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**On the space of the analytic discs which are transversal to a smooth real hypersurface of $C^n$**

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# On the Space of the Analytic Discs which are Transversal to a Smooth Real Hypersurface of $\mathbf{C}^n$.

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## 0. – Introduction.

This paper deals with analytic discs $\varphi: D \to \mathbf{C}^n$ continuous on $\bar{D}$ and their position with respect to a real, closed, smooth hypersurface $S$. One can consider in the Banach space $B$ of all analytic discs the open subset $\mathcal{A}$ of those $\varphi$ such that $\varphi^{-1}S$ is compact. $\mathcal{A}$ has an open dense subset $\mathcal{A}_0$: the discs $\varphi \in \mathcal{A}$ which are transversal to $S$.

For a fixed continuous function $f: S \to \mathbf{C}$, $\operatorname{Res} f: \varphi \mapsto \int_{\varphi^{-1}S} f \circ \varphi \, d\zeta$ is a well defined function $\mathcal{A}_0 \to \mathbf{C}$ and turns out to be holomorphic when $f$ is a CR function (th. 2.3). Conversely if the Levi form of $S$ does not vanish in any open subset of $S$ then the holomorphicity of $\operatorname{Res} f$ implies that $f$ is CR.

Exactly those discs in $\mathcal{A}$ which are tangent to $S$ belong to the complementary $\Theta$ of $\mathcal{A}_0$ in $\mathcal{A}$. $\Theta$ can have very singular parts but $\{(\zeta, \varphi) \in D \times \Theta,$ $\varphi$ is tangent at $\zeta\}$ is a smooth submanifold of $D \times \mathcal{A}$ of real codimension 3.

Its projection by $D \times \mathcal{A} \to \mathcal{A}$ is obviously $\Theta$ so one sees that $\Theta$ has an open dense subset $\Theta_0$ which is a smooth hypersurface of $\mathcal{A}$: the elements of $\Theta_0$ are those discs in $\mathcal{A}$ which are tangent for a unique value $\zeta \in D$ and this is a Morse tangency.

Of course $\operatorname{Res} f$ is continuous across $\Theta_0$.

If we fix a component of $\mathcal{A}_0$ then, for any $\varphi_0$, $\varphi_1$ in it, the 1-dimensional manifolds $\varphi_0^{-1}S$ and $\varphi_1^{-1}S$ are isotopically imbedded in the unit disc $D$ and therefore isomorphic. However the isomorphism $\varphi_0^{-1}S \to \varphi_1^{-1}S$ is not canonic because it depends on the path joining $\varphi_0$ with $\varphi_1$.

One can make a covering and avoid this indeterminacy.

We prefer to avoid the covering.

If $\varphi$ moves to another component of $\mathcal{A}_0$ crossing $\Theta_0$ then the manifold

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$\varphi^{-1}S$ undergoes a simple catastrophe which can be of four types: the death of a component, the split of one component in two components or their reverses.

The type of the catastrophe only depends on the component of $\Theta_0$ we cross.

For simplicity we prefer to deal with pairs $(\varphi, \gamma)$ with $\gamma =$ compact component of $\varphi^{-1}S$. Those are *contours*. This spares us the trouble of following simultaneously several components of $\varphi^{-1}S$ which can have different behaviours.

The second element $\gamma$ is topologized by the usual distance between compact parts of a metric space (in this case $D$).

We take much advantage from the fact that $(\varphi, \gamma) \mapsto \varphi$ is a local homeomorphism when restricted to the set $A$ of the transversal contours so that $A$ becomes an analytic Banach manifold.

The topological boundary of $A$ has also an open dense smooth part $\partial A$: the pairs $(\varphi, \gamma)$ for which $\varphi$ has only one tangency point on $\gamma$ and this point is Morse. $\partial A$ is modelled on $\Theta_0$.

A component of $\partial A$ can be elliptic or hyperbolic according to the type of the Morse tangency.

Those contours which can be brought to collapse to a point have particular interest in the literature.

This is the union of those components of $A$ which touch the ellyptic part of $\partial A$.

Of course Res $f$ vanishes for them because it vanishes on the ellyptic part of $\partial A$.

We prefer to define a more general kind of contours: those which satisfy the following

PROPERTY (P). *The transversal contour* $(\varphi_0, \gamma_0)$ *satisfies the property* (P) *if it can be connected by a continuous family* $(\varphi_t, \gamma_t)$, $0 \leqslant t \leqslant 1$, *transversal for* $0 \leqslant t < 1$, *to a contour* $(\varphi_1, \gamma_1)$ *such that for all transversal contours* $(\varphi, \gamma)$ *in a neighbourhood* $U$ *of* $(\varphi_1, \gamma_1)$, *each* CR *function on* $S$ *is the uniform limit on* $\varphi(\gamma)$ *of functions which are holomorphic in a neighbourhood of* $\varphi(\Delta)$, *continuous up to* $\varphi(\gamma)$. *Here* $\Delta \subset\subset D$ *is the interior of* $\gamma$.

One can also say that if $R$ is the set of contours satisfying the condition required to $(\varphi, \gamma)$ above then the contours which fullfill (P) are the union of those components of $A$ which meet the topological interior of $R$.

Baouendi-Treves theorem [1] ensures that suitably small contours satisfy (P).

We shall also call (P)-contours the contours which satisfy (P) and also speak about (P)-components of $A$.

REMARK 0.1. There exist (P)-contours on $S$ unless the Levi form of $S$ vanishes identically. But in this case there exist on $S$ no contour at all.

Res $f$ vanishes also on (P)-contours.

One can modify the integral Res $f$ by the introduction of a holomorphic function $h: D \to \mathbf{C}$ and consider $G(\varphi, \gamma) = \int f[\varphi(\zeta)]h(\zeta)\, d\zeta$ which also depends holomorphically on $(\varphi, \gamma)$ and vanishes on (P)-contours. Those have therefore the property that $f \circ \varphi$ extends holomorphically in the interior $\Delta$ of $\gamma$ by the Cauchy formula.

This formula defines a function $C$ on an open subset of $A \times D$: the triplets $(\varphi, \gamma, \zeta_0)$ with $(\varphi, \gamma) \in A$ and $\zeta_0$ in the interior of $\gamma$; this is

$$C(\varphi, \gamma, \zeta_0) = \frac{1}{2\pi i}\int_\gamma \frac{f[\varphi(\zeta)]}{\zeta - \zeta_0}\, d\zeta \, .$$

We prove in section 3 that $C$ is holomorphic in all its arguments.

It is a delicate question whether $C$ extends only $f \circ \varphi$ from $\gamma$ inside $\Delta$ or also $f$ from $\varphi(\gamma)$ inside $\varphi(\Delta)$.

For one needs to compare $C$ with the evaluation map $v: (\varphi, \gamma, \zeta_0) \to \varphi(\zeta_0)$ which sends $A \times D$ into $\mathbf{C}^n$.

A first encouraging result obtained in section 3 is that at least on the (P)-components of $A$ we have

$$dC \wedge dv = 0 \, .$$

In orther words if we vary continuously a disc throught a point $z \in \mathbf{C}^n$ then the value in $z$ of the Cauchy extension of $f$ stays fixed.

Contrarily this value can change if we perform the Cauchy integral along two different components of $\varphi^{-1}S$.

At the end of the paper we give a partial answer to the question above for those (P)-contours $(\varphi, \gamma)$ such that $\varphi$ is *proper* on the interior $\Delta$ of $\gamma$, i.e. $\varphi(\gamma) \cap \varphi(\Delta) = \emptyset$ .

For such a contour $(\varphi, \gamma)$ we find that there is a $\varepsilon > 0$ such that the set of discs $\{\varphi + w\}$ with $w \in \mathbf{C}^n$, $|w| < \varepsilon$ gives a one valued extension of CR functions with Cauchy formula, holomorphic in a neighbourhood of $\varphi(\Delta)$.

## 1. – Analytic contours.

An *analytic disc* is a holomorphic map $\varphi: D \to \mathbf{C}^n$ of the unit disc $D$ into $\mathbf{C}^n$, continuous up to the boundary of $D$.

Analytic discs form a Banach space $B$ with the usual norm

$$\|\varphi\| = \max_{\bar{D}} |\varphi| .$$

This norm controls the derivatives of $\varphi$ on compact subsets of $D$.

This is due to the Cauchy estimates

$$\sup_{\Delta} |\varphi^{(k)}| \leqslant C\|\varphi\| , \qquad \text{with } \Delta \subset\subset D \tag{1.1}$$

and $C$ depends only on $k \geqslant 0$ and $\Delta$.

Therefore the evaluation $v^{(k)}(\varphi, \zeta) = \varphi^{(k)}(\zeta)$, $k = 0, 1, 2, \ldots$, of the $k$-th derivative is a continuous map $B \times D \to \mathbf{C}^n$.

Actually $v^{(k)}$ is analytic; its differential $dv^{(k)}$ at $(\varphi_0, \zeta_0)$ is given by

$$dv^{(k)}(\varphi, \zeta) = \varphi^{(k)}(\zeta_0) + \zeta\varphi_0^{(k+1)}(\zeta_0) . \tag{1.2}$$

Let now $S \equiv \{r = 0\}$ be a smooth real hypersurface in an open subset $\Omega \in \mathbf{C}^n$. The real function $r$ is of class $C^\infty$ in $\Omega$ and $dr \neq 0$ on $S$.

Def. A disc $\varphi \in B$ is said to be *tangent* to $S$ at $\zeta_0 \in D$ if $\zeta_0$ is a critical zero for $r \circ \varphi$ and *Morse tangent* if $\zeta_0$ is a Morse critical zero.

Proposition 1.1.

$$\mathcal{T} \equiv \{(\varphi, \zeta) \in B \times D \mid \varphi \text{ is tangent to } S \text{ at } \zeta\}$$

*is a closed smooth submanifold of* $B \times D$ *of real codimension* 3.

*Moreover the pairs* $(\varphi, \zeta) \in \mathcal{T}$ *for which* $\varphi$ *is Morse-tangent at* $\zeta$ *are exactly those* $(\varphi, \zeta) \in \mathcal{T}$ *whose neighboudhood in* $\mathcal{T}$ *is projected diffeomorphically by* $\pi: B \times D \to B$ *onto a smooth submanifold* $\Theta_0$ *of* $B$. $\Theta_0$ *is a real hypersurface which disconnects locally* $B$.

Proof. $\mathcal{T}$ is the zero set of the function $F: B \times D \to \mathbf{R} \times \mathbf{C}$ given by

$$F(\varphi, \zeta) = \{r[\varphi(\zeta)] , \; dr[\varphi(\zeta)]/d\zeta\} .$$

All we must prove are the following statements

(1) $dF$ *is surjective at the points of* $\mathcal{T}$,

(2) $\ker dF$ *at* $(\varphi_0, \zeta_0)$ *is projected isomorphically onto* $B$ *by* $\pi$ *iff* $(\varphi_0, \zeta_0)$ *is Morse.*

There is

$$F = \left\{r\circ v, \sum_{j=1}^{n}(r_j\circ v)v'_j\right\},$$

$$dF = \left\{dr\circ dv, \sum_{j=1}^{n}[(dr_j\circ dv)v'_j + (r_j\circ v)\, dv'_j]\right\};$$

where $v_j$ is the $j$-th component of the evaluation $v$, $r_j = \partial r/\partial z_j$, and the differentials, as well as the evaluations $v'_j$, are computed at $(\varphi_0, \zeta_0)$.

Take $u, w$ in $\mathbf{C}^n$. $dF$ computed for $(u + (\zeta - \zeta_0)w, 0)$ is

$$dF = \left\{\sum_{j=1}^{n} r_j u_j,\ \alpha(u) + \sum_{j=1}^{n} r_j w_j\right\}$$

where $\alpha(u)$ is a linear function.

We can choose first $u$ such that the first component does not vanish and secondly $w$ in order to annihilate the second component.

This generates $\mathbf{R}\times\{0\}$. Also we generate $\{0\}\times\mathbf{C}$ taking $u = 0$ and letting $w$ free. This proves (1).

The statement (2) can be reworded by requiring that $dF(0, \zeta) = 0$ must imply $\zeta = 0$, if and only if $(\varphi_0, \zeta_0)$ is Morse.

If we set $\varrho = r\circ\varphi_0$ we have at $(\varphi_0, \zeta_0)$

$$dF(0, \zeta) = \left(0, \zeta\varrho_{\zeta\zeta}(\zeta_0) + \bar{\zeta}\varrho_{\zeta\bar{\zeta}}(\zeta_0)\right)$$

which is annihilated by some $\zeta \neq 0$ if and only if the Hessian determinant of $\varrho$ at $\zeta_0$ vanishes, i.e. $\zeta_0$ is *not* a Morse critical point for $\varrho$. □

DEFINITION. An *analytic contour* (or simply a *contour*) of $S$ is a pair $(\varphi, \gamma)$ of an analytic disc $\varphi$ and a compact component $\gamma$ of $\varphi^{-1}S$.

It is easy to check that there exist contours in $S$ unless the Levi form of $S$ vanishes identically.

Let $K$ be the metric space of the compact parts of the unit disc $D$. For $\gamma, \gamma'$ in $K$ there is $d(\gamma, \gamma') = \inf\{\varepsilon | \gamma \text{ is in the } \varepsilon\text{-nhb of } \gamma' \text{ and viceversa}\}$. The set of analytic contours is considered as a metric subspace of $B\times K$.

$$d[(\varphi, \gamma), (\varphi', \gamma')] = \|\varphi - \varphi'\| + d(\gamma, \gamma') .$$

DEFINITIONS 1.1.

(i) A contour $(\varphi, \gamma)$ is said to be *transversal* if $\varphi$ is transversal to $S$ at the points of $\gamma$.

$(\varphi, \gamma)$ is said to be *Morse* if $r\circ\varphi$ has on $\gamma$ a unique critical zero $\zeta_0$ which is nondegenerate.

(Of course $\varphi$ is tangent to $S$ at $\zeta_0$).

(ii) *Let $A$ be the set of transversal contours and $\partial A$ the set of Morse contours.*

We want to show now that the standard projection $\Pi\colon B\times K \to B$ induces a local homeomorphism $A \to B$ and $A$ turns out to be an analytic Banach manifold with smooth boundary $\partial A$.

It is a trivial matter that if $\varphi_0$ is transversal to $S$ along the component $\gamma_0$ of $\varphi_0^{-1}S$, then a component $\gamma(\varphi)$ of $\varphi^{-1}S$ is determined by $\gamma_0$, and $\gamma(\varphi)$ varies continuously, when $\varphi$ is near $\varphi_0$.

A more precise set up of this fact is necessary for later use.

Set $B(\varphi_0, \delta) = \{\varphi\in B, \|\varphi-\varphi_0\| < \delta\}$.

PROPOSITION 1.2. *For each transversal contour $(\varphi_0, \gamma_0)$ there are positive constants $\varepsilon$, $\delta$, $C$, $C_1$ and a smooth map*

$$\chi\colon B(\varphi_0, \delta)\times\gamma_0 \to D$$

*such that, if we set $\gamma(\varphi) = \chi(\{\varphi\}, \gamma_0)$, for each $\varphi\in B(\varphi_0, \delta)$ there is*

(i) $\gamma = \gamma(\varphi)$ *if and only if $\gamma$ is a component of $\varphi^{-1} S$ and $d(\gamma_0, \gamma) < \varepsilon$,*

(ii) $\chi(\varphi, \cdot)\colon \gamma_0 \to \gamma(\varphi)$ *is an embedding of $\gamma_0$ in $D$ and reduces to the identity when $\varphi = \varphi_0$,*

(iii) $\sup_{\zeta\in\gamma_0} |\chi(\varphi, \zeta) - \zeta| \leqslant C\|\varphi-\varphi_0\|$,

(iv) *let $\Delta(\varphi)$ be the open, relatively compact subset of $D$ having $\gamma(\varphi)$ as boundary and let $\Delta(\varphi_0, \varphi)$ be the symmetric difference*

$$[\Delta(\varphi_0) - \Delta(\varphi)] \cup [\Delta(\varphi) - \Delta(\varphi_0)].$$

*There is*

$$\sup_{\zeta\in\Delta(\varphi_0,\varphi)} |r[\varphi(\zeta)]| \leqslant C_1\|\varphi - \varphi_0\|.$$

PROOF. We use the notation $\varphi = (\varphi_1, \ldots, \varphi_n)$ for $\varphi\in B$ and $r_j = \partial r/\partial z_j$ for $j = 1, \ldots, n$.

Fix $\psi\in B$ with $\|\psi\| = 1$ and set

$$R(t, \zeta) = r[\varphi_0(\zeta) + t\psi(\zeta)].$$

The particular structure of the functions

$$R_\zeta(t,\zeta) = \sum_{j=1}^{n} r_j[\varphi_0(\zeta) + t\psi(\zeta)][\varphi'_{0j}(\zeta) + t\psi'_j(\zeta)]\,,$$
$$R_t(t,\zeta) = 2\,\mathrm{Re}\left\{\sum_{j=1}^{n} r_j[\varphi_0(\zeta) + t\psi(\zeta)]\psi_j(\zeta)\right\}$$

and the Cauchy estimates (1.1) imply that there are positive constants $\varepsilon$, $\delta$, $c$ only depending on $(\varphi_0, \gamma_0)$ *and not on* $\psi$, such that there is

$$|R_\zeta| > c\,, \quad \text{for } |t| < \delta\,,\ \zeta \in E_\varepsilon\,, \tag{1.3}$$

where $E_\varepsilon$ is the $\varepsilon$-neighbourhood of $\gamma_0$. Similarly $R_t$, $R_{\zeta\zeta}$, $R_{\zeta\bar\zeta}$, $R_{t\zeta}$ are bounded from above in $]-\delta, \delta[\times E_\varepsilon$ independently of $\psi$, $\|\psi\| = 1$.

If we set $F = -R_t/2R_\zeta$, the same can be affirmed, by (1.3), for $F$, $F_\zeta$, $F_{\bar\zeta}$.

After a possible reduction of $\varepsilon$ and $\delta$ there exists in $]-\delta, \delta[\times E_\varepsilon$ a solution $f(t,\zeta)$ of the system

$$\begin{cases} f_t(t,\zeta) = F[t, f(t,\zeta)] \\ f(0,\zeta) = \zeta\,. \end{cases}$$

Since $R_\zeta[t, f(t,\zeta)]f_t(t,\zeta) = -R_t[t, f(t,\zeta)]/2$ is real, there is

$$dR[t, f(t,\zeta)]/dt = 0\,.$$

Hence, reducing again $\varepsilon$ and $\delta$, we can say that $f(t,\cdot)$ sends $\gamma_0$ onto a component of the set $R(t,\cdot) = 0$ by an embedding. The map $\chi$ is now defined by setting

$$\chi(\varphi_0 + t\psi, \zeta) = f(t,\zeta)\,.$$

We also get from the previous remarks the inequalities

$$\begin{cases} |\chi(\varphi,\zeta) - \zeta| \leqslant C\|\varphi - \varphi_0\| \\ |\chi_\zeta(\varphi,\zeta) - 1| \leqslant C\|\varphi - \varphi_0\|\,, \quad \text{for } (\varphi,\zeta) \in B(\varphi_0,\delta)\times E_\varepsilon \\ |\chi_{\bar\zeta}(\varphi,\zeta)| \leqslant C\|\varphi - \varphi_0\| \end{cases} \tag{1.5}$$

with $C = C(\varphi_0)$.

We can now conclude that (ii) and the « only if » part of (i) hold, (iii) is a consequence of $(1.5)_1$.

We prove now the « if » part of (i).

Since $\varphi$ is transversal to $S$ along $\gamma$, then $\gamma$ is a smooth contour. Consider the open set $\Delta$ having $\gamma - \gamma(\varphi)$ as oriented boundary.

Since $\Delta \subset E_\varepsilon$, then, by (1.3), $r\circ\varphi$ has nonvanishing gradient in $\Delta$ but vanishes at the boundary. This is only possible when $\Delta = \emptyset$.

Finally we prove (iv). Note that $\delta$ can be supposed to be smaller than 1. Set

$$C_1 = \sup\{|r_j(z)|,\ j = 1, \ldots, n,\ |z| \leqslant 1 + \|\varphi_0\|\}.$$

Consider the above functions $R$ and $f$ relatively to the choice $\psi = (\varphi - \varphi_0)/\|\varphi - \varphi_0\|$ and fix $\zeta^* \in \Delta(\varphi, \varphi_0)$.

We must prove

$$R(\|\varphi - \varphi_0\|, \zeta^*) \leqslant C_1\|\varphi - \varphi_0\| .$$

To that purpose it is sufficient to prove

$$(1) \qquad R(t^*, \zeta^*) = 0 , \qquad \text{for some } 0 < t^* < \|\varphi - \varphi_0\| ,$$

$$(2) \qquad R_t(t, \zeta^*) \leqslant C_1\|\varphi - \varphi_0\| , \qquad \text{for each } 0 \leqslant t \leqslant \|\varphi - \varphi_0\| .$$

Now (2) is immediately obtained by derivation. Furthermore, since $\gamma_0 = f(0, \gamma_0)$ and $\gamma(\varphi) = f(\|\varphi - \varphi_0\|, \gamma_0)$ have different winding number with respect to $\zeta^*$, then it must be $\zeta^* = f(t^*, \zeta_0^*)$ for some $\zeta_0^* \in \gamma_0$ and some $0 < t^* < \|\varphi - \varphi_0\|$.

Set $\varphi^* = \varphi_0 + t^*(\varphi - \varphi_0)/\|\varphi - \varphi_0\|$.

Since $\gamma(\varphi^*) = f(t^*, \gamma_0)$ is a component of $\varphi^{*-1}S$, as proved in (i), there is $r[\varphi^*(\zeta^*)] = 0$ which is exactly (1). □

As a consequence of Prop. 1.2 (i) we have that each transversal contour $(\varphi_0, \gamma_0)$ has a neighbourhood

$$N \equiv \{(\varphi, \gamma(\varphi))\}_{\varphi \in B(\varphi_0, \delta)}$$

in $A$ which projects homeomorphically onto $B(\varphi_0, \delta)$ by $\Pi\colon B \times K \to B$.

*The set $A$ of the transversal contours will be considered henceforth as a Banach analytic manifold with the cards $(N, \Pi)$ defined above.*

Let now $(\varphi_0, \gamma_0)$ be a Morse contour, tangent at the point $\zeta_0$.

If $\varphi \in B$ and $\zeta \in D$ are sufficiently near to $\varphi_0$ and $\zeta_0$ respectively and if $\varphi$ is tangent at $\zeta$, then $\varphi$ is also Morse tangent at $\zeta$ and the component $\gamma$ of $\zeta$ in $\varphi^{-1}S$ is arbitrarily close to $\gamma_0$.

Therefore, with the notation of prop. 1.1, we have a local homeomorphism $\Phi\colon \partial A \to \mathcal{T}$ defined by $(\varphi, \gamma) \mapsto (\varphi, \zeta)$.

Now since $\pi \circ \Phi = \Pi$, there is locally $\Pi\partial A = \Theta_0$.

We resume those considerations in the following

THEOREM 1.1. *The projection $\Pi: (\varphi, \gamma) \mapsto \varphi$ gives the set $A$ of transversal contours the structure of a Banach analytic manifold with the set $\partial A$ of Morse contours as smooth boundary.*

REMARK 1.1. $\partial A$ has two disconnected parts: the ellyptic and hyperbolic Morse contours according to the type of the Morse critical zero at the tangency point.

DEFINITION 1.2. A contours which is connectable in $A \cup \partial A$ with the ellyptic part of the boundary $\partial A$ is called a *collapsing contour.*

Of course the set of collapsing contours is open and closed in $A \cup \partial A$.

Roughly speaking $(\varphi, \gamma)$ is collapsing if $\gamma$ can be reduced to a point by a continuous motion which keeps $\varphi$ transversal to $S$ along $\gamma$.

Notice that as a consequence of the Baouendi-Treves theorem [1] which says that entires functions are locally dense in CR functions one has that collapsing contours are (P)-contours (Definition in section 0).

## 2. – Integration of CR functions.

A function $f$ of class $C^1$ on $S$ is said to be a CR function if, for each point $z_0$ of $S$ and each complex line $l$ tangent to $S$ at $z_0$, $f|_l$ has complex derivative at $z_0$.

If we write $\lambda \mapsto z_0 + \lambda v$ for the line, this derivative is

$$f'_v(z_0) = \sum_{j=1}^{n} f_{z_j}(z_0) v_j . \tag{2.1}$$

Hence

$$f(z^0 + \lambda v) = f(z^0) + \lambda f'_v(z_0) + O(|\lambda|^2) , \qquad \lambda \in \mathbf{C} . \tag{2.2}$$

Traces on $S$ of functions which are holomorphic on $\Omega$ or even only on one side of $S$ are of course CR functions.

It is well known that $f$ has a smooth extension to $\mathbf{C}^n$ such that

$$|f_{\bar{z}_1}|^2 + \ldots + |f_{\bar{z}_n}|^2 = O(|r|)^k \tag{2.3}$$

holds for all $k \geq 0$.

For this particular extension formula (2.2) holds also for a nontangent vector $v$.

*We will always refer implicitly to such an extension. We fix now a* CR *function* $f$ *on* $S$ and a function $h(\zeta, \zeta_0)$, holomorphic in both variables in the set $\{\zeta \neq \zeta_0\}$ and such that, for all $\varrho > 0$, $h$ and its derivatives are bounded for $|\zeta - \zeta_0| > \varrho$.

Our aim is to study the integral

$$F(\varphi, \gamma, \zeta_0) = \int_\gamma f[\varphi(\zeta)]h(\zeta, \zeta_0)\, d\zeta\,. \tag{2.4}$$

We begin by a local study.

THEOREM 2.1. *Let* $\varphi_0$ *be a fixed disc transversal to* $S$ *along the component* $\gamma_0$ *of* $\varphi_0^{-1}S$ *so that, according with prop.* 1.2, *for* $\|\varphi - \varphi_0\| < \delta$, $\delta > 0$ *sufficiently small, the component* $\gamma(\varphi)$ *of* $\varphi^{-1}S$ *close to* $\gamma_0$ *is well defined. Let* $h(\zeta, \zeta_0)$ *be a holomorphic function as above, then*

$$F(\varphi, \zeta_0) = \int_{\gamma(\varphi)} f[\varphi(\zeta)]h(\zeta, \zeta_0)\, d\zeta \tag{2.5}$$

*is a holomorphic function of the pair* $(\varphi, \zeta_0)$ *in the set*

$$\{\|\varphi - \varphi_0\| < \delta,\ \zeta_0 \in \Delta(\varphi) = \textit{the interior of } \gamma(\varphi)\}\,.$$

*Moreover the differential* $d_0F$ *of* $F$ *at* $(\varphi_0, \zeta_0)$ *is given by*

$$d_0F(\varphi_1, \zeta_1) = \zeta_1 \int_{\gamma_0} f[\varphi_0(\zeta)]h_{\zeta_0}(\zeta, \zeta_0)\, d\zeta + \sum_{j=1}^{n} \int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\varphi_{1j}(\zeta)h(\zeta, \zeta_0)\, d\zeta\,. \tag{2.6}$$

PROOF. Fix $\zeta^* \in \Delta(\varphi_0)$. It is not restrictive to prove the holomorphicity at $(\varphi_0, \zeta^*)$. From prop. 1.2 one sees that, if $\delta$ is small enough, then there exists $\varrho > 0$ such that $\|\varphi - \varphi_0\| < \delta$, $|\zeta_0 - \zeta^*| < \varrho$ and $\zeta \in \gamma(\varphi)$ imply $|\zeta - \zeta_0| > c$ with $c > 0$. Hence $F(\varphi, \zeta^*)$ is well defined for $\|\varphi - \varphi_0\| < \delta$ and $|\zeta_0 - \zeta^*| < \varrho$, moreover $h$ in the integral (2.5) is bounded with the derivatives so that (2.5) can be derived under the sign. This proves $\zeta_0$-holomorphicity of $F$.

In addition we note that the first term on the right in (2.6) is the differential of $F$ in the $\zeta_0$-direction.

In order to prove the $\varphi$-holomorphicity of $F$ we apply the Green formula in the form

$$\int_{\gamma(\varphi) - \gamma_0} \omega = \int_{\Delta(\varphi) - \Delta(\varphi_0)} \bar{\partial}\omega - \int_{\Delta(\varphi_0) - \Delta(\varphi)} \bar{\partial}\omega \tag{+}$$

where $\omega$ is the integrand in (2.5) so that

$$\delta\omega = \sum_{j=1}^{n} f_{\bar{z}_j}[\varphi(\zeta)]\bar{\varphi}_j'(\zeta)h(\zeta, \zeta_0)d\bar{\zeta}\wedge d\zeta .$$

Notice that the area of the set $\Delta(\varphi, \varphi_0)$ defined in prop. 1.2 (iv) stays bounded.

For $\zeta \in \Delta(\varphi, \varphi_0)$, $|\varphi'(\zeta)|$ is also bounded by Cauchy inequalities, and also, up to make $\delta$ smaller only in dependence of $(\varphi_0, \gamma_0, \zeta_0)$, $|\zeta - \zeta_0| > c > 0$.

Hence the integrands on the right in $(+)$ are estimated by $|f_{\bar{z}}[\varphi(\zeta)]|$.

Now $C$ will be a constant depending only on $\varphi_0$ and $\delta$ and can be different in different estimates.

Since $\varphi(\zeta)$ varies is a compact set, (2.3) gives for $k = 2$

$$|f_{\bar{z}_j}[\varphi(\zeta)]| < Cr^2[\varphi(\zeta)] , \quad \text{for } \zeta \in \Delta(\varphi, \varphi_0) , \ j = 1, \ldots, n ,$$

so that, application of prop. 1.2 (iv) yields

$$|f_{\bar{z}_j}[\varphi(\zeta)]| < C\|\varphi - \varphi_0\|^2 , \quad \text{for } \zeta \in \Delta(\varphi, \varphi_0) , \ j = 1, \ldots, n ,$$

and finally, by $(+)$

$$\left|F(\varphi, \zeta_0) - \int_{\gamma_0} f[\varphi(\zeta)]h(\zeta, \zeta_0)d\zeta\right| < C\|\varphi - \varphi_0\|^2 .$$

In the new integral $\zeta$ varies in the fixed closed curve $\gamma_0$ so that $\varphi_0(\zeta) \in S$.

Therefore we can apply (2.2) and (2.1) with $z_0 = \varphi_0(\zeta)$, $\lambda v = \varphi(\zeta) - \varphi_0(\zeta)$, taking also account of the compactness of $\gamma_0$ for the $\zeta$-uniformity of the estimate, we obtain

$$\left|f[\varphi(\zeta)] - f[\varphi_0(\zeta)] - \sum_{j=1}^{n} f_{z_j}[\varphi_0(\zeta)][\varphi_j(\zeta) - \varphi_{0j}(\zeta)]\right| < C\|\varphi - \varphi_0\|^2 , \quad \text{for } \zeta \in \gamma_0$$

and hence

$$|F(\varphi, \zeta_0) - F(\varphi_0, \zeta_0) - L(\zeta_0, \varphi - \varphi_0)| < C\|\varphi - \varphi_0\|^2$$

with

$$L(\zeta_0, \varphi - \varphi_0) = \sum_{j=1}^{n} \int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\,[\varphi_j(\zeta) - \varphi_{0j}(\zeta)]\,h(\zeta, \zeta_0)\,d\zeta .$$

This is the second term in (2.6) with $\varphi_1 = \varphi - \varphi_0$.

The continuity of $L$ as $\mathbf{C}$-linear function of $\varphi - \varphi_0$ is evident. □

THEOREM 2.2. *For each function $h$ holomorphic in the unit disc and each* CR *function $f$ on $S$ the integral*

$$G(\varphi, \gamma) = \int_{\gamma} f[\varphi(\zeta)]h(\zeta)d\zeta$$

*defines an analytic function on the complex manifold $A$ of the transversal contours. Moreover $G(\varphi, \gamma)$ vanishes if $(\varphi, \gamma)$ satisfies condition* (P)

PROOF. The first part is a rewording of th. 2.1 in the case that $h$ does not depend on $\zeta_0$. The second part follows immediately from the vanishing of $G$ on the open set $U$ mentioned in *property*-(P) and the fact that (P)-contours are open-closed in $A$ □

We are now able to prove a kind of Morera's theorem for CR functions. Let the disc $\varphi: D \to \mathbf{C}^n$ be transversal to $S$ and $\varphi^{-1}S$ compact.

Consider the integral

$$\text{Res}_{\varphi} f = \frac{1}{2\pi i} \int_{\varphi^{-1}S} f \circ \varphi \, d\zeta$$

as a function of $\varphi$ defined on an open subset of $B$.

THEOREM 2.3. *For each* CR *function $f$ on $S$* $\text{Res}_{\varphi} f$ *depends holomorphically on $\varphi$.*

*Conversely if the Levi form of $S$ does not vanish in any open region then all $C^1$ functions $f$ on $S$ such that* $\text{Res}_{\varphi} f$ *depends holomorphically on $\varphi$ are* CR *functions.*

REMARK 2.4. Of course th. 2.3 can be restated as the equivalence between the property of $f$ of being CR and the vanishing of the integral

$$\int_{\gamma} f \circ \varphi d\zeta$$

for all (P)-contours $(\varphi, \gamma)$.

Note that next proof holds also for such a statement.

This recalls better the classical set up of Morera's theorem. Notice also that the (P) assumption replaces the requirement in the classical Morera's theorem that $\gamma$ must be homotopic to a point.

Also the hypothesis $f \in C^1$ can be easily relaxed to continuity.

The hypothesis on the Levi form depends on the impossibility to inspect Levi flat hypersurfaces with analytic discs. See remark 0.1.

PROOF OF TH. 2.3. The first part is a trivial particular case of th. 2.2 for $h = 1$.

So we must only prove that, for each $z \in S$, there is

$$\sum_{j=1}^{n} f_{\bar{z}_j}(z)\,\bar{v}_j = 0 \quad \text{if} \quad \sum_{j=1}^{n} r_{z_j}(z)\,v_j = 0 . \tag{1}$$

We can assume that the Levi form of $S$ is nonzero at $z$ and since the complex tangent space (whose equation is $(1)_2$) is generated by the vectors $v$ for which

$$\sum_{jk=1}^{n} r_{z_j\bar{z}_k}(z)\,v_j\bar{v}_k \neq 0$$

it is sufficient to prove $(1)_1$ for such a $v$.

By a straightforward changement of coordinates and possibly of the sign of $r$, we may write

$$z = (0, \dots, 0)\,, \quad v = (1, 0, \dots, 0)\,, \quad r = |z_1|^2 - x_n + g(z_1, \dots, z_{n-1}, y_n)$$

with $g(z_1, 0, \dots, 0) = 0(|z_1|^3)$.

Now the contours $(\varphi_t, \gamma_t)$ with

$$\varphi_t(\zeta) = (\zeta, 0, \dots, 0, t)\,, \quad \gamma_t \equiv \{r[\varphi_t(\zeta)] \equiv |\zeta|^2 - t + O(|\zeta|^3) = 0\}$$

are transversal for $0 < t \ll 1$ because $|r_\zeta[\varphi_t(\zeta)]| > |t|^{\frac{1}{2}} + O(|t|)$ on $\gamma_t$.

Moreover $(\varphi_t, \gamma_t)$ collapses for $t = 0$ hence $(\varphi_t, \gamma_t)$ satisfies (P) for $0 < t \ll 1$ by remark 2.2.

So by theorem 2.2 we have that $\mathrm{Res}_{\varphi_t} f$ vanishes for small $t \geq 0$.

Applying the Green formula we obtain

$$\frac{1}{t} \int\limits_{|\zeta| + O(|\zeta|^{3/2}) < t^{1/2}} f_{\bar{z}_j}(\zeta, 0, \dots, 0, t)\, d\xi\, d\eta = 0\,, \qquad \text{for } 0 < t \ll 1$$

with $\zeta = \xi + i\eta$.

Going to the limit for $t \downarrow 0$ we obtain $f_{\bar{z}_1}(0) = 0$ which is $(1)_1$ for our particular $v$. □

Finally we give a result which will be useful in next section.

THEOREM 2.4. *Let $(\varphi, \gamma)$ be a (P)-contour and let be $f$ a CR function on $S$. Then $f \circ \varphi$ extends holomorphically to the interior of $\gamma$ in $D$.*

PROOF. Take $h(\zeta)=\zeta^k$, $k=0,1,\dots$ in theorem 2.2 and have

$$\int_\gamma f[\varphi(\zeta)]\zeta^k d\zeta = 0\,.$$

The theorem follows from the density of the $\zeta$-polynomials in the space of the functions which are holomorphic in a neighbourhood of the closure of the domain in $D$ bounded by $\gamma$ (classical Runge theorem).

## 3. – Cauchy formula.

If we look at the (P)-property in section 0 we see immediately that the contour $(\varphi_1, \gamma_1)$ mentioned there has the property that CR functions extend holomorphically to a neighbourhood of $\varphi_1(\bar{\Delta}_1)$ ($\Delta_1$ is the interior of $\gamma_1$).

This follows from the hypothesis on $(\varphi_1, \gamma_1)$ and the maximum principle.

This property could be no longer satisfied by the $P$-contour $(\varphi_0, \gamma_0)$.

We want to show in this section that the continuation of CR functions is also possible if $\varphi_0$ is a proper map when restricted to $\Delta_0$. *Notice that* $\varphi_0$ *is not supposed to be injective.*

This will be performed by the Cauchy formula.

Set $h(\zeta, \zeta_0) = [2\pi i(\zeta-\zeta_0)]^{-1}$ in theorem 2.1 so that the integral (2.5) becomes

$$C(\varphi, \gamma, \zeta_0) = (2\pi i)^{-1}\int_\gamma f[\varphi(\zeta)](\zeta-\zeta_0)^{-1}d\zeta$$

with $\zeta_0 \in \Delta$: the interior of $\gamma$.

$C$ gives the extension of $f\circ\varphi$ mentioned in theorem 2.4.

We want to find out to which extent this theorem yields an extension of $f$.

Next theorem in fact says that, at least locally (i.e. for $\varphi$ near $\varphi_0$), $C(\varphi, \gamma, \zeta_0)$ depends only on $\varphi(\zeta_0)$ when $(\varphi, \gamma)$ is a (P)-contour.

Consider the evaluation map $v: B\times D \to \mathbf{C}^n$ given by $v(\varphi, \zeta) = \varphi(\zeta)$. The differential $d_0 v$ of $v$ at $(\varphi_0, \zeta_0)$ is $d_0 v(\varphi_1, \zeta_1) = \varphi_1(\zeta_0) + \zeta_1\varphi_0'(\zeta_0)$.

Let $d_0 C$ be the differential of $C$ at $(\varphi_0, \gamma_0, \zeta_0)$.

THEOREM 3.1. *If* $(\varphi_0, \gamma_0)$ *is a* (P)*-contour then there is*

$$d_0 C(\varphi_1, \zeta_1) = (2\pi i)^{-1}\sum_{j=1}^n \int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\, d_0 v_j(\varphi_1, \zeta_1)\, d\zeta/(\zeta-\zeta_0)\,.$$

*in particular there is*

$$dC\wedge dv = 0$$

*on the* (P)*-components of* $A$.

PROOF. Considering the extension in the interior of $\gamma_0$ of the function $f\circ\varphi_0$ given by th. 2.4 and comparing two expressions of its derivative, we have

$$\int_{\gamma_0} f[\varphi_0(\zeta)]\,d\zeta/(\zeta-\zeta_0)^2=\sum_{j=1}^{n}\int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\varphi'_{0j}(\zeta)\,d\zeta/(\zeta-\zeta_0)\,.$$

Combining with (2.6) we obtain

$$\begin{aligned}2\pi i\,d_0C(\varphi_1,\zeta_1)&=\sum_{j=1}^{n}\int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\varphi_{1j}(\zeta)\,d\zeta/(\zeta-\zeta_0)+\zeta_1\int_{\gamma_0} f[\varphi_0(\zeta)]\,d\zeta/(\zeta-\zeta_0)^2\\&=\sum_{j=1}^{n}\int_{\gamma_0} f_{z_j}[\varphi_0(\zeta)]\,d_0v_j(\varphi_1,\zeta_1)\,d\zeta/(\zeta-\zeta_0)\,.\qquad\square\end{aligned}$$

An immediate consequence is the

COROLLARY 3.1. *Let $\varphi_t$ be a smooth family of analytic discs, $0\leqslant t\leqslant 1$, which are transversal to $S$ along the component $\gamma_t$ of $\varphi_t^{-1}S$ which depends continuously on $t$ and let $\varphi_t(\zeta_t)=z$ be a fixed point of $\mathbf{C}^n$ with $\zeta_t$ in the interior of the contour $\gamma_t$. If some of the $(\varphi_t,\gamma_t)$ (and hence all) is a* (P)*-contour then the value assigned to $f$ at $z$ by the Cauchy formula, on the disc $\varphi_t$ is independent of $t$.*

Next extension theorem will be proved combining this corollary and the following simple

LEMMA 3.1. *Let $\varphi$ be a disc transversal to $S$ along the component $\gamma$ of $\varphi^{-1}S$, $\Delta$ the interior of $\gamma$.*

*Assume that $\varphi$ is proper when restricted to $\Delta$, i.e.*

$$\varphi\gamma\cap\varphi\Delta=\emptyset\,. \tag{3.1}$$

*Let $\delta>0$ be small enough so that $\gamma(\psi)$ is defined according to prop.* 1.2, *when*

$$\|\psi-\varphi\|<\delta\,.$$

*There exists $\varepsilon=\varepsilon(\varphi,\delta)>0$ such that when*

$$z=\varphi(\zeta_0)+w_0=\varphi(\zeta_1)+w_1$$

*is verified, with $\zeta_0,\zeta_1\in\Delta$ and $|w_0|,|w_1|<\varepsilon$, then there exists a smooth family $\varphi_t$, $0\leqslant t\leqslant 1$, of transversal discs and a smooth curve $\zeta(t)$ in $D$ such that*

(i) $\|\varphi-\varphi_t\|<\delta$

(ii) $\varphi_0(\zeta)=\varphi(\zeta)+w_0,\quad \varphi_1(\zeta)=\varphi(\zeta)+w_1$

(iii) $\zeta(t) \in \Delta(\varphi_t)$

(iv) $\varphi_t[\zeta(t)] = z$ .

Notice that it might be impossible to impose $\zeta(0) = \zeta_0$, $\zeta(1) = \zeta_1$ as further conditions.

PROOF. Notice first that from (3.1) follows that for each neighbourhood $U$ of $\varphi_{|\Delta}^{-1}\varphi(\zeta_0)$ there is a neighbourhood $V$ of $\varphi(\zeta_0)$ such that $\varphi_{|\Delta}^{-1}V \subset U$.

Set $\varphi_{|\Delta}^{-1}\varphi(\zeta_0) = \{\zeta_0, Z_1, \ldots, Z_k\}$. $\delta$ can be assumed so small that there are discs $D_0, D_1, \ldots, D_k \subset\subset \Delta$ centered at $\zeta_0, Z_1, \ldots, Z_k$ such that when $\|\psi - \varphi\| < \delta$ then $D_0 \cup D_1 \cup \ldots \cup D_k \subset\subset \Delta(\psi)$.

Those discs can be further shrunk so that

$$(+) \qquad |\varphi(\zeta) - \varphi(\zeta_0)| < \delta/2 , \quad \text{for } \zeta \in D_0 \cup \ldots \cup D_k .$$

Take $U = D_0 \cup \ldots \cup D_k$ in the above argument and $\{z \in \mathbf{C}^n, |z - \varphi(\zeta_0)| < \delta_1\}$ for the corresponding $V$ with $\delta_1 > 0$.

Set finally $\varepsilon = \min(\delta_1/2, \delta/2)$. Since $|\varphi(\zeta_1) - \varphi(\zeta_0)| = |w_0 - w_1| < 2\varepsilon \leqslant \delta_1$, the point $\zeta_1$ is in one of the discs, say $D_h$, so that $\zeta(t) = (1-t)Z_h + t\zeta_1$ is also in $D_h$. (Notice now that in general $\zeta(0) = Z_h \neq \zeta_0$).

Set finally $\varphi_t(\zeta) = \varphi(\zeta) - \varphi[\zeta(t)] + \varphi(\zeta_0) + w_0$. Now (ii) and (iv) are trivially verified. Moreover, by (+),

$$\|\varphi_t - \varphi\| \leqslant |\varphi[\zeta(t)] - \varphi(\zeta_0)| + |w_0| < \delta/2 + \delta/2 = \delta ,$$

which proves (i), and (iii) follows from $\zeta(t) \subset D_h$ and from the fact that $D_h$ is in $\Delta(\varphi_t)$ because of (i). □

We can prove now an extension result for CR functions.

THEOREM 3.2. *Let the analytic disc $\varphi$ be transversal to $S$ along the compact component $\gamma$ of $\varphi^{-1}S$, let $\Delta$ be the interior of $\gamma$ in $D$. Assume that $(\varphi, \gamma)$ satisfies* (P) *and that $\varphi$ is proper on $\Delta$ (i.e.* (3.1)*).*

*Then there exists a neighbourhood $N$ of $\varphi(\Delta)$ in $\mathbf{C}^n$ such that $\partial N \cap S$ is open in $S$ and non empty, with the property that for each* CR *function $f$ on $S$ there is a continuous function $\hat{f}$ on $N \cup (\partial N \cap S)$, holomorphic in $N$, equal to $f$ in $\partial N \cap S$.*

PROOF. Set $N = \{z \in \mathbf{C}^n | z = \varphi(\zeta) + w, \zeta \in \Delta(\varphi + w), |w| < \varepsilon\}$, with $\varepsilon$ suitably small, positive. From prop. 1.2 and transversality of $\varphi$ we have easily that $\{z \in \mathbf{C}^n | z = \varphi(\zeta) + w, \zeta \in \gamma(\varphi + w)\}$ is equal to $\partial N \cap S$ and is open is $S$. Combining lemma 3.1 and cor. 3.1 (with $\zeta(t) = \zeta_t$) it turns out that the holomorphic extension of $f \circ (\varphi + w)$ from $\gamma(\varphi + w)$ in its interior

$\Delta(\varphi + w)$ defines a function $\hat{f}$ in $N \cup (\partial N \cap S)$ which is continuous and equals $f$ in $\partial N \cup S$.

We prove now that $\hat{f}$ is holomorphic in $N$ by a standard argument. From $\partial f[\varphi(\zeta) + w]/\partial\bar{\zeta} = 0$ we have the $\zeta$-holomorphicity of $\partial f[\varphi(\zeta) + w]/\partial\overline{w}_j = f_{\bar{z}_j}[\varphi(\zeta) + w]$ on $\Delta(\varphi + w)$, but this last function vanishes at the boundary $\gamma(\varphi + w)$ of $\Delta(\varphi + w)$ by (2.3) and hence vanishes on $\Delta(\varphi + w)$. In other words $f_{\bar{z}_j}$ vanishes identically in $N$, $j = 1, \ldots, n$. □

REMARK 2.1. The hypersurface $S$ can have a piece inside $N$. On that pieces our extension $\hat{f}$ can differ from the given function $f$.

## REFERENCES

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