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Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method

Periodic and Stationary Solutions for Compressible Navier-Stokes Equations Via a Stability Method.

ALBERTO VALLI (*)

1. – Introduction.

This paper deals with some problems concerning the motion of a viscous compressible barotropic fluid.

The equations which describe the motion are (see for instance Serrin [16])

$$(1.1) \quad \begin{cases} \varrho[\dot{v} + (v \cdot \nabla)v - b] = - \nabla p + \mu \, \Delta v + (\zeta + \tfrac{1}{3}\mu) \, \nabla \operatorname{div} v & \text{in } Q_T, \\ \dot{\varrho} + v \cdot \nabla \varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_T, \\ v|_{\partial \Omega} = 0 & \text{on } \Sigma_T, \\ v|_{t=0} = v_0 & \text{in } \Omega, \\ \varrho|_{t=0} = \varrho_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$, $Q_T \equiv]0, T[\times \Omega$, $\Sigma_T \equiv]0, T[\times \partial \Omega$, $0 < T \leqslant \infty$, $\varrho = \varrho(t, x)$ is the density of the fluid, $v = v(t, x)$ the velocity, $b = b(t, x)$ the (assigned) external force field and $p = p(\varrho)$ the pressure, which is assumed to be a known function of the density ϱ. The constants μ and ζ are the viscosity coefficients, which satisfy the thermodynamic restrictions

$$(1.2) \quad \mu > 0, \quad \zeta \geqslant 0;$$

finally, $v_0 = v_0(x)$ and $\varrho_0 = \varrho_0(x) > 0$ are the initial velocity and initial density respectively.

In the last thirty years, several papers have appeared concerning these equations, first about the problem of uniqueness (see Graffi [6], Serrin [19]),

and later about the problem of (local in time) existence (see Nash [13], Itaya [9], Vol'pert-Hudjaev [26] for the Cauchy problem in $\mathbf{R}^3$; Solonnikov [20], Tani [22], Valli [25] for the Cauchy-Dirichlet problem in a general domain Ω). Only recently the first global existence results have been proved by Matsumura-Nishida, both in the whole space $\mathbf{R}^3$ and in a bounded domain Ω (see [10], [11], [12]), under the assumption that the data of the problem are small enough.

In this paper we are concerned with some global properties of the solutions. In particular we obtain some new a-priori estimates and a stability result which enables us to prove the existence of periodic solutions and of stationary solutions under the assumption that the external force field is small enough. Results of this type are essentially new. To our knowledge, only two other results about stationary solutions have been obtained: Matsumura-Nishida [11], [12] proved that there exists an equilibrium solution $\varrho = \tilde{\varrho}(x)$, $v = 0$ when the external force field is the *gradient of a time-independent function*; Padula [14] found a stationary solution when the ratio ζ/μ is *large enough* (and, as usual, in both these papers the external force field is supposed to be small). It is worthy of noting that in our theorems we don't need any assumption of this type.

The paper is subdivided in seven parts. After this introduction, in § 2 we prove the local existence of a solution of problem (1.1) (see Theorem 2.4). The proof is obtained by a fixed point argument, and the estimates which give the result are simpler than those usually employed (see [20], [22], [25]). Moreover, only *one* (necessary) compatibility condition must be verified, that is $v_0|_{\partial\Omega} = 0$. In § 3 we remark that the solution is unique in a suitable class of functions (see (3.6)), by modifying a little the proof given in [24]. In § 4 we get the global a-priori estimates for the solution. Here the crucial points are essentially two: *first*, to obtain estimates which balance the linear terms $p_1 \nabla\sigma$ and $\bar{\varrho} \operatorname{div} v$ (see (4.1) and (4.2)) with each other (this idea appeared for the first time in Matsumura-Nishida [11]); *second*, to obtain estimates for the derivative in t, and not of integral type, in such a way that we can prove that for $b = 0$ and small initial data the solution decreases in t in suitable norms. This fact gives that the solution is global in time under the assumptions that b belongs to $L^\infty(\mathbf{R}^+; H^1(\Omega))$ and that b and the initial data are small enough (see Theorem 4.12). We don't need any hypothesis about the norm of b in $L^2(\mathbf{R}^+; H^1(\Omega))$. It is clear that this is an essential point for proving the existence of periodic solutions. Finally, we remark that the proof of the first step is simpler than that contained in [11], [12], and it requires less assumptions. In § 5 we prove an asymptotic stability result, which is an essential tool for the following arguments (see Theorem 5.2 and Theorem 5.3). The proof is given by an energy

method, and some results about Stokes's problem are utilized. In § 6 we prove the existence of a periodic solution subjected to a periodic force field (see Theorem 6.1). It is well known from the papers of Serrin [17], [18] that a stability result plus suitable global estimates permit the construction of the periodic solution, which is unique and stable in a neighborhood of zero. In § 7 we show that if b is independent of t, then there exists a stationary solution, which is obtained by taking the limit of periodic solutions as the period goes to zero (see Theorem 7.1). The solution constructed in this way is consequently stable and unique in a neighborhood of zero.

We want to remark now that some natural generalizations of our problem present a few difficulties. For instance, in the non-barotropic case (i.e. the pressure $p = p(\varrho, \theta)$ is a function of the density ϱ and of the absolute temperature θ) the nonlinear terms in the equation of conservation of energy are quadratic in Dv. By modifying a little the proof, we can still obtain the same results of § 2 and § 3. Moreover, we can get an a-priori estimate like (4.42), but we are not able to control the nonlinear terms in Dv to obtain (4.49). Observe also that for a parabolic equation with Dirichlet boundary condition it is possible to estimate in a « right » way the time derivative of the norm of the solution in $L^2(\Omega)$ and in $H^1(\Omega)$, while this appears difficult (may be false) for the norm in $H^k(\Omega)$, $k \geqslant 2$. Hence, if we study our problem in higher norms, in such a way that Sobolev spaces are Banach algebras, we can control the nonlinear terms in Dv but we are not able to obtain an estimate like (4.42).

Another generalization may be concerned with the viscosity coefficients, which in general can be variable functions of ϱ and θ. Also in this case some problems appear since the second order elliptic operator in $(1.1)_1$ has now coefficients which are not regular enough to apply the standard elliptic regularization methods. Moreover, uniqueness is obtained in classes of functions which do not seem to contain the (eventual) solution.

Finally, we observe that our stability result may give the possibility of a deeper investigation of the numerical approximations of the solution (see for instance the recent results of Heywood-Rannacher [7], [8], [15], concerning incompressible Navier-Stokes equations).

In this paper we assume that

$$(1.3) \qquad m \equiv \min_{\overline{\Omega}} \varrho_0(x) > 0 \, ,$$

and we set

$$(1.4) \qquad M \equiv \max_{\overline{\Omega}} \varrho_0(x) \, , \quad \bar{\varrho} \equiv \frac{1}{\operatorname{vol} \Omega} \int_{\Omega} \varrho_0(x) \, dx \, .$$

Obviously one has

(1.5) $m \leqslant \bar{\varrho} \leqslant M$.

We will denote the norm in $H^k(\Omega)$ (the usual Sobolev space) by $\|\cdot\|_k$ for $k + 1 \in \mathbf{N}$; the norm in $L^q(0, T; H^k(\Omega))$ by

$$[\,\cdot\,]_{q;\,k;\,T},$$

for $1 \leqslant q \leqslant \infty$, $k + 1 \in \mathbf{N}$, $0 < T \leqslant \infty$; the norm in $L^q(0, T; L^s(\Omega))$ by

$$\|\|\cdot\|\|_{q;\,s;\,T},$$

for $1 \leqslant q \leqslant \infty$, $1 \leqslant s \leqslant \infty$, $0 < T \leqslant \infty$. The norm in $L^\infty(0, T; X)$ and in $C^0([0, T]; X)$ are denoted in the same way. Moreover, it is useful to remark for $C^0([0, \infty]; X)$ we mean $C_0^B(\mathbf{R}^+; X)$, the space of continuous and bounded functions from $\mathbf{R}^+$ to X.

Finally, we recall that from the classical results of Agmon-Douglis-Nirenberg [1] one has that for $k \in \mathbf{N}$ the norms $\|\mu \varDelta v + (\zeta + \frac{1}{3}\mu) \nabla \operatorname{div} v\|_k$ and $\|v\|_{k+2}$ are equivalent, since $\mu \varDelta + (\zeta + \frac{1}{3}\mu) \nabla \operatorname{div}$ is a strongly elliptic system.

For reasons which will be clear in the sequel, we rewrite problem (1.1) in a new form, by the change of variables

(1.6) $\sigma \equiv \varrho - \bar{\varrho}$

and obtain

$$
(1.7) \quad
\begin{cases}
(\sigma + \bar{\varrho})[\dot{v} + (v \cdot \nabla)v - b] = -\nabla[p(\sigma + \bar{\varrho})] + \\
\qquad\qquad\qquad\qquad + \mu \varDelta v + (\zeta + \frac{1}{3}\mu) \nabla \operatorname{div} v & \text{in } Q_T, \\
\dot{\sigma} + v \cdot \nabla \sigma + \sigma \operatorname{div} v + \bar{\varrho} \operatorname{div} v = 0 & \text{in } Q_T, \\
v|_{\partial\Omega} = 0 & \text{on } \Sigma_T, \\
v|_{t=0} = v_0 & \text{in } \Omega, \\
\sigma|_{t=0} = \varrho_0 - \bar{\varrho} & \text{in } \Omega.
\end{cases}
$$

Problem (1.1) and (1.7) are obviously equivalent.

2. – Local existence.

The results of this paragraph are strictly related to the paper of Beirão da Veiga [2], where the author shows the existence of a solution for the equations which describe the motion of a non-homogeneous viscous incompressible fluid in the presence of diffusion. We begin by considering the following linear problem

(2.1)
$$\begin{cases} \tilde\varrho \dot v + Av = F & \text{in } Q_T, \\ v \quad\quad = 0 & \text{on } \Sigma_T, \\ v(0) \quad = v_0 & \text{in } \Omega, \end{cases}$$

where

$$A \equiv -\mu\varDelta - \beta\,\nabla\,\mathrm{div}\,,$$

$$\beta \equiv \zeta + \tfrac13\mu\,,$$

and $\tilde\varrho$ and F are known functions, $0 < T \leqslant \infty$.

The first lemma concerns the existence of a unique solution of (2.1).

LEMMA 2.1. *Let $\partial\Omega \in C^2$, $\tilde\varrho \in L^\infty(Q_T)$, $0 < m/2 \leqslant \tilde\varrho(t,x) \leqslant 2M$ a.e. in Q_T, $F \in L^2(0, T; L^2(\Omega))$ and $v_0 \in H_0^1(\Omega)$. Then there exists a unique solution v of (2.1) such that $v \in L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H_0^1(\Omega))$, $\dot v \in L^2(0, T; L^2(\Omega))$ and*

(2.2)
$$\mu[Dv]_{\infty;0;T}^2 + \frac{m}{32M^2}[Av]_{2;0;T}^2 + \frac{m}{2}[\dot v]_{2;0;T}^2$$
$$\leqslant 2(\mu + 3\beta)\|Dv_0\|_0^2 + 2\left(\frac{4}{m} + \frac{m}{16M^2}\right)[F]_{2;0;T}^2\,.$$

PROOF. We start by proving the a-priori bound (2.2). Multiply (2.1)$_1$ by $\dot v + \varepsilon_0 Av$ and integrate in Ω. One has

$$\int\tilde\varrho|\dot v|^2 + \varepsilon_0\int\tilde\varrho\dot v\cdot Av + \int\dot v\cdot Av + \varepsilon_0\int|Av|^2 = \int F\cdot\dot v + \varepsilon_0\int F\cdot Av$$

(here and in the sequel we will omit the conventional volume infinitesimal; moreover the integral is understood to be extended over Ω).

Then integrating by parts the third term, we obtain

$$\frac{m}{2}\int|\dot v|^2 + \frac{\mu}{2}\frac{d}{dt}\int|Dv|^2 + \frac{\beta}{2}\frac{d}{dt}\int|\mathrm{div}\,v|^2 + \varepsilon_0\int|Av|^2$$
$$\leqslant \|F\|_0\,\|\dot v\|_0 + \varepsilon_0\|F\|_0\,\|Av\|_0 + 2\varepsilon_0 M\|\dot v\|_0\,\|Av\|_0\,.$$

Moreover

$$\|F\|_0 \|\dot{v}\|_0 \leqslant \frac{m}{8} \|\dot{v}\|_0^2 + \frac{2}{m} \|F\|_0^2 \,,$$

$$\varepsilon_0 \|F_0\| \|Av\|_0 \leqslant \frac{\varepsilon_0}{4} \|Av\|_0^2 + \varepsilon_0 \|F\|_0^2 \,,$$

$$2\varepsilon_0 M \|\dot{v}\|_0 \|Av\|_0 \leqslant 4\varepsilon_0 M^2 \|\dot{v}\|_0^2 + \frac{\varepsilon_0}{4} \|Av\|_0^2 \,,$$

and choosing $\varepsilon_0 = m/32M^2$ one has

$$\frac{m}{2} \|\dot{v}\|_0^2 + \mu \frac{d}{dt} \|Dv\|_0^2 + \beta \frac{d}{dt} \|\text{div } v\|_0^2 + \frac{m}{32M^2} \|Av\|_0^2 \leqslant \left(\frac{4}{m} + \frac{m}{16M^2}\right) \|F\|_0^2 \,.$$

By integrating on $[0, T]$ and by using the estimate

$$\|\text{div } v_0\|_0^2 \leqslant 3 \|Dv_0\|_0^2 \,,$$

one gets (2.2).

Now we can prove the existence of a solution of (2.1) by a continuity method. We will follow exactly the proof given in [2], and we present it here only for completeness.

First of all, if $\tilde{\varrho}$ is a positive constant, say $\tilde{\varrho} = \bar{\varrho}$, the existence of the solution of (2.1) is well known, since A is a strongly elliptic operator, and consequently generates an analytic semigroup in $L^2(\Omega)$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Define

$$\mathcal{K} \equiv \{v | v \in L^2(0, T; D(A)), \dot{v} \in L^2(0, T; L^2(\Omega))\} \,,$$

$$\mathcal{Y} \equiv L^2(0, T; L^2(\Omega)) \times H_0^1(\Omega) \,,$$

and

$$\|v\|_{\mathcal{K}}^2 \quad \equiv \text{left hand side of equation (2.2)},$$

$$\|(F, v_0)\|_{\mathcal{Y}}^2 \equiv \text{right hand side of equation (2.2)}.$$

Set

$$\varrho_\alpha \equiv (1 - \alpha)\bar{\varrho} + \alpha\tilde{\varrho} \,, \quad \alpha \in [0, 1] \,, \quad \text{and} \quad T_\alpha \equiv (1 - \alpha)\overline{T} + \alpha\tilde{T} \,,$$

where

$$\overline{T}v \equiv (\bar{\varrho}\dot{v} + Av, v(0)) \,,$$

$$\tilde{T}v \equiv (\tilde{\varrho}\dot{v} + Av, v(0)) \,,$$

i.e. $T_\alpha v = (\varrho_\alpha \dot{v} + Av, v(0))$.

Clearly, ϱ_α satisfies the assumptions of Lemma 2.1 for each $\alpha \in [0, 1]$. Finally, denote by

$$\gamma \equiv \left\{ \alpha \in [0, 1] \,|\, T_\alpha v = (F, v_0) \text{ is uniquely solvable in } \mathcal{K} \text{ for each } (F, v_0) \in \mathcal{Y} \right\}.$$

We have seen that $0 \in \gamma$. Let us verify that γ is open and closed.

γ *is open.* Let $\alpha_0 \in \gamma$. From (2.2) we know that $T_{\alpha_0}^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{K})$ and

$$\| T_{\alpha_0}^{-1} \|_{\mathcal{Y}, \mathcal{K}} \leqslant 1.$$

Equation $T_{\alpha_0 + \varepsilon} v = (F, v_0)$ can be written in the form

$$(2.3) \qquad T_{\alpha_0}^{-1}(F, v_0) = T_{\alpha_0}^{-1} T_{\alpha_0 + \varepsilon} v = [I - \varepsilon T_{\alpha_0}^{-1}(\overline{T} - \tilde{T})] v.$$

Since

$$\| T_{\alpha_0}^{-1}(\overline{T} - \tilde{T}) \|_{\mathcal{K}, \mathcal{K}} \leqslant \| \overline{T} - \tilde{T} \|_{\mathcal{K}, \mathcal{Y}},$$

equation (2.3) is solvable for $|\varepsilon| < \| \overline{T} - \tilde{T} \|_{\mathcal{K}, \mathcal{Y}}^{-1}$.

γ *is closed.* Let $\alpha_n \to \alpha_0$, $\alpha_n \in \gamma$. From (2.2) we have

$$\| T_{\alpha n}^{-1}(F, v_0) \|_{\mathcal{K}} \leqslant \| (F, v_0) \|_{\mathcal{Y}}.$$

Set $v_n \equiv T_{\alpha n}^{-1}(F, v_0)$. $\mathcal{K}$ is a Hilbert space, hence there exists a subsequence v_{n_k} such that $v_{n_k} \rightharpoonup v$ weakly in $\mathcal{K}$, $v \in \mathcal{K}$. Moreover one verifies easily that $\overline{T} v_{n_k} \rightharpoonup \overline{T} v$ and $\tilde{T} v_{n_k} \rightharpoonup \tilde{T} v$ weakly in $\mathcal{Y}$; hence $T_{\alpha n_k} v_{n_k} \rightharpoonup T_{\alpha_0} v$ weakly in $\mathcal{Y}$, and $(F, v_0) = T_{\alpha n_k} v_{n_k} = T_{\alpha_0} v$. This proves that $\alpha_0 \in \gamma$. $\square$

The second lemma gives some stronger estimates. From now on in this paragraph each constant c, c_i, C_i will depend at most on $\Omega, \mu, \zeta, \bar{\varrho}, m$ and M, and not on T. Other possible dependences will be explicitly pointed out.

LEMMA 2.2. *Let* $\partial \Omega \in C^3$, $\tilde{\varrho} \in L^\infty(Q_T)$, $0 < m/2 \leqslant \tilde{\varrho}(t, x) \leqslant 2M$ *a.e. in* Q_T, $0 < m \leqslant \tilde{\varrho}(0, x) \leqslant M$ *a.e. in* Ω, $\nabla \tilde{\varrho} \in L^4(0, T; L^6(\Omega))$, $\dot{\tilde{\varrho}} \in L^2(0, T; L^3(\Omega))$, $F \in L^2(0, T; H^1(\Omega))$, $\dot{F} \in L^2(0, T; H^{-1}(\Omega))$ *and* $v_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. *Then the solution* v *of* (2.1) *is such that* $v \in L^2(0, T; H^3(\Omega)) \cap C^0([0, T]; H^2(\Omega))$, $\dot{v} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ *and*

$$(2.4) \qquad [v]_{\infty; 2; T}^2 + [v]_{2; 3; T}^2 + [\dot{v}]_{\infty; 0; T}^2 + [\dot{v}]_{2; 1; T}^2$$

$$\leqslant c_1 \Big\{ [F]_{2; 1; T}^2 + [F]_{\infty; 0; T}^2 + \big([\dot{F}]_{2; -1; T}^2 + \| v_0 \|_2^2 + \| F(0) \|_0^2 \big)$$

$$\cdot \big(1 + \|| \nabla \tilde{\varrho} \||_{4; 6; T}^4 + \|| \dot{\tilde{\varrho}} \||_{2; 3; T}^2 \big) \exp \big(c_1 \|| \dot{\tilde{\varrho}} \||_{2; 3; T}^2 \big) \Big\}.$$

PROOF. Take the derivative in t of $(2.1)_1$. One obtains

$$\tilde{\varrho}\ddot{v} + \dot{\tilde{\varrho}}\dot{v} + A\dot{v} = \dot{F} \qquad \text{in } Q_T,$$

and moreover $\dot{v}|_{\partial\Omega} = 0$ for each $t \in [0, T]$.

Hence $\dot{v} = V$ satisfies the equation

$$(2.5) \qquad \begin{cases} \tilde{\varrho}\dot{V} + AV + \dot{\tilde{\varrho}}V = \dot{F} & \text{in } Q_T, \\[2mm] V = 0 & \text{on } \Sigma_T, \\[2mm] V(0) = V_0 \equiv \dfrac{1}{\tilde{\varrho}(0)}\left[F(0) - Av_0\right] & \text{in } \Omega. \end{cases}$$

Multiply $(2.5)_1$ by V and integrate in Ω: one gets

$$\int \tilde{\varrho}\dot{V}\cdot V + \int AV\cdot V + \int \dot{\tilde{\varrho}}|V|^2 = \langle \dot{F}, V\rangle_{H^{-1}; H^1_0},$$

hence

$$(2.6) \qquad \frac{1}{2}\frac{d}{dt}\int \tilde{\varrho}|V|^2 + \mu\int|DV|^2 + \beta\int|\text{div } V|^2$$
$$\leqslant \frac{1}{\mu}\|\dot{F}\|^2_{-1} + \frac{\mu}{4}\|DV\|^2_0 + \frac{1}{2}\int|\dot{\tilde{\varrho}}|\,|V|^2.$$

From the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ one has

$$(2.7) \qquad \int|\dot{\tilde{\varrho}}|\,|V|^2 \leqslant c\|\dot{\tilde{\varrho}}\|_{L^3(\Omega)}\|V\|_0\|DV\|_0 \leqslant \frac{\mu}{2}\|DV\|^2_0 + c\|\dot{\tilde{\varrho}}\|^2_{L^3(\Omega)}\int\tilde{\varrho}|V|^2.$$

From (2.6), (2.7) and Gronwall's lemma we have

$$(2.8) \qquad [V]^2_{\infty;0;T} \leqslant c\left([\dot{F}]^2_{2;-1;T} + \|v_0\|^2_2 + \|F(0)\|^2_0\right)\exp\left(c\|\dot{\tilde{\varrho}}\|^2_{2;3;T}\right),$$

and integrating (2.6) in $[0, T]$

$$(2.9) \quad [V]^2_{2;1;T} \leqslant c\left([\dot{F}]^2_{2;-1;T} + \|v_0\|^2_2 + \|F(0)\|^2_0\right)\left(1 + \|\dot{\tilde{\varrho}}\|^2_{2;3;T}\right)\exp\left(c\|\dot{\tilde{\varrho}}\|^2_{2;3;T}\right).$$

Now recall that

$$V = \dot{v} = \frac{1}{\tilde{\varrho}}(F - Av);$$

hence from (2.8) and (2.9)

$$(2.10) \qquad [Av]^2_{\infty;0;T} \leqslant c\left([F]^2_{\infty;0;T} + \text{right hand side of (2.8)}\right),$$

$$(2.11) \qquad [Av]^2_{2;1;T} \leqslant c\left([F]^2_{2;1;T} + [\nabla\tilde{\varrho}\cdot V]^2_{2;0;T} + \text{right hand side of (2.9)}\right).$$

On the other hand,

$$\|\nabla\tilde{\varrho}\cdot V\|_0^2 \leqslant c\|\nabla\tilde{\varrho}\|_{L^6(\Omega)}^2\|V\|_0\|DV\|_0 \leqslant c\big(\|DV\|_0^2 + \|\nabla\tilde{\varrho}\|_{L^6(\Omega)}^4\|V\|_0^2\big),$$

and from (2.8), (2.9)

(2.12) $[\nabla\tilde{\varrho}\cdot V]_{2;\,0;\,T}^2$

$\leqslant c\{\text{right hand side of (2.9)} + \|\!|\nabla\tilde{\varrho}\|\!|_{4;\,6;\,T}^4 \text{ (right hand side of (2.8))}\}.$ □

Finally (2.8), (2.9), (2.10), (2.11) and (2.12) give estimate (2.4).   □

Consider now the linear problem

(2.13) $\begin{cases} \dot{\sigma} + \tilde{v}\cdot\nabla\sigma + \sigma\,\mathrm{div}\,\tilde{v} + \bar{\varrho}\,\mathrm{div}\,\tilde{v} = 0 & \text{in } Q_T, \\ \sigma(0) = \sigma_0 & \text{in } \Omega, \end{cases}$

where $\tilde{v}$ and σ_0 are known functions.

We obtain the following lemma:

LEMMA 2.3. *Let* $\partial\Omega \in C^1$, $\tilde{v} \in L^1(0,\,T;\,H^3(\Omega))$, $\tilde{v}\cdot n = 0$ *on* Σ_T, *and* $\sigma_0 \in H^2(\Omega)$ *with* $\int\sigma_0 = 0$. *Then there exists a unique solution* σ *of* (2.13) *such that* $\sigma \in C^0([0,\,T];\,H^2(\Omega))$, $\int\sigma = 0$ *for each* $t \in [0,\,T]$ *and*

(2.14) $[\sigma]_{\infty;\,2;\,T} \leqslant c_2\big(\|\sigma_0\|_2 + 1\big)\exp\big(c_2[\tilde{v}]_{1;\,3;\,T}\big).$

If in addition $\tilde{v} \in C^0([0,\,T];\,H^2(\Omega))$, *then* $\dot{\sigma} \in C^0([0,\,T];\,H^1(\Omega))$ *and*

(2.15) $[\dot{\sigma}]_{\infty;\,1;\,T} \leqslant c_3[\tilde{v}]_{\infty;\,2;\,T}\big(\|\sigma_0\|_2 + 1\big)\exp\big(c_2[\tilde{v}]_{1;\,3;\,T}\big).$

PROOF. The existence of the solution follows from the method of characteristics.

Moreover, from (2.13)$_1$ one has

$$\frac{d}{dt}\int\sigma = -\int\mathrm{div}\,(\sigma\tilde{v} + \varrho\tilde{v}) = 0,$$

and from (2.13)$_2$

$$\int\sigma(0) = \int\sigma_0 = 0.$$

Hence $\int\sigma = 0$, and we have only to prove the a-priori estimate (2.14).

At first, take the gradient of $(2.13)_1$, multiply by $\nabla\sigma$ and integrate in Ω. By integrating by parts the term $\int[(\tilde{v}\cdot\nabla)\,\nabla\sigma]\cdot\nabla\sigma$, one has

$$(2.16) \qquad \frac{1}{2}\frac{d}{dt}\int|\nabla\sigma|^2 \leqslant 2\int|D\tilde{v}|\,|\nabla\sigma|^2 + \int|\sigma|\,|\nabla\sigma|\,|\nabla\operatorname{div}\tilde{v}| + \bar{\varrho}\int|\nabla\operatorname{div}\tilde{v}|\,|\nabla\sigma|\,.$$

Moreover

$$(2.17) \qquad \int|\sigma|\,|\nabla\sigma|\,|\nabla\operatorname{div}\tilde{v}| \leqslant \|\nabla\operatorname{div}\tilde{v}\|_{L^3(\Omega)}\|\sigma\|_{L^6(\Omega)}\|\nabla\sigma\|_0 \leqslant c\|\nabla\operatorname{div}\tilde{v}\|_{L^3(\Omega)}\|\nabla\sigma\|_0^2$$

since $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\int\sigma = 0$.
In the same way, for the second derivatives one has

$$(2.18) \qquad \frac{1}{2}\frac{d}{dt}\int|D^2\sigma|^2 \leqslant c\left(\|D\tilde{v}\|_{L^\infty(\Omega)}\|D^2\sigma\|_0^2 + \int|\sigma|\,|D^2\operatorname{div}\tilde{v}|\,|D^2\sigma|\right.$$
$$\left. + \int|D^2\tilde{v}|\,|\nabla\sigma|\,|D^2\sigma|\right) + \bar{\varrho}\int|D^2\operatorname{div}\tilde{v}|\,|D^2\sigma|\,,$$

and

$$(2.19) \qquad \int|\sigma|\,|D^2\operatorname{div}\tilde{v}|\,|D^2\sigma| \leqslant \|\sigma\|_{L^\infty(\Omega)}\|D^2\operatorname{div}\tilde{v}\|_0\|D^2\sigma\|_0$$
$$\leqslant c\|D^2\operatorname{div}\tilde{v}\|_0\|\nabla\sigma\|_1\|D^2\sigma\|_0\,,$$

$$(2.20) \qquad \int|D^2\tilde{v}|\,|\nabla\sigma|\,|D^2\sigma| \leqslant \|D^2\tilde{v}\|_{L^3(\Omega)}\|\nabla\sigma\|_{L^6(\Omega)}\|D^2\sigma\|_0$$
$$\leqslant c\|D^2\tilde{v}\|_{L^3(\Omega)}\|\nabla\sigma\|_1\|D^2\sigma\|_0\,.$$

Hence by adding (2.16) and (2.18), taking into account (2.17), (2.19) and (2.20), one gets

$$(2.21) \qquad \frac{1}{2}\frac{d}{dt}\|\nabla\sigma\|_1^2 \leqslant c\|\tilde{v}\|_3\|\nabla\sigma\|_1^2 + \bar{\varrho}\|\operatorname{div}\tilde{v}\|_2\|\nabla\sigma\|_1$$

and consequently (2.14), which follows from Gronwall's lemma.

We have now to show that σ belongs to $C^0([0, T]; H^2(\Omega))$ and not only to $L^\infty(0, T; H^2(\Omega))$. But this is easily proved by observing that the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}U(t, s, x) = \tilde{v}(t, U(t, s, x)) & \text{in } Q_T\,, \\[2mm] U(s, s, x) = x & \text{in } \Omega\,, \end{cases}$$

belongs to $C^0([0, T]\times[0, T]; H^3(\Omega))$. Consequently the representation formula

$$\sigma(t, x) = \sigma_0\big(U(0, t, x)\big) - \int\limits_0^t (\sigma\operatorname{div}\tilde{v} + \bar{\varrho}\operatorname{div}\tilde{v})\big(s, U(s, t, x)\big)\,ds$$

gives the result. (See also the proof of Lemma A.3 and A.4 in Bourguignon-Brezis [3] for similar calculations).

Finally, the proof of (2.15) is trivial. ☐

We are now in a position to prove the local existence of a solution to problem (1.1).

Take $0 < T < \infty$ and define

$$
\begin{aligned}
(2.22) \quad R_T \equiv \Big\{ & (\tilde{v}, \tilde{\sigma}) \,|\, \tilde{v} \in C^0([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\
& \dot{\tilde{v}} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
& \tilde{\sigma} \in L^\infty(0, T; H^2(\Omega)), \; \dot{\tilde{\sigma}} \in L^\infty(0, T; H^1(\Omega)), \\
& [\tilde{v}]^2_{\infty;2;T} + [\tilde{v}]^2_{2;3;T} + [\dot{\tilde{v}}]^2_{\infty;0;T} + [\dot{\tilde{v}}]^2_{2;1;T} \leqslant B_1, \\
& \tilde{v}(0) = v_0 \text{ in } \Omega, \; \tilde{v} = 0 \text{ on } \Sigma_T, \\
& [\tilde{\sigma}]_{\infty;2;T} \leqslant B_1, \; [\dot{\tilde{\sigma}}]_{\infty;1;T} \leqslant B_2, \\
& \tilde{\sigma}(0) = \varrho_0 - \bar{\varrho}, \; 0 < \frac{m}{2} \leqslant \tilde{\sigma}(t, x) + \bar{\varrho} \leqslant 2M \text{ a.e. in } Q_T \Big\},
\end{aligned}
$$

where B_1 and B_2 will be choosen in the sequel (see (2.23), (2.24), (2.25)).

If B_1 is large enough, it is clear that $R_T \neq \emptyset$ for each $0 < T < \infty$: in fact, let v^* be the solution of

$$
\begin{cases}
\dot{v}^* - \Delta v^* = 0 & \text{in } Q_\infty, \\
v^* \qquad\; = 0 & \text{on } \Sigma_\infty, \\
v^*(0) \quad\; = v_0 & \text{in } \Omega,
\end{cases}
$$

which satisfies

$$
[v^*]^2_{\infty;2;\infty} + [v^*]^2_{2;3;\infty} + [\dot{v}^*]^2_{\infty;0;\infty} + [\dot{v}^*]^2_{2;1;\infty} \leqslant c_4 \|v_0\|^2_2 .
$$

If we take

$$
(2.23) \qquad\qquad B_1 > \max \big\{ c_4 \|v_0\|^2_2, \; \|\varrho_0 - \bar{\varrho}\|_2 \big\} ,
$$

then $(v^*, \varrho_0 - \bar{\varrho}) \in R_T$ for each $0 < T < \infty$. From now on B_1 will satisfy (2.23). Consider now the map Φ defined in R_T in this way:

$$
\Phi \colon \; (\tilde{v}, \tilde{\sigma}) \to (v, \sigma) ,
$$

where v and σ are the solution of (2.1) and (2.13), respectively, with

$$F \equiv \tilde{F} + \tilde{\varrho} b \,,$$

$$\tilde{F} \equiv -\tilde{\varrho}(\tilde{v}\cdot\nabla)\tilde{v} - p_\varrho(\tilde{\varrho})\,\nabla\tilde{\varrho} \,,$$

$$\tilde{\varrho} \equiv \tilde{\sigma} + \bar{\varrho} \,,$$

$$\sigma_0 \equiv \varrho_0 - \bar{\varrho} \,,$$

and $p_\varrho \equiv dp/d\varrho$.

We want to prove that Φ has a fixed point in R_T for T small enough. This fixed point will be clearly a solution of problem (1.7).

To see this, we use estimates (2.4), (2.14) and (2.15). We have easily that

$$[F]^2_{2;\,1;\,T} \leqslant C_1(B_1)\{(1 + \|p_\varrho\|^2_{C^1})\,T + [b]^2_{2;\,1;\,T}\} \,,$$

$$[F]^2_{\infty;\,0;\,T} \leqslant 4(\|\tilde{F}(0)\|^2_0 + T[\overset{\cdot}{\tilde{F}}]^2_{2;\,0;\,T}) + 8M^2[b]^2_{\infty;\,0;\,T} \,,$$

$$\|\tilde{F}(0)\|^2_0 \leqslant c_5(\|v_0\|^4_2 + \|p_\varrho\|^2_{C^0}\,\|\nabla\varrho_0\|^2_0) \,,$$

$$[\overset{\cdot}{\tilde{F}}]^2_{2;\,0;\,T} \leqslant C_2(B_1, B_2)\{(1 + \|p_\varrho\|^2_{C^1})\,T + 1\} \,,$$

$$[\overset{\cdot}{F}]^2_{2;\,-1;\,T} \leqslant C_3(B_1, B_2)\{(1 + \|p_\varrho\|^2_{C^1})\,T + [b]^2_{2;\,0;\,T} + [\overset{\cdot}{b}]^2_{2;\,-1;\,T}\} \,,$$

$$\|\!|\nabla\tilde{\varrho}\|\!|^4_{4;\,6;\,T} + \|\!|\overset{\cdot}{\tilde{\varrho}}\|\!|^2_{2;\,3;\,T} \leqslant C_4(B_1, B_2)\,T \,.$$

Furthermore, from (2.14) and (2.15) we have

$$[\sigma]_{\infty;\,2;\,T} \leqslant c_2(\|\sigma_0\|_2 + 1)\exp[C_5(B_1)\,T^{\frac{1}{2}}] \,,$$

$$[\overset{\cdot}{\sigma}]_{\infty;\,1;\,T} \leqslant c_3 B_1^{\frac{1}{2}}(\|\sigma_0\|_2 + 1)\exp[C_5(B_1)\,T^{\frac{1}{2}}] \,.$$

Hence, if we take

$$(2.24) \qquad B_1 > \max\left\{6c_1c_5(\|v_0\|^4_2 + \|p_\varrho\|^2_{C_0}\,\|\nabla\varrho_0\|^2_0) + 10c_1M^2\|b(0)\|^2_0 \right.$$
$$\left. + c_1\|v_0\|^2_2,\; c_2(\|\sigma_0\|_2 + 1)\right\} \,,$$

$$(2.25) \qquad B_2 > c_3 B_1^{\frac{1}{2}}(\|\sigma_0\|_2 + 1) \,,$$

and T small enough, we get

$$(2.26) \qquad [v]^2_{\infty;\,2;\,T} + [v]^2_{2;\,3;\,T} + [\overset{\cdot}{v}]^2_{\infty;\,0;\,T} + [\overset{\cdot}{v}]^2_{2;\,1;\,T} \leqslant B_1 \,,$$

$$(2.27) \qquad [\sigma]_{\infty;\,2;\,T} \leqslant B_1 \,,$$

$$(2.28) \qquad [\overset{\cdot}{\sigma}]_{\infty;\,1;\,T} \leqslant B_2 \,.$$

Observe also that the assumptions on $\tilde{\sigma}$ imply that $\ddot{v} \in L^2(0, T; H^{-1}(\Omega))$, hence by interpolation $\dot{v} \in C^0([0, T]; L^2(\Omega))$.

Finally, one has

$$[\sigma - \sigma_0]_{\infty; 1; T} \leqslant T[\dot{\sigma}]_{\infty; 1; T} \leqslant TB_2 \,,$$

and by well known interpolation results

$$\|\sigma - \sigma_0\|_{C^0(\bar{Q}_T)} \leqslant c[\sigma - \sigma_0]_{\infty; 1; T}^{\frac{1}{3}}[\sigma - \sigma_0]_{\infty; 2; T}^{\frac{2}{3}} \leqslant cT^{\frac{1}{3}}B_2^{\frac{1}{3}}(B_1 + \|\sigma_0\|_2)^{\frac{2}{3}} \,.$$

Hence if T is small enough

(2.29) $$\frac{m}{2} \leqslant \sigma(t, x) + \bar{\varrho} \leqslant 2M \qquad \text{in } \bar{Q}_T \,.$$

We have proved in this way that $\Phi(R_T) \subset R_T$ for T small enough, say $T = T^* > 0$.

Now we utilize Schauder's fixed point theorem. Clearly R_{T*} is convex and it is easily seen that it is closed in $X \equiv C^0([0, T^*]; H^1(\Omega)) \times C^0([0, T^*]; H^1(\Omega))$. Moreover, from Ascoli's theorem R_{T*} is relatively compact in X. Hence we need only to prove that Φ is continuous in X. Suppose that $(\tilde{v}_n, \tilde{\sigma}_n) \in R_{T*}$, $(\tilde{v}_n, \tilde{\sigma}_n) \to (\tilde{v}, \tilde{\sigma})$ in X and set $(v_n, \sigma_n) \equiv \Phi(\tilde{v}_n, \tilde{\sigma}_n)$, $(v, \sigma) \equiv \Phi(\tilde{v}, \tilde{\sigma})$. Take the difference between the equations for (v_n, σ_n) and (v, σ), multiply by $v_n - v$ and $\sigma_n - \sigma$, respectively, and integrate in Ω. By an energy argument and by Gronwall's lemma it is easily seen that (v_n, σ_n) converge to (v, σ) in $C^0([0, T^*]; L^2(\Omega))$. From the compactness of $\overline{R_{T*}}$, (v_n, σ_n) converge indeed to (v, σ) in X. Hence Φ is continuous, and it has a fixed point, which is the solution of problem (1.7) in Q_{T*}.

We have proved the following theorem.

THEOREM 2.4. *Let $\partial\Omega \in C^3$, $b \in L^2_{\text{loc}}(\mathbf{R}^+; H^1(\Omega))$, $\dot{b} \in L^2_{\text{loc}}(\mathbf{R}^+; H^{-1}(\Omega))$, $p \in C^3$, $v_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\varrho_0 \in H^2(\Omega)$, $0 < m \leqslant \varrho_0(x) \leqslant M$ on $\bar{\Omega}$. Then there exist $T^* > 0$ small enough, $v \in L^2(0, T^*; H^3(\Omega)) \cap C^0([0, T^*]; H^2(\Omega))$ with $\dot{v} \in L^2(0, T^*; H^1(\Omega)) \cap C^0([0, T]^*; L^2(\Omega))$, $\varrho \in C^0([0, T^*]; H^2(\Omega))$ with $\dot{\varrho} \in C^0 \cdot ([0, T^*]; H^1(\Omega))$, $\varrho(t, x) > 0$ in $\bar{Q}_{T*}$ such that (v, ϱ) is a solution of (1.1) in Q_{T*}.*

REMARK 2.5. It is useful to observe that if $b \in L^\infty(\mathbf{R}^+; H^1(\Omega))$ with $\dot{b} \in L^\infty(\mathbf{R}^+; H^{-1}(\Omega))$ and if

$$\|v_0\|_2^2 + \|\sigma_0\|_2^2 \leqslant D \,, \qquad 0 < E \leqslant \inf_{\bar{\Omega}} (\sigma_0 + \bar{\varrho}) \leqslant \sup_{\bar{\Omega}} (\sigma_0 + \bar{\varrho}) \leqslant F \,,$$

then the instant T^* in Theorem 2.4 depends only on $\Omega, \mu, \zeta, \bar{\varrho}, \|p_\varrho\|_{C^1}^2$, $[b]_{\infty; 1; \infty}$, $[\dot{b}]_{\infty; -1; \infty}$ and on D, E, F.

3. – Uniqueness.

A uniqueness theorem for problem (1.1) is proved in [24] (see also the papers of Graffi [6] and Serrin [19]). However our solution does not belong to the class of functions in which uniqueness is shown to hold, since

$$\nabla\tilde\varrho \notin L^2\big(0,\, T^*;\, L^\infty(\Omega)\big), \quad D^2 v \notin L^1\big(0,\, T^*;\, L^\infty(\Omega)\big) \text{ and } b \notin L^1\big(0,\, T^*;\, L^\infty(\Omega)\big).$$

But one can modify a little the proof of [24], and in this way it is possible to obtain a better result.

Suppose that (v, ϱ) and $(\tilde v, \tilde\varrho)$ are two solutions of (1.1) in Q_T, $0 < T \leqslant \infty$ and set, as in [24], $u \equiv \tilde v - v$, $\eta \equiv \tilde\varrho - \varrho$. u and η satisfy the following equations

$$(3.1) \qquad \tilde\varrho[\dot u + (\tilde v\cdot\nabla)u + (u\cdot\nabla)v] + p_\varrho(\tilde\varrho)\nabla\eta + [p_\varrho(\tilde\varrho) - p_\varrho(\varrho)]\nabla\varrho$$
$$+ \eta[\dot v + (v\cdot\nabla)v - b] - \mu\,\varDelta u - \beta\,\nabla\operatorname{div} u = 0 \qquad \text{in } Q_T,$$

$$(3.2) \qquad \dot\eta + \tilde v\cdot\nabla\eta + u\cdot\nabla\varrho + \tilde\varrho\operatorname{div} u + \eta\operatorname{div} v = 0 \qquad \text{in } Q_T.$$

Multiply (3.1) by u, (3.2) by $\tilde\varrho\eta$ and integrate in Ω. As in [24] it is easy to obtain

$$(3.3) \qquad \frac{1}{2}\frac{d}{dt}\int\tilde\varrho|u|^2 + \mu\int|Du|^2 + \beta\int|\operatorname{div} u|^2 \leqslant \int\tilde\varrho|u|^2\,|Dv|$$

$$+ \|p_\varrho\|_{C^1}\int|\eta|\,|u|\,|\nabla\varrho| + \int|\eta|\,|u|\,(|\dot v| + |v|\,|Dv| + |b|)$$

$$+ \|p_{\varrho\varrho}\|_{C^0}\int|\nabla\tilde\varrho|\,|\eta|\,|u| + \|p_\varrho\|_{C^0}\int|\eta|\,|\operatorname{div} u|,$$

$$(3.4) \qquad \frac{1}{2}\frac{d}{dt}\int\tilde\varrho\eta^2 \leqslant \int\tilde\varrho|\eta|\,|u|\,|\nabla\varrho| + \int\tilde\varrho^2\,|\eta|\,|\operatorname{div} u| + \int\tilde\varrho\eta^2\,|\operatorname{div} v|.$$

We want to apply Gronwall's lemma. The last term in (3.3), the second and the third term in (3.4) can be estimated by

$$(3.5) \qquad \varepsilon\int|\operatorname{div} u|^2 + c\big(\|p_\varrho\|_{C^0}^2 + \|\tilde\varrho(t)\|_{L^\infty(\Omega)}^3 + \|\operatorname{div} v(t)\|_{L^\infty(\Omega)}\big)\int\tilde\varrho\eta^2.$$

The other terms are of this type:

$$\int(|u| + |\eta|)\,|u|\,|g|,$$

for a suitable function g which can be easily calculated. Hence we have

$$\int\big((|u| + |\eta|)\big)|u| \, |g| \leqslant \big(\|u(t)\|_0 + \|\eta(t)\|_0\big)\|u(t)\|_{L^6(\Omega)}\|g(t)\|_{L^3(\Omega)}$$
$$\leqslant \varepsilon\int|Du|^2 + c\|g(t)\|_{L^3(\Omega)}^2\Big(\int\tilde{\varrho}|u|^2 + \int\tilde{\varrho}\eta^2\Big).$$

Consequently, looking at the expression which gives g and at (3.5), we conclude that we have obtained uniqueness in the class

$$(3.6) \quad \begin{cases} \inf_{\Omega_T}\varrho > 0\,, \quad \varrho \in L^\infty(Q_T)\,, \quad \nabla\varrho \in L^2(0,T;L^3(\Omega))\,, \\[2mm] v \in L^\infty(Q_T)\,, \quad Dv \in L^2(0,T;L^3(\Omega))\,, \quad \dot{v} \in L^2(0,T;L^3(\Omega))\,, \\[2mm] \operatorname{div} v \in L^1(0,T;L^\infty(\Omega))\,, \end{cases}$$

under the assumptions $p \in C^2$, $b \in L^2(0,T;L^3(\Omega))$.

The solution obtained in Theorem 2.4 belongs to this class for $T = T^*$; consequently the fixed point constructed in § 2 is unique.

4. – Global existence.

We want to obtain now an a-priori estimate for $[v]_{\infty;\,2;\,\infty}$ and $[\sigma]_{\infty;\,2;\,\infty}$. This will be done under the assumption that the initial data (v_0, σ_0) and the external force field b are small enough. The results of this section draw their inspiration from the paper of Matsumura-Nishida [11]. However we need to obtain better estimates than those contained there, since our aim is to prove successively the existence of periodic solutions. In fact, if the external force field $b \in L^2(R^+;H^1(\Omega))$ with $\dot{b} \in L^2(R^+;H^{-1}(\Omega))$, then it looks possible to adapt the methods of [11] to obtain an a-priori estimate for v and σ in $L^\infty(R^+;H^2(\Omega))$. But this estimate seems to fail if $b \in L^2_{\text{loc}}(R^+;H^1(\Omega))$ only $\big($and $\dot{b} \in L^2_{\text{loc}}(R^+;H^{-1}(\Omega))$ of course$\big)$, and this is exactly the case of a periodic b. We begin with some lemmas. Here and in the sequel of this paragraph we suppose that v and σ are a solution of the following problem

$$(4.1) \quad \begin{cases} \dot{v} + \bar{A}v + p_1\nabla\sigma \quad = f \quad \text{in } Q_T\,, \\[2mm] v \quad\quad\quad\quad\quad\quad = 0 \quad \text{on } \Sigma_T\,, \\[2mm] v(0) \quad\quad\quad\quad\quad = v_0 \quad \text{in } \Omega\,, \end{cases}$$

$$(4.2) \quad \begin{cases} \dot{\sigma} + v\cdot\nabla\sigma + \bar{\varrho}\operatorname{div} v = f^0 \quad \text{in } Q_T\,, \\[2mm] \sigma(0) \quad\quad\quad\quad\quad = \sigma_0 \quad \text{in } \Omega\,, \end{cases}$$

where

$$\overline{A} \equiv - \bar{\mu}\varDelta - \bar{\beta}\,\nabla\,\mathrm{div}\,,$$

$$\bar{\mu} \equiv \mu/\bar{\varrho}\,, \qquad \bar{\beta} \equiv \beta/\bar{\varrho}\,,$$

$$p_1 \equiv p_\varrho(\bar{\varrho})/\bar{\varrho} > 0\,,$$

and

$$(4.3) \qquad f \equiv -(v\cdot\nabla)\,v - \bar{\mu}\frac{\sigma}{\sigma + \bar{\varrho}}\varDelta v - \bar{\beta}\frac{\sigma}{\sigma + \bar{\varrho}}\nabla\,\mathrm{div}\,v$$
$$+ \left[p_1 - \frac{p_\varrho(\sigma + \bar{\varrho})}{\sigma + \bar{\varrho}}\right]\nabla\sigma + b\,,$$

$$(4.4) \qquad f^0 \equiv -\sigma\,\mathrm{div}\,v\,.$$

Hence v and $\varrho = \sigma + \bar{\varrho}$ are solutions of problem (1.1) in Q_T, $0 < T \leqslant \infty$. We suppose that v and ϱ belong to the classes of functions obtained in Theorem 2.4. Moreover we assume that $\partial\varOmega \in C^4$ and that

$$(4.5) \qquad \frac{\bar{\varrho}}{4} \leqslant \sigma(t, x) + \bar{\varrho} \leqslant 3\bar{\varrho} \qquad \text{in } Q_T\,.$$

Finally, from now on in this paragraph each constant c, c_i, $\bar{c}_i$ will depend at most on $\varOmega$, μ, ζ, $\bar{\varrho}$, p.

LEMMA 4.1. *One has, for each* $0 < \varepsilon < 1$

$$(4.6) \qquad \bar{\mu}\frac{d}{dt}\|Dv\|_0^2 + \bar{\beta}\frac{d}{dt}\|\mathrm{div}\,v\|_0^2 + \|\overline{A}v\|_0^2 \leqslant c(\|\sigma\|_1^2 + \|f\|_0^2)\,,$$

$$(4.7) \qquad \frac{d}{dt}\|\nabla\sigma\|_1^2 \leqslant \frac{c}{\varepsilon}\big(\|\mathrm{div}\,v\|_2^2 + \|\sigma\|_2^2 + \|\sigma\|_2^4\big) + \varepsilon\|v\|_3^2\,.$$

PROOF. (4.7) follows at once from (2.21).
For (4.6), multiply (4.1)$_1$ by $\overline{A}v$ and integrate in $\varOmega$: one has

$$\frac{1}{2}\bar{\mu}\frac{d}{dt}\|Dv\|_0^2 + \frac{1}{2}\bar{\beta}\frac{d}{dt}\|\mathrm{div}\,v\|_0^2 + \|\overline{A}v\|_0^2$$
$$\leqslant (p_1\|\nabla\sigma\|_0 + \|f\|_0)\|\overline{A}v\|_0 \leqslant \frac{1}{2}(p_1\|\nabla\sigma\|_0 + \|f\|_0)^2 + \frac{1}{2}\|\overline{A}v\|_0^2\,,$$

and consequently (4.6).   □

Moreover, from $(4.2)_1$ one obtains at once

$$(4.8) \qquad \|\dot{\sigma}\|_1^2 \leqslant c\left(\|v\|_2^2 + \|f^0\|_1^2 + \|v\|_2^2 \|\sigma\|_2^2\right).$$

We can obtain more interesting estimates if we add equation $(4.1)_1$ to equation $(4.2)_1$.

LEMMA 4.2. *One has, for each* $0 < \varepsilon < 1$

$$(4.9) \qquad \frac{1}{2}\frac{d}{dt}\|v\|_0^2 + \frac{1}{2}\frac{p_1}{\bar{\varrho}}\frac{d}{dt}\|\sigma\|_0^2 + \frac{1}{2}\bar{\mu}\|Dv\|_0^2 + \bar{\beta}\|\mathrm{div}\, v\|_0^2$$

$$\leqslant \frac{c}{\varepsilon}\left(\|f\|_{-1}^2 + \|f^0\|_0^2 + \|\sigma\|_0^4\right) + \varepsilon\|\sigma\|_0^2 + \varepsilon\|v\|_3^2,$$

$$(4.10) \qquad \frac{1}{2}\frac{d}{dt}\|\dot{v}\|_0^2 + \frac{1}{2}\frac{p_1}{\bar{\varrho}}\frac{d}{dt}\|\dot{\sigma}\|_0^2 + \frac{1}{4}\bar{\mu}\|D\dot{v}\|_0^2 + \bar{\beta}\|\mathrm{div}\,\dot{v}\|_0^2$$

$$\leqslant \frac{c}{\varepsilon}\left(\|\dot{f}\|_{-1}^2 + \|\dot{f}^0\|_0^2 + \|\sigma\|_2^4 + \|\dot{\sigma}\|_0^4\right) + \varepsilon\|\dot{\sigma}\|_0^2 + \varepsilon\|v\|_3^2.$$

PROOF. Multiply $(4.1)_1$ by v and $(4.2)_1$ by $(p_1/\bar{\varrho})\sigma$, integrate in Ω and add these two equations. Since

$$\int p_1 \nabla\sigma\cdot v = -\int p_1\sigma\,\mathrm{div}\, v,$$

one has

$$\frac{1}{2}\frac{d}{dt}\|v\|_0^2 + \frac{1}{2}\frac{p_1}{\bar{\varrho}}\frac{d}{dt}\|\sigma\|_0^2 + \bar{\mu}\|Dv\|_0^2 + \bar{\beta}\|\mathrm{div}\, v\|_0^2$$

$$\leqslant c\left(\|f\|_{-1}^2 + \frac{1}{\varepsilon}\|f^0\|_0^2 + \int|\mathrm{div}\, v|\sigma^2\right) + \varepsilon\|\sigma\|_0^2 + \frac{\bar{\mu}}{2}\|Dv\|_0^2.$$

Moreover

$$\int|\mathrm{div}\, v|\sigma^2 \leqslant c\|v\|_3\|\sigma\|_0^2 \leqslant \frac{c}{\varepsilon}\|\sigma\|_0^4 + \varepsilon\|v\|_3^2,$$

hence (4.9) is obtained.

By taking the derivative in t of $(4.1)_1$, $(4.1)_2$ and $(4.2)_1$ one gets the same estimates for $\dot{v}$ and $\dot{\sigma}$, the only difference being in the behaviour of the non-linear term

$$\frac{p_1}{\bar{\varrho}}\int\frac{\partial}{\partial t}(v\cdot\nabla\sigma)\,\dot{\sigma} = \frac{p_1}{\bar{\varrho}}\int(\dot{v}\cdot\nabla\sigma)\,\dot{\sigma} + \frac{p_1}{\bar{\varrho}}\int(v\cdot\nabla\dot{\sigma})\,\dot{\sigma}.$$

We have

$$\frac{p_1}{\varrho}\left|\int(\dot{v}\cdot\nabla\sigma)\,\dot{\sigma}\right|\leqslant c\|\dot{v}\|_1\|\sigma\|_2\|\dot{\sigma}\|_0\leqslant\frac{\bar{\mu}}{4}\|D\dot{v}\|_0^2+c\|\sigma\|_2^4+c\|\dot{\sigma}\|_0^4,$$

$$\frac{p_1}{\varrho}\left|\int(v\cdot\nabla\dot{\sigma})\,\dot{\sigma}\right|=\frac{p_1}{2\varrho}\left|\int\operatorname{div}v\dot{\sigma}^2\right|\leqslant c\|v\|_3\|\dot{\sigma}\|_0^2\leqslant\varepsilon\|v\|_3^2+\frac{c}{\varepsilon}\|\dot{\sigma}\|_0^4,$$

and consequently (4.10).　　　□

From the results of Cattabriga [4] on Stokes's problem (see also Temam [23], pag. 33; Giaquinta-Modica [5]), one obtains at once that:

LEMMA 4.3. *v and σ satisfy*

$$(4.11)\qquad\|\sigma\|_1^2+\|v\|_2^2\leqslant c\big(\|\operatorname{div}v\|_1^2+\|\dot{v}\|_0^2+\|f\|_0^2\big)\,,$$

$$(4.12)\qquad\|\sigma\|_2^2+\|v\|_3^2\leqslant c\big(\|\operatorname{div}v\|_2^2+\|D\dot{v}\|_0^2+\|f\|_1^2\big)\,.$$

Hence from (4.6)-(4.12) we obtain, by choosing ε small enough,

$$(4.13)\qquad\bar{\mu}\frac{d}{dt}\|v\|_1^2+\bar{\mu}\frac{p_1}{\varrho}\frac{d}{dt}\|\sigma\|_2^2+\bar{\beta}\frac{d}{dt}\|\operatorname{div}v\|_0^2+c_6\frac{d}{dt}\|\dot{v}\|_0^2$$

$$+c_6\frac{p_1}{\varrho}\frac{d}{dt}\|\dot{\sigma}\|_0^2+c\big(\|v\|_3^2+\|\sigma\|_2^2+\|\dot{v}\|_1^2+\|\dot{\sigma}_1^2\|\big)$$

$$\leqslant c\big(\|\operatorname{div}v\|_2^2+\|f\|_1^2+\|f^\circ\|_1^2+\|\dot{f}\|_{-1}^2+\|\dot{f}^\circ\|_0^2+\|\sigma\|_2^4+\|v\|_2^2\|\sigma\|_2^2+\|\dot{\sigma}\|_0^4\big)\,.$$

It is clear now that the crucial term to estimate is

$$\|\operatorname{div}v\|_2^2\,.$$

We will see that, as in Matsumura-Nishida [11], [12], the interior estimates and those concerning the tangential derivatives of div v follow by adding equation $(4.1)_1$ to equation $(4.2)_1$ as in Lemma 4.2 (we can integrate by parts and balance the term $p_1\nabla\sigma$ by the term $\bar{\varrho}\operatorname{div}v$); the estimates concerning the normal derivatives of div v follow by observing that $\Delta v\cdot n$ and $\nabla\operatorname{div}v\cdot n$ are essentially equal on $\partial\Omega$. (Here and in the sequel n is the unit outward normal vector to $\partial\Omega$). Let us begin with the interior estimates. Take the gradient of $(4.1)_1$ and $(4.2)_1$, and then multiply the first equation by χ_0^2Dv and the second equation by $(p_1/\bar{\varrho})\chi_0^2\nabla\sigma$, where $\chi_0\in C_0^\infty(\Omega)$. Finally, integrate in Ω and add the two equations. By repeating the same procedure also for the second derivatives, one obtains

LEMMA 4.4. *v and σ satisfy, for each* $0 < \delta < 1$

(4.14)
$$\frac{d}{dt} \|\chi_0 Dv\|_0^2 + \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \|\chi_0 \nabla\sigma\|_0^2 + \bar{\mu} \|\chi_0 D^2 v\|_0^2 + \bar{\beta} \|\chi_0 \nabla \operatorname{div} v\|_0^2$$

$$\leqslant \frac{c}{\delta} \left(\|v\|_1^2 + \|f\|_0^2 + \|f^0\|_1^2 + \|\sigma\|_1^4 \right) + \delta \|\sigma\|_1^2 + \delta \|v\|_3^2,$$

(4.15)
$$\frac{d}{dt} \|\chi_0 D^2 v\|_0^2 + \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \|\chi_0 D^2 \sigma\|_0^2 + \bar{\mu} \|\chi_0 D^3 v\|_0^2 + \bar{\beta} \|\chi_0 D^2 \operatorname{div} v\|_0^2$$

$$\leqslant \frac{c}{\delta} \left(\|v\|_2^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4 \right) + \delta \|\sigma\|_2^2 + \delta \|v\|_3^2.$$

PROOF. One has, by integrating by parts as usual

$$\frac{1}{2} \frac{d}{dt} \|\chi_0 Dv\|_0^2 + \frac{1}{2} \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \|\chi_0 \nabla\sigma\|_0^2 + \bar{\mu} \|\chi_0 D^2 v\|_0^2 + \bar{\beta} \|\chi_0 \nabla \operatorname{div} v\|_0^2$$

$$\leqslant c \left(\int |\chi_0| |\nabla\chi_0| |D^2 v| |Dv| + \int |\chi_0| |\nabla\chi_0| |\nabla \operatorname{div} v| |Dv| + \int |\chi_0| |\nabla\chi_0| |\nabla\sigma| |Dv| \right.$$

$$+ \int \chi_0^2 |f| |D^2 v| + \int |\chi_0| |\nabla\chi_0| |f| |Dv| + \int \chi_0^2 |f^0| |\nabla\sigma| + \int \chi_0^2 |Dv| |\nabla\sigma|^2$$

$$\left. + \int |\chi_0| |\nabla\chi_0| |v| |\nabla\sigma|^2 \right) \leqslant c \left(\frac{1}{\delta} \|v\|_1^2 + \|f\|_0^2 + \frac{1}{\delta} \|f^0\|_1^2 + \frac{1}{\delta} \|\sigma\|_1^4 \right)$$

$$+ \frac{\bar{\mu}}{2} \|\chi_0 D^2 v\|_0^2 + \frac{\bar{\beta}}{2} \|\chi_0 \nabla \operatorname{div} v\|_0^2 + \frac{\delta}{2} \|\nabla\sigma\|_0^2 + \frac{\delta}{2} \|v\|_3^2,$$

hence (4.14).

In the same way one obtains also (4.15), the only difference being the behaviour of the non linear term

$$\frac{p_1}{\bar{\varrho}} \left| \int \chi_0^2 D^2 (v \cdot \nabla\sigma) \cdot D^2 \sigma \right| \leqslant c \int \left[|D^2 v| |\nabla\sigma| |D^2 \sigma| + (|v| + |Dv|) |D^2 \sigma|^2 \right]$$

$$\leqslant c \|v\|_3 \|\sigma\|_2^2 \leqslant \frac{c}{\delta} \|\sigma\|_2^4 + \frac{\delta}{2} \|v\|_3^2. \qquad \square$$

Let us obtain now the estimates on the boundary. We proceed essentially as in [11], [12], but the proof that we present here looks simpler. We choose as local coordinates the isothermal coordinates $\lambda_s(\psi, \varphi)$ (see for instance Spivak [21], pag. 460). We can cover the boundary $\partial\Omega$ by a finite number of bounded open set $W_s \subset \mathbf{R}^3$, $s = 1, 2, \ldots, L$, such that each

point x of $W_s \cap \Omega$ can be written as

$$(4.16) \qquad x = \Lambda_s(\psi, \varphi, r) \equiv \lambda_s(\psi, \varphi) + rn\big(\lambda_s(\psi, \varphi)\big) \, .$$

From the assumption on $\partial\Omega$, the map Λ_s is a diffeomorphism of class C^3 if r is small enough. From now on we will omit the suffix s. By the properties of the isothermal coordinates one can choose as orthonormal system

$$(4.17) \qquad e_1 \equiv \frac{\lambda_\psi}{|\lambda_\psi|} \, , \qquad e_2 \equiv \frac{\lambda_\varphi}{|\lambda_\varphi|} \, , \qquad e_3 \equiv n \circ \lambda = e_1 \wedge e_2$$

(where $\lambda_\psi \equiv \partial\lambda/\partial\psi$, $\lambda_\varphi \equiv \partial\lambda/\partial\varphi$). Moreover it is easily seen that

$$(4.18) \qquad J \equiv \det \mathit{Jac}\Lambda = (\Lambda_\psi \wedge \Lambda_\varphi) \cdot e_3$$
$$= |\lambda_\psi| \, |\lambda_\varphi| + (\alpha|\lambda_\varphi| + \beta' |\lambda_\psi|) \, r + (\alpha\beta' - \beta\alpha') \, r^2 \, ,$$

where $\alpha \equiv (e_3)_\psi \cdot e_1$, $\beta \equiv (e_3)_\psi \cdot e_2$, $\alpha' \equiv (e_3)_\varphi \cdot e_1$, $\beta' \equiv (e_3)_\varphi \cdot e_2$. Hence J is positive for r small enough, and $J \in C^2$.

As $(\mathit{Jac}\Lambda^{-1}) \circ \Lambda = (\mathit{Jac}\Lambda)^{-1}$ one gets also the following relations, which will be useful in the sequel:

$$(4.19) \qquad [\nabla(\Lambda^{-1})^1] \circ \Lambda = (1/J)(\Lambda_\varphi \wedge e_3) \, ,$$

$$(4.20) \qquad [\nabla(\Lambda^{-1})^2] \circ \Lambda = (1/J)(e_3 \wedge \Lambda_\psi) \, ,$$

$$(4.21) \qquad [\nabla(\Lambda^{-1})^3] \circ \Lambda = (1/J)(\Lambda_\psi \wedge \Lambda_\varphi) = e_3 \, .$$

We can now rewrite equations $(4.1)_1$ and $(4.2)_1$ in $U \equiv \Lambda^{-1}(W \cap \Omega)$: we set for simplicity $y \equiv (\psi, \varphi, r)$ and we have

$$(4.22) \qquad \dot{V}^j - \bar{\mu}a_{ki}D_k(a_{si}D_s V^j) - \bar{\beta}a_{kj}D_k(a_{si}D_s V^i) + p_1 a_{kj}D_k S = F^j$$
$$\text{in }]0, T[\times U \, ,$$

$$(4.23) \qquad \dot{S} + V^j a_{kj}D_k S + \bar{\varrho}a_{kj}D_k V^j = F^0 \qquad \text{in }]0, T[\times U \, ,$$

where

$$V(t, y) \equiv v\big(t, \Lambda(y)\big) \, , \qquad S(t, y) \equiv \sigma\big(t, \Lambda(y)\big) \, ,$$

$$F(t, y) \equiv f\big(t, \Lambda(y)\big) \, , \qquad F^0(t, y) \equiv f^0\big(t, \Lambda(y)\big) \, , \qquad a_{ki} = a_{ki}(y) \in C^2 \, .$$

is the entry (k, i) of $(Jac\Lambda)^{-1} = (Jac\Lambda^{-1}) \circ \Lambda$ ($Jac\Lambda$ has the term $D_j \Lambda^i$ in the i-th row, j-th column), and $D_k \equiv \partial/\partial y_k$. Here and in the sequel we adopt the Einstein convention about summation over repeated indices. To obtain the estimates for tangential derivatives, one applies D_τ, $\tau = 1, 2$, to (4.22) and (4.23), and then multiplies by $J\chi^2 D_\tau V^j$ and $(p_1/\bar{\varrho})J\chi^2 D_\tau\sigma$, respectively, and integrates in Ω. Here χ belongs to $C_0^\infty(\Lambda^{-1}(W))$. Then one repeats the same procedure for the second tangential derivatives. The calculations are long and involved, but essentially are the same employed in Lemma 4.4. One can observe however that in the integration by parts one utilizes

$$(4.24) \qquad D_i(Ja_{ij}) = 0 \qquad \text{for each } j = 1, 2, 3 ,$$

and

$$(4.25) \qquad \chi D_\tau V = 0 , \qquad \chi D_\tau D_\xi V = 0 \text{ on } \partial U , \qquad \tau, \xi = 1, 2 .$$

Moreover, one has

$$\frac{\bar{\mu}}{2} \chi^2 J \sum_{i,j} \left(\sum_k a_{ki} D_k D_\tau V^j \sum_h a_{hi} D_h D_\tau V^j \right)$$
$$= \frac{\bar{\mu}}{2} \chi^2 J \sum_j \sum_{k,h} b_{kh} D_k D_\tau V^j D_h D_\tau V^j \geqslant c_7 \chi^2 |D_y D_\tau V|^2 ,$$

since the matrix $B \equiv (Jac\Lambda)^{-1t}(Jac\Lambda)^{-1}$, with entries $b_{kh} \equiv \sum_i a_{ki} a_{hi}$, is uniformly strongly elliptic in $\bar{U}$. Hence one has obtained the following lemma:

LEMMA 4.5. V and S satisfy, for each $0 < \delta < 1$

$$(4.26) \qquad \frac{d}{dt} \int_U J\chi^2 |D_\tau V|^2 + \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \int_U J\chi^2 |D_\tau S|^2 + c_8 \int_U J\chi^2 |D_\tau D_y V|^2$$

$$+ c_9 \int_U J\chi^2 |D_\tau(a_{kj} D_k V^j)|^2 \leqslant \frac{c}{\delta} \left(\|v\|_1^2 + \|f\|_0^2 + \|f^0\|_1^2 + \|\sigma\|_1^4 \right) + \delta\|\sigma\|_1^2 + \delta\|v\|_3^2 ,$$
$$\tau = 1, 2 ,$$

$$(4.27) \qquad \frac{d}{dt} \int_U J\chi^2 |D_\xi D_\tau V|^2 + \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \int_U J\chi^2 |D_\xi D_\tau S|^2 + c_8 \int_U J\chi^2 |D_\xi D_\tau D_y V|^2$$

$$+ c_9 \int_U J\chi^2 |D_\xi D_\tau(a_{kj} D_k V^j)|^2 \leqslant \frac{c}{\delta} \left(\|v\|_2^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4 \right) + \delta\|\sigma\|_2^2 + \delta\|v\|_3^2 ,$$
$$\tau, \xi = 1, 2 .$$

Let us consider now the normal derivative.

Take the normal derivative of $(4.2)_1$ and then multiply by $(\bar{\mu} + \bar{\beta})/\bar{\varrho}$. Then take the scalar product of $(4.1)_1$ by n and add these two equations.

In this way one gets

$$(4.28) \qquad \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} \frac{\partial}{\partial t}\left(\frac{\partial \sigma}{\partial n}\right) + p_1 \frac{\partial \sigma}{\partial n} = \bar{\mu}(\Delta v \cdot n - \nabla \operatorname{div} v \cdot n)$$

$$+ f \cdot n - \dot{v} \cdot n + \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} \nabla f^0 \cdot n - \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} \nabla(v \cdot \nabla \sigma) \cdot n.$$

The term $\Delta v \cdot n - \nabla \operatorname{div} v \cdot n$ does not contain second order normal derivatives $\partial^2 v^j / \partial n^2$. In fact in local coordinates equation (4.28) becomes

$$(4.29) \qquad \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} D_3 \dot{S} + p_1 D_3 S = \bar{\mu}\{a_{ki} D_k(a_{si} D_s V) \cdot e_3 - D_3(a_{si} D_s V^i)\}$$

$$+ F \cdot e_3 - \dot{V} \cdot e_3 + \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} D_3 F^0 - \frac{\bar{\mu} + \bar{\beta}}{\bar{\varrho}} D_3(V^i a_{si} D_s S),$$

and one has

$$a_{ki} D_k(a_{si} D_s V) \cdot e_3 - D_3(a_{si} D_s V^i) = a_{3i} a_{3i} D_3 D_3 V \cdot e_3$$

$$+ a_{3i}(D_3 a_{3i}) D_3 V \cdot e_3 + a_{\tau i} D_\tau(a_{si} D_s V) \cdot e_3 + a_{3i} D_3(a_{\tau i} D_\tau V) \cdot e_3$$

$$- a_{3i} D_3 D_3 V^i - (D_3 a_{3i}) D_3 V^i - D_3(a_{\tau i} D_\tau V^i),$$

where τ runs only over 1 and 2.

From (4.21) we have that $a_{3i} = e_3^i$, and from (4.19) and (4.20) we get $a_{1i} a_{3i} = 0$, $a_{2i} a_{3i} = 0$. Hence one obtains

$$(4.30) \qquad a_{ki} D_k(a_{si} D_s V) \cdot e_3 - D_3(a_{si} D_s V^i) = a_{3i}(D_3 a_{3i}) D_3 V \cdot e_3$$

$$+ a_{\tau i} D_\tau(a_{si}) D_s V \cdot e_3 + a_{\tau i} a_{\xi i} D_\tau D_\xi V \cdot e_3 + a_{3i} D_3(a_{\tau i}) D_\tau V \cdot e_3 -$$

$$- (D_3 a_{3i}) D_3 V^i - (D_3 a_{\tau i}) D_\tau V^i - a_{\tau i} D_3 D_\tau V^i,$$

where τ and ξ run only over 1 and 2.

Multiply now (4.29) by $J\chi^2 D_3 S$ and integrate in U. Then one obtains that

LEMMA 4.6. V and S satisfy, for each $0 < \delta < 1$

$$(4.31) \qquad \frac{\bar{\mu} + \bar{\beta}}{2\bar{\varrho}} \frac{d}{dt} \int_U J\chi^2 |D_3 S|^2 + \frac{p_1}{2} \int_U J\chi^2 |D_3 S|^2 \leqslant c \int_U J\chi^2 |D_\tau D_y V|^2$$

$$+ \frac{c}{\delta}\left(\|v\|_1^2 + \|\dot{v}\|_0^2 + \|f\|_0^2 + \|f^0\|_1^2 + \|\sigma\|_1^4\right) + \delta \|v\|_3^2, \qquad \tau = 1, 2.$$

PROOF. After what we have already seen, we have only to consider the non-linear term

$$\left| \int_U D_3(V^i a_{si} D_s S) J \chi^2 D_3 S \right| = \left| \int_U (D_3 V^i a_{si} + V^i D_3 a_{si}) D_s S J \chi^2 D_3 S \right|$$

$$+ \frac{1}{2} \int_U V^i a_{si} D_s (D_3 S)^2 J \chi^2 \Bigg| = \left| \int_U (D_3 V^i a_{si} + V^i D_3 a_{si}) D_s S J \chi^2 D_3 S \right|$$

$$- \frac{1}{2} \int_U D_s V^i a_{si} (D_3 S)^2 J \chi^2 - \int_U V^i a_{si} (D_3 S)^2 J \chi D_s \chi \Bigg| \leqslant c \|v\|_3 \|\sigma\|_1^2. \quad \Box$$

For the second derivatives, one can proceed in the same way, and one obtains

LEMMA 4.7. V ans S satisfy, for each $0 < \delta < 1$

$$(4.32) \qquad \frac{\bar{\mu} + \bar{\beta}}{2\bar{\varrho}} \frac{d}{dt} \int_U J \chi^2 |D_\xi D_3 S|^2 + \frac{p_1}{2} \int_U J \chi^2 |D_\xi D_3 S|^2$$

$$\leqslant c \int_U J \chi^2 |D_\tau D_\xi D_y V|^2 + \frac{c}{\delta} \left(\|v\|_2^2 + \|\dot{v}\|_1^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4 \right) + \delta \|v\|_3^2,$$
$$\tau, \xi = 1, 2,$$

$$(4.33) \qquad \frac{\bar{\mu} + \bar{\beta}}{2\bar{\varrho}} \frac{d}{dt} \int_U J \chi^2 |D_3 D_3 S|^2 + \frac{p_1}{2} \int_U J \chi^2 |D_3 D_3 S|^2$$

$$\leqslant c \int_U J \chi^2 |D_\tau D_3 D_y V|^2 + \frac{c}{\delta} \left(\|v\|_2^2 + \|\dot{v}\|_1^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4 \right) + \delta \|v\|_3^2,$$
$$\tau = 1, 2.$$

Finally, from (4.22) and (4.30) we have that

$$(4.34) \qquad (\bar{\mu} + \bar{\beta}) D_3(a_{kj} D_k V^j)$$
$$= p_1 D_3 S + \dot{V} \cdot e_3 - F \cdot e_3 - \bar{\mu} \{\text{right hand side of } (4.30)\} .$$

Hence by Lemma 4.6 and Lemma 4.7 we get

LEMMA 4.8. V and S satisfy, for each $0 < \delta < 1$

$$(4.35) \qquad \frac{\bar{\mu} + \bar{\beta}}{2\bar{\varrho}} \frac{d}{dt} \int_U J \chi^2 |D_3 S|^2 + \frac{(\bar{\mu} + \bar{\beta})^2}{4p_1} \int_U J \chi^2 |D_3(a_{kj} D_k V^j)|^2$$

$$\leqslant c_{10} \int_U J \chi^2 |D_\tau D_y V|^2 + \frac{c}{\delta} \left(\|v\|_1^2 + \|\dot{v}\|_0^2 + \|f\|_0^2 + \|f^0\|_1^2 + \|\sigma\|_1^4 \right) + \delta \|v\|_3^2,$$
$$\tau = 1, 2 ,$$

(4.36) $$\frac{\bar{\mu}+\bar{\beta}}{2\bar{\varrho}}\frac{d}{dt}\int_{U}J\chi^2\,|D_\xi D_3 S|^2 + \frac{(\bar{\mu}+\bar{\beta})^2}{4p_1}\int_{U}J\chi^2\,|D_\xi D_3(a_{kj}D_k V^j)|^2$$

$$\leqslant c_{10}\int_{U}J\chi^2\,|D_\tau D_\xi D_y V|^2 + \frac{c}{\delta}\left(\|v\|_2^2 + \|\dot{v}\|_1^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4\right) + \delta\|v\|_3^2\,,$$

$$\tau,\,\xi = 1,2\,,$$

(4.37) $$\frac{\bar{\mu}+\bar{\beta}}{2\bar{\varrho}}\frac{d}{dt}\int_{U}J\chi^2\,|D_3 D_3 S|^2 + \frac{(\bar{\mu}+\bar{\beta})^2}{4p_1}\int_{U}J\chi^2\,|D_3 D_3(a_{kj}D_k V^j)|^2$$

$$\leqslant c_{10}\int_{U}J\chi^2\,|D_\tau D_3 D_y V|^2 + \frac{c}{\delta}\left(\|v\|_2^2 + \|\dot{v}\|_1^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4\right) + \delta\|v\|_3^2\,,$$

$$\tau = 1,2\,.$$

PROOF. One has only to consider (4.34), from which $D_3(a_{kj}D_k V^j)$ is estimated by $D_3 S$. Then the thesis follows from (4.31), (4.32), (4.33). $\quad\square$

We need another estimate which is given by Stokes's problem in local coordinates.

Consider the equations

(4.38) $$\begin{cases} -\bar{\mu}\varDelta[(\chi D_\tau V)\circ\varLambda^{-1}] + p_1\nabla[(\chi D_\tau S)\circ\varLambda^{-1}] = H & \text{in } W\cap\varOmega\,, \\[2mm] \bar{\varrho}\,\text{div}\,[(\chi D_\tau V)\circ\varLambda^{-1}] \qquad\qquad\quad = K & \text{in } W\cap\varOmega\,, \\[2mm] (\chi D_\tau V)\circ\varLambda^{-1} \qquad\qquad\qquad\qquad = 0 & \text{on } \partial(W\cap\varOmega)\,. \end{cases}$$

One can easily calculate H and K by writing the problem in local coordinates in U, and by using equation $(4.1)_1$ and $(4.2)_1$. Hence one gets, in local coordinates:

LEMMA 4.9. V *satisfies*

(4.39) $$\int_{U}J\chi^2\,|D_\tau D_y^2 V|^2 \leqslant c\int_{U}J\chi^2\,|D_\tau D_y(a_{kj}D_k V^j)|^2$$

$$+ c\left(\|v\|_2^2 + \|\sigma\|_1^2 + \|f\|_1^2 + \|\dot{v}\|_1^2\right)\,, \qquad \tau = 1,2\,.$$

PROOF. One has only to apply the results of Cattabriga [4] (see also Temam [23], pag. 33; Giaquinta-Modica [5]) to problem (4.38), and obtains

$$\int_{W\cap\varOmega}|D^2[(\chi D_\tau V)\circ\varLambda^{-1}]|^2 \leqslant c\left(\|H\|_0^2 + \|K\|_1^2\right)\,.$$

By evaluating $\|H\|_0$ and $\|K\|_1$, a straightforward calculation gives (4.39). $\quad\square$

We are now in a position to obtain the estimate for $\|\text{div } v\|_2^2$. By adding (4.14), (4.26) and (4.35) we obtain for each $0 < \delta < 1$

$$(4.40) \qquad \frac{d}{dt}[v]_1^2 + \frac{d}{dt}|[\sigma]|_1^2 + \|\|\text{div } v\|\|_1^2 < \frac{c}{\delta}\left(\|v\|_1^2 + \|\dot{v}\|_0^2 + \|f\|_0^2 + \|f^0\|_1^2 + \|\sigma\|_1^4\right)$$
$$+ \delta\|\sigma\|_1^2 + \delta\|v\|_3^2,$$

where $[\cdot]_k$ is a sum of L^2-norms concerning only interior and tangential derivatives of order k (hence it can be estimated by $\|\cdot\|_k$), where $|[\cdot]|_k$ and $\|\|\cdot\|\|_k$ are norms equivalent to the $H^k(\Omega)$-norm.

If necessary one can write exactly these norms by adding (4.14), (4.26) and (4.35), and then by estimating the term

$$c_{10}\int_U J\chi^2 |D_\tau D_y V|^2$$

which appears in the right hand side of (4.35), by means of (4.26) again. On the other hand, by using also (4.9) and (4.10), where we choose $\varepsilon = \delta^2$, we get

$$(4.41) \qquad \bar{c}_1\frac{d}{dt}\|v\|_0^2 + \bar{c}_2\frac{d}{dt}\|\sigma\|_0^2 + \frac{\bar{c}_3}{\delta}\frac{d}{dt}\|\dot{v}\|_0^2 + \frac{\bar{c}_3}{\delta}\frac{p_1}{\varrho}\frac{d}{dt}\|\dot{\sigma}\|_0^2$$

$$+ \frac{d}{dt}[v]_1^2 + \frac{d}{dt}|[\sigma]|_1^2 + \|\|\text{div } v\|\|_1^2$$

$$< \frac{c}{\delta^2}\left(\|f\|_0^2 + \|f^0\|_1^2 + \|\dot{f}\|_{-1}^2 + \|\dot{f}^0\|_0^2 + \|\sigma\|_2^4 + \|\dot{\sigma}\|_0^4\right) + \delta\|\dot{\sigma}\|_0^2 + \delta\|\sigma\|_1^2 + \delta\|v\|_3^2.$$

In an analogous way, by adding (4.15), (4.27), (4.36) and (4.37), and by using (4.39), we get for each $0 < \delta < 1$

$$\frac{d}{dt}[v]_2^2 + \frac{d}{dt}|[\sigma]|_2^2 + \|\|\text{div } v\|\|_2^2$$

$$\frac{c}{\sqrt{\delta}}\left(\|v\|_2^2 + \|\dot{v}\|_1^2 + \|\sigma\|_1^2 + \|f\|_1^2 + \|f^0\|_2^2 + \|\sigma\|_2^4\right) + \sqrt{\delta}\|\sigma\|_2^2 + \sqrt{\delta}\|v\|_3^2.$$

From (4.11) we have that

$$\|\sigma\|_1^2 + \|v\|_2^2 < c\left(\|\text{div } v\|_1^2 + \|\dot{v}\|_0^2 + \|f\|_0^2\right);$$

moreover we can estimate $\|\dot{v}\|_1^2$ by means of (4.10), where we choose $\varepsilon = \delta$. In conclusion, by using also (4.41) and (4.13), and by choosing δ small enough,

we have

(4.42)
$$\frac{d}{dt}\,]\,|v|\,[_1^2 + \frac{d}{dt}\,]\,|\sigma|\,[_2^2 + \bar{c}_4\frac{d}{dt}\,\|\dot{v}\|_0^2 + \bar{c}_4\frac{p_1}{\varrho}\frac{d}{dt}\,\|\dot{\sigma}\|_0^2$$

$$+\ \bar{c}_5\frac{d}{dt}\,[v]_2^2 + \|v\|_3^2 + \|\sigma\|_2^2 + \|\dot{v}\|_1^2 + \|\dot{\sigma}\|_1^2$$

$$\leqslant c\big(\|f\|_1^2 + \|f^0\|_2^2 + \|\dot{f}\|_{-1}^2 + \|\dot{f}^0\|_0^2 + \|\sigma\|_2^4 + \|v\|_2^2\,\|\sigma\|_2^2 + \|\dot{\sigma}\|_0^4\big)$$

where $]\,|\cdot|\,[_k$ is a norm equivalent to the $H^k(\Omega)$-norm. Now it is sufficient to estimate the norm of f and f^0. We have easily, by (4.3) and (4.4)

(4.43) $\|f\|_1^2 \leqslant c\big(\|v\|_1^2\,\|v\|_3^2 + \|\sigma\|_2^2\,\|v\|_3^2 + \|\sigma\|_2^4\,\|v\|_3^2 + \|\sigma\|_2^4 + \|\sigma\|_2^6 + \|b\|_1^2\big)\,,$

(4.44) $\|\dot{f}\|_{-1}^2 \leqslant c\big(\|\dot{v}\|_0^2\,\|v\|_2^2 + \|\dot{\sigma}\|_0^2\,\|v\|_3^2 + \|\sigma\|_2^2\,\|\dot{v}\|_1^2$

$$+\ \|\sigma\|_2^4\,\|\dot{v}\|_1^2 + \|\dot{\sigma}\|_0^2\,\|\sigma\|_2^2 + \|\dot{\sigma}\|_0^2\,\|\sigma\|_2^4 + \|\dot{b}\|_{-1}^2\big)\,,$$

(4.45) $\|f^0\|_2^2 \leqslant c\|\sigma\|_2^2\,\|v\|_3^2\,,$

(4.46) $\|\dot{f}^0\|_0^2 \leqslant c\big(\|\dot{\sigma}\|_0^2\,\|v\|_3^2 + \|\sigma\|_2^2\,\|\dot{v}\|_1^2\big)\,.$

Define

(4.47) $\varphi(t) \equiv \,]\,|v(t)|\,[_1^2 + \,]\,|\sigma(t)|\,[_2^2 + \bar{c}_4\|\dot{v}(t)\|_0^2 + \bar{c}_4\dfrac{p_1}{\varrho}\,\|\dot{\sigma}(t)\|_0^2 + \bar{c}_5[v(t)]_2^2\,.$

From (4.43)-(4.46), we can write (4.42) as

(4.48) $\dot{\varphi} + \|v\|_3^2 + \|\sigma\|_2^2 + \|\dot{v}\|_1^2 + \|\dot{\sigma}\|_1^2$

$$\leqslant c\|v\|_3^2\big(\|v\|_1^2 + \|\sigma\|_2^2 + \|\sigma\|_2^4 + \|\dot{v}\|_0^2 + \|\dot{\sigma}\|_0^2\big) +$$

$$+\ c\|\sigma\|_2^2\big(\|\sigma\|_2^2 + \|\sigma\|_2^4 + \|\dot{\sigma}\|_0^2\big) +$$

$$+\ c\|\dot{v}\|_1^2\big(\|\sigma\|_2^2 + \|\sigma\|_2^4\big) + c\,\|\dot{\sigma}\|_0^2\big(\|\sigma\|_2^4 + \|\dot{\sigma}\|_0^2\big) +$$

$$+\ c\big(\|b\|_1^2 + \|\dot{b}\|_{-1}^2\big)\,,$$

that is, for each $t \in [0,\,T]$,

(4.49) $\dot{\varphi} \leqslant -\big(\|v\|_3^2 + \|\sigma\|_2^2 + \|\dot{v}\|_1^2 + \|\dot{\sigma}\|_1^2\big)\,[1 - \bar{c}_6(\varphi + \varphi^2)]$

$$+\ \bar{c}_6\big(\|b\|_1^2 + \|\dot{b}\|_{-1}^2\big)\,,$$

where $\bar{c}_6 \geqslant 1$.

Moreover it is easily seen that

(4.50) $\|v(t)\|_3^2 + \|\sigma(t)\|_2^2 + \|\dot{v}(t)\|_1^2 + \|\dot{\sigma}(t)\|_1^2 \geqslant \bar{c}_7\varphi(t)\,,\qquad \forall t \in \mathbb{R}^+\,,$

where $\bar{c}_7$ does not depend on t, $\bar{c}_7 \leqslant 1$.

From these estimates one obtains

LEMMA 4.10. *Let $\partial\Omega \in C^4$ and let v and σ be solutions of* (4.1), (4.2) *in Q_T belonging to the classes of functions obtained in Theorem 2.4. Suppose moreover that* (4.5) *holds in $\bar{Q}_T$, and that*

$$(4.51) \qquad \varphi(0) \leqslant \frac{\gamma}{\bar{c}_6}, \qquad \gamma \in \left]0, \tfrac{1}{2}\right],$$

$$(4.52) \qquad [b]^2_{\infty;1;\infty} + [\dot{b}]^2_{\infty;-1;\infty} < \frac{1}{4}\frac{\bar{c}_7}{\bar{c}_6^2}\gamma .$$

Then one has

$$(4.53) \qquad \varphi(t) \leqslant \frac{\gamma}{\bar{c}_6} \qquad \text{for each } t \in [0, T].$$

PROOF. Suppose that (4.53) is not true, and set

$$\bar{t} \equiv \inf\left\{t \in [0, T] \,|\, \bar{c}_6\varphi(t) > \gamma\right\}.$$

Clearly one has $\bar{c}_6\varphi(\bar{t}) = \gamma$. Then from (4.49) and (4.50) we get

$$\dot{\varphi}(\bar{t}) \leqslant -\frac{1}{4}\frac{\bar{c}_7}{\bar{c}_6}\gamma + \bar{c}_6\big([b]^2_{\infty;1;\infty} + [\dot{b}]^2_{\infty;-1;\infty}\big),$$

that is from (4.52) $\dot{\varphi}(\bar{t}) < 0$. This gives a contradiction, hence (4.53) holds. $\square$

On the other hand, from Sobolev's embedding theorem $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, one sees that there exists a constant $\bar{c}_8$ small enough such that, if $\varphi(t) \leqslant \bar{c}_8$, then

$$(4.54) \qquad \frac{\bar{\varrho}}{2} \leqslant \sigma(t, x) + \bar{\varrho} \leqslant \frac{3}{2}\bar{\varrho} \quad \text{in } \bar{\Omega}.$$

Finally, one gets

LEMMA 4.11. *There exists a constant $\bar{c}_9$ such that, if $\bar{\varrho}/2 \leqslant \sigma(t, x) + \bar{\varrho} \leqslant \frac{3}{2}\bar{\varrho}$ in $\bar{\Omega}$, then*

$$(4.55) \qquad \|v(t)\|_2^2 + \|\sigma(t)\|_2^2 \leqslant \bar{c}_9\{\varphi(t) + \varphi(t)^3 + \|b(t)\|_0^2\}.$$

PROOF. One has, from (4.1)$_1$

$$\|v(t)\|_2^2 + \|\sigma(t)\|_2^2 \leqslant c\big[\varphi(t) + \|Av(t)\|_0^2\big]$$

$$\leqslant c\big[\varphi(t) + \|\dot{v}(t)\|_0^2 + \|\sigma(t)\|_1^2 + \|(v(t)\cdot\nabla)v(t)\|_0^2 + \|b(t)\|_0^2\big],$$

and from Sobolev's embedding theorem $H^1(\Omega) \hookrightarrow L^6(\Omega)$

$$\int |(v(t) \cdot \nabla) v(t)|^2 \leqslant c \|v(t)\|_1^3 \|v(t)\|_2 \leqslant \varepsilon \|v(t)\|_2^2 + \frac{c}{\varepsilon} \|v(t)\|_1^6.$$

Hence, by choosing ε small enough, one gets (4.55). □

We can prove now the existence of global solution of (1.7), under the condition that the initial data and the external force field are small enough.

Suppose that

$$(4.56) \qquad\qquad \varphi(0) \leqslant \min\left(\frac{1}{2\bar{c}_6}, \bar{c}_8\right) \equiv \bar{c}_{10},$$

and

$$(4.57) \qquad\qquad [b]_{\infty;1;\infty}^2 + [\dot{b}]_{\infty;-1;\infty}^2 < \frac{1}{4} \frac{\bar{c}_7}{\bar{c}_6} \bar{c}_{10}.$$

From (4.54) we have that

$$(4.58) \qquad\qquad \frac{\bar{\varrho}}{2} \leqslant \inf_{\bar{\Omega}}(\sigma_0 + \bar{\varrho}) \leqslant \sup_{\bar{\Omega}}(\sigma_0 + \bar{\varrho}) \leqslant \frac{3}{2}\bar{\varrho},$$

and from Lemma 4.11

$$(4.59) \qquad\qquad \|v_0\|_2^2 + \|\sigma_0\|_2^2 \leqslant \bar{c}_9 \left\{ \bar{c}_{10} + \bar{c}_{10}^3 + \frac{1}{4} \frac{\bar{c}_7}{\bar{c}_6} \bar{c}_{10} \right\}.$$

Hence by Theorem 2.4 (see also Remark 2.5) we find a solution of (1.7) in Q_{T^*}, where T^* depends only on Ω, μ, ζ, $\bar{\varrho}$, $\|p_\varrho\|_{c^3}$ and on the constants which appear in (4.57), (4.58) and (4.59). Moreover σ satisfies

$$\frac{\bar{\varrho}}{4} \leqslant \sigma(t, x) + \bar{\varrho} \leqslant 3\bar{\varrho} \quad \text{in } \bar{Q}_{T^*},$$

as it is clear by looking at the proof of the existence of a fixed point in Theorem 2.4 (see in particular (2.22)). Hence, by Lemma 4.10 we get that

$$\varphi(t) \leqslant \bar{c}_{10} \qquad \text{in } [0, T^*],$$

and consequently, by (4.54),

$$\frac{\bar{\varrho}}{2} \leqslant \sigma(T^*, x) + \bar{\varrho} \leqslant \frac{3}{2}\bar{\varrho} \quad \text{in } \bar{\Omega}.$$

Moreover, by Lemma 4.11.

$$\|v(T^*)\|_2^2 + \|\sigma(T^*)\|_2^2 \leqslant \bar{c}_9 \left\{ \bar{c}_{10} + \bar{c}_{10}^3 + \frac{1}{4} \frac{\bar{c}_7}{\bar{c}_6} \bar{c}_{10} \right\}.$$

We can apply again Theorem 2.4, and we find a solution in $[T^*, 2T^*]$, since $v(T^*, x)$ and $\sigma(T^*, x)$ satisfy estimates (4.58) and (4.59) as $v_0(x)$ and $\sigma_0(x)$.

We can repeat this argument in each interval $[0, nT^*]$, $n \in \mathbf{N}$, and consequently we obtain:

THEOREM 4.12. *Let*

$$\partial \Omega \in C^4, \qquad b \in L^\infty(\mathbf{R}^+; H^1(\Omega)), \qquad \dot{b} \in L^\infty(\mathbf{R}^+; H^{-1}(\Omega)),$$

$$p \in C^3, \qquad p_\varrho > 0, \qquad v_0 \in H^2(\Omega) \cap H_0^1(\Omega), \qquad \varrho_0 \in H^2(\Omega)$$

and assume that (4.56) and (4.57) hold. Then there exist

$$v \in L^2_{\text{loc}}(\mathbf{R}^+; H^3(\Omega)) \cap C_B^0(\mathbf{R}^+; H^2(\Omega))$$

with $\dot{v} \in L^2_{\text{loc}}(\mathbf{R}^+; H^1(\Omega)) \cap C_B^0(\mathbf{R}^+; L^2(\Omega)),$

$$\varrho \in C_B^0(\mathbf{R}^+; H^2(\Omega))$$

with $\dot{\varrho} \in C_B^0(\mathbf{R}^+; H^1(\Omega))$ *such that* (v, ϱ) *is a solution of (1.1) in* Q_∞.

Moreover we have $\varphi(t) \leqslant \bar{c}_{10}$ *in* $\mathbf{R}^+$, *and consequently (4.58) and (4.59) hold for each* $t \in \mathbf{R}^+$.

Finally, if in addition $b \in L^2(\mathbf{R}^+; H^1(\Omega))$, $\dot{b} \in L^2(\mathbf{R}^+; H^{-1}(\Omega))$, *then one has also that*

$$v \in L^2(\mathbf{R}^+; H^3(\Omega)), \qquad \dot{v} \in L^2(\mathbf{R}^+; H^1(\Omega)),$$

$$\varrho - \bar{\varrho} = \sigma \in L^2(\mathbf{R}^+; H^2(\Omega)), \qquad \dot{\varrho} \in L^2(\mathbf{R}^+; H^1(\Omega)),$$

and their norms can be estimated by integrating (4.49) in $\mathbf{R}^+$.

We can also remark that if $\|v_0\|_2^2 + \|\sigma_0\|_2^2$ and $[b]_{\infty;1;\infty}^2 + [\dot{b}]_{\infty;-1;\infty}^2$ are small enough, we have that (4.56) and (4.57) are satisfied.

5. – Stability.

Suppose now that the assumptions of Theorem 4.12 are satisfied, and let (v_1, σ_1) and (v_2, σ_2) be two solutions of (1.1) in Q_∞ corresponding to two

different initial data such that $\int \sigma_0^{(1)} = \int \sigma_0^{(2)} = 0$. We suppose indeed that both these initial data satisfy

$$(5.1) \qquad\qquad \varphi(0) \leqslant \min\left(\frac{\gamma}{\bar{c}_6}, \bar{c}_8\right) \equiv N,$$

and

$$(5.2) \qquad\qquad [b]_{\infty;1;\infty}^2 + [\dot{b}]_{\infty;-1;\infty}^2 < \frac{1}{4}\frac{\bar{c}_7}{\bar{c}_6} N,$$

where $\gamma \in]0, \frac{1}{2}]$ will be specified later (see (5.22) and (5.33)).

From Lemma 4.11 we know that the corresponding solutions (v_1, σ_1) and (v_2, σ_2) satisfy

$$(5.3) \qquad \|v_i(t)\|_2^2 + \|\sigma_i(t)\|_2^2 \leqslant \bar{c}_9 \left\{ N + N^3 + \frac{1}{4}\frac{\bar{c}_7}{\bar{c}_6} N \right\} \equiv R, \quad i = 1, 2,$$

for each $t \in \mathbf{R}^+$, and from (4.49) we easily get that

$$(5.4) \qquad \frac{1}{4}\int_0^t \left(\|v_i\|_3^2 + \|\sigma_i\|_2^2 + \|\dot{v}_i\|_1^2 + \|\dot{\sigma}_i\|_1^2 \right)$$

$$\leqslant N + \bar{c}_6 \int_0^t \left(\|b\|_1^2 + \|\dot{b}\|_{-1}^2 \right) \leqslant N\left(1 + \frac{\bar{c}_7}{4} t \right), \quad i = 1, 2.$$

By choosing γ small enough, we can have that N and R are small as we need. From now on we will assume that $R \leqslant 1$.

Set now $w \equiv v_1 - v_2$, $\eta \equiv \sigma_1 - \sigma_2$. We want to prove that all the solutions of (1.1) are asymptotically equivalent; more precisely, we will prove that

$$\|w(t)\|_0^2 + \|\eta(t)\|_0^2 \leqslant c\left(\|w(0)\|_0^2 + \|\eta(0)\|_0^2 \right) \exp\left(-\varepsilon t \right), \qquad t \in \mathbf{R}^+.$$

First of all, we write the equations for w and η

$$(5.5) \qquad \begin{cases} \dot{w} + \bar{A}w + p_1 \nabla\eta = f_1 - f_2 & \text{in } Q_\infty, \\ w = 0 & \text{on } \Sigma_\infty, \\ w(0) = w_0 \equiv v_0^{(1)} - v_0^{(2)} & \text{in } \Omega, \end{cases}$$

$$(5.6) \qquad \begin{cases} \dot{\eta} + \bar{\varrho}\,\mathrm{div}\, w = f_1^0 - f_2^0 - v_1 \cdot \nabla\eta - w \cdot \nabla\sigma_2 & \text{in } Q_\infty, \\ \eta(0) = \eta_0 \equiv \sigma_0^{(1)} - \sigma_0^{(2)} & \text{in } \Omega, \end{cases}$$

where

$$(5.7) \qquad f_1 - f_2 = -(w \cdot \nabla) v_1 - (v_2 \cdot \nabla w) + \frac{\eta}{(\sigma_1 + \bar{\varrho})(\sigma_2 + \bar{\varrho})} A v_1$$

$$+ \frac{\sigma_2}{(\sigma_2 + \bar{\varrho}) \bar{\varrho}} A w + \left[\frac{p_\varrho(\sigma_2 + \bar{\varrho})}{\sigma_2 + \bar{\varrho}} - \frac{p_\varrho(\sigma_1 + \bar{\varrho})}{\sigma_1 + \bar{\varrho}} \right] \nabla \sigma_1 + \left[p_1 - \frac{p_\varrho(\sigma_2 + \bar{\varrho})}{\sigma_2 + \bar{\varrho}} \right] \nabla \eta ,$$

$$(5.8) \qquad f_1^0 - f_2^0 = -\eta \operatorname{div} v_1 - \sigma_2 \operatorname{div} w .$$

Observe also that we can write

$$(5.9) \qquad f_1^0 - f_2^0 - v_1 \cdot \nabla \eta - w \cdot \nabla \sigma_2 = -\operatorname{div}(\eta v_1) - \operatorname{div}(\sigma_2 w) .$$

First of all, we recall that in this paragraph each constant c, $\bar{C}_i$ will depend at most on Ω, μ, ζ, $\bar{\varrho}$, p. Then, by following the same calculations employed in Lemma 4.2 we get

LEMMA 5.1. *One has, for each* $0 < \varepsilon < 1$

$$(5.10) \qquad \frac{1}{2} \frac{d}{dt} \|w\|_0^2 + \frac{1}{2} \frac{p_1}{\varrho} \frac{d}{dt} \|\eta\|_0^2 + \frac{\bar{\mu}}{2} \|Dw\|_0^2 + \bar{\beta} \|\operatorname{div} w\|_0^2$$

$$\leqslant \frac{c}{\varepsilon} \left(\|f_1 - f_2\|_{-1}^2 + \|f_1^0 - f_2^0\|_0^2 \right) + \varepsilon \|\eta\|_0^2 + \frac{c}{\varepsilon} R \|w\|_1^2 + \frac{c}{\varepsilon} \|v_1\|_3^2 \|\eta\|_0^2 .$$

PROOF. We have only to estimate the non-linear terms

$$-\frac{p_1}{\varrho} \int v_1 \cdot \nabla \eta \, \eta - \frac{p_1}{\varrho} \int w \cdot \nabla \sigma_2 \, \eta .$$

By integrating by parts the first term we have

$$\frac{p_1}{\varrho} \left| \int v_1 \cdot \nabla \eta \, \eta \right| = \frac{p_1}{2\varrho} \left| \int \operatorname{div} v_1 \eta^2 \right| \leqslant c \|v_1\|_3 \|\eta\|_0^2 \leqslant \frac{c}{\varepsilon} \|v_1\|_3^2 \|\eta\|_0^2 + \frac{\varepsilon}{2} \|\eta\|_0^2 .$$

The second gives

$$\frac{p_1}{\varrho} \left| \int w \cdot \nabla \sigma_2 \, \eta \right| \leqslant c \|w\|_1 \|\sigma_2\|_2 \|\eta\|_0 \leqslant \frac{c}{\varepsilon} \|w\|_1^2 \|\sigma_2\|_2^2 + \frac{\varepsilon}{2} \|\eta\|_0^2 \leqslant \frac{c}{\varepsilon} R \|w\|_1^2 + \frac{\varepsilon}{2} \|\eta\|_0^2 . \quad \square$$

On the other hand we have

$$\|f_1 - f_2\|_{-1}^2 \leqslant c \big[\left(\|v_1\|_2^2 + \|v_2\|_2^2 \right) \|w\|_0^2 + \|v_1\|_3^2 \|\eta\|_0^2$$

$$= \left(\|\sigma_2\|_2^2 + \|\sigma_2\|_2^4 \right) \|w\|_1^2 + \|\sigma_1\|_2^2 \|\eta\|_0^2 + \left(\|\sigma_2\|_2^2 + \|\sigma_2\|_2^4 \right) \|\eta\|_0^2 \big]$$

$$\leqslant c \left(R + \|v_1\|_3^2 \right) \|\eta\|_0^2 + c R \|w\|_1^2 ,$$

$$\|f_1^0 - f_2^0\|_0^2 \leqslant c \|v_1\|_3^2 \|\eta\|_0^2 + c \|\sigma_2\|_2^2 \|w\|_1^2 \leqslant c \|v_1\|_3^2 \|\eta\|_0^2 + c R \|w\|_1^2 ;$$

hence from (5.10)

(5.11) $\quad \dfrac{1}{2}\dfrac{d}{dt}\|w\|_0^2 + \dfrac{1}{2}\dfrac{p_1}{\bar{\varrho}}\dfrac{d}{dt}\|\eta\|_0^2 + \dfrac{\bar{\mu}}{2}\|Dw\|_0^2 + \bar{\beta}\|\mathrm{div}\,v\|_0^2$

$$\leqslant \dfrac{c}{\varepsilon}\big(R + \|v_1\|_3^2\big)\|\eta\|_0^2 + \dfrac{c}{\varepsilon}R\|w\|_1^2 + \varepsilon\|\eta\|_0^2 .$$

Now we want to estimate $\|\eta\|_0^2$. Let z be the solution of Stokes's problem

(5.12) $\quad \begin{cases} \Delta z - \nabla\pi = 0 & \text{in } Q_\infty , \\ \mathrm{div}\,z \quad = \eta & \text{in } Q_\infty , \\ z \quad = 0 & \text{on } \Sigma_\infty , \end{cases}$

which exists since $\int\eta = 0$ for each $t \in \mathbb{R}^+$ (see Cattabriga [4]; Temam [23], pag. 35; Giaquinta-Modica [5]). We have that

$$z \in C_B^0\big(\mathbb{R}^+; H^3(\Omega)\big) ,$$

and that

(5.13) $$\|z\|_1^2 \leqslant c\|\eta\|_0^2 .$$

Multiply now $(5.5)_1$ by z and integrate in Ω; one has, by (5.13)

(5.14) $\quad p_1\|\eta\|_0^2 = \int[\dot{w} + \bar{A}w - (f_1 - f_2)]\cdot z$

$$\leqslant \int\dot{w}\cdot z + c\big(\|\bar{A}w\|_{-1} + \|f_1 - f_2\|_{-1}\big)\|\eta\|_0$$

$$\leqslant \int\dot{w}\cdot z + c\big(\|w\|_1^2 + \|f_1 - f_2\|_{-1}^2\big) + \dfrac{p_1}{2}\|\eta\|_0^2 .$$

On the other hand, we have

$$\int\dot{w}\cdot z = \dfrac{d}{dt}\int w\cdot z - \int w\cdot\dot{z} .$$

By taking the time derivative of (5.12) we get that $\dot{z}$ is the solution of

$$\begin{cases} \Delta\dot{z} - \nabla\dot{\pi} = 0 & \text{in } Q_\infty , \\ \mathrm{div}\,\dot{z} \quad = \dot{\eta} & \text{in } Q_\infty , \\ \dot{z} \quad = 0 & \text{on } \Sigma_\infty , \end{cases}$$

and from $(5.6)_1$, (5.9)

$$\dot{\eta} = - \operatorname{div} W \, ,$$

where

(5.15) $$W \equiv \eta v_1 + \sigma_2 w + \bar{\varrho} w \, .$$

Let (V, P) be the solution of Stokes's problem

$$\begin{cases} \varDelta V - \nabla P = w & \text{in } Q_\infty \, , \\ \operatorname{div} V \quad = 0 & \text{in } Q_\infty \, , \\ V \qquad\quad = 0 & \text{on } \Sigma_\infty \, , \end{cases}$$

which satisfies

(5.16) $$\|V\|_2 + \|\nabla P\|_0 \leqslant c \|w\|_0 \, .$$

Hence, be recalling that $W|_{\partial\Omega} = 0$

$$\left| \int w \cdot \dot{z} \right| = \left| \int (\varDelta V - \nabla P) \cdot \dot{z} \right| = \left| \int V \cdot \varDelta \dot{z} - \int P \operatorname{div} W \right|$$
$$= \left| \int V \cdot \nabla \dot{\pi} + \int \nabla P \cdot W \right| = \left| \int \nabla P \cdot W \right| \leqslant c \|w\|_0 \|W\|_0 \, ,$$

and estimating W by (5.15) one gets

(5.17) $$\left| \int w \cdot \dot{z} \right| \leqslant c \|w\|_0^2 + cR \|\eta\|_0^2 \, ,$$

where we have used the fact that $\|v_1\|_2^2 \leqslant R$, $\|\sigma_2\|_2^2 \leqslant R < 1$.

Then we multiply (5.14) by δ, $0 < \delta < 1$ and recalling the estimate for $\|f_1 - f_2\|_{-1}^2$ we obtain

(5.18) $$\delta \frac{p_1}{2} \|\eta\|_0^2 - \delta \frac{d}{dt} \int w \cdot z \leqslant c \delta \|w\|_1^2 + c \delta (R + \|v_1\|_3^2) \|\eta\|_0^2 \, .$$

By adding (5.11) and (5.18) and by choosing $\varepsilon = \delta p_1/4$, we get

(5.19) $$\frac{1}{2} \frac{d}{dt} \|w\|_0^2 + \frac{1}{2} \frac{p_1}{\bar{\varrho}} \frac{d}{dt} \|\eta\|_0^2 - \delta \frac{d}{dt} \int w \cdot z$$

$$+ \frac{\bar{\mu}}{2} \|Dw\|_0^2 + \bar{\beta} \|\operatorname{div} w\|_0^2 + \delta \frac{p_1}{4} \|\eta\|_0^2$$

$$\leqslant \bar{C}_1 \left(\delta + \frac{1}{\delta} \right) (R + \|v_1\|_3^2) \|\eta\|_0^2 + \bar{C}_1 \delta \|Dw\|_0^2 + \frac{1}{\delta} \bar{C}_1 R \|Dw\|_0^2 \, .$$

Now we integrate (5.19) in $(0, t)$. Since by (5.13)

$$\delta \left| \int w \cdot z \right| \leqslant \bar{C}_2 \delta (\|w\|_0^2 + \|\eta\|_0^2) \,,$$

we get

$$(5.20) \qquad \left(\frac{1}{2} - \bar{C}_2 \delta \right) \|w(t)\|_0^2 + \left(\frac{1}{2} \frac{p_1}{\varrho} - \bar{C}_2 \delta \right) \|\eta(t)\|_0^2 + \frac{\bar{\mu}}{2} \int_0^t \|Dw\|_0^2$$

$$+ \bar{\beta} \int_0^t \|\operatorname{div} w\|_0^2 + \delta \frac{p_1}{4} \int_0^t \|\eta\|_0^2 \leqslant \bar{C}_3 (\|w_0\|_0^2 + \|\eta_0\|_0^2)$$

$$+ \bar{C}_1 \left(\delta + \frac{1}{\delta} \right) R \int_0^t \|\eta\|_0^2 + \bar{C}_1 \left(\delta + \frac{1}{\delta} \right) \int_0^t \|v_1\|_3^2 \|\eta\|_0^2$$

$$+ \bar{C}_1 \delta \int_0^t \|Dw\|_0^2 + \frac{1}{\delta} \bar{C}_1 R \int_0^t \|Dw\|_0^2 \,.$$

We choose

$$(5.21) \qquad \delta \equiv \frac{1}{2} \min \left(\frac{1}{2\bar{C}_2}, \frac{p_1}{2\bar{C}_2 \varrho}, \frac{\bar{\mu}}{2\bar{C}_1}, 1 \right),$$

and

$$(5.22) \qquad R \leqslant \frac{1}{2} \min \left(\frac{\delta^2 p_1}{4\bar{C}_1(1 + \delta^2)}, \frac{\bar{\mu}}{4} \frac{\delta}{\bar{C}_1}, 1 \right),$$

(more precisely we choose γ in (5.1) such that (5.22) is satisfied) and obtain

$$(5.23) \qquad \|w(t)\|_0^2 + \|\eta(t)\|_0^2 + \int_0^t (\|Dw\|_0^2 + \|\eta\|_0^2)$$

$$\leqslant \bar{C}_4 \left(\|w_0\|_0^2 + \|\eta_0\|_0^2 + \int_0^t \|v_1\|_3^2 \|\eta\|_0^2 \cdot \right)$$

Set

$$(5.24) \qquad \psi(t) \equiv \|w(t)\|_0^2 + \|\eta(t)\|_0^2 + \int_0^t (\|Dw\|_0^2 + \|\eta\|_0^2) \,.$$

Estimate (5.23) gives

$$\psi(t) \leqslant \bar{C}_4 \left[\psi(0) + \int_0^t \|v_1(s)\|_3^2 \psi(s) \, ds \right] ;$$

hence by Gronwall's lemma and by (5.4)

$$(5.25) \qquad \psi(t) \leqslant \bar{C}_4 \psi(0) \exp\left(\bar{C}_4 \int_0^t \|v_1(s)\|_3^2 \, ds\right) \leqslant \bar{C}_4 \psi(0) \exp\left[4\bar{C}_4 N\left(1 + \frac{\bar{c}_7}{4} t\right)\right],$$
$$t \in \mathbf{R}^+ \,.$$

Consider now

$$(5.26) \qquad \overline{w}(t, x) \equiv \exp(\alpha t) w(t, x) \,, \quad \bar{\eta}(t, x) \equiv \exp(\alpha t) \eta(t, x) \,, \qquad 0 < \alpha < 1 \,,$$

which satisfy

$$(5.27) \qquad \begin{cases} \dot{\overline{w}} + \overline{A}\overline{w} + p_1 \nabla \bar{\eta} = \alpha \overline{w} + (\bar{f}_1 - \bar{f}_2) & \text{in } Q_\infty \,, \\ \overline{w} = 0 & \text{on } \Sigma_\infty \,, \\ \overline{w}(0) = w_0 & \text{in } \Omega \,, \end{cases}$$

$$(5.28) \qquad \begin{cases} \dot{\bar{\eta}} + \bar{\varrho} \operatorname{div} \overline{w} = \alpha \bar{\eta} + \bar{f}_1^0 - \bar{f}_2^0 - v_1 \cdot \nabla \bar{\eta} - \overline{w} \cdot \nabla \sigma_2 & \text{in } Q_\infty \,, \\ \bar{\eta}(0) = \eta_0 & \text{in } \Omega \,, \end{cases}$$

where

$$(5.29) \qquad \bar{f}_1 - \bar{f}_2 \equiv -(\overline{w} \cdot \nabla) v_1 - (v_2 \cdot \nabla)\overline{w} + \frac{\bar{\eta}}{(\sigma_1 + \bar{\varrho})(\sigma_2 + \bar{\varrho})} A v_1$$
$$+ \frac{\sigma_2}{\bar{\varrho}(\sigma_2 + \bar{\varrho})} A \overline{w} + \exp(\alpha t) \left[\frac{p_\varrho(\sigma_2 + \bar{\varrho})}{\sigma_2 + \bar{\varrho}} - \frac{p_\varrho(\sigma_1 + \bar{\varrho})}{\sigma_1 + \bar{\varrho}}\right] \nabla \sigma_1$$
$$+ \left[p_1 - \frac{p_\varrho(\sigma_2 + \bar{\varrho})}{\sigma_2 + \bar{\varrho}}\right] \nabla \bar{\eta} \,,$$

$$(5.30) \qquad \bar{f}_1^0 - \bar{f}_2^0 \equiv -\bar{\eta} \operatorname{div} v_1 - \sigma_2 \operatorname{div} \overline{w} \,.$$

We can proceed exactly as we have already done for w and η, and we obtain (5.20) for $\overline{w}$ and $\bar{\eta}$, the only difference being the presence of the term

$$c\alpha(\|\overline{w}\|_0^2 + \|\bar{\eta}\|_0^2)$$

on the right hand side of (5.20). This term can be obviously estimated by $\bar{C}_5 \alpha(\|D\overline{w}\|_0^2 + \|\bar{\eta}\|_0^2)$.

We choose

$$(5.31) \qquad \alpha \equiv \frac{1}{2} \min\left(\frac{\bar{\mu}}{8\bar{C}_5}, \frac{\delta p_1}{8\bar{C}_5}, 1\right),$$

and we get (5.23) for $\overline{w}$ and $\overline{\eta}$, with a different constant $\overline{C}_6$. Hence we get (5.25), that is

$$\|\overline{w}(t)\|_0^2 + \|\overline{\eta}(t)\|_0^2 + \int_0^t (\|D\overline{w}\|_0^2 + \|\overline{\eta}\|_0^2) \leqslant \overline{C}_6 (\|w_0\|_0^2 + \|\eta_0\|_0^2)$$
$$\cdot \exp\left[4\overline{C}_6 N\left(1 + \frac{\overline{c}_7}{4}t\right)\right],$$

and by multiplying by $\exp(-2\alpha t)$

$$(5.32) \qquad \|w(t)\|_0^2 + \|\eta(t)\|_0^2 + \int_0^t \exp[-2\alpha(t-s)]\,[\|Dw(s)\|_0^2 + \|\eta(s)\|_0^2]\,ds$$

$$\leqslant \overline{C}_6 \exp[4\overline{C}_6 N](\|w_0\|_0^2 + \|\eta_0\|_0^2) \exp[-(2\alpha - \overline{c}_7\,\overline{C}_6 N)\,t], \qquad t \in \mathbf{R}^+.$$

We have thus obtained

THEOREM 5.2. *Let*

$$\partial\Omega \in C^4, \qquad b \in L^\infty(\mathbf{R}^+; H^1(\Omega)), \qquad \dot{b} \in L^\infty(\mathbf{R}^+; H^{-1}(\Omega)), \qquad p \in C^3,$$

$$p_\varrho > 0, \qquad v_0^{(i)} \in H^2(\Omega) \cap H_0^1(\Omega), \qquad \varrho_0^{(i)} \in H^2(\Omega), \qquad \int \varrho_0^{(i)} = \overline{\varrho}\,\mathrm{vol}\,(\Omega),$$

$i = 1, 2$, *and assume that* (5.1) *and* (5.2) *hold for* $i = 1, 2$, *with* γ *small enough in such a way that* (5.22) *is satisfied and that*

$$(5.33) \qquad\qquad\qquad \overline{c}_7\,\overline{C}_6 N < 2\alpha,$$

where α *is defined in* (5.31). *Let* (v_i, ϱ_i) *be the solution of* (1.1) *corresponding to the initial data* $(v_0^{(i)}, \varrho_0^{(i)})$, $i = 1, 2$. *Then the difference* (w, η) *between* (v_1, ϱ_1) *and* (v_2, ϱ_2) *satisfies* (5.32), *and goes to zero (in the mean) as* $t \to \infty$.

More precisely, by looking at the proof, we see indeed that we have obtained the following theorem:

THEOREM 5.3. *Let* (v_1, ϱ_1) *and* (v_2, ϱ_2) *be two solutions of* (1.1) *in* Q_∞, *such that* $\int \varrho_1 = \int \varrho_2 = \overline{\varrho}\,\mathrm{vol}\,(\Omega)$ *and*

$$(5.34) \qquad\qquad \|v_i(t)\|_2^2 + \|\sigma_i(t)\|_2^2 \leqslant B_3, \qquad i = 1, 2, \ t \in \mathbf{R}^+,$$

$$(5.35) \qquad\qquad \int_0^t \|v_1\|_3^2 \leqslant B_4 + B_5 t, \qquad\qquad t \in \mathbf{R}^+,$$

where B_3 satisfies (c_Ω is the norm of immersion $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$)

(5.36)
$$B_3 \leqslant \frac{1}{2} \min \left(\frac{\delta^2 p_1}{4\bar{C}_1(1+\delta^2)}, \frac{\bar{\mu}\delta}{4\bar{C}_1}, \frac{\bar{\varrho}^2}{2c_\Omega^2}, 1 \right)$$

(see (5.22)), and B_5 satisfies

(5.37)
$$\bar{C}_6 B_5 < 2\alpha$$

(see (5.33)). Then the difference (w, η) between (v_1, ϱ_1) and (v_2, ϱ_2) satisfies for $t \in \mathbb{R}^+$

(5.38)
$$\|w(t)\|_0^2 + \|\eta(t)\|_0^2 + \int_0^t \exp\left[-2\alpha(t-s)\right]\left[\|Dw(s)\|_0^2 + \|\eta(s)\|_0^2\right]ds$$

$$\leqslant \bar{C}_6 \exp(\bar{C}_6 B_4)(\|w(0)\|_0^2 + \|\eta(0)\|_0^2) \exp\left[-(2\alpha - \bar{C}_6 B_5)t\right],$$

and goes to zero (in the mean) as $t \to \infty$.

6. – Periodic solutions.

In order to obtain the existence of periodic solutions we will follow the approach of Serrin [17], which concerns periodic solutions for incompressible Navier-Stokes equations. Suppose that the external force field is periodic of period T, and assume that

(6.1)
$$[b]_{\infty;1;T}^2 + [b]_{\infty;-1;T}^2 < \frac{1}{4}\frac{\bar{c}_7}{\bar{c}_6} N,$$

where N is defined in (5.1) and γ is chosen in such a way that (5.22) and (5.33) are satisfied.

Let (v^*, σ^*) be the solution of (1.7) with zero initial data. From Theorem 4.12 we have that $\varphi(v^*, \sigma^*)(t) \leqslant N$ in $\mathbb{R}^+$, $\bar{\varrho}/2 \leqslant \sigma^*(t, x) + \bar{\varrho} \leqslant \frac{3}{2}\bar{\varrho}$ in $\bar{Q}_\infty$ and $\|v^*(t)\|_2^2 + \|\sigma^*(t)\|_2^2 \leqslant R$ in $\mathbb{R}^+$. $\Big($Here and in the sequel $\varphi(V, S)$ means the quantity in the right hand side of (4.47) in which we take $v = V$, $\sigma = S$, and $\dot{V}$, $\dot{S}$ are obtained by equation $(4.1)_1$, $(4.2)_1$; moreover we can think that the ratio $\bar{c}_7/\bar{c}_6$ is such that

$$\varphi(v^*, \sigma^*)(0) = \bar{c}_4 \|b(0)\|_0^2 \leqslant \frac{\bar{c}_4\bar{c}_7}{4\bar{c}_6} N \leqslant N.\Big)$$

Define

(6.2)
$$\Phi_n(x) \equiv v^*(nT, x), \qquad \Psi_n(x) \equiv \sigma^*(nT, x).$$

We want to prove that Φ_n and Ψ_n are Cauchy's sequences in $L^2(\Omega)$. Set, for each $m, n \in \mathbf{N}$, $m > n$:

(6.3) $\tilde{v}(t, x) \equiv v^*\big(t + (m-n)T, x\big)\,, \qquad \tilde{\sigma}(t, x) \equiv \sigma^*\big(t + (m-n)T, x\big)\,.$

Since b is periodic, $\tilde{v}$ and $\tilde{\sigma}$ are the solution of (1.7) with initial data $v^*\big((m-n)T, x\big)$ and $\sigma^*\big((m-n)T, x\big)$. Moreover these initial data satisfy

$$\varphi(\tilde{v}, \tilde{\sigma})(0) = \varphi(v^*, \sigma^*)\big((m-n)T\big) \leqslant N\,, \qquad \forall m, n \in \mathbf{N}\,.$$

By Theorem 5.2 we have

$$\|v^*(t) - \tilde{v}(t)\|_0^2 + \|\sigma^*(t) - \tilde{\sigma}(t)\|_0^2 \leqslant \bar{C}_6 \exp\left(4\bar{C}_6 N\right) R \exp\left(-\lambda t\right),$$

$$\lambda \equiv 2\alpha - \bar{c}_7 \bar{C}_6 N > 0$$

that is, for $t = nT$,

(6.4) $\|\Phi_n - \Phi_m\|_0^2 + \|\Psi_n - \Psi_m\|_0^2 \leqslant \bar{C}_6 \exp\left(4\bar{C}_6 N\right) R \exp\left(-\lambda T n\right)$

for each $m, n \in \mathbf{N}$, $m > n$. Hence Φ_n and Ψ_n are Cauchy's sequences in $L^2(\Omega)$, and we set

(6.5) $\Phi \equiv \lim_n \Phi_n\,, \qquad \Psi \equiv \lim_n \Psi_n\,.$

Define now

(6.6) $F \equiv \big\{(f, g) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H^2(\Omega)\,|\,\varphi(f, g)(0) \leqslant N\big\}\,.$

We have that $(\Phi_n, \Psi_n) \in F$, and moreover

(6.7) $\|\Phi_n\|_2^2 + \|\Psi_n\|_2^2 \leqslant R\,, \qquad \forall n \in \mathbf{N}\,.$

Hence we can select a subsequence (Φ_{n_k}, Ψ_{n_k}) which converges weakly in $H^2(\Omega) \times H^2(\Omega)$, and, by Rellich's theorem, strongly in $H^s(\Omega) \times H^s(\Omega)$ $(0 \leqslant s < 2)$ to (Φ, Ψ). In particular, by Sobolev's embedding theorem, (Φ_{n_k}, Ψ_{n_k}) converges uniformly to (Φ, Ψ).

It is easily seen now that F is closed in $L^2(\Omega)$; we only observe that, by using equations $(4.1)_1$ and $(4.2)_1$, one proves that $\Psi_n \to \Psi$ in $L^2(\Omega)$ and $\dot{\Phi}_n \rightharpoonup \dot{\Phi}$ weakly in $L^2(\Omega)$.

Hence $(\Phi, \Psi) \in F$, and from Theorem 4.12 we can obtain a global solution (v, σ) which has (Φ, Ψ) as initial datum.

We prove that this solution (v, σ) is periodic. In fact, set

$$(6.8) \qquad \bar{v}(t, x) \equiv v^*(t + nT, x) , \qquad \bar{\sigma}(t, x) \equiv \sigma^*(t + nT, x) .$$

Since b is periodic, $\bar{v}$ and $\bar{\sigma}$ are the solution of (1.7) with initial data Φ_n and Ψ_n. Then by Theorem 5.2

$$(6.9) \quad \|v(t) - \bar{v}(t)\|_0^2 + \|\sigma(t) - \bar{\sigma}(t)\|_0^2 \leqslant \bar{C}_6 \exp\,(4\bar{C}_6 N)\big(\|\Phi - \Phi_n\|_0^2 + \|\Psi - \Psi_n\|_0^2\big)$$

$$\times \exp\,(-\lambda t) , \qquad \lambda \equiv 2\alpha - \bar{c}_7 \bar{C}_6 N > 0 , \ t \in \mathbf{R}^+ .$$

Putting $t = T$ in (6.9) we get

$$\|v(T) - \Phi_{n+1}\|_0^2 + \|\sigma(t) - \Psi_{n+1}\|_0^2 \leqslant$$

$$\leqslant \bar{C}_6 \exp\,(4\bar{C}_6 N) \exp\,(-\lambda T)\big(\|\Phi - \Phi_n\|_0^2 + \|\Psi - \Psi\|_0^2\big).$$

Taking the limit as $n \to \infty$ we have

$$v(T) = \Phi = v(0) , \qquad \sigma(T) = \Psi = \sigma(0) .$$

We have thus obtained the following theorem

THEOREM 6.1. *Let* $\partial\Omega \in C^4$, $b \in L^\infty\big(R^+; H^1(\Omega)\big)$, $\dot{b} \in L^\infty\big(\mathbf{R}^+; H^{-1}(\Omega)\big)$, $p \in C^3$, $p_\varrho > 0$. *Assume moreover that* b *is periodic of period* T *and that* (5.2) *holds, with* γ *small enough in such a way that* (5.22) *and* (5.33) *are satisfied. Then there exists a periodic solution* (v, σ) *of period* T *of problem* (4.1)$_1$, (4.1)$_2$ *and* (4.2)$_1$ *and such that* $\int\sigma = 0$. *Moreover* (v, σ) *is asymptotically stable and unique among any other solution* $(\underset{\sim}{v}, \underset{\sim}{\sigma})$ *of* (4.1)$_1$, (4.1)$_2$ *and* (4.2)$_1$ *which satisfies* (5.34) *and* (5.36), *and such that* $\int\underset{\sim}{\sigma} = 0$.

7. – Stationary solutions.

Suppose now that b is independent of t, $b \in H^1(\Omega)$, and

$$(7.1) \qquad\qquad\qquad \|b\|_1^2 < \frac{1}{4} \frac{\bar{c}_7}{\bar{c}_6} N ,$$

where N is defined in (5.1), and γ is small enough in such a way that (5.22) and (5.33) hold.

Since b is periodic of any period $T > 0$, by Theorem 6.1 there exists a solution (v_1, σ_1) of (4.1)$_1$, (4.1)$_2$ and (4.2)$_1$ of period 1. Moreover, it is the unique solution of period 1 having initial data which satisfy (5.1).

On the other hand, by Theorem 6.1 we know that there exists a solution (v_2, σ_2) of period $\frac{1}{2}$, having initial data which satisfy (5.1). Hence we must have $v_1 = v_2$, $\sigma_1 = \sigma_2$. Going on in this way, we see that (v_1, σ_1) is periodic of any period $1/n$, n a positive integer, hence of any rational period. This gives that $\big(v_1(t), \sigma_1(t)\big)$ is constant for $t \in \mathbb{Q}$, and consequently by a continuity argument we get that (v_1, σ_1) is independent of t. Hence we have proved

THEOREM 7.1. *Let* $\partial\Omega \in C^4$, $b \in H^1(\Omega)$, $p \in C^3$, $p_\varrho > 0$. *Assume moreover that* (7.1) *holds, with* γ *small enough in such a way that* (5.22) *and* (5.33) *hold. Then there exists a time-independent solution* (v_1, σ_1) *of* $(4.1)_1$, $(4.1)_2$ *and* $(4.2)_1$ *and such that* $\int \sigma_1 = 0$. *Moreover* (v_1, σ_1) *is asymptotically stable and unique among any other solution* $(v, \underset{\sim}{\sigma})$ *of* $(4.1)_1$, $(4.1)_2$ *and* $(4.2)_1$ *which satisfies* (5.34) *and* (5.36), *and such that* $\int \underset{\sim}{\sigma} = 0$.

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