formally,

$$\partial_j\big(a_{ij}(\nabla u)\,\partial_i w_k\big) = 0$$

where

$$a_{ij}(p) = f_{p_i p_j}(p) \tag{1.9}$$

is uniformly elliptic matrix. Thus the Nash-de Giorgi estimate should give a C^α estimate on $\partial_k u$ for some $\alpha \in (0, 1)$.

In order to derive the C^α estimate rigorously, we approximate u on ∂G by smooth functions $\tilde{u}_m$ (in the L^2 trace norm). By [6; Theorem 14.8] there exists a unique solution u_m of

$$\mathscr{L}u_m = 0 \qquad \text{in } G\,,$$

$$u_m = \tilde{u}_m \qquad \text{on } \partial G$$

with $u_m \in C^{2+\alpha}(\bar{G})$. The formal argument given above can be applied to u_m. It gives

$$|u_m|^G_{1+\alpha} \leqslant C_m$$

as well as

$$|u_m|^K_{1+\alpha} \leqslant C$$

where K is any compact subset of G (C is independent of m). Since u_m and the minimizer of

$$\int_G F(|\nabla v|^2)\,, \qquad v = \tilde{u}_m \text{ on } \partial G$$

coincide by Lemma 1.4, it follows that for some constants $C, \tilde{C}$

$$\int_G |\nabla u_m|^2 \leq C \int_G F(|\nabla u_m|^2) \leq \tilde{C}.$$

Thus, for a subsequence,

$$u_m \to \tilde{u} \quad \text{weakly in } H^{1,2}(G),$$

$$u_m \to u \quad \text{in } L^2(\partial G) \text{ (in the trace sense)}.$$

It follows, again by Lemma 1.4, that $\tilde{u} = u$ and thus $u \in C^{1+\alpha}(K)$. Then the coefficients $a_{ij}(\nabla u)$ are Hölder continuous so that by elliptic estimates w_k is of class $C^{1+\alpha}$.

DEFINITION 1.8. We define functions ψ, Φ by

$$\text{[1.10]} \qquad \begin{cases} \psi(p) = pf_p - f(p) = 2|p|^2 F'(|p|^2) - F(|p|^2), \\ \Phi(s) = 2sF'(s) - F(s), \end{cases}$$

so that $\Phi(|p|^2) = \psi(p)$.

THEOREM 1.9. *If u is a local minimum then*

$$\lim_{\varepsilon \to 0} \int_{\partial\{u > \varepsilon\}} (\Phi(|\nabla u|^2) - \lambda^2)\eta \cdot \nu = 0$$

for any $\eta \in H_0^{1,2}(\Omega; R^n)$, where ν is the outward normal.

The proof of this very weak formulation of the free boundary condition is similar to the proof of Theorem 2.5 in [1].

2. – Regularity and nondegeneracy.

We set

$$\fint_A v = \frac{1}{\mu(A)} \int_A v \, d\mu.$$

Since we shall use this notation mostly when either A is a ball $B_r(x^0) = \{|x - x^0| < r\}$ and $\mu = L^n$ or A is a sphere $\partial B_r(x^0)$ and $\mu = H^{n-1}$, we often omit in the notation the measure $d\mu$.

LEMMA 2.1. *Any local minimum u is in $C^\alpha(\Omega)$, for any $0<\alpha<1$.*

PROOF. Let $B_r \subset \Omega$ and let v be the solution of

$$
\begin{aligned}
&\int_{B_r} \nabla\zeta\cdot f_p(\nabla v) = 0 \quad \text{for all } \zeta\in C_0^\infty(B_r)\,, \\
&v-u\in H_0^{1,2}(B_r)\,;
\end{aligned}
\tag{2.1}
$$

its existence follows by minimization. Then

$$
\begin{aligned}
\int_{B_r}\big(f(\nabla u)+\lambda^2 I_{\{u>0\}}\big) &\leqslant \int_{B_r}\big(f(\nabla v)+\lambda^2 I_{\{u>0\}}\big)\\
&= \int_{B_r}\big(f(\nabla v)+\lambda^2\big)
\end{aligned}
\tag{2.2}
$$

since $v>0$ in B_r, by the maximum principle. By convexity (1.2),

$$f(\nabla u)\geqslant f(\nabla v)+f_p(\nabla v)\cdot\nabla(u-v)+\beta|\nabla(u-v)|^2\,.$$

Integrating this relation and using (2.1) with $\zeta=u-v$, we find, after comparing with (2.2), that

$$\int_{B_r}|\nabla(u-v)|^2\leqslant C\int_{B_r}\lambda^2 I_{\{u=0\}}\,. \tag{2.3}$$

We can now use the method of Morrey [11; Th. 5.3.6] in order to deduce the assertion of the lemma.

In the sequel we denote positive constants depending only on β and n by C or c.

Set

$$E_+=\{u>0\}\;,\qquad E_0=\{u=0\}\;.$$

By Lemma 2.1, E_+ is open and E_0 is closed in Ω.

Set

$$d(x)=\operatorname{dist}(x,E_0)\;.$$

LEMMA 2.2. *Suppose $x_0\in\Omega$, $d(x_0)<\frac12\operatorname{dist}(x_0,\partial\Omega)$. Then*

$$u(x_0)\leqslant C\lambda d(x_0)$$

where C is a constant depending only on β, n.

PROOF. We assume that

$$u(x_0) > Md(x_0) \tag{2.4}$$

and derive an upper bound on M. By scaling (i.e., by considering

$$\tilde{u}(x') = \frac{1}{r}\, u\big(x_0 + r(x' - x_0)\big)\,, \quad r = d(x_0)\Big)$$

we may assume that $d(x_0) = 1$. Since $\mathscr{L}u = 0$ in $B_1(x_0)$, by Harnack's inequality [6; p. 189] and (2.4) we have

$$\inf_{B_{3/4}(x_0)} u \geqslant cu(x_0) \geqslant cM\,. \tag{2.5}$$

Let y be a point in $\partial B_1(x_0) \cap E_0$. We define a function v by (2.1) with B_r replaced by $B_1(y)$. Then $v \geqslant u$ in $B_1(y)$ and (cf. (2.3))

$$\int_{B_1(y)} |\nabla(u - v)|^2 \leqslant C \int_{B_1(y)} \lambda^2 I_{\{u=0\}}\,. \tag{2.6}$$

Recalling (2.5) we have

$$v \geqslant cM \quad \text{in } B_{\frac{3}{4}}(x_0) \cap B_1(y)$$

and then, by Harnack's inequality [6; p. 189],

$$v \geqslant C^* \text{ in } B_{\frac{1}{2}}(y)\,, \qquad C^* = cM\,. \tag{2.7}$$

We take for simplicity $y = 0$ and introduce the function

$$w(x) = C^*\big(\exp(-\mu|x|^2) - \exp(-\mu)\big)\,.$$

We compute

$$\begin{aligned} \mathscr{L}w(x) &= \nabla\cdot f_p\big(\nabla w(x)\big) = \nabla\cdot f_p\big(-2\mu x C^* \exp(-\mu|x|^2)\big) \\ &= f_{p_j p_i}\big(-2\mu x C^* \exp(-\mu|x|^2)\big)(4\mu^2 x_i x_j - 2\mu\delta_{ij})\, C^* \exp(-\mu|x|^2)\,. \end{aligned} \tag{2.8}$$

Hence

$$\mathscr{L}w(x) > 0 \quad \text{if } \tfrac{1}{2} < |x| < 1 \text{ and } \mu \text{ sufficiently large}\,.$$

Thus

$$w \text{ is an } \mathscr{L}\text{-subsolution in } B_1 \setminus B_{\frac{1}{2}}\,,$$

$$w = 0 \text{ on } \partial B_1\,,$$

and, by (2.7),

$$w \leqslant C^* \leqslant v \quad \text{on } \partial B_{\frac{1}{2}}.$$

Since $\mathscr{L}v < \mathscr{L}w$, the proof of the maximum principle gives

$$v(x) \geqslant C^*\big(\exp(-\mu|x|^2) - \exp(-\mu)\big) \geqslant cC^*(1-|x|) \quad \text{in } B_1 \setminus B_{\frac{1}{2}}.$$

Recalling (2.7) we then have,

$$v(x) \geqslant cM(1-|x|) \quad \text{in } B_1. \tag{2.9}$$

Take two disjoint balls $B_{\frac{1}{8}}(y_i)$ $(i = 1, 2)$ in $B_{\frac{1}{2}}(y)$. Let ξ vary on $\partial B_1(y)$ and denote by $l_i(\xi)$ the largest segment with endpoints ξ, $\eta_i(\xi)$, going from ξ into y_i, such that $\eta_i(\xi) \notin B_{\frac{1}{8}}(y_i)$ and $u(\eta_i(\xi)) = 0$ $\big(\eta_i(\xi) = \xi$ if $u(\xi) > 0\big)$. Denote by S_i the union of all the segments $l_i(\xi)$, and let $S = S_1 \cup S_2$.

As in [1; Lemma 1.3] we get, using (2.9),

$$cM|S| \leqslant \int_S |\nabla(u-v)| \leqslant |S|^{\frac{1}{2}}\Big\{\int_S |\nabla(u-v)|^2\Big\}^{\frac{1}{2}},$$

so that

$$M^2|S| \leqslant C\int_S |\nabla(u-v)|^2 \leqslant C\int_{B_1(y)} \lambda^2 I_{\{u=0\}}$$

where (2.6) was used. Since $S \supset \{u=0\} \cap B_1(y)$, we deduce that $M^2 \leqslant C\lambda^2$ and the assertion of the lemma follows.

THEOREM 2.3. *$u \in C^{0,1}(\Omega)$; moreover, for any domain $D \subset\subset \Omega$ containing a free boundary point the Lipschitz coefficient of u in D is estimated by $C\lambda$ where C depends on n, β, D and Ω only.*

PROOF. Suppose $d(x) < \frac{1}{3}\operatorname{dist}(x, \partial\Omega)$. By Lemma 2.2 applied to

$$\tilde{u}(x') = \frac{1}{d(x)}\, u\big(x + d(x)x'\big)$$

we have

$$\tilde{u}(x') \leqslant C\lambda \quad \text{in } B_1.$$

By elliptic estimates (*e.g.* Lemma 1.7) we then get

$$|\nabla \tilde{u}(0)| \leqslant C\lambda,$$

that is, $|\nabla u(x)| \leqslant C\lambda$. Thus, for any domain $D \subset\subset \Omega$, $|\nabla u(x)|$ is bounded in $D \cap E_+ \cap N$ where N is a small neighborhood of the free boundary. Since further $u \in C^{2+\alpha}$ in E^+, it follows that $u \in C^{0,1}(\Omega)$.

To prove the second part of the theorem, consider any domains $D \subset\subset D' \subset\subset \Omega$, D containing free boundary points. We shall prove that

$$|\nabla u(x)| \leqslant C\lambda \tag{2.10}$$

for all $x \in D \cap E_+$ with C depending only on n, β, D, D' and Ω.

Let $r_0 = \frac{1}{3}\operatorname{dist}(D', \partial\Omega)$ $x \in D'$. We argue as in [1; Theorem 4.3]. Since D' is connected and not contained in E_+, we find a sequence of points $x_0, \ldots, x_k$ in D' (k depending only on D' and Ω) with

$$x_j \in B_{r_0/2}(x_{j-1}) \qquad \text{for } j = 1, \ldots, k\,,$$

such that $x_0 = x$, $B_{r_0}(x_j)$ is contained in E_+ for $j = 0, \ldots, k-1$, and such that $B_{r_0}(x_k)$ is not contained in E_+. By Lemma 2.2,

$$u(x_k) \leqslant C\lambda r_0\,.$$

Since u is a $\mathscr{L}$-solution in each $B_{r_0}(x_j)$, $j = 0, \ldots, k-1$, we have, by Harnack's inequality [6; p. 189],

$$u(x_{j+1}) \geqslant c u(x_j)$$

Inductively we then obtain

$$u(x) = u(x_0) \leqslant C\lambda r_0 \qquad \text{for all } x \in D'\,.$$

Now let $x \in D \cap E_+$. If $d(x) > r_1 \equiv \frac{1}{2}\operatorname{dist}(D, \partial D')$, then

$$|\nabla u(x)| \leqslant \frac{C}{r_1} \sup_{D'} u \leqslant C\lambda$$

by the uniform estimate on u in D'. On the other hand, if $d(x) < r_1$ then $|\nabla u(x)| \leqslant C\lambda$ by Lemma 2.2. This completes the proof of the theorem.

As a consequence of Theorem 2.1 we shall prove:

LEMMA 2.4. *For any domain $D \subset\subset \Omega$ there exists a constant C (depending only on n, β, D and Ω) such that for any absolute (local) minimum u and for* *any (small) ball* $B_r \subset D$

$$\text{(2.11)} \qquad \text{if } \frac{1}{r} \fint_{\partial B_r} u \geqslant C\lambda \quad \text{then } u > 0 \text{ in } B_r .$$

PROOF. If B_r contains a free boundary point then $|\nabla u| \leqslant C\lambda$ in B_r, by the proof of Theorem 2.3, with C depending on n, β, D and Ω only. Since u vanishes at some point of B_r we conclude that

$$u \leqslant C\lambda r \quad \text{in } \overline{B_r} ,$$

which contradicts (2.11) if C in (2.11) is large enough.

We next state a *nondegeneracy lemma.*

LEMMA 2.5. *For any* $p > 1$ *and for any* $0 < \varkappa < 1$ *there exists a constant* $c_\varkappa$ *such that for any global (local) minimum and for any (small) ball* $B_r \subset \Omega$

$$\text{(2.12)} \qquad \text{if } \frac{1}{r}\Big(\fint_{B_r} u^p\Big)^{1/p} \leqslant c_\varkappa \lambda \qquad \text{then } u = 0 \text{ in } B_{\varkappa r}.$$

PROOF. Take for simplicity $r = 1$; otherwise we work with a scaled function. Set

$$\varepsilon = \frac{1}{\sqrt{\varkappa}} \sup_{B_{\sqrt{\varkappa}}} u .$$

By the L^∞ estimate of [6; p. 184],

$$\varepsilon < C\Big(\fint_{B_1} u^p\Big)^{1/p} .$$

Consider the function

$$v = \begin{cases} C_1 \varepsilon\big(\exp[-\mu|x|^2] - \exp[-\mu\varkappa^2]\big) & \text{in } B_{\sqrt{\varkappa}} \setminus B_\varkappa \\ 0 \quad \text{in } B_\varkappa , \end{cases}$$

By (2.8), $\mathscr{L} v < 0$ in $B_{\sqrt{\varkappa}} \setminus B_\varkappa$ if μ is small enough; that is, v is a supersolution. We choose C_1 such that

$$v = \sqrt{\varkappa}\varepsilon \geqslant u \quad \text{on } \partial B_{\sqrt{\varkappa}} .$$

Then $\min(u, v)$ is an admissible function and therefore

$$J(u) \leqslant J\big(\min(u, v)\big) ,$$

which gives

$$\int_{B_\varkappa}(f(\nabla u)) + \lambda^2 I_{\{u>0\}} \leqslant \int_{B_{\sqrt{\varkappa}}\setminus B_\varkappa}\bigl(f(\nabla \min(u, v)) + \lambda^2 I_{\{\min(u,v)>0\}}\bigr)$$

or

$$\begin{aligned}\int_{B_{\sqrt{\varkappa}}}(f(\nabla u) + \lambda^2 I_{\{u>0\}}) &\leqslant \int_{B_{\sqrt{\varkappa}}\setminus B_\varkappa}\bigl(f(\nabla \min(u, v)) - f(\nabla u)\bigr)\\ &\leqslant \int_{B_{\sqrt{\varkappa}}\setminus B_\varkappa}\nabla\bigl(\min(u, v) - u\bigr)\cdot f_p(\nabla \min(u, v))\\ &= -\int_{(B_{\sqrt{\varkappa}}\setminus B_\varkappa)\cap\{u>v\}}\nabla \max(u - v, 0)\cdot f_p(\nabla v)\\ &\leqslant \int_{\partial B_\varkappa} u f_p(\nabla v)\cdot \nu ,\end{aligned}$$

where we have used the fact that v is a supersolution. Since also

$$\frac{\partial v}{\partial \nu} \leqslant C\varepsilon \qquad \text{on } \partial B_\varkappa$$

we find that

$$\int_{B_\varkappa}(f(\nabla u) + \lambda^2 I_{\{u>0\}}) \leqslant C\varepsilon\int_{\partial B_\varkappa} u . \tag{2.13}$$

On the other hand we can estimate the last integral by (cf. [1])

$$\frac{C}{\lambda}\left(\frac{\varepsilon}{\lambda} + 1\right)\int_{B_\varkappa}(|\nabla u|^2 + \lambda^2 I_{\{u>0\}}) \leqslant \frac{C}{\lambda}\left(\frac{\varepsilon}{\lambda} + 1\right)\int_{B_\varkappa}(f(\nabla u) + \lambda^2 I_{\{u>0\}})$$

Taking ε/λ sufficiently small we then obtain

$$\int_{B_\varkappa}(|\nabla u|^2 + \lambda^2 I_{\{u>0\}}) = 0 ,$$

that is $u = 0$ in $B_\varkappa$.

REMARK 2.6. Lemma 2.5 remains true if B_r is not contained in Ω provided $u = 0$ in $B_r \cap \partial\Omega$. Lemma 2.4 also remains true if B_r is not contained in Ω

provided $B_{\varkappa r} \subset \Omega$, $u = 0$ on $B_r \cap \partial\Omega$, and $B_r \cap \partial\Omega$ is smooth, that is any point of $\partial B_r/B_{\varepsilon r}(\partial\Omega)$ can be connected with the center of B_r by a tube with thickness of order εr and length of order r). Then the statement (2.11) is replaced by:

$$\frac{1}{r} \fint_{\partial B_r \setminus B_{\varepsilon r}(\partial\Omega)} u \geqslant C(\varepsilon, \varkappa)\lambda \quad \text{which implies } u > 0 \text{ in } B_{\varkappa r}\,,$$

for any given $\varepsilon > 0$, $\varkappa < 1$; Theorem 2.3 is also valid in $B_r \cap \bar{\Omega}$.

Corollary 2.7. *For any* domain $D \subset\subset \Omega$ *there exist positive constants* c, C *such that if* $B_r(x)$ *is a ball in* $D \cap \{u > 0\}$ *touching* $\partial\{u > 0\}$, *then*

$$cr \leqslant u(x) \leqslant Cr\,.$$

Theorem 2.8. *For any* $D \subset\subset \Omega$ *there exists a constant* c, $0 < c < 1$, *such that for any absolute* (*local*) *minimum* u *and for any* (*small*) *ball* $B_r \subset D$ *with center in the free boundary,*

$$c < \frac{\mathcal{L}^n(B_r \cap \{u > 0\})}{\mathcal{L}^n(B_r)} < 1 - c\,; \tag{2.14}$$

c depends on D, but not on λ.

Proof. By Lemma 2.5 there exists $y \in B_r$ with $u(y) > c\lambda r$. Using Lipschitz continuity we get

$$\fint_{\partial B_{\varkappa r}(y)} u \geqslant \frac{c\lambda}{2} r$$

if $\varkappa$ is small enough. Hence

$$\frac{1}{\varkappa r} \fint_{\partial B_{\varkappa r}(y)} u \geqslant \frac{c\lambda}{2\varkappa}\,,$$

which implies, by Lemma 2.4, that $u > 0$ in $B_{\varkappa r}(y)$. This gives the lower estimate in (2.14).

Next, let v be as in (2.1). Then, by (2.3),

$$\lambda^2 \int_{B_r} I_{\{u>0\}} \geqslant c \int_{B_r} |\nabla(u - v)|^2 \geqslant \frac{c}{r^2} \int_{B_r} |u - v|^2\,, \tag{2.15}$$

where Poincaré's inequality was used. If $y \in B_{\varkappa r}$ ($\varkappa$ small) then, by Harnack's inequality and Lemma 2.5,

$$v(y) \geqslant c\Big(\fint_{B_{r/2}} u^2\Big)^{1/2} \geqslant c\lambda r\,.$$

By Lipschitz continuity of u we also have

$$u(y) \leqslant C\lambda\varkappa r\,.$$

Therefore

$$(v-u)(y) \geqslant c\lambda r$$

if $\varkappa$ is small enough. Substituting this into the right-hand side of (2.15) we find that

$$\int_{B_r} I_{\{u=0\}} \geqslant \frac{c}{\lambda^2 r^2}\int_{B_{\varkappa y}} |u-v|^2 \geqslant \frac{c}{r^2}\, r^{n+2} = cr^n$$

and the upper estimate in (2.14) follows.

REMARK 2.9. Theorem 2.8 implies that $\mathfrak{L}^n(\partial\{u>0\}) = 0$.

3. – The measure $\Lambda = \mathscr{L}u$ and the function q.

In Theorem 1.9 we proved that the free boundary condition

$$\Phi(|\nabla u|^2) = \lambda^2$$

is satisfied in a very weak sense. Since $\Phi(0) = 0$ and (by (1.2)) $\Phi' \geqslant c > 0$ there is a unique value $\lambda^* > 0$ with

$$\Phi(\lambda^{*2}) = \lambda^2\,. \tag{3.1}$$

Hence the free boundary condition can be written as

$$|\nabla u| = \lambda^*\,,$$

or

$$-f_p(\nabla u)\cdot\nu = -2\frac{\partial u}{\partial\nu}F'(|\nabla u|^2) = Q\,, \tag{3.2}$$

if

$$Q \equiv 2\lambda^* F'(\lambda^{*2})\,. \tag{3.3}$$

We shall prove (3.3) in the sense that

$$\nabla\cdot f_p(\nabla u) = QH^{n-1}\llcorner\partial\{u>0\} \text{ in } \Omega\,,$$

where, of course, we need the fact that the free boundary $\partial\{u>0\}$ is $(n-1)$-dimensional. This we will show first.

By Lemma 1.2,

$$\nabla\cdot f_p(\nabla u) = \Lambda$$

where Λ is a positive Radon measure supported on the free boundary

$$\Gamma = \partial\{u>0\}\cap\Omega\,.$$

In the sequel we shall not indicate the explicit dependence of constants upon λ.

THEOREM 3.1. *For any $D\subset\subset\Omega$ there exist positive constants c, C such that for any ball $B_r\subset D$ with center in the free boundary*

$$cr^{n-1}\leqslant\int_{B_r} d\Lambda\leqslant Cr^{n-1}\,. \tag{3.4}$$

PROOF. Let $\zeta\in C_0^\infty(\Omega)$, $\zeta\geqslant 0$. Then

$$\int_\Omega \zeta d\Lambda = \int_\Omega \nabla\zeta\cdot f_p(\nabla u)\,.$$

Approximating I_{B_r} from below by suitable test functions ζ we get, using the Lipschitz continuity u, that for almost all r,

$$\int_{B_r} d\Lambda = \int_{\partial B_r} \nu\cdot f_p(\nabla u)\,dH^{n-1}\leqslant Cr^{n-1}\,.$$

To establish the left-hand side of (3.4) we may normalize by taking $r=1$. Set

$$\tilde{\mathscr{L}}v = \nabla\cdot(a\nabla v) \qquad \big(a(x) = F'(|\nabla u(x)|^2)\big)\,.$$

This is a linear selfadjoint uniformly elliptic operator. Denote its Green's function in B_1 with pole x by G_x. By [6; p. 184], if

$$\tilde{\mathscr{L}}v = f \quad \text{in } B_1\,,$$
$$v = 0 \quad \text{on } \partial B_1$$

then

$$|v|_{L^\infty(B_{1/2})} \leqslant C|f|_{L^p(B_1)} \quad \text{if } p > \frac{n}{2}. \tag{3.5}$$

Representing v in terms of Green's function we deduce that (3.5) for all f means

$$\sup_{x\in B_{1/2}} \int_{B_1} G_x^q \leqslant C \qquad \left(\frac{1}{q} + \frac{1}{p} = 1\right). \tag{3.6}$$

Now let $x \in B_{\frac{1}{2}} \setminus \bar{B}_{\frac{1}{4}}$. Then $\tilde{\mathscr{L}} G_x = 0$ in $B_{\frac{1}{4}}$ and, by (3.6), G_x is bounded in $L^q(B_{\frac{1}{4}})$. Hence, by elliptic estimates [6; p. 184]

$$G_x(y) \leqslant C \quad \text{if } y \in B_{\frac{1}{8}},\ x \in B_{\frac{1}{2}} \setminus B_{\frac{1}{4}},$$

and by the symmetry of Green's function

$$G_y(x) \leqslant C \quad \text{if } y \in G_{\frac{1}{8}},\ x \in B_{\frac{1}{2}} \setminus B_{\frac{1}{4}}.$$

Similarly

$$\begin{aligned} &G_y(x) \leqslant C \quad \text{if } x \in D_1,\ y \in D_2,\ \operatorname{dist}(D_1, D_2) \geqslant \delta, \\ &D_1 \cup D_2 \subset B_1, \qquad\qquad \operatorname{dist}(D_2, \partial B_1) \geqslant \delta \end{aligned} \tag{3.7}$$

where C is a constant depending on δ but not on D_1, D_2.

Now let w be the solution of

$$\tilde{\mathscr{L}} w = 0 \quad \text{in } B_1,$$
$$w = u \quad \text{on } \partial B_1.$$

By nondegeneracy, for any $0 < \varkappa < 1$ there exists a point $y \in B_\varkappa$ with $u(y) > c\varkappa$. By Lipschitz continuity,

$$u > 0 \quad \text{in } B_{c(\varkappa)}(y)$$

with $c(\varkappa)$ sufficiently small; hence also $d\Lambda = 0$ in $B_{c(\varkappa)}(y)$. Since

$$\tilde{\mathscr{L}}(w - u) = \Lambda, \qquad w = 0 \text{ on } \partial B_1,$$

we can write

$$(w - u)(y) = \int_{B_1} G_y\, d\Lambda = \int_{B_1 \setminus B_{c(\varkappa)}(y)} G_y\, d\Lambda \leqslant C(\varkappa) \int_{B_1} d\Lambda \tag{3.8}$$

where we have used (3.7) with $\delta = \delta(\varkappa)$.

By the maximum principle $w \geqslant u$; therefore, by nondegeneracy (for $p > 1$)

$$\Big(\fint_{B_{1/2}} w^p\Big)^{1/p} \geqslant \Big(\fint_{B_{1/2}} u^p\Big)^{1/p} \geqslant c \qquad (c > 0)\,.$$

Using Harnack's inequality [6; p. 184], if $p < n/(n-2)$, we get

$$w(y) \geqslant c\Big(\fint_{B_1} w^p\Big)^{1/p} \geqslant c\,.$$

Also, since u vanishes at the center of B_1, by Lipschitz continuity,

$$u(y) \leqslant C\varkappa\,.$$

Choosing $\varkappa$ small enough we get

$$w(y) - u(y) \geqslant c - C\varkappa > \frac{c}{2}\,,$$

and recalling (3.8) we obtain

$$\int_{B_1} d\Lambda \geqslant c\,.$$

The next theorem follows easily from Theorem 3.1, precisely as in [1].

THEOREM 3.2 (*Representation theorem*). *Let u be a local minimum. Then*:

(1) $H^{n-1}(D \cap \partial\{u > 0\}) < \infty$ *for every* $D \subset\subset \Omega$.

(2) *There is a Borel measure* q_u *such that*

$$\mathscr{L}u = q_u H^{n-1} \llcorner \partial\{u > 0\}\,,$$

that is, for every $\zeta \in C_0^\infty(\Omega)$

$$-\int_{\Omega} \nabla\zeta \cdot f_p(\nabla u) = \int_{\Omega \cap \partial\{u>0\}} \zeta q_u\, dH^{n-1}.$$

(3) *For any* $D \subset\subset \Omega$ *there exist positive constants* c, C *such that for every ball* $B_r(x) \subset D$ *with* $x \in \partial\{u > 0\}$,

$$c \leqslant q_u(x) \leqslant C\,, \qquad c r^{n-1} \leqslant H^{n-1}(B_r(x) \cap \partial\{u > 0\}) \leqslant C r^{n-1}\,.$$

From (1) it follows [5; 4.5.11] that the set $A = \{u > 0\}$ has a finite perimenter locally in Ω, that is, $\mu_u \equiv -\nabla I_A$ is a Borel measure. We denote by $\partial_{\text{red}} A$ the reduced boundary of A, *i.e.*, the set of points $x \in \Omega$ for which the normal $\nu_u(x)$ of A at x exists and $|\nu_u(x)| = 1$; see [5] [7].

We shall deal with blow up sequences

$$u_m(x) = \frac{1}{\varrho_m} u(x_m + \varrho_m x)\,, \qquad u(x_m) = 0\,, \qquad x_m \to x_0 \in \Omega\,.$$

Since $|\nabla u| \leqslant C$, for a subsequence,

$$u_m \to u_0 \quad \text{uniformly in } C^\alpha_{\text{loc}}, \text{ for all } \alpha < 1\,, \tag{3.9}$$

$$\nabla u_m \to \nabla u_0 \quad \text{weakly star in } L^\infty_{\text{loc}}\,, \tag{3.10}$$

We also have:

$$\partial\{u_m > 0\} \to \partial\{u_0 > 0\} \quad \text{locally in the Hausdorff distance}\,, \tag{3.11}$$

$$I_{\{u_m > 0\}} \to I_{\{u_0 > 0\}} \quad \text{in } L^1_{\text{loc}}\,, \tag{3.12}$$

$$\nabla u_m \to \nabla u_0 \quad \text{a.e.} \tag{3.13}$$

The proof of (3.11) follows from (3.9) and the nondegeneracy. The same arguments show that if $x_m \in \partial\{u_m > 0\}$ then $x_0 \in \partial\{u_0 > 0\}$.

To prove (3.12) let $x \in \partial\{u_0 > 0\} \cap \Omega$. Then there exists a sequence $y_m \in \partial\{u_m > 0\}$ such that $y_m \to x$. By nondegeneracy (Lemma 2.5 with $p = 2$)

$$\frac{1}{r}\Big(\fint_{\partial B_r(y_m)} u_m^2\Big)^{1/2} \geqslant c\,.$$

Hence

$$\frac{1}{r}\Big(\fint_{\partial B_r(x)} u_0^2\Big)^{1/2} \geqslant c\,.$$

Since u_0 is clearly also in $C^{0,1}$, the proof of Theorem 2.8 applies to u_0. Consequently

$$\mathcal{L}^n(\partial\{u_0 > 0\}) = 0\,.$$

Combining this fact with (3.11), the assertion (3.12) follows.

By elliptic estimates

$$\nabla u_m \to \nabla u_0 \quad \text{uniformly in compact subsets of } \{u_0 > 0\}$$

and

$$\nabla\cdot f_p(\nabla u_0) = 0 \quad \text{in } \{u_0 > 0\}\,. \tag{3.14}$$

Thus in order to prove (3.13) it remains to show that

$$\nabla u_m \to \nabla u_0 \quad \text{a.e. on } \{u_0 = 0\}\,. \tag{3.15}$$

In the set $\{u_0 = 0\}$ a.a. points x_0 have density 1; denote the set of such points by S. We claim that if $x_0 \in S$ then

$$u_0(x_0 + x) = o(|x|)\,. \tag{3.16}$$

Indeed, if $u_0(y) > \gamma r$ for some $y \in B_r(x_0)$, $\gamma > 0$ and a sequence $r \to 0$, then (by the Lipschitz continuity of u_0)

$$u_0 > \frac{\gamma}{2} r \qquad \text{in } B_{c\gamma r}(y)$$

for some small $c > 0$. This means that $\{u_0 > 0\}$ has positive density at x_0, contradicting $x_0 \in S$.

From (3.16) we deduce that, for any $\varepsilon > 0$,

$$\frac{u_m}{r} < \varepsilon \qquad \text{in } B_r(x_0) \text{ for small } r\,,$$

provided m is large enough, say $m \geqslant m_0(\varepsilon, r)$. By nondegeneracy it then follows that $u_m = 0$ in $B_{r/2}(x_0)$ and, consequently, $u_0 = 0$ in a neighborhood of x_0. Thus the set S is open. Furthermore, the above argument shows that $u_m = u_0$ in any compact subset of S, if m is large enough. This completes the proof of (3.15).

In order to identify the function q_u in Theorem 3.2, we need the following two important statements about the minimum.

LEMMA 3.3. *If $u(x_m) = 0$, $x_m \to x_0 \in \Omega$, then any blow up limit u_0 with respect to $B_{\varrho_m}(x_m)$ is absolute minimum for J in any ball.*

PROOF. Let

$$u_m(x) = \frac{1}{\varrho_m} u(x_m + \varrho_m x)$$

and suppose that (3.9), (3.10) hold. Then also (3.11)-(3.13) are satisfied.

Set $D = B_R(0)$ and take any v, $v - u_0 \in H_0^{1,2}(D)$, $\eta \in C_0^\infty(D)$, $0 \leqslant \eta \leqslant 1$. Let

$$v_m = v + (1-\eta)(u_m - u_0).$$

Since $v_m = u_m$ on ∂D,

$$\int_D \big(f(\nabla u_m) + \lambda^2 I_{\{u_m>0\}}\big) \leqslant \int_D \big(f(\nabla v_m) + \lambda^2 I_{\{v_m>0\}}\big). \tag{3.17}$$

Since $|\nabla u_m| \leqslant C$, $\nabla u_m \to \nabla u_0$ a.e.,

$$\int_D f(\nabla u_m) \to \int_D f(\nabla u_0) \quad \text{as } m \to \infty.$$

Similarly

$$\int_D f(\nabla v_m) \to \int_D f(\nabla v).$$

Noting also that

$$I_{\{v_m>0\}} \leqslant I_{\{v>0\}} + I_{\{\eta<1\}},$$

we obtain from (3.17)

$$\int_D \big(f(\nabla u_0) + \lambda^2 I_{\{u_0>0\}}\big) \leqslant \int_D \big(f(\nabla u) + \lambda^2 I_{\{v>0\}}\big) + \gamma(\eta)$$

where $\gamma(\eta) = \lambda^2 \int_D I_{\{\eta<1\}}$. Choosing a sequence of η's with $\gamma(\eta) \to 0$, the assertion follows.

Next we prove an estimate for $|\nabla u|$ at the free boundary from above (see [1; Remark 6.4]), which will also be used in Section **4**, where we prove the corresponding Hölder estimate.

LEMMA 3.4. *Let u be a local minimum and let λ^* be defined by* (3.1). *If $x_0 \in \{u > 0\}$ then*

$$\limsup_{x \to x_0,\, u(x)>0} |\nabla u(x)| = \lambda^* \tag{3.18}$$

PROOF. Denote the left-hand side of (3.18) by γ. Then there exists a sequence z_k such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to \gamma.$$

Denote by y_k the nearest point to z_k on $\partial\{u > 0\}$ and set $\varrho_k = |z_k - y_k|$.

Consider a blow up sequence about $B_{\varrho_k}(y_k)$ with limit u_0, such that

$$\frac{z_k - y_k}{\varrho_k} \to - e_n .$$

Then u_0 is $\mathscr{L}$-subsolution in R^n and $\mathscr{L}u_0 = 0$ in $\{u_0 > 0\}$. By Lemma 3.3 the blow-up limit u_0 is an absolute minimum, hence $B_1(-e_n)$ is contained in $\{y_0 > 0\}$ by (3.12). Moreover $u_0(0) = 0$ and

$$\begin{aligned} &|\nabla u_0| \leqslant \gamma \qquad \text{in } \{u_0 > 0\} , \\ &|\nabla u_0(-e_0)| = \gamma \end{aligned}$$

which implies that $\gamma > 0$.

Choose a unit vector e such that

$$|\nabla u_0(-e_n)| = - \nabla u_0(-e_n) \cdot e$$

and consider the function $\partial u_0/\partial e$. We have (see the proof of Lemma 1.7)

$$\mathscr{L}_0\left(\frac{\partial u_0}{\partial e}\right) = 0 \qquad \text{in } \{u_0 > 0\}$$

where $\mathscr{L}_0$ is a uniformly divergence-form elliptic operator. Also

$$\text{(3.19)} \qquad \begin{aligned} &\frac{\partial u_0}{\partial e} \geqslant -\gamma \qquad \text{in } B_1(-e_n) , \\ &\frac{\partial u_0(-e_n)}{\partial e} = -\gamma . \end{aligned}$$

Applying the maximum principle we conclude from (3.19) that

$$\frac{\partial u_0(x)}{\partial e} = -\gamma \qquad \text{or } u_0(x) = -\gamma x \cdot e + c \text{ for } x \in B_1(-e_n) .$$

Since $u_0(0) = 0$ and $u_0 > 0$ in $B_1(-e_n)$, the constant c is equal to zero and $e = e_n$. Thus

$$u_0(x) = -\gamma x_n \qquad \text{for } x \in B_1(-e_n) .$$

By continuation the same argument shows that

$$\text{(3.20)} \qquad u_0(x) = -\gamma x_n \qquad \text{whenever } x_n \leqslant 0 .$$

We next claim that

$$u_0 = 0 \quad \text{in some strip } \{0 < x_n < \varepsilon_0\}. \tag{3.21}$$

Indeed, suppose this is not true, and define

$$s = \limsup_{\substack{x_n \downarrow 0,\, u_0(x', x_n) > 0 \\ x' \in R^{n-1}}} \frac{\partial u_0(x', x_n)}{\partial x_n}.$$

By Lipschitz continuity of u, u_0 is uniformly Lipschitz continuous and thus $s < \infty$. Suppose $s > 0$ and let

$$\frac{\partial u_0}{\partial x_n}(y_k, h_k) \to s, \qquad h_k \downarrow 0.$$

Consider a blow up sequence with respect to $B_{h_k}(y_k, 0)$ with limit u_{00}. Arguing as before we conclude that

$$u_{00}(x', x_n) = s x_n \quad \text{for all } x_n > 0.$$

But we also have (by (3.20))

$$u_{00}(x', x_n) = -\gamma x_n \quad \text{for all } x_n < 0.$$

Thus, u_{00} is a minimum (Lemma 3.3), and any point $(x', 0)$ is a free boundary point. It follows that the set $\{u_{00} = 0\}$ has density zero at any point $(x', 0)$, a contradiction to Theorem 2.8.

We have thus proved that $s = 0$ and, consequently, $u_0(x', x_n) = o(x_n)$ if $x_n \downarrow 0$. Hence, for any $\varepsilon > 0$,

$$\frac{1}{r}\Big(\fint_{B_r(x_0)} u_0^2\Big)^{1/2} < \varepsilon$$

for any $x_0 = (y_0, h_0)$, $r = h_0$, with h_0 small enough. It follows, by non-degeneracy, that $u_0 = 0$ in some strip $\{0 < x_n < \varepsilon_0\}$, which contradicts the assumption that (3.21) was not satisfied.

Having proved (3.20), (3.21) we can now apply Theorem 1.9 to deduce that, on the free boundary $\{x_n = 0\}$, $|\nabla u_0|$ coincides with λ^*, *i.e.*, $\gamma = \lambda^*$.

In the next statement we use the following notation (see [5; 3.1.21]).

For any set E and $x_0 \in E$ we define the (topological) *tangent plane* of E

at x_0 by

$$\operatorname{Tan}(E, x_0) = \left\{v;\ v = \lim_{m\to 0} r_m v_m,\ r_m > 0,\ x_0 + v_m \in E,\ v_m \to 0\right\}.$$

THEOREM 3.5 (*Identification of q_u*). *Let $x_0 \in \partial_{\text{red}}\{u > 0\}$ and suppose that*

$$\theta^{*n-1}\left(H^{n-1} \llcorner \partial\{u > 0\}, x_0\right) \leqslant 1 .$$

Then $\operatorname{Tan}\left(\partial\{u > 0\}, x_0\right) = \{x;\ x\cdot\nu_u(x_0) = 0\}$. *If, in addition,*

$$\int\limits_{B_r(x_0)\cap\partial\{n>0\}} |q_u - q_u(x_0)|\, dH^{n-1} = o(r^{n-1}) \qquad \textit{as } r \to 0$$

then, $q_u(x_0) = Q$ and, as $x \to 0$,

$$u(x_0 + x) = \lambda^* \max\{-x\cdot\nu_u(x_0), 0\} + o(|x|)\ , \tag{3.22}$$

where λ^*, Q are defined as in (3.1) *and* (3.2).

PROOF. Take for simplicity $\nu_u(x_0) = e_n$. Let u_m be a blow up sequence with respect to balls $B_{\varrho_m}(x_0)$, with blow up limit u_0. Then $I_{\{u_m>0\}}$ converges in L^1_{loc} to $I_{\{u_0>0\}}$ (by (3.12)) and to $I_{\{u_n>0\}}$ since,

$$\int\limits_{B_r(x)} |I_{\{u>0\}} - I_{\{y:(y-r_0)\cdot\nu_u(x_0)>0\}}| = o(r^n) \qquad \text{as } r \to 0\,,$$

if $\nu_u(x_0)$ is the normal to $\{u > 0\}$ at x_0. It follows that $u_0 = 0$ in $\{x_n \geqslant 0\}$ and $u_0 > 0$ a.e. in $\{x_n < 0\}$.

In order to show that $u_0 > 0$ in $\{x_n < 0\}$, we proceed as in [1; p. 121] to deduce that

$$H^{n-1}\left(\partial\{u_m > 0\} \cap D\right) \to 0 \qquad \text{if } m \to \infty,$$

for any $D \subset\subset \{x_n < 0\}$. We then conclude that for any $\zeta \in C_0^\infty(D)$

$$-\int\limits_D f_p(\nabla u_0)\cdot\nabla\zeta \leftarrow -\int\limits_D f_p(\nabla u_m)\cdot\nabla\zeta = \int\limits_{\partial\{u_m>0\}} \zeta(x)\, q_u(x_0 + \varrho_m x)\, dH^{n-1}(x) \to 0 \tag{3.23}$$

where (3.13) was used. Thus u_0 is a solution of $\mathscr{L}u_0 = 0$ and it is therefore smooth in $\{x_n < 0\}$ (Lemma 1.7). The maximum principle now implies that $u_0 > 0$ in $\{x_n < 0\}$.

Now let

$$\zeta(x) = \min\left(2\left(1-\frac{|x_n|}{2}\right), 1\right)\, \eta(x_1, \ldots, x_{n-1})$$

where $\eta \in C_0^\infty(B'_r)$, $\eta \geqslant 0$ (B'_r is an $(n-1)$-dimensional ball with radius r). Proceeding as in [1; p. 121] and using also (3.13), we get

$$(3.24)\qquad \int_{R^n} f_p(\nabla u_0)\cdot\nabla\zeta = -\lim_{m\to\infty}\int_{R^n} f_p(\nabla u_m)\cdot\nabla\zeta$$
$$= \lim_{m\to\infty}\int_{\partial\{u_m>0\}} \zeta q_u(x_0+\varrho_m x)\, dH^{n-1} = q_u(x_0)\int_{R^{n-1}}\eta\, dH^{n-1}\,.$$

Notice that, unlike in (3.23), the test functions here are not supported just in $\{x_n<0\}$. From (3.24) we deduce the boundary conditions

$$(3.25)\qquad f_p(\nabla u_0)\cdot e_n = q_u(x_0) \quad \text{ on } \{x_n = 0\}\,,$$

in the sense that

$$-\int_{B_r\cap\{x_n>0\}} f_p(\nabla u_0)\cdot\nabla\zeta = q_u(x_0)\int_{B'_r}\zeta(x', 0)\, dH^{n-1} \qquad \forall\zeta\in C_0^1(B_r)\,.$$

Since $\mathscr{L}u_0 = 0$ in $\{x_n<0\}$, by boundary regularity for elliptic equations (see proof of Lemma 1.7) it follows that u_0 satisfies (3.25) in the classical sense.

Since furthermore u_0 is an absolute minimum (Lemma 3.3) we deduce from Theorem 1.9 that

$$(3.26)\qquad \frac{\partial u_0}{\partial\nu} = -\lambda^* \quad \text{ on } \partial\{x_n<0\}\,,$$

or $q_u(x_0) = Q$.

We have to show that u_0 is a half plane solution with slope λ^*. To prove this we use Lemma 3.4, which tells us that

$$0 < u_0(x) \leqslant -\lambda^* x_n \quad \text{ if } x_n < 0\,,$$

and $\tilde{\mathscr{L}}u_0 = 0$ in $\{u_0<0\}$, where

$$\tilde{\mathscr{L}}w = f_{p_ip_j}(\nabla u_0)\,\partial_{ij}w\,.$$

Therefore, the function

$$v(x) \equiv -\lambda^* x_n - u_0(x)$$

is nonnegative in $\{x_0 < 0\}$ and $\tilde{\mathscr{L}}v = 0$. Hence, if v is positive at some point in $\{x_n < 0\}$, it must be positive everywhere in $\{x_n < 0\}$, by the maximum principle, and, further,

$$\frac{\partial v}{\partial \nu} < 0 \qquad \text{on } \partial\{x_n < 0\}\,,$$

that is, $\partial u_0/\partial \nu > -\lambda^*$, which contradicts (3.26). Consequently $v = 0$ in $\{x_n < 0\}$, that is,

$$u_0(x) = \begin{cases} -\lambda^* x_n & \text{for } x_n \leqslant 0\,, \\ 0 & \text{for } x_n \geqslant 0\,. \end{cases}$$

Since finally the blow up sequence was arbitrary whereas the limit is unique, the assertion (3.22) follows.

Since the conclusion of Theorem 3.5 holds for H^{n-1} a.a. $x_0 \in \partial_{\text{red}}\{u > 0\}$ we obtain:

THEOREM 3.6. *For H^{n-1} a.a. x in $\partial_{\text{red}}\{u > 0\}$*

$$q_u(x) = Q\,.$$

REMARK 3.7. From the positive density property (Theorem 2.8 and [5; 4.5.6 (3)] it follows that $H^{n-1}(\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$. Hence

$$q_u = Q \qquad H^{n-1} \text{ a.e.} \quad \text{on } \partial\{u > 0\}. \tag{3.27}$$

4. - Estimates on $|\nabla u|$.

For the regularity theory of the free boundary we need the following strengthened version of Lemma 3.4.

THEOREM 4.1. *Let u be a local minimum. For any $D \subset\subset \Omega$ there exist positive constants C, $\alpha < 1$ depending only on β, λ and D such that, for any ball $B_r(x) \subset D$ which intersects the free boundary,*

$$\sup_{B_r(x)} |\nabla u(x)| \leqslant \lambda^* + C r^{\alpha} \tag{4.1}$$

PROOF. We have

$$a_{ij}(\nabla u)\,\partial_{ij} u = 0 \qquad \text{in } \{u > 0\}$$

where the uniformly positive and bounded matrix a_{ij} is defined as in (1.9).

Therefore, by [6; pp. 270-271], the function $v = |\nabla u|^2$ satisfies

$$\mathcal{M}v \equiv \partial_i(\exp(\gamma v) a_{ij}(\nabla u)\partial_j v) \geq 0 \quad \text{in } \{u > 0\} \tag{4.2}$$

where γ is some positive constant. For any $\varepsilon > 0$, the function

$$U_\varepsilon = \left(|\nabla u|^2 - (\lambda^*)^2 - \varepsilon\right)^+$$

is $\mathcal{M}$-subsolution in $\{u > 0\}$ and, by Lemma 3.4, it vanishes in a neighborhood of the free boundary. We extend U_ε by 0 into $\{u = 0\}$ and set

$$h_\varepsilon(r) = \sup_{B_r} U_\varepsilon$$

for any r, where the origin is taken to be a free boundary point. Then $h_\varepsilon(r) - U_\varepsilon$ is $\mathcal{M}$-supersolution in the entire ball B_r and

$$\begin{aligned} h_\varepsilon(r) - U_\varepsilon &\geq 0 && \text{in } B_r \\ &= h_\varepsilon(r) && \text{in } \tilde{B}_r = B_r \cap \{u = 0\}\ . \end{aligned}$$

By [6; p. 193] with $1 \leq p < n/(n-2)$,

$$\inf_{B_{r/2}} [h_\varepsilon(r) - U_\varepsilon(x)] \geq cr^{-n/p}\|h_\varepsilon - U_\varepsilon\|_{L^p(B_r)} \geq ch_\varepsilon(r)\,,$$

since $\mathcal{L}^n(\tilde{B}_r) \geq cr^n$ by the positive density property (Theorem 2.8). Taking $\varepsilon \to 0$ we get

$$\inf_{B_{r/2}} [h_0(r) - U_0] \geq ch_0(r) \qquad (0 < c < 1)$$

or

$$\sup_{B_{r/2}} U_0 \leq (1 - c)h_0(r)\,.$$

Thus

$$h_0\left(\frac{r}{2}\right) \leq (1 - c)h_0(r)\,,$$

from which we deduce that $h_0(x) \leq Cs^\alpha$ for some $C > 0$, $0 < \alpha < 1$, and (4.1) follows.

REMARK 4.2. If, in Theorem 4.1, $B_r(x) \subset B_R(x) \subset D$ then

$$\sup_{B_r} |\nabla u| \leq \mu + C\left(\frac{r}{R}\right)^\alpha. \tag{4.3}$$

Indeed, this follows by applying Theorem 4.1 to $\tilde{u}(x') = (1/R)u(x + Rx')$.

THEOREM 4.3. *Let $n = 2$ and assume that u is any local minimum and that $D \subset\subset \Omega$. Then for any small ball $B_r \subset D$ with center in the free boundary,*

$$\fint_{B_r \cap \{u>0\}} (\lambda^{*2} - |\nabla u|^2)^+ \leq \frac{C}{\log(1/r)}. \tag{4.4}$$

PROOF. For any $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$, $\varepsilon > 0$, the function

$$v \equiv u - \min(u, \varepsilon\zeta) = \max(u - \varepsilon\zeta, 0)$$

is admissible, so that $J(u) \leq J(v)$, or

$$\int_{\{0<u\leq\varepsilon\zeta\}} \lambda^2 = \int [f(\nabla v) - f(\nabla u)] \equiv I. \tag{4.5}$$

We have

$$f(\nabla v) - f(\nabla u) = \int_0^1 f_p(\nabla u + t\nabla(v-u)) \cdot \nabla(v-u)\, dt$$

$$= f_p(\nabla u) \cdot \nabla(v-u) + \int_0^1 dt \int_0^t \nabla(v-u) f_{pp}(\nabla u + s\nabla(v-u))\, \nabla(v-u)\, ds.$$

On the set $\{u > \varepsilon\zeta\}$ we have $v - u = -\varepsilon\zeta$, hence

$$\int_{\{u>\varepsilon\zeta\}} \int_0^1 dt \int_t^1 \nabla(v-u) f_{pp}(\nabla u + s\nabla(v-u))\, \nabla(v-u)\, ds \leq C\varepsilon^2 \int_{\{u>\varepsilon\zeta\}} |\nabla\zeta|^2.$$

On the set $\{u \leq \varepsilon\zeta\}$ we have $v = 0$ and thus

$$\int_{\{u\leq\varepsilon\zeta\}} \int_0^1 dt \int_0^t \nabla(v-u) f_{pp}(\nabla u + s\nabla(v-u))\, ds = \int_{\{u\leq\varepsilon\zeta\}} \int_0^1 dt \int_t^1 \nabla u f_{pp}((1-s)\nabla u)\, \nabla u\, ds.$$

Next

$$0 = -f(0) = f(\nabla u) - \int_0^1 f_p((1-t)\nabla u) \cdot (\nabla u)\, dt$$

$$= f(\nabla u) - f_p(\nabla u) \cdot \nabla u + \int_0^1 dt \int_0^t \nabla u f_{pp}((1-s)\nabla u)(-\nabla u)\, ds.$$

Combining these facts we find that

$$I \leqslant \int_{\{u \leqslant \varepsilon\zeta\}} \left(f_p(\nabla u)\cdot\nabla u - f(\nabla u)\right) + C\varepsilon^2 \int_{\{u > \varepsilon\zeta\}} |\nabla\zeta|^2 + \int_{\Omega} f_p(\nabla u)\,\nabla(v-u)\,.$$

In the last integral the integrand vanishes on the set $\{u = 0\}$. Since also $v - u = 0$ on $\partial\{u > 0\}$ and $\nabla\cdot f_p(\nabla u) = 0$ on $\{u > 0\}$, this integral vanishes. (for a rigorous proof approximate $v - u$ by $-\min(u - \delta, \varepsilon\zeta)$ for $\delta > 0$.) We thus obtain from (4.5) and (3.1), (1.10),

$$\int_{\{0<u\leqslant\varepsilon\zeta\}} \Phi(\lambda^{*2}) \leqslant \int_{\{0<u\leqslant\varepsilon\zeta\}} \Phi(|\nabla u|^2) + C\varepsilon^2 \int_{\{u>\varepsilon\zeta\}} |\nabla\zeta|^2\,. \tag{4.6}$$

This is true for any $n \geqslant 2$.

Now we specialize to $n = 2$, and take $B_r \subset B_\varrho \subset B_R$ with center x_0 in the free boundary. Thus $u \leqslant Cr$ in B_r. Choosing $\varepsilon = Cr$ and

$$\zeta(x) = \begin{cases} \dfrac{\log(\varrho/|x - x_0|)}{\log(\varrho/r)} & \text{for } x \in B_\varrho \setminus B_r\,, \\ 1 & \text{for } x \in B_r\,, \end{cases}$$

we get

$$\int_{B_r \cap \{u>0\}} \left(\Phi(\lambda^{*2}) - \Phi(|\nabla u|^2)\right)^+ \leqslant \int_{B_\varrho} \left(\Phi(|\nabla u|^2) - \Phi(\lambda^*)\right)^+ + \frac{Cr^2}{\log(\varrho/r)}\,.$$

Using Remark 4.2 and the estimate $c \leqslant \Phi' \leqslant C$ (c, C positive) we find that

$$\frac{1}{r^2} \int_{B_r \cap \{u>0\}} (\lambda^{*2} - |\nabla u|^2)^+ \leqslant C\left(\frac{\varrho}{r^2}\right)^2 \left(\frac{\varrho}{R}\right)^\alpha + \frac{C}{\log(\varrho/r)}\,.$$

Choosing $R = \varrho^{\frac{1}{2}}$, $\varrho = r^\varepsilon$ with $\varepsilon > 1$, $(2 + \alpha/2)\varepsilon > 2$, we get (4.4).

COROLLARY 4.4. *Let $n = 2$ and let u be a local minimum. Then every blow up limit of u at $x_0 \in \partial\{u > 0\} \cap \Omega$ is a half plane solution with slope λ^* in some neighborhood of the origin.*

PROOF. – From Theorems 4.1, 4.3 we see that, for any blow up limit u_0, with respect to balls $B_{\varrho_m}(x_0)$,

$$|\nabla u_0| = \lambda^* \quad \text{in } \{u_0 > 0\}\,.$$

Now let us use the fact that $v = |\nabla u_0|^2$ is an $\mathcal{M}$-subsolution (see (4.2)), which implies (if γ is large enough in (4.2)) that

$$\mathcal{M}v \geqslant c|D^2 u_0|^2$$

for some small positive constant c. Since $v = \lambda^{*2}$ we conclude that u_0 is linear in each connected component of $\{u_0 > 0\}$, therefore

$$u_0(x) = q_0 \max(-x\cdot\nu_0, s_0) + q_1 \max(x\cdot\nu_0, s_1)$$

for some unit vector ν_0 and some numbers $q_k > 0$, $0 \leqslant s_k \leqslant \infty$, $k = 0, 1$. Since the origin is a free boundary point, we may assume that $s_0 = 0$.

Since $\{u_0 = 0\}$ has positive density at the origin (by Lemma 3.3 and Theorem 2.8) we must have $s_1 > 0$. By Theorem 1.9 we also have $q_0 = \lambda^*$.

5. – Flat free boundary points.

DEFINITION 5.1. Let $0 < \sigma_+, \sigma_- \leqslant 1$ and $\tau > 0$. We say that u belongs to class $F(\sigma_+, \sigma_-; \tau)$ in $B_\varrho = B_\varrho(0)$ if u is a local minimum in $B_\varrho(0)$ with $0 \in \partial\{u > 0\}$, and

$$u(x) = 0 \qquad \text{for } x_n \geqslant \sigma_+ \varrho\,, \tag{5.1}$$

$$u(x) \geqslant -\lambda^*(x_n + \sigma_-\varrho) \qquad \text{for } x_n \leqslant -\sigma_-\varrho\,, \tag{5.2}$$

$$|\nabla u| \leqslant \lambda^*(1+\tau) \qquad \text{in } B_\varrho\,. \tag{5.3}$$

If the origin is replaced by x_0 and the direction of flatness e_n is replaced by a unit vector ν, then we say that u belongs to the class $F(\sigma_+, \sigma_-; \tau)$ in $B_\varrho(x_0)$ in direction ν.

THEOREM 5.1. *There exists a positive constant $C = C(n)$ such that, for any small $\sigma > 0$, if $u \in F(\sigma, 1; \sigma)$ in B_ϱ then $u \in F(2\sigma, C\sigma; \sigma)$ in $B_{\varrho/2}$.*

Thus flatness from above implies flatness from below.

In proving the theorem we shall need:

LEMMA 5.2. *If B is a ball in $\{u = 0\}$ touching $\partial\{u > 0\}$ at x_0, then*

$$\limsup_{x\to x_0} \frac{u(x)}{\operatorname{dist}(x, B)} = \lambda^*\,. \tag{5.4}$$

PROOF. Denote the left-hand side by l and let $y_k \to x_0$, $u(y_k) > 0$, $d_k = \operatorname{dist}(y_k, B)$,

$$\frac{u(y_k)}{d_k} \to l\,.$$

By nondegeneracy $l > 0$. Consider a blow up sequence u_k with respect to $B_{d_k}(x_k)$ where x_k is a point on ∂B with $|x_k - y_k| = d_k$.

For a subsequence,

$$\frac{x_k - y_k}{d_k} \to e\,, \qquad u_k \to u_0\,.$$

We claim that

$$u_0(x) = l \max(-x\cdot e, 0)\,. \tag{5.5}$$

Indeed, we have

$$\tilde{\mathscr{L}} u_0 = 0 \quad \text{ in } \{u_0 > 0\}$$

where

$$\tilde{\mathscr{L}} w \equiv f_{p_i p_j}(\nabla u_0)\partial_{ij} w$$

is a uniformly elliptic operator. Thus

$$\tilde{\mathscr{L}}(u_0(x) + lx\cdot e) = 0 \quad \text{ for } x \in \{u_0 > 0\}$$

and by construction $u_0(x) + lx\cdot e \leqslant 0$ when $x\cdot e < 0$, and $u_0(x) + lx\cdot e = 0$ for $x = -e$. The maximum principle yields that (5.5) holds for x near $-e$, and then by analytic continuation, the same argument shows that (5.5) holds if $x\cdot e < 0$. If $x\cdot e \geqslant 0$ then $u_0(x) = 0$ by construction, so that again (5.5) holds.

Since u_0 is an absolute minimum by Lemma 3.3, we deduce from (5.5) and Theorem 1.9 that $l = \lambda^*$.

Proof of Theorem 5.1. By homogeneity we may assume that $\varrho = 1$, $\lambda^* = 1$. Let

$$\begin{aligned} \eta(y) &= \exp\left(-\frac{9|y|^2}{1 - 9|y|^2}\right) && \text{for } |y| < \frac{1}{3} \\ &= 0 && \text{for } |y| > \frac{1}{3}, \end{aligned}$$

and choose $s \geqslant 0$ maximal such that

$$B_1 \cap \{u > 0\} \subset D \equiv D_\sigma \equiv \{x \in B_1;\ x_n < \sigma - s\eta(x')\}$$

where $x = (x', x_n)$. Thus there exists a point

$$z \in B_{\frac{1}{2}} \cap \partial D \cap \partial\{u > 0\}\,.$$

Also, $s \leqslant \sigma$ since $0 \in \partial\{u > 0\}$.

We will prove an estimate for $u(\xi)$ from below for points $\xi \in \partial B_{\frac{3}{4}}$, $\xi_n < -\frac{1}{2}$. For that consider the solution v of

$$\begin{aligned} \mathscr{L}v = \nabla\cdot\big(f_p(\nabla v)\big) = 0 \quad & \text{in} \quad D\setminus\overline{B(\xi)}\,, \\ v = 0 \quad & \text{on } \partial D \cap B_1\,, \\ v = (1+\sigma)(\sigma - x_n) \quad & \text{on } \partial D\setminus B_1\,, \\ v = -(1-\varkappa\sigma)x_n \quad & \text{on } \partial B(\xi)\,, \end{aligned}$$

where $B(\xi) = B_{\frac{1}{10}}(\xi)$ and $\varkappa$ is a large positive constant to be chosen later. Such a function exists by minimization, and by elliptic estimates up to the boundary (see proof of Lemma 1.7) v is smooth in $D\setminus B(\xi)$ (possibly not at the corners).

We claim that

$$u(x_\xi) \geqslant v(x_\xi) \qquad \text{for some } x_\xi \in \partial B(\xi) \tag{5.6}$$

provided $\varkappa$ is a sufficiently large constant. Indeed, otherwise we have $u \leqslant v$ on $\partial B(\xi)$; also $u \leqslant v$ on ∂D by (5.3) (with $\tau = \sigma$). Therefore $u \leqslant v$ on $\partial\big(D\setminus B(\xi)\big)$. Then by the maximum principle (Lemma 1.4) also $u \leqslant v$ on $\overline{D}\setminus\overline{B(\xi)}$. It follows from Lemma 5.2 that

$$1 \leqslant \limsup_{x\to z} \frac{u(x)}{\operatorname{dist}(x, \partial D)} \leqslant \partial_{-\nu} v(z)\,. \tag{5.7}$$

On the other hand we have an estimate

$$\partial_{-\nu} v(z) \leqslant (\lambda + C\sigma) - c\varkappa\sigma \tag{5.8}$$

with constants C, c independent of $\varkappa$ and σ. Before we show this, let us finish the proof of the theorem. Choosing $\varkappa$ large enough we see that (5.8) contradicts (5.7), consequently (5.6) is true.

Using this and (5.3) we find that for any $x \in B_{\frac{1}{5}}(\xi)$

$$\begin{aligned} u(x) &\geqslant u(x_\xi) - \frac{3}{10}(1+\sigma) \geqslant v(x_\xi) - \frac{3}{10}(1+\sigma) \\ &\geqslant -x_{\xi_n} - \frac{3}{10} - C\sigma \geqslant \frac{3}{10} - C\sigma > 0 \end{aligned}$$

if σ is small enough. Thus $u > 0$ in $B_{\frac{1}{5}}(\xi)$ and, consequently, $\mathscr{L}u = 0$ in $B_{\frac{1}{5}}(\xi)$.

Set

$$w(x) = (1+\sigma)(\sigma - x_n) - u(x) .$$

Then $w \geqslant 0$ in $B_{\frac{1}{8}}(\xi)$ (by (5.3)) and

$$w(x_\xi) \leqslant - x_{\xi_n} - v(x_\xi) + C\sigma \leqslant C\sigma .$$

Since

$$0 = -\mathscr{L}u = f_{p_i p_j}(\nabla u)\,\partial_{ij} w \qquad \text{in } B_{\frac{1}{8}}(\xi) ,$$

we get by Harnack's inequality,

$$w(\xi) \leqslant C w(x_\xi) \leqslant C\sigma ,$$

or

$$u(\xi) \geqslant - \xi_n - C\sigma \qquad (\xi \in \partial B_{\frac{3}{4}},\ \xi_n < -\tfrac{1}{2}) .$$

We now integrate along vertical lines and, using (5.3), obtain

$$u(\xi + \alpha e_n) \geqslant u(\xi) - (1+\sigma)\alpha \geqslant -(\xi_n + \alpha) - C\sigma$$

which implies the asserted flatness from below, i.e., $u \in F(2\sigma, C\sigma; \sigma)$ in $B_{\frac{1}{2}}$.

To complete the proof we have to establish (5.8), where the first term on the right comes from elliptic estimates and the values of v on ∂D, and the second term from the maximum principle due to the values of v on $\partial B(\xi)$. More precisely, we will estimate v from above by a $\mathscr{L}$-supersolution $v_1 - \varkappa\sigma v_2$, where

$$\tilde{\mathscr{L}} w = a_{ij}\partial_{ij} w , \qquad a_{ij}(x) = f_{p_i p_j}(\nabla v(x)) ,$$

and with v_1, v_2 defined as follows. First

$$v_1 = \frac{\gamma_1}{\mu_1}\big(1 - \exp(-\mu_1 d_1)\big) \qquad \text{in } D,$$

where

$$d_1(x) = - x_n + \sigma - s\eta(x') ,$$

and with positive constants μ_1, γ_1 depending on σ. Then

$$1 \leqslant |\nabla d_1| \leqslant 1 + C\sigma , \qquad |\nabla^2 d_1| \leqslant C\sigma ,$$

hence

$$\tilde{\mathscr{L}} v_1 = \gamma_1 \exp(-\mu_1 d_1) a_{ij}(\partial_{ij} d_1 - \mu_1 \partial_i d_1 \partial_j d_1)$$
$$\leqslant \gamma_1 \exp(-\mu_1 d_1)(C\sigma - c\mu_1) < 0$$

if

$$\mu_1 = C_1 \sigma , \qquad C_1 \text{ large enough} .$$

For $x \in D$

$$v_1(x) \geqslant \gamma_1 d_1(x)(1 - C\mu_1) \geqslant (1 + 2\sigma) d_1(x) \tag{5.9}$$

if $\gamma_1 = 1 + C_2\sigma$, C_2 large enough, and (σ small)

$$(1 + 2\sigma) d_1(x) \geqslant v(x) \qquad \text{for } x \in \partial(D \setminus B(\xi)) .$$

Since $\tilde{\mathscr{L}} v = 0$ the maximum principle yields

$$v_1 \geqslant v \qquad \text{in } D \setminus B(\xi) . \tag{5.10}$$

At the point z we compute

$$|\nabla v_1(z)| = \gamma_1 |\nabla d_1| \leqslant 1 + C\sigma .$$

We define v_2 depending on ξ by

$$v_2 = \frac{\gamma_2}{\mu_2}(\exp(\mu_2 d_2) - 1) \qquad \text{in } \tilde{D} \setminus B(\xi)$$

with constants γ_2, μ_2. Here $\tilde{D} \subset D$ is a domain with smooth boundary containing

$$D \setminus B_{\frac{1}{10}}(\{(x', 0);\, x' \in R^{n-1},\, |x'| = 1\}) ,$$

and d_2 is a function in $C^2(D \setminus B(\xi))$ satisfying

$$d_2 = 0 \qquad \text{on } \partial\tilde{D} ,$$
$$d_2 = 1 \qquad \text{on } \partial B(\xi) ,$$
$$|\nabla d_2| \geqslant c > 0 \qquad \text{in } \tilde{D} \setminus \overline{B(\xi)} .$$

Then

$$\tilde{\mathscr{L}} v_2 = \gamma_2 \exp(\mu_2 d_2) a_{ij}(\partial_{ij} d_2 + \mu_2 \partial_i d_2 \partial_j d_2)$$
$$\geqslant \gamma_2 \exp(\mu_2 d_2)(-C + c\mu_2) > 0$$

if μ_2 is large enough. Then choose γ_2 such that

$$v_2 = 1 \quad \text{on } \partial B(\xi) .$$

At the point z

$$|\nabla v_2(z)| = \gamma_2 |\nabla d_2(z)| \geqslant c > 0 .$$

Thus the function

$$w \equiv v_1 - \varkappa\sigma v_2$$

is a $\tilde{\mathscr{L}}$-supersolution in $\tilde{D}\setminus B(\xi)$ with (see (5.10))

$$w = v_1 \geqslant v \quad \text{on } \partial\tilde{D} ,$$

and for $x \in \partial B(\xi)$ (by (5.9))

$$w(x) \geqslant d_1(x) - \varkappa\sigma \geqslant -(1 - \varkappa\sigma)x_n = v(x)$$

Therefore $w \geqslant v$ on $\partial(\tilde{D}\setminus B(\xi))$ and the maximum principle yields $w \geqslant v$ in $\tilde{D}\setminus B(\xi)$; in particular,

$$\begin{aligned}\partial_{-\nu} v(z) \leqslant \partial_{-\nu} w(z) &= |\nabla v_1(z)| - \varkappa\sigma |\nabla v_2(z)| \\ &\leqslant (1 + C\sigma) - \varkappa\sigma ,\end{aligned}$$

which is the estimate (5.8) we had to prove.

We shall denote points in R^n by (y, h) with $y \in R^{n-1}$, and balls in R^{n-1} by B'_ϱ, or $B'_\varrho(y)$.

LEMMA 5.3 (*Non-homogeneous blow up*). *Let* $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ *in* B_{ϱ_k} *with* $\sigma_k \to 0$, $\tau_k \sigma_k^{-2} \to 0$. *Set, for* $y \in B'_1$,

$$f_k^+(y) = \sup\{h;\ (\varrho_k y, \sigma_k \varrho_k h) \in \partial\{u_k > 0\}\} ,$$

$$f_k^-(y) = \inf\{h;\ (\varrho_k y, \sigma_k \varrho_k h) \in \partial\{u_k > 0\}\} .$$

Then, for a subsequence,

$$f(y) \equiv \limsup_{\substack{z\to y\\ k\to\infty}} f_k^+(z) = \liminf_{\substack{z\to y\\ k\to\infty}} f_k^-(z) \qquad \textit{for all } y \in B'_1 .$$

Further, $f_k^+ \to f$, $f_k^- \to f$ *uniformly,* $f(0) = 0$ *and* f *is continuous.*

The proof which is based on Theorem 5.1 is identical with the proof of [1; Lemma 7.3].

LEMMA 5.4. *f is subharmonic.*

PROOF. For simplicity we take $\varrho_k = 1$, $\lambda^* = 1$. If the assertion is not true then there is a ball $B'_\sigma(y_0) \subset\subset B'_1$ and a harmonic function g in a neighborhood of this ball such that

$$g > f \text{ on } \partial B'_\varrho(y_0) \text{ and } f(y_0) > g(y_0) .$$

We proceed as in [1], setting

$$Z = B'_\varrho(y_0) \times R , \qquad Z^\pm(\varphi) = \{(y, h) \in Z ; h \gtrless \varphi(y)\} ,$$
$$Z_0(\varphi) = \{(y, h) \in Z ; h = \varphi(y)\} ,$$

and letting $d_\delta(Z^+(\sigma_k g))$ be a test function which converges (as $\delta \to 0$) to the characteristic function of $Z^+(\sigma_k g)$ (say, $d_\delta(A)(x) = \min\{(1/\delta)\,\mathrm{dist}\cdot(x, R^n/A), 1\}$).

We have, by Theorems 3.2 and 3.6,

$$-\int_\Omega f_p(\nabla u_k)\cdot\nabla d_\delta(Z^+(\sigma_k g)) = \int_{\partial_{\mathrm{red}}\{u_k>0\}} Q d_\delta(Z^+(\sigma_k g))\, dH^{n-1}.$$

Taking $\sigma \to 0$ (and assuming that $Z^0(\sigma_k g) \cap \partial\{u_k > 0\}$ has H^{n-1} measure zero; otherwise we replace g by $g + c$ for some suitable small c), we get

$$\int_{\partial(Z^+(\sigma_k g))\cap\{u_k>0\}} f_p(\nabla u_k)\cdot\nu\, dH^{n-1} = \int_{Z^+(\sigma_k g)\cap\partial_{\mathrm{red}}\{u_k>0\}} Q\, dH^{n-1}.$$

Using the fact that $f_p(\nabla u_k) = 2F'(|\nabla u_k|^2)\nabla u_k$ and (3.3), (5.3), we see that the integrand on the left-hand side is bounded by $Q(1 + C\tau_k)$. Hence

$$\text{(5.11)}\quad H^{n-1}(Z^+(\sigma_k g) \cap \partial_{\mathrm{red}}\{u_k > 0\}) \leqslant (1 + C\tau_k) H^{n-1}(Z^0(\sigma_k g) \cap \{u_k > 0\}).$$

The set

$$E_k = \{u_k > 0\} \cup Z^-(g_k g)$$

has finite perimeter in the cylinder Z, with

$$\text{(5.12)}\quad H^{n-1}(Z \cap \partial_{\mathrm{red}} E_k) \leqslant H^{n-1}(Z^+(\sigma_k g) \cap \partial_{\mathrm{red}}\{u_k > 0\}) + H^{n-1}(Z^0(\sigma_k g) \cap \{u_k = 0\}) .$$

By the excess area estimate of [1; p. 136]

$$H^{n-1}(Z \cap \partial_{\mathrm{red}} E_k) \geqslant H^{n-1}(Z^0(\sigma_k g)) + c\sigma_k^2$$

for k large enough. Substituting this into (5.12) and using (5.11), we get a contradiction to the relation $\tau_k = o(\sigma_k^2)$ which was assumed in Lemma 5.3.

The following lemma will be needed later on in proving further regularity of the function $f(y)$.

LEMMA 5.5. *Let w be a function satisfying:*

$\Delta w = 0$ *in* $B_1^- \equiv B_1 \cap \{h < 0\}$,

$w(y, 0) = g(y)$ *in the sense that* $w(y, h)$ *as a function of* y *converges to* g *in* L^1 *as* $h \uparrow 0$,

g *is subharmonic and continuous in* B_1', $g(0) = 0$,

$w(0, h) \leqslant C|h|$,

$w \geqslant - C$.

Then

$$\int_0^{1/2} \frac{1}{r^2} \int_{\partial B_r'} g(y)\, dH^{n-2} \leqslant C_0 \tag{5.13}$$

where C_0 is a constant dpending only on C.

Notice that the integrand is nonnegative, since g is subharmonic and $g(0) = 0$.

PROOF. Denote by $G_{(y',h')}$ the Green function for Δ in B_1^- with singularity at (y', h'). Then

$$C|h'| \geqslant w(0, h') = - \int_{\partial B_1^- \cap \{h=0\}} g(y) \frac{\partial}{\partial \nu} G_{(y',h')}(y, 0)\, dy - \int_{\partial B_1^- \cap \{h<0\}} w \frac{\partial}{\partial \nu} G_{(y',h')} dH^{n-1} .$$

Here it is assumed that w has L^1 boundary values on $\partial B_1^- \cap \{h < 0\}$ (otherwise replace B_1 by B_{1-c} with a suitable small positive c). The function

$$\zeta(y', h') = - \int_{\partial B_1^- \cap \{h<0\}} \frac{\partial}{\partial \nu} G_{(y',h')}\, dH^{n-1}$$

is harmonic in B_1^-, with boundary values 1 on $\partial B_1^- \cap \{h' < 0\}$, and 0 on

$\partial B_1^- \cap \{h'=0\}$. Hence

$$\zeta(0, h') \leqslant C|h'| .$$

It follows that

$$-\int_{B_1'} g(y) \frac{\partial}{\partial \nu} G_{(0,h')}(y, 0)\, dy \leqslant C|h'| . \tag{5.14}$$

Since the Poisson kernel in (5.14) depends only on $|y|$, say,

$$-\frac{\partial}{\partial \nu} G_{(0,h')}(y, 0) = \varphi_{h'}(|y|)$$

this estimate can be written as

$$\int_0^1 \frac{r^{n-2}\varphi_{h'}(r)}{|h'|}\Big(\fint_{\partial B_r'} g\, dH^{n-2}\Big) dr \leqslant C,$$

where the average of g over $\partial B_r'$ is nonnegative, since g is subharmonic with $g(0) = 0$. One easily computes that

$$\begin{aligned}\frac{\varphi_{h'}(r)}{|h'|} &= c\big((r^2 + h'^2)^{-n/2} - (h'^2 r^2 + 1)^{-n/2}\big)\\ &\nearrow c(r^{-n} - 1)\end{aligned}$$

as $|h'| \searrow 0$, where c is a constant depending only on n. Thus we obtain

$$\int_0^1 \frac{1 - r^n}{r^2}\Big(\fint_{\partial B_r'} g\, dH^{n-2}\Big) dr \leqslant C ,$$

which implies (5.13).

LEMMA 5.6. *There exists a positive constant C such that, for any $y \in B'_{r/2}$,*

$$\int_0 \frac{1}{r^2}\Big(\int_{\partial B_r'(y)} f - f(y)\Big) \leqslant C . \tag{5.15}$$

Here f is the function appearing in Lemma 5.4. Lemma 5.6 is analogous to Lemma 7.6 in [1]. Once this lemma is proved we shall be able to proceed as in [1] to establish regularity of the free boundary.

PROOF. For simplicity we take $\varrho_k = 1$, $\lambda^* = 1$. For large k the u_k are of class $F(8\sigma_k, 8\sigma_k; \tau_k)$ in $B_{\frac{1}{4}}(y, \sigma_k f_k^+(y))$. Therefore it suffices to prove the lemma for $y = 0$.

Set

$$w_k(y, h) = \frac{u_k(y, h) + h}{\sigma_k} \tag{5.16}$$

Since the free boundary of u_k lies in the strip $|x_n| \leqslant c\sigma_k$ and since $|\nabla u_k| \leqslant 1 + C\tau_k$, $\tau_k \leqslant \sigma_k$, we have and $w_k \leqslant C$ in B_1^-. The flatness assumption also implies that $w_k \geqslant -C$ in B_1^-. Thus

$$|w_k| \leqslant C \quad \text{in } B_1^- . \tag{5.17}$$

LEMMA 5.7. *For a subsequence,*

$$\lim_{k\to\infty} w_k \equiv w \quad \textit{exists everywhere in } B_1^- ,$$

the convergence is uniform in compact subsets of B_1^-, *and* w *satisfies*:

$$a_{ij}\partial_{ij} w = 0 \quad \textit{in } B_1^- \tag{5.18}$$

where

$$a_{ij} = f_{p_i p_j}(-\lambda^* e_n) = F'(\lambda^{*2})\delta_{ij} + 2F''(\lambda^{*2})\lambda^{*2}\delta_{in}\delta_{jn} ,$$

$$w(0, h) \leqslant 0 , \tag{5.19}$$

$$w(y, 0) = f(y) \tag{5.20}$$

in the sense that $\lim_{h\uparrow 0} w(y, h) = f(y)$,

$$|w| \leqslant C . \tag{5.21}$$

Observe that the proof of Lemma 5.5 extends (by affine transformation) to the linear elliptic operator in (5.18). Hence, once Lemma 5.7 is proved, Lemma 5.5 can be applied to the function f, so that (5.15) follows.

PROOF OF LEMMA 5.7. The u_k satisfy

$$\nabla\cdot f_p(\nabla u_k) = 0 \quad \text{in } B_1 \cap \{h < -\sigma_k\} .$$

Writing this equation in non-divergence form, it is clear that

$$\mathscr{L}_k w_k \equiv f_{p_i p_j}(\nabla u_k)\partial_{ij} w_k = 0 \tag{5.22}$$

if $h < -\sigma_k$. From the flatness assumptions on the u_k and Lemma 1.7 we conclude that, for a subsequence,

$$u_k(y, h) \to -h$$

in C^2 in compact subsets of B_1^-. Recalling (5.17) we conclude that also $w_k \to w$ in C^2 in compact subsets of B_1^-, and w satisfies (5.18). Clearly (5.21) is also valid.

Since.

$$-\frac{\partial}{\partial h} w_k = -\frac{1}{\sigma_k}\left(\frac{\partial}{\partial h} u_k + 1\right) \leqslant \frac{|\nabla u_k| - 1}{\sigma_k} \leqslant \frac{\tau_k}{\sigma_k} \tag{5.23}$$

and $w_k(0, 0) = 0$, we have for $h \leqslant 0$,

$$w_k(0, h) \leqslant |h| \frac{\tau_k}{\sigma_k} \to 0 .$$

Thus $w(0, h) \leqslant 0$, and (5.19) is proved. It remains to establish (5.20).

We first show that, for any small $\delta > 0$ and any large constant K

$$w_k(y, h\sigma_k) \to f(y) \qquad \text{uniformly for } y \in B'_{1-\delta}, \ -K \leqslant h \leqslant -1 . \tag{5.24}$$

By Lemma 5.3 it suffices to prove

$$w_k(y, h\sigma_k) - f_k^+(y) \to 0 . \tag{5.25}$$

From (5.23) we obtain

$$\begin{aligned} w_k(y, h\sigma_k) - f_k^+(y) &\leqslant w_k\big(y, \sigma_k f_k^+(y)\big) - f_k^+(y) + \big(f_k^+(y) - h\big)\tau_k/\sigma_k \\ &= \big(f_k^+(y) - h\big)\tau_k/\sigma_k \leqslant (1 + K)\tau_k/\sigma_k \to 0 . \end{aligned} \tag{5.26}$$

In order to show (5.25) from below we take any sequence $y_k \in B'_{1-\delta}$, $-K \leqslant h_k \leqslant -1$, and consider u_k in $B_{R\sigma_k}(x_k)$, where x_k is the free boundary point

$$x_k = \big(y_k, \sigma_k f_k^+(y_k)\big)$$

and R any large constant. We know that

$$u_k \in F(\tilde{\delta}_k, 1; \tau_k) \quad \text{ in } B_{R\sigma_k}(x_k) ,$$

if

$$\tilde{\delta}_k = \frac{1}{R} \sup_{y \in B'_{R\sigma_k}(y_k)} \big(f_k^+(y) - f_k^+(y_k)\big) ,$$

where $\tilde{\delta}_k \to 0$ by Lemma 5.3. It follows from the flatness theorem (Theorem 5.1) that

$$u_k \in F(2\delta_k, C\delta_k; \tau_k) \quad \text{in } B_{(R/2)\sigma_k}(x_k),$$

if

$$\delta_k = \max(\tilde{\delta}_k, \tau_k).$$

Hence for any h with $|h| < R/2$

$$u_k(x_k + h\sigma_k e_n) \geq -\left(h\sigma_k + C\delta \cdot \frac{R}{2}\sigma_k\right),$$

that is,

$$w_k(x_k + h\sigma_k e_n) - f_k^+(y_k) = \frac{u_k(x_k + h\sigma_k e_n) + h\sigma_k}{\sigma_k} \geq -C\frac{R}{2}\delta_k \to 0,$$

which together with (5.26) proves (5.25).

For any $\varepsilon > 0$ choose a C^3 function g_ε such that

$$f - 2\varepsilon \leq g_\varepsilon \leq f - \varepsilon \quad \text{on } B_1', \tag{5.27}$$

and let u_ε be the solution of (see (5.18))

$$\begin{aligned} \mathscr{L}_0 u_\varepsilon &\equiv a_{ij}\partial_{ij}u_\varepsilon = 1 && \text{in } B_1^-, \\ u_\varepsilon &= g_\varepsilon && \text{on } \partial B_{1-\delta}^- \cap \{h = 0\}, \\ u_\varepsilon &= \inf_{B_1^-} w && \text{on } \partial B_{1-\delta}^- \cap \{h < 0\} \end{aligned} \tag{5.28}$$

with δ as in (5.24).

By (5.24) and (5.27),

$$w_k > u_\varepsilon \quad \text{on } \partial\big(B_{1-\delta}^- \cap \{h < -K\sigma_k\}\big) \tag{5.29}$$

for any large constant K (independent of δ, ε), provided k is sufficiently large (depending on ε, K).

The function w_k is a bounded function (uniformly with respect to k) satisfying a uniformly elliptic equation (uniformly also with respect to k) in $B_1^- \cap \{h \leq -\sigma_k\}$. By elliptic estimates (see the proof of Lemma 1.7) we deduce that

$$|\nabla w_k| \leq \frac{C}{K\sigma_k} \qquad B_{1-\delta}^- \cap \{h \leq -K\sigma_k\}$$

where C is a constant independent of k, K. Consequently

$$\begin{aligned}\mathscr{L}_k u_\varepsilon &= \big(f_{p_ip_j}(\nabla u_k) - f_{p_ip_j}(-e_n)\big)\partial_{ij}u_\varepsilon + 1\\ &\geqslant 1 - \frac{C}{K}\|u_\varepsilon\|_{C^{1,1}(B^-_{1-\delta})} > 0 \qquad \text{in } B^-_{1-\delta}\cap\{h < -K\sigma_k\}\end{aligned}$$

if K is large enough (depending on δ and ε). Thus $\mathscr{L}_k u_\varepsilon \geqslant \mathscr{L}_k w_k$. Recalling (5.29) and applying the maximum principle, we get $u_\varepsilon \leqslant w_k$ in $B^-_{1-\delta}\cap\{h\leqslant -K\sigma_k\}$ if k is large enough. It follows that $w(y,h)\geqslant u_\varepsilon(y,h)$ in $B^-_{1-\delta}$ and, consequently,

$$\underline{\lim}_{h\downarrow 0}\, w(y,h) \geqslant g_\varepsilon(y) \geqslant f(y) - 2\varepsilon\,.$$

Similarly by working with a solution of

$$\begin{aligned}\mathscr{L}_0\tilde u_\varepsilon &= -1 && \text{in } B^-_1\,,\\ \tilde u_\varepsilon &= \tilde g_\varepsilon && \text{on } \partial B^-_1\cap\{h=0\}\ \ (f+\varepsilon\leqslant \tilde g_\varepsilon \leqslant f+2\varepsilon)\,,\\ \tilde u_\varepsilon &= \sup_{B^-_1} w && \text{on } \partial B^-_1 \cap\{h<0\}\,,\end{aligned}$$

we obtain $\overline{\lim}_{h\uparrow 0}\, w(y,h)\leqslant f(y)+2\varepsilon$. Since ε is arbitrary. (5.20) follows.

6. – Smoothness of the free boundary.

Using Theorem 5.1 and Lemma 5.6 we can now proceed precisely as in [1] and establish that f is «better than» Lipschitz continuous (Lemma 7.8 of [1]) and that the flatness condition implies better flatness in a smaller ball (Lemmas 7.9, 7.10). Using also Theorem 4.1 we obtain as in [1; Theorem 8.1]:

THEOREM 6.1. *Suppose u is a local minimum and $D \subset\subset \Omega$. Then there exist positive constants $\alpha, \beta, \sigma_0, \tau_0, C$ such that if*

$$u\in F(\sigma, 1;\infty) \quad \text{in } B_\varrho(x_0)\subset D \text{ in direction } \nu \tag{6.1}$$

with $\sigma\leqslant\sigma_0$, $\varrho\leqslant\tau_0\sigma^{2/\beta}$, then

$$B_{\varrho/4}(x_0)\cap\partial\{u>0\} \quad \text{is a } C^{1,\alpha} \text{ surface}\,,$$

more precisely, a graph in direction ν of a $C^{1,\alpha}$ function, and, for any x_1, x_2

on this surface,

$$|\nu(x_1) - \nu(x_2)| \leqslant C\sigma \left|\frac{x_1 - x_2}{\varrho}\right|^{\alpha}.$$

We refer to (6.1) as the *flatness condition.*

From the first part of the proof of Theorem 3.5 we see that if $x_0 \in \partial_{\text{red}}\{u > 0\}$ then

$$u \in F(\sigma_\varrho, 1; \infty) \quad \text{in } B_\varrho(x_0) \text{ in direction } \nu_u(x_0)$$

with $\sigma_\varrho \to 0$ for $\varrho \to 0$. Hence, applying Theorem 6.1 we obtain:

THEOREM 6.2. *Let u be a local minimum. Then $\partial_{\text{red}}\{u > 0\}$ is a $C^{1,\alpha}$ surface locally in Ω, and the remainder of $\partial\{u > 0\}$ has H^{n-1} measure zero.*

For $n = 2$, any blow up sequence u_k with respect to balls $B_{\varrho_k}(x_0)$ $(x_0 \in \partial\{u > 0\})$ has a subsequence which is convergent to a linear function $\lambda^* \max\{-x\cdot\nu, 0\}$, at least in a neighborhood of the origin, by Corollary 4.4. Using (3.11), (3.12) we see that

$$u \in F(\sigma_k, 1; \infty) \quad \text{in } B_{\varrho_k}(x_0) \text{ in direction } \nu\,,$$

with $\sigma_k \to 0$ as $k \to \infty$. Hence Theorem 6.1 can be applied. We get:

THEOREM 6.3. *Let $n = 2$ and let u be a local minimum. Then $\partial\{u > 0\}$ is a $C^{1,\alpha}$ curve locally in Ω.*

Higher regularity follow from [8-10]; in particular, if $F(t)$ is analytic (C^∞) then

$$\text{(6.2)} \qquad \begin{aligned} &\text{for } n \geqslant 3\,, \quad \partial_{\text{red}}\{u > 0\} \text{ is locally analytic } (C^\infty);\\ &\text{for } n = 2\,, \quad \partial\{u > 0\} \quad \text{is locally analytic } (C^\infty)\,. \end{aligned}$$

REMARK 6.4. All the results of this paper extend with minor changes to the case where λ is a function $\lambda(x)$, bounded, uniformly positive and Hölder continuous. In the definition of flatness we require that

$$\operatorname*{osc}_{B_\varrho} \lambda^*(x) \leqslant \lambda^*(0)\,\tau$$

where $\lambda^*(x)$ is defined by (3.1) with $\lambda = \lambda(x)$. If λ and F are in $C^{k,\alpha}$ then $\partial_{\text{red}}\{u > 0\}$ (for $n \geqslant 3$) and $\partial\{u > 0\}$ (for $n = 2$) belong to $C^{k+1,\alpha}$; if λ and F are analytic (C^∞) then (6.2) holds.

Finally, all the results extend to the functional (0.2) provided $F(x, t)$ is smooth enough.

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