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Bi-Extensions Associated to Divisors on Abelian Varieties and Theta Functions.

MAURIZIO CANDILERA - VALENTINO CRISTANTE (*)

Introduction.

The comparison of various theories about theta functions is a hard subject because of the different techniques used by various authors. For this reason we give here a method which, in addition to giving new theta functions, allows us to unify the construction techniques of the principal existing theories.

The methods which we will use in this paper are completely algebraic; however they are inspired by the classical theory of theta functions in the following way:

Let A be an abelian variety over the field \mathbf{C} of complex numbers; we will denote by V_A the universal covering space of A , by $\pi_A: V_A \rightarrow A$ the canonical projection and by \mathcal{M}_A the algebra of meromorphic functions on V_A of the form f/g where f and g are entire functions on V_A and g isn't identically zero.

Now π_A induces an embedding of the group $\mathbf{C}(A)^*$ of the rational functions on A different from zero into the multiplicative group \mathcal{M}_A^* of \mathcal{M}_A . From this map, passing to the quotient $\mathbf{C}(A)^*/\mathbf{C}^*$, we get a homomorphism:

$$j_A: \mathbf{C}(A)^*/\mathbf{C}^* \rightarrow \mathfrak{G}_A$$

where $\mathfrak{G}_A = \mathcal{M}_A^*/Q_A$, and Q_A is the subgroup of \mathcal{M}_A^* formed by the non-zero multiples of second degree characters (quadratic exponentials) from V_A to \mathbf{C}^* . The group $\mathbf{C}(A)^*/\mathbf{C}^*$ is isomorphic to the group of principal divisors on A hence; to assert the existence in \mathcal{M}_A^* of theta functions associated to

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the divisors of A is the same as saying that j_A extends uniquely to the group \mathcal{D}_A of divisors on A , and that this extension \tilde{j}_A is functorial with respect to A .

Working over $A \times A \times A$ we may give a more explicit description of \tilde{j} . We denote by $p_i: A \times A \times A \rightarrow A$ ($i = 1, 2, 3$) the i -th projection, and, if $f: B \rightarrow A$ is a morphism, we denote by f^* the corresponding morphism from \mathcal{M}_A to \mathcal{M}_B . Now, if X is in \mathcal{D}_A , the divisor

$$Y = (p_1 + p_2 + p_3)^{-1}X - (p_1 + p_2)^{-1}X - (p_1 + p_3)^{-1}X - (p_2 + p_3)^{-1}X + p_1^{-1}X + p_2^{-1}X + p_3^{-1}X$$

of $A \times A \times A$ is principal. So that, if g is a representative in \mathcal{M}_A^* of the coset $\tilde{j}_A X$, from the functoriality of \tilde{j} we find that the element

$$(*) \quad F = \frac{(p_1 + p_2 + p_3)^*g(p_1^*g)(p_2^*g)(p_3^*g)}{(p_1 + p_2)^*g(p_1 + p_3)^*g((p_2 + p_3)^*g)}$$

of $\mathcal{M}_{A \times A \times A}$ is an equation of Y . It's clear that the coset of F in $j_{A \times A \times A}(\mathbf{C}(A \times A \times A)^*/\mathbf{C}^*)$ depends only on $\tilde{j}_A X$ and not on the representative g . On the other hand, if we fix an equation F of Y , the equation $(*)$ determines g up to a second degree character; hence, in order to construct \tilde{j}_A , it is sufficient to give an element of \mathcal{M}_A^*/Q_A which satisfies $(*)$ for each divisor X .

This viewpoint suggests the following problem: given an algebraically closed field k , and an abelian variety A over k , find a $k(A)$ -algebra \mathcal{C}_A , functorial with respect to A , such that for each X in \mathcal{D}_A the equation $(*)$ has solution.

We remark here that we are in fact interested only in a multiplicative subgroup of \mathcal{C}_A where we can find the solutions of $(*)$ as X varies in \mathcal{D}_A .

If the characteristic of k is equal to zero, one may choose as \mathcal{C}_A the field of quotients of the completion of the local ring of the identity point of A . This is done in [1]. If k has positive characteristic, the previous field is, in general, too small, as one can see in [2] and also in [6]. In this case, one may use the field of quotients of the completion of the perfect closure of the local ring at the identity point of A . In the last two cited papers the ground field is supposed to be perfect, and this suffices only if the previous completion doesn't have multiplicative components. In chapter two we make up for this error by showing that, if k is algebraically closed, there exist always solutions of $(*)$ in \mathcal{C}_A .

The case of positive characteristic is also studied in [4], where is shown that, if A is ordinary, one may choose for \mathcal{C}_A a suitable algebra over the affine algebra of the étale component of Barsotti-Tate group of A .

This paper, after the first section, which has a general character, is divided in two parts: the first one, which regards the theta functions on the Barsotti-Tate group and consists of sections 2, 3, 6 and 8; the second one composed by sections 4, 5, 7 and the appendix, which regards the theory of theta functions on the Tate space.

These two parts are substantially independent one from the other, so that the order in the sequence of the sections is motivated only by the hope of making the reading easier. For instance, in section 3 we reconsider the étale case, but in chapter 4, before considering the theta functions on the global Barsotti-Tate group of A , we begin to construct the ℓ -adic theory, with ℓ different from p (= characteristic of k). This is done for two reasons:

i) the direct comparison between the two situations seems to us really interesting. In fact, in the étale case, as already shown in [4], if X is a positive divisor, $(*)$ has solutions which are functions on the group of p -torsion points. While, in general, we don't have solutions which are functions on the group G of ℓ -torsion points. In order to find solutions we must change from G to the corresponding Tate space $V = \varprojlim (G \xleftarrow{\ell} G \xleftarrow{\ell} \dots)$.

ii) the computations which, in section 6, give the theta functions on the global Barsotti-Tate group, will be incomparably clarified by the analogous computation used in the ℓ -adic situation of sections 4 and 5.

In general, our methods in constructing solutions of $(*)$ are elementary: that is we don't use structure theorems, but, starting out from F , we construct directly a solution of $(*)$. In this way one sees easily the automorphy factors of solutions and, by an examination of these, one sees that, if X is positive and totally symmetric, there exists a unique solution g of $(*)$ such that $g(p) = g(-p)$ for p in V . Such a solution, if $\ell = 2$, is the function $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of [11], which Mumford calls the Riemann theta function.

This relation between the solutions of $(*)$ and the theory of Mumford theta functions is the answer to a question which Barsotti raised in 1978 during the Journées de Géométrie Algébrique of Rennes (cfr. [2]). In the appendix, after an elementary construction of differential nature of solutions for $(*)$ in the case of characteristic zero, we give a short comparison between ℓ -adic theta functions and classical ones. And probably, it shouldn't be difficult to find also the relations between our theta functions and the ones of Tate and Morikawa, but we will leave this for another time.

Before finishing this introduction is a pleasure for us to thank F. Baldassarri, I. Barsotti, L. Breen and F. Sullivan for their useful suggestions and stimulating conversations.

1. – Bi-extensions.

Let A be an abelian variety over the algebraically closed field k , and let C_A be a $k(A)$ -algebra which solves the problem stated on the introduction. In order to look for the solutions in C_A of the functional equation (*), we must investigate the structure of F with more accuracy.

In our study the notion of bi-extension of groups will be very useful. This concept was introduced by D. Mumford in 1968 for studying the deformations of polarized Barsotti-Tate groups, with the aim of lifting abelian varieties to characteristic zero [10].

Here we treat particular bi-extensions, therefore we will use an *ad hoc* definition. The reader interested in a general theory of bi-extensions, should read the paper of A. Grothendieck [8].

Some notation: we denote by $k(x)$ the field of the rational functions on A and by x_i the element p_i^*x of $k(x) \otimes \dots \otimes k(x)$. For instance, if p_1 and p_2 are respectively the first and the second projection of $A \times A$ onto A , we will write $k(x_1, x_2)$ for $k(A \times A)$. Moreover we denote by $f(x_1 + x_2)$ the rational function $(p_1 + p_2)^*f$.

It's a well-known fact that a (commutative) extension of A by the multiplicative group G_m is determined by a (symmetric) rational factor set over $A \times A$ with values in G_m , that is an element $f(x_1, x_2)$ of $k(x_1, x_2)$ subject to the following conditions:

$$f(x_1, x_2) = f(x_2, x_1)$$

$$f(x_1 + x_2, x_3)f(x_1, x_2) = f(x_1, x_2 + x_3)f(x_2, x_3) .$$

Now we consider the subfield $k(x_3)$ of $k(x_1, x_2, x_3)$; we denote by $A_{k(x_3)}$ the abelian variety over $k(x_3)$ which is obtained by extension of the ground field. To any commutative extension of $A_{k(x_3)}$ by G_m , there corresponds an element $g(x_1, x_2, x_3)$ of $k(x_1, x_2, x_3)$. Hence, to each point P of A such that $g(x_1, x_2, P)$ is defined, there corresponds an extension. In this case we'll say that g determines an algebraic family of extensions of A by G_m parametrized by A . If we suppose g symmetric, that is $g(x_1, x_2, x_3) = g(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3})$ for each permutation $\sigma \in \mathfrak{S}_3$, then we can say that the family of extensions \mathcal{F} , determined by g , is a *bi-extension of A by G_m* and g is a *co-cycle of the bi-extension*.

Summing up: an element g of $k(x_1, x_2, x_3)$ is a co-cycle of a bi-extension of A by G_m if and only if

$$(1.1) \quad g(x_1, x_2, x_3) = g(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}), \quad \text{for each } \sigma \in \mathfrak{S}_3;$$

$$(1.2) \quad g(x_1 + x_2, x_3, x_4)g(x_1, x_2, x_4) = g(x_1, x_2 + x_3, x_4)g(x_2, x_3, x_4).$$

The bi-extensions we treat in this paper arise in the way described by the following

(1.3) PROPOSITION. *Let X be a divisor on A ; if we denote by $p_i: A \times A \times A \rightarrow A$ ($i = 1, 2, 3$) the i -th projection and by Y the divisor on $A \times A \times A$*

$$Y = (p_1 + p_2 + p_3)^{-1}X + p_1^{-1}X + p_2^{-1}X + p_3^{-1}X - (p_1 + p_2)^{-1}X - (p_1 + p_3)^{-1}X - (p_2 + p_3)^{-1}X,$$

then Y is principal and each equation of it is a co-cycle of a bi-extension of A by G_m .

PROOF. It's well known that Y is principal (cfr., for instance, p. 91 of [9]). Now, let F be an equation of Y . It is symmetric, because $F(x_1, x_2, x_3)$ and $F(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3})$ are equations of the same divisor and they assume the same values over the points of the diagonal of $A \times A \times A$ in which both are defined. Again it is easy to show that F satisfies the condition (1.2). In fact each side of the equality is an equation of the divisor over $A \times A \times A \times A$:

$$(p_1 + p_2 + p_3 + p_4)^{-1}X - (p_1 + p_2 + p_3)^{-1}X - (p_1 + p_4)^{-1}X - (p_2 + p_4)^{-1}X - (p_3 + p_4)^{-1}X + p_1^{-1}X + p_2^{-1}X + p_3^{-1}X + 2(p_4^{-1}X),$$

so their ratio is a constant. In order to show that it is equal to 1, we observe that there exists a divisor X' linearly equivalent to X whose support doesn't contain the identity point e_A of A . Hence

$$F(x_1, x_2, x_3) = F'(x_1, x_2, x_3) \frac{f(x_1 + x_2 + x_3)f(x_1)f(x_2)f(x_3)}{f(x_1 + x_2)f(x_1 + x_3)f(x_2 + x_3)},$$

where F' is regular on $e_{A \times A \times A}$ and $(f) = X - X'$. From this we deduce that our ratio is $F'(e_A, e_A, e_A)^2 / F'(e_A, e_A, e_A)^2$, Q.E.D.

The bi-extension described in the above proposition is called *associated to X* , and F is said to be a *co-cycle of X* .

Now, if a contravariant functor associating to each abelian variety A $k(A)$ -algebra \mathcal{C}_A is given, and if $f: B \rightarrow A$ is a morphism of abelian varieties, we denote by f^* the corresponding morphism of \mathcal{C}_A to \mathcal{C}_B . In this situation, given a divisor X on A , we say that an element g of \mathcal{C}_A is a *theta of X* , if the element

$$F = \frac{((p_1 + p_2 + p_3)^*g)(p_1^*g)(p_2^*g)(p_3^*g)}{((p_1 + p_2)^*g)((p_1 + p_3)^*g)((p_2 + p_3)^*g)}$$

of $\mathcal{C}_{A \times A \times A}$ is a co-cycle of X . Here we identify $k(x_1, x_2, x_3)$ with its image in $\mathcal{C}_{A \times A \times A}$.

A co-cycle $G(x_1, x_2, x_3) \in k(x_1, x_2, x_3)$ of a bi-extension \mathcal{F} is a *co-boundary* if there exists an element $f(x_1, x_2)$ of $k(x_1, x_2)$ such that

$$(1.4) \quad \begin{cases} f(x_1, x_2) = f(x_2, x_1) \\ \frac{f(x_1 + x_2, x_3)}{f(x_1, x_3)f(x_2, x_3)} = G(x_1, x_2, x_3). \end{cases}$$

Let X be a divisor of A algebraically equivalent to zero; in symbols: $X \equiv 0$. It's well-known that the divisor

$$Z = (p_1 + p_2)^{-1}X - p_1^{-1}X - p_2^{-1}X$$

is principal on $A \times A$. Hence each co-cycle F of X is a co-boundary. We will see later that there is a converse for this proposition.

Finally, if \mathcal{C}_A is an algebra over the affine algebra \mathcal{R}_A of a (formal) group G_A , and if there exists a theta g of X in (the image of) \mathcal{R}_A , then we'll say that *the pull-back of X to G_A is principal* and that g is its equation; or, in brief, that g is a *theta of X over G_A* .

2. - Theta functions on the Barsotti-Tate group; the local case.

In this chapter we are dealing again with the subject matter of [6]. Let k be an algebraically closed field with characteristic $p > 0$. Let A be an abelian variety over k , and R_A the affine algebra of the local component G_t of the Barsotti-Tate group of A . R_A is endowed with its topology as local ring.

Following the notation of the introduction, we choose as \mathcal{C}_A the field $\text{Frac}(\mathcal{R}_A)$, where \mathcal{R}_A is the completion of $\mathcal{R}_A^0 = \varinjlim (R_A \xrightarrow{2^i} R_A \xrightarrow{2^i} \dots)$. Of course $i_A: k(A) \rightarrow \mathcal{C}_A$ is the natural embedding.

Now, if X is a divisor over A , we are looking for a theta of X in \mathcal{C}_A . The answer lies in the following theorem.

(2.1) THEOREM. *Let X be a divisor over A and F a co-cycle of X . Then the equation*

$$(2.2) \quad F = \frac{((p_1 + p_2 + p_3)^*\vartheta)(p_1^*\vartheta)(p_2^*\vartheta)(p_3^*\vartheta)}{((p_1 + p_2)^*\vartheta)((p_1 + p_3)^*\vartheta)((p_2 + p_3)^*\vartheta)}$$

has a solution ϑ in $\text{Frac}(\mathcal{R}_A)$.

Let us establish some conventions about notations: if $(t) = (t_1, \dots, t_g)$ is a regular set of parameters for R_A , that is $R_A = k[[t]]$. Then for each element $f = f(t)$ of \mathcal{C}_A , we denote by $f(t_1 + t_2)$ the element $(p_1 + p_2)^*f$ of $\mathcal{C}_{A \times A}$.

We may reduce the problem to an easier one, by appealing to the following lemma

(2.3) LEMMA. *Let $\varphi(t_1, t_2)$ be an element of $\mathcal{R}[[t_2]]$ such that (with the notation of (2.1))*

$$(2.4) \quad F = \frac{\varphi(t_1, t_2 + t_3)}{\varphi(t_1, t_2)\varphi(t_1, t_3)}.$$

Then the equation (2.2) has a solution in $\text{Frac}(\mathcal{R}_A)$.

PROOF. A geometric proof of the above lemma is contained in [6]; here we'll give an algebraic proof. In this way we'll show explicitly that the existence of solutions for the equation (2.2) depends only on the fact that F is a co-cycle of a bi-extension. First remark: the equation (2.2) has solutions if and only if the analogous equation, obtained by substituting X with a linearly equivalent divisor X' has solutions, hence, we may suppose that e_A doesn't belong to the support of X ; in this case F is invertible in $R \otimes_k R \otimes_k R$ ($R = R_A$) and will suppose also that $F(0, t_2, t_3) = 1$.

It follows from this, that each solution $\varphi(t_1, t_2)$ of (2.4) satisfies the equality $\varphi(t_1, 0) = 1$, and that we may choose it in such a way that $\varphi(0, t_2) = 1$.

If we put

$$(2.5) \quad \mu(t_1, t_2, t_3) = \frac{\varphi(t_1, t_2 + t_3)\varphi(t_2, t_3)}{\varphi(t_1 + t_2, t_3)\varphi(t_1, t_2)},$$

and we remember that F satisfies (1.4); then, from (2.4) and (2.5), we deduce that

$$\mu(t_1, t_2, t_3)\mu(t_1, t_2, t_4) = \mu(t_1, t_2, t_3 + t_4).$$

Therefore μ is a multiplicative element of $(\mathcal{R} \widehat{\otimes} \mathcal{R})[[t_3]]$. But any such element lies in $k[[t_3]]$ because k is algebraically closed. Hence we get

$$\mu = \mu(0, 0, t_3) = \frac{\varphi(0, t_3)\varphi(0, t_3)}{\varphi(0, t_3)\varphi(0, 0)} = 1.$$

From this and (2.5), we deduce that if φ satisfies (2.4), then it also satisfies

$$(2.6) \quad F = \frac{\varphi(t_1 + t_2, t_3)}{\varphi(t_1, t_3)\varphi(t_2, t_3)},$$

and

$$(2.7) \quad \varphi(t_1 + t_2, t_3)\varphi(t_1, t_2) = \varphi(t_1, t_2 + t_3)\varphi(t_2, t_3),$$

Now we set

$$\psi(t_1, t_2) = (\varphi(t_1, t_2)\varphi(t_2, t_1))^{\frac{1}{2}},$$

where $y^{\frac{1}{2}}$ means the binomial series of y (when p is different from 2). Because F is symmetric, by a comparison of (2.4) and (2.6), we get

$$(2.8) \quad F = \frac{\psi(t_1 + t_2, t_3)}{\psi(t_1, t_3)\psi(t_2, t_3)}$$

Hence, if (2.4) has a solution $\varphi \in \mathcal{R}[[t_2]]$, then there exists also a symmetric solution ψ in $\mathcal{R} \otimes \mathcal{R}$. This one satisfies the equation (2.7); therefore it is a symmetric, multiplicative factor set. As a consequence if we denote by G the group $\varprojlim (G_i \xleftarrow{p^i} G_i \xleftarrow{p^i} \dots)$, where G_i is the local component of the Barsotti-Tate group of A , then to ψ there corresponds a commutative extension of G by the multiplicative group G_m . In other words ψ defines a hyperalgebra $\mathcal{E}_A = k[[t, s]]^{1/p^\infty}$, where $\mathcal{R} = k[[t]]^{1/p^\infty}$ and $s_1 + s_2 = (1 + s_1) \cdot (1 + s_2)\psi(t_1, t_2) - 1$. This is a bidomain, in the sense of Barsotti; hence, from $[MA]$, it follows that there exists an element y of \mathcal{E}_A such that $y_1 + y_2 = y_1 + y_2 - y_1y_2$ and $\mathcal{E}_A = k[[t, y]]^{1/p^\infty}$. Now it's clear that, if we put

$$\vartheta = \frac{1 + s}{1 + y},$$

we get that

$$(2.9) \quad \psi(t_1, t_2) = \frac{\vartheta(t_1 + t_2)}{\vartheta(t_1)\vartheta(t_2)}.$$

But any solution ϑ of (2.9) is a solution of (2.2), as follows from (2.8), and this concludes the proof, Q.E.D.

A remark about φ : the equation (2.7) implies that φ as an element of $\mathcal{R} \widehat{\otimes} \mathcal{R}$, is also a multiplicative factor set, in general non-commutative. Hence it determines a non-trivial extension of G by G_m . We call this extension the (local) *Mumford group of X* .

At this point we have reduced the proof of (2.1), to the solution of the equation (2.4).

(2.10) PROPOSITION. *Let $F \in k[[t_1, t_2, t_3]]$ be the co-cycle of a bi-extension such that $F(0, t_2, t_3) = 1$. Then there exists $\varphi(t_1, t_2) \in R[[t_2]]$ which satisfies the equation (2.4).*

We begin the proof with an observation and a lemma. The local component G_t of the Barsotti-Tate group of A splits into the product of G_m^f and G_r where G_m is the multiplicative group, f the separable codimension of A , and G_r is a radical group. In order to use this fact we need the following

(2.11) LEMMA. *Let $F \in k[[t_1, t_2, t_3]]$ be a co-cycle of a bi-extension. If G_t splits into the product of two subgroups G_1 and G_2 , and if we denote by $k[[t^{(1)}, t^{(2)}]]$ the corresponding splitting of the affine algebra $k[[t]]$ of G_t . Then F splits as follows:*

$$F(t_1, t_2, t_3) = F_1(t_1, t_2^{(1)}, t_3^{(1)})F_2(t_1, t_2^{(2)}, t_3^{(2)}) \frac{\sigma(t_1, t_2 + t_3)}{\sigma(t_1, t_2)\sigma(t_1, t_3)},$$

where F_1 (resp. F_2) is the co-cycle of the extension of $(G_1)_R$ (resp. $(G_2)_R$) by G_m^m induced by F , and $\sigma(t_1, t_2)$ is an element of $R[[t_2]]$.

PROOF. Looking at the decomposition of R we may write:

$$F(t_1, t_2, t_3) = F(t_1, t_2^{(1)} + t_2^{(2)}, t_3^{(1)} + t_3^{(2)}).$$

Now F is a co-cycle of an extension of $(G_t)_R = (G_1)_R \times (G_2)_R$ by G_m . Hence, if we put

$$F_1 = F(t_1, t_2^{(1)}, t_3^{(1)}), \quad F_2 = F(t_1, t_2^{(2)}, t_3^{(2)})$$

and $\sigma(t_1, t_2) = F(t_1, t_2^{(1)}, t_2^{(2)})$, by (1.2) we get the result, Q.E.D.

PROOF. (2.10) According to the last lemma we may split this proof in two parts: the first one for $G_t = G_m$ and the second one for $G_t = G_r$.

i) $G_t = G_m$: Because a group of multiplicative type over an algebraically closed field is isomorphic to a product of multiplicative groups, a solution $\varphi(t_1, t_2)$ for the equation (2.4) surely exists in $(\overline{\text{Frac } R})[[t_2]]$. We will prove that such a solution φ belongs to $R[[t_2]]$.

A simple remark shows that each solution $\varphi(t_1, t_2)$ of (2.4) also solves the equation

$$(2.12) \quad \prod_{j=1}^{p-1} F(t_1, t_2, jt_2) = \frac{\varphi(t_1, pt_2)}{\varphi(t_1, t_2)^p}.$$

Moreover, if t_2 is a multiplicative parameter, we may rewrite (2.12) in this way:

$$(2.12)' \quad \prod_{j=1}^{p-1} F(t_1, t_2, jt_2) = \frac{\varphi(t_1, t_2^p)}{\varphi(t_1, t_2)^p}.$$

Now, let φ_1, φ_2 be two solutions of (2.12)'. Then

$$\frac{\varphi_1(t_1, t_2)}{\varphi_2(t_1, t_2)} = \sum_{i=0}^{\infty} a_i t_2^i$$

satisfies the equality:

$$\sum_{i=0}^{\infty} a_i t_2^{pi} = \sum_{i=0}^{\infty} a_i^p t_2^{pi}.$$

Therefore the coefficients a_i are in the Galois field \mathbb{F}_p ; hence if one solution of (2.12)' belongs to $R[[t_2]]$, so do the other ones. And now we are done because we can exhibit a solution of (2.12)' in $R[[t_2]]$. In fact, if we put:

$$\prod_{j=1}^{p-1} F(t_1, t_2, jt_2) = 1 + \sum_{n>0} B_n(t_1) t_2^{pn}$$

and

$$\varphi(t_1, t_2) = \sum_{n \geq 0} X_n(t_1) t_2^n$$

we find that the equation (2.12)' is equivalent to

$$(2.13)_n \quad X_n(t_1)^p + \sum_{i=0}^{n-1} X_i(t_1) B_{n-i}(t_1) = X_n(t_1).$$

If we solve (2.13) in $k[[t_1]] = R$, for $m < n$, then equation (2.13)_n is an Artin-Schreier equation with coefficients in $k[[t_1]]$ and hence with solution in $k[[t_1]]$. One can see easily that $X_0 = 1$ is a solution for the (2.13)₀ and this concludes the first part of the proof.

ii) $G_t = G_r$. A complete proof for this situation is given in [6]. Here we give another proof in which we construct a solution of (2.4) as a limit of algebraic functions (over \mathcal{A}).

If $\varphi(t_1, t_2)$ is a solution of (2.4) in $(\text{Frac } R)[[t_2]]$, then by an easy computation, we get the following equations

$$(2.14) \quad \prod_{j=1}^{p^r-1} F(t_1, t_2, jt_2) = \frac{\varphi(t_1, p^r t_2)}{\varphi(t_1, t_2)^{p^r}},$$

for each integer $r \geq 0$. Now, if we denote by V the *Verschiebung* (shift) of R , for each x in the augmentation ideal R^+ of R , the sequence $(V^r x)_{r \in \mathbb{N}}$ converges to 0 in the local topology of R . As a consequence, if we observe that

$$\varphi(t_1, t_2) \equiv 1 \pmod{(\text{Frac } R)[[t_2]]^+},$$

we have that

$$\varphi(t_1, p^r t_2)^{p^{-r}} \equiv 1 \pmod{V^r(\text{Frac } R)[[t_2]]^+},$$

therefore we conclude that

$$\varphi(t_1, t_2) = \lim_{r \rightarrow \infty} \prod_{j=1}^{p^r-1} F(t_1, t_2, j t_2)^{-p^{-r}}$$

in $\mathcal{R}[[t_2]]$, Q.E.D.

3. – Theta functions on the Barsotti-Tate group of A ; the étale case.

In this section we are dealing with the subject matter of [4]. So we'll show how to construct the theta functions on the étale component $G_{\text{ét}}$ of the Barsotti-Tate group of A . We limit ourselves to the case of an ordinary abelian variety, this meaning that the local component G_l of the Barsotti-Tate group of A is of multiplicative type, and also that the set of points of $G_{\text{ét}}$ (which we also call $G_{\text{ét}}$) is a dense subgroup of A in the Zariski topology. This density is of capital importance in order to reconstruct the divisor from its theta.

In the general case, that is when A isn't ordinary, we must use the group $G_{\text{ét}} \times G_r$ in place of $G_{\text{ét}}$ and use arguments similar to lemma (2.11) for the construction of the theta: this is done in section 6. Now we are interested in the difference between theta functions on the group of p -torsion on points and theta functions on the group of the l -torsion points, where l is a prime integer different from the characteristic of the ground field k .

According to the introduction we must fix a $k(A)$ -algebra $i_A: k(A) \rightarrow \mathbb{C}_A$ where the theta functions will live. In order to do this we consider $G_{\text{ét}}$ endowed with the topology induced by the Zariski topology of A and we denote by \mathcal{F}_A the set of the k -valued functions defined on (some) open set of $G_{\text{ét}}$. Now we put on \mathcal{F}_A an equivalence relation \sim by decreeing $f_1 \sim f_2$ if and only if they coincide on an open set contained in the intersection of the domains of f_1 and f_2 . Because of the density of $G_{\text{ét}}$ in A , $G_{\text{ét}}$ is an irreducible space and hence \sim is really an equivalence relation over \mathcal{F}_A , and the quotient set \mathcal{F}_A/\sim acquires a natural structure of $k(A)$ -algebra. This is \mathbb{C}_A .

Another way of defining it, is the following: let $C(U)$ be the set of k -valued functions defined on the open set U . $C(U)$ is a ring, and a straightforward verification shows that

$$C_A = \varinjlim C(U),$$

as U varies in the class of the Zariski open-sets of $G_{\text{ét}}$.

Really we will only deal with the multiplicative group \mathcal{U}_A of the invertible elements of C_A ; namely the elements representable by a function different from zero on some open set. Clearly C_A depends functorially on A . In fact, if $\varphi: B \rightarrow A$ is a morphism of abelian varieties and \bar{f} is an element of C_A represented by a function f of $C(U)$, then $\varphi^*(\bar{f})$ is the element of C_B represented by $f \circ \varphi$ on $\varphi^{-1}U$.

Now we begin our construction; first we consider a positive divisor X on A .

(3.1) PROPOSITION. *Let X and F be respectively a positive divisor of A and one of its co-cycles. Then:*

i) *For each positive integer n , the element*

$$(3.2) \quad \sigma_n(x) = \prod_{1 \leq i, j \leq p^n - 1} F(x, ix, jix)$$

of $k(x)$ is regular on each point of the subgroup $G_{\text{ét}}(n)$ of $G_{\text{ét}}$, where $G_{\text{ét}}(n) = \{P \in G_{\text{ét}} \mid p^n P = e_A\}$.

ii) *If we put $\Theta_n(P) = \sigma_n(P)^{1/p^{2n}}$ for P in $G_{\text{ét}}(n)$, we get $\Theta_n(P) = \Theta_m(P)$ for each P in $G_{\text{ét}}(\inf(n, m))$. From this it follows that there exists a unique function $\Theta: G_{\text{ét}} \rightarrow k$ such that the restriction of Θ to $G_{\text{ét}}(n)$ is equal to Θ_n for each n .*

iii) *If we denote by $\bar{\Theta}$ the element of C_A determined by Θ , then*

$$(3.3) \quad \bar{F} = \frac{((p_1 + p_2 + p_3)^* \bar{\Theta})(p_1^* \bar{\Theta})(p_2^* \bar{\Theta})(p_3^* \bar{\Theta})}{((p_1 + p_2)^* \bar{\Theta})((p_1 + p_3)^* \bar{\Theta})((p_2 + p_3)^* \bar{\Theta})},$$

where \bar{F} is the image of F in $C_{A \times A \times A}$:

PROOF. Let X' be a divisor linearly equivalent to X and whose support doesn't meet $G_{\text{ét}}(n)$. Therefore $X = X' + (f)$, where f is regular on the points of $G_{\text{ét}}(n)$. Let F_X and $F_{X'}$ be respectively co-cycles of X and X' such that

$$F_X = F_{X'} \frac{((p_1 + p_2 + p_3)^* f)((p_1^* f)(p_2^* f)(p_3^* f))}{((p_1 + p_2)^* f)((p_1 + p_3)^* f)((p_2 + p_3)^* f)}.$$

If we define σ'_n using $F_{X'}$ in (3.2), then we obtain

$$\sigma_n(P) = \sigma'_n(P) f^{p^n}(P),$$

for each point P of $G_{\text{ét}}(n)$ where both terms are defined.

Now, because $F_{X'}$ is regular and invertible over $G_{\text{ét}}(n)^3$, the same holds also for σ'_n over $G_{\text{ét}}(n)$, so that i) follows from the last formula. Using these arguments in the proof of ii), we may suppose that $\text{supp } X$ doesn't meet $G_{\text{ét}}(n)$ for some n much greater than 0. In this situation F defines a map $F: G_{\text{ét}}(n)^3 \rightarrow k^*$; and we assume F such that $F(e_A, e_A, e_A) = 1$; hence, as consequence of (1.2), it follows that $F(e_A, P, Q) = 1$ for each P, Q in $G_{\text{ét}}(n)$.

If we fix P in $G_{\text{ét}}(n)$, the map

$$F_P: G_{\text{ét}}(n)^2 \rightarrow k^* \\ (Q, R) \mapsto F(P, Q, R)$$

is a symmetric factor set and hence it defines an extension of $G_{\text{ét}}(n)$ by k^* . Any such extension is trivial, so there is a map $\varphi_n(P, -): G_{\text{ét}}(n) \rightarrow k^*$ such that:

$$(3.4) \quad F(P, Q, R) = \frac{\varphi_n(P, Q + R)}{\varphi_n(P, Q)\varphi_n(P, R)},$$

where $P, Q, R \in G_{\text{ét}}(n)$. This map is unique because there's no non-trivial homomorphism from $G_{\text{ét}}(n)$ to k^* . Moreover we can calculate this map directly from (3.4). In fact, using (3.4) recursively, we get

$$(3.5) \quad \varphi_n(P, rQ) = \varphi_n(P, Q)^r \prod_{j=1}^{r-1} F(P, Q, jQ),$$

for each integer $r \geq 0$. But, $p^n Q = e_A$ for each Q in $G_{\text{ét}}(n)$ and $\varphi_n(P, e_A) = 1$, therefore we obtain for each Q in $G_{\text{ét}}(n)$,

$$(3.6) \quad \varphi_n(P, Q) = \prod_{j=1}^{p^n-1} F(P, Q, jQ)^{-1/p^n}.$$

We will show that φ_n also satisfies the following

$$(3.7) \quad F(P, Q, R) = \frac{\varphi_n(P + Q, R)}{\varphi_n(P, R)\varphi_n(Q, R)}.$$

Using (3.6) and (1.2), we get

$$\begin{aligned} \frac{\varphi_n(P + Q, R)}{\varphi_n(P, R)\varphi_n(Q, R)} &= \prod_{j=1}^{p^n-1} \left(\frac{F(P + Q, R, jR)}{F(P, R, jR)F(Q, R, jR)} \right)^{-1/p^n} \\ &= \prod_{j=1}^{p^n-1} \left(\frac{F(P, Q + R, jR)F(Q, R, jR)}{F(P, Q, jR)F(P, R, jR)F(Q, R, jR)} \right)^{-1/p^n} \\ &= \prod_{j=1}^{p^n-1} \left(\frac{F(P, Q, (j + 1)R)F(P, R, jR)}{F(P, Q, R)F(P, Q, jR)F(P, R, jR)} \right)^{-1/p^n} = F(P, Q, R). \end{aligned}$$

By a comparison of (3.4) with (3.7), it follows that for any P , the map

$$Q \mapsto \frac{\varphi_n(P, Q)}{\varphi_n(Q, P)}$$

is a homomorphism from $G_{\text{ét}}(n)$ to k^* ; therefore $\varphi_n(P, Q) = \varphi_n(Q, P)$ for each P, Q in $G_{\text{ét}}(n)$. Moreover φ_n is a multiplicative symmetric factor set, as we can see equating (3.4) and (3.7). Therefore, reasoning as above, we get a unique $\Theta_n: G_{\text{ét}}(n) \rightarrow k^*$ such that

$$(3.8) \quad \varphi_n(P, Q) = \frac{\Theta_n(P + Q)}{\Theta_n(P)\Theta_n(Q)},$$

this is given by the formula

$$(3.9) \quad \Theta_n(P) = \prod_{j=1}^{p^n-1} \varphi_n(P, jP)^{-1/p^n} = \prod_{1 \leq i, j \leq p^n-1} F(P, iP, jP)^{1/p^{2n}};$$

this concludes part ii). From (3.7) and (3.8) we get

$$(3.10) \quad F_n(P, Q, R) = \frac{\Theta_n(P + Q + R)\Theta_n(P)\Theta_n(Q)\Theta_n(R)}{\Theta_n(P + Q)\Theta_n(P + R)\Theta_n(Q + R)},$$

for each (P, Q, R) in

$$U_n = G_{\text{ét}}(n)^3 \setminus \left(G_{\text{ét}}(n)^3 \cap \text{supp} \left((p_1 + p_2)^{-1}X + (p_1 + p_3)^{-1}X + (p_2 + p_3)^{-1}X \right) \right)$$

where F_n is the map induced by F on U_n . U_n is an open set, hence this means that $\bar{\Theta}$ satisfies (3.3) and this gives iii), and concludes the proof.

(3.11) REMARK. *The last arguments of the preceding proof show that Θ is the unique element of $\bar{\Theta}$ defined at all points of $G_{\text{ét}}$ and whose restriction to $G_{\text{ét}}(n)$ satisfies (3.10) for each integer n . We mean this element whenever we speak of a theta of X defined over all $G_{\text{ét}}$.*

Before we state the next result, we make another observation: it follows from the previous construction that Θ is equal to 0 only on the points of $G_{\text{ét}} \cap \text{supp } X$; hence $\bar{\Theta}$ belongs to the multiplicative group \mathcal{U}_A of invertible elements of \mathcal{C}_A .

(3.12) THEOREM. *We denote by $\bar{\Theta}_X$ the element of \mathcal{U}_A/k^* determined by Θ , as defined in (3.1). Then the map $X \mapsto \bar{\Theta}_X$, defined on the semigroup of positive divisors, extends in a unique way to a homomorphism*

$$\tilde{j}_A: \mathcal{D}_A \rightarrow \mathcal{U}_A/k^*,$$

where we denote by \mathcal{D}_A the group of divisors of A . Moreover, the restriction j_A of \tilde{j}_A to the subgroup of principal divisors of A coincides with the natural embedding of $k(A)/k^*$ into \mathcal{U}_A/k^* .

PROOF. Looking at the proof (3.1), one sees that the map $X \mapsto \bar{\Theta}_X$, defined on the semigroup of positive divisors, is a homomorphism. Now if X is a divisor of A (not necessarily positive), we may write $X = X_0 - X_1$, where X_0 and X_1 are positive divisors. Then we put $\bar{\Theta}_X = \bar{\Theta}_{X_0} \bar{\Theta}_{X_1}^{-1}$. Hence the map $X \rightarrow \bar{\Theta}_X$ gives the unique extension of the preceding homomorphism to a homomorphism of the group \mathcal{D}_A .

Now let X be a principal divisor and f one of its equations. By (3.2) one sees that we may choose two functions $\Theta_{X_0}, \Theta_{X_1}$ such that $f(P) = \Theta_{X_0}(P)\Theta_{X_1}(P)^{-1}$ for each P on a suitable open set U of $G_{\text{ét}}$. This shows that f and $\Theta_{X_0}\Theta_{X_1}^{-1}$ define the same element of \mathcal{U}_A ; this means that j_A is the natural embedding, Q.E.D.

Another remark: Let X be a divisor of A , and $X = X_0 - X_1$ where X_0 and X_1 do not have common prime components. When we speak about a theta of X on $G_{\text{ét}} \setminus (G_{\text{ét}} \cap \text{supp } X_1)$, we refer to the function defined by $P \mapsto \Theta_{X_0}(P)\Theta_{X_1}(P)^{-1}$, where Θ_{X_0} and Θ_{X_1} are theta functions defined over all $G_{\text{ét}}$ (cfr. (3.11)). This function is clearly determined up to a multiplicative constant.

Now we show how one may obtain information about the polarization associated to X directly from a theta of X .

First of all we describe a natural action of $G_{\text{ét}}$ on \mathcal{U}_A (and therefore on \mathcal{U}_A/k^*): let c be an element of \mathcal{U}_A determined by a function η defined on $G_{\text{ét}}$; if P is a point of $G_{\text{ét}}$ we put $c_P = \tilde{\eta}_P$, where η_P is the function on $G_{\text{ét}}$ defined by $Q \rightarrow \eta(P + Q)$. The map from \mathcal{U}_A to \mathcal{U}_A which sends c to c_P is clearly bijective ($c \rightarrow c_{-P}$ is its inverse). We state some properties of this action in the following lemma.

(3.13) LEMMA. *Let c be an element of \mathcal{U}_A/k^* we denote by G and H respectively, the subgroups of \mathcal{U}_A/k^* spanned by the sets $I_G = \{c_P c_Q^{-1} \mid P, Q \in G_{\text{ét}}\}$ and $I_H = \{cc_{P+Q} c_P^{-1} c_Q^{-1} \mid P, Q \in G_{\text{ét}}\}$, and by $[cc_P^{-1}]$ the coset determined by cc_P^{-1} in G/H . Then the map*

$$\varphi_c: G_{\text{ét}} \rightarrow G/H,$$

defined by $P \mapsto [cc_P^{-1}]$, is a surjective homomorphism.

PROOF. Let P, Q be points of $G_{\text{ét}}$; then one has

$$(cc_P^{-1})(cc_Q^{-1}) = (cc_{P+Q} c_P^{-1} c_Q^{-1})(cc_{P+Q}^{-1}),$$

which means $[cc_P^{-1}][cc_Q^{-1}] = [cc_{P+Q}^{-1}]$. Hence φ_c is a homomorphism. Moreover

$$\prod_{i=1}^r c_{P_i} c_{Q_i}^{-1} \equiv cc_{\sum_{i=1}^r (P_i - Q_i)}^{-1} \pmod{H},$$

and this states the surjectivity of φ_c , Q.E.D.

If the above element c is determined by $\bar{\mathcal{O}}_X$, where X is a divisor of A , then from the comparison of G and H we obtain information about X . For instance, if we denote by G' and H' the pre-image of G and H in \mathcal{U}_A , we can prove the following proposition:

(3.14) PROPOSITION. *Let X be a divisor on A and G, H, G', H' defined as above. Then X is algebraically equivalent to 0 if and only if G' is contained in $i_A(k(A))$ and the subfields $k(G'), k(H')$ of \mathcal{C}_A , spanned by G' and H' over $i_A k$, coincide.*

PROOF. As H' is contained in $i_A k(A)$, if $k(G') = k(H')$ it is clear that $X - X_P$ is principal for each P in $G_{\text{ét}}$. Hence, from the density of $G_{\text{ét}}$ in A , it follows that $X - X_P$ is principal for each P of A ; therefore $X \equiv 0$.

Conversely, let X be algebraically equivalent to 0. Then G' is contained in $i_A k(A)$ and hence one has the following inclusions:

$$k \subseteq k(H') \subseteq k(G') \subseteq i_A k(A).$$

From this and from the fact that $k(A)$ is finitely generated over k , we deduce that $k(G')$ is finitely generated over $K(H')$. Moreover, from (3.13), we get that $p^n P = e_A$ implies that $(\bar{\mathcal{O}}/\bar{\mathcal{O}}_P)^{p^n}$ belongs to H . Hence there exists a positive integer m such that $k(G')^{p^m} \subseteq k(H)$. But $G_{\text{ét}}$ is a p -divisible group, therefore for each P in $G_{\text{ét}}$ there exists Q such that $P = p^m Q$, and this implies that $(\bar{\mathcal{O}}/\bar{\mathcal{O}}_P) = (\bar{\mathcal{O}}/\bar{\mathcal{O}}_Q)^{p^m} h$ for some h in H' , Q.E.D.

The group G/H is also important in the construction of the Riemann form associated to X . This is our next goal.

We denote by V_A the p -adic Tate space of A , that is:

$$V_A = \{(P_0, P_1, \dots) : P_i \in G_{\text{ét}}, pP_{i+1} = P_i, i = 0, 1, 2, \dots\},$$

and by T_A the sub- \mathbb{Z}_p -module of V_A defined by

$$T_A = \{(P_0, P_1, \dots) \mid (P_0, P_1, \dots) \in V_A \text{ and } P_0 = e_A\}.$$

Now, if X is a divisor of A , we denote by $\check{V}_{A,X}$ and $\check{T}_{A,X}$ respectively, the vector space

$$\check{V}_{A,X} = \{([\bar{\theta}/\bar{\theta}_{P_0}], [\bar{\theta}/\bar{\theta}_{P_1}], \dots) \mid (P_0, P_1, \dots) \in V_A\}$$

and the corresponding sub- \mathbb{Z}_p -module. Because each element $[\bar{\theta}/\bar{\theta}_P]$ of G/H may be read as a point of the étale group of the dual variety \tilde{A} , the map $V_A \rightarrow \check{V}_{A,X}$, defined by $(P_0, P_1, \dots) \rightarrow ([\bar{\theta}/\bar{\theta}_{P_0}], [\bar{\theta}/\bar{\theta}_{P_1}], \dots)$ determines a linear map $\lambda_X: V_A \rightarrow V_{\tilde{A}}$. The homomorphism λ_X depends only on the class of X with respect to algebraic equivalence, hence the map $X \rightarrow \lambda_X$ induces a homomorphism from the Severi group of A to the (additive) group $\text{Hom}(V_A, V_{\tilde{A}})$.

Instead of giving a direct verification of these fact, we link λ_X to the Riemann form φ_X introduced in [MA]. Hence, the properties of λ_X will be consequences of the analogous properties of φ_X . This is also a more economical procedure; in fact the following arguments are essentially contained in [MA].

Let \mathcal{R}_A and \mathcal{R}_A be as in section 2; an element m in \mathcal{R}_A is *multiplicative* if $(p_1 + p_2)^*m = (p_1^*m)(p_2^*m)$. The set N such elements is a \mathbb{Q}_p -module. Now we construct a homomorphism from $\check{V}_{A,X}$ to N .

First we remark that if $y = ([\bar{\theta}/\bar{\theta}_{P_0}], [\bar{\theta}/\bar{\theta}_{P_1}], \dots)$ is an element of $\check{V}_{A,X}$, for each j there exists a representative y_j of $[\bar{\theta}/\bar{\theta}_{P_j}]$ in \mathcal{U}_A subject to the following conditions: $y_j = \bar{\theta}_{Q_j}/\bar{\theta}_{P_j+Q_j}$, where θ_{Q_j} and $\theta_{P_j+Q_j}$ are both defined in a neighborhood of e_A and they coincide at e_A .

The two elements

$$(3.15) \quad y_{j+1}^p y_j^{-1} = (\bar{\theta}\bar{\theta}_{P_{j+1}}^{-1})^p (\bar{\theta}\bar{\theta}_{P_j}^{-1})^{-1}$$

and

$$(\bar{\theta}\bar{\theta}_{P_{j+1}}^{-1})^p (\bar{\theta}\bar{\theta}_{P_j}^{-1}) = \prod_{i=1}^{p-1} (\bar{\theta}\bar{\theta}_{(i+1)P_j}) (\bar{\theta}_{P_{j+1}} \bar{\theta}_{iP_{j+1}})^{-1}$$

are both contained in the image by i_A of $k(A)^*$ in \mathcal{U}_A . Hence, from $\Theta_{\mathcal{Q}_j}(e_A) = \Theta_{P_j+Q_j}(e_A)$, one has

$$(3.16) \quad y_j^p y_j^{-1} \equiv 1 \pmod{i_A \mathcal{O}_{e_A}^+},$$

and therefore

$$(3.17) \quad y_{j+1}^{p^{j+1}} y_j^{-p^j} \equiv 1 \pmod{i_A \mathcal{O}_{e_A}^{+p^j}}.$$

Now, because of the periodicity of P , there exists a positive integer $r = r(y)$ such that $y_j^{p^{j+rr}}$ belongs $i_A \mathcal{O}_{e_A}$, for each j . As a consequence (3.17) implies that the sequence $(y_j^{p^j})_{j \in \mathbb{N}}$ converges in $i_A \mathcal{R}_A^0$, and because its limit depends only on y , we put

$$\varrho(y) = \lim_{j \rightarrow \infty} y_j^{p^j}.$$

Now, because $\Theta_{\mathcal{Q}_j} \Theta_{P_j+Q_j}^{-1}$ is a theta of a divisor algebraically equivalent to zero, one has

$$((p_1 + p_2)^* y_j)(p_1^* y_j)^{-1}(p_2 y_j)^{-1} \in i_{A \times A} k(A \times A);$$

hence $\lim_{j \rightarrow \infty} ((p_1 + p_2)^* y_j)^{p^j} (p_1^* y_j)^{-p^j} (p_2^* y_j)^{-p^j} = 1$, and this, in view of the continuity of $(p_1 + p_2)^*$, p_1^* , p_2^* , implies

$$(p_1 + p_2)^* \varrho(y) = (p_1^* \varrho(y))(p_2^* \varrho(y)).$$

That is, $\varrho(y)$ is a multiplicative element in \mathcal{R}_A . This implies that $\varrho(y) \equiv 1 \pmod{\mathcal{R}_A^{0+}}$, hence (cfr. [MA]). $\log \{\varrho(y)\}$ exists, and it is a canonical bivector. In this way, for each divisor X , we have a map $(P_0, P_1, \dots) \rightarrow \log \{\varrho(y)\}$ from V_A to the canonical module M_i of canonical bivectors with components in \mathcal{R}_A .

Now, looking at the results of [MA], which show how V_A is canonically isomorphic to the module $\tilde{M}_{\text{ét}}$, dual of the module of canonical bivectors with components in $\mathcal{R}_{\text{ét}}$, we obtain an homomorphism $\varphi_X: \tilde{M}_{\text{ét}} \rightarrow M_i$; this is the map φ_X^π of [MA].

Because A is an ordinary abelian variety, φ_X is sufficient to determine the class of X with respect to algebraic equivalence.

(3.18) PROPOSITION. *The correspondence that associates the homomorphism φ_X to the divisor X of A , defines a homomorphism*

$$\varphi: \mathcal{D}_A \rightarrow \text{Hom}(\tilde{M}_{\text{ét}}, M_i).$$

The kernel of φ is the subgroup of the divisors algebraically equivalent to zero.

PROOF. The first statement is a consequence of (3.12) and of the construction of ϱ . The proof of the second statement is similar to the one given in [MA]. If $X \equiv 0$, then for each P in $G_{\text{ét}}$, $\overline{\Theta}\overline{\Theta}_P^{-1}$ is an element of $i_A k(A)$, so the y_i 's which appear in the construction of ϱ are all in $i_A k(A)$. As a consequence, $\varinjlim_{i \rightarrow \infty} y_i^{p^i} = 1$, therefore $\log \{\varrho(y)\} = 0$. Conversely, $\log \{\varrho(y)\} = 0$ implies $\varinjlim_{i \rightarrow \infty} y_i^{p^i} = 1$, so

$$y_i^{p^i} = \lim_{r \rightarrow \infty} \prod_{j=0}^r y_{i+j}^{p^{i+j}} y_{i+j+1}^{-p^{i+j+1}}.$$

From this and from (3.17), since for some integer $s > 0$, $y^{p^{i+s}}$ is in \mathcal{O}_{e_A} , we deduce that $y_i^{p^{i+s}}$ is in $\mathcal{O}_{e_A} \cap R^{p^{i+s}}$; hence $y_i \in k(A)$. But each y_i is in $k(A)$, and this means that $X - X_P$ is principal for each P in $G_{\text{ét}}$ and finally this implies $X \equiv 0$ (cfr. (3.16)), Q.E.D.

4. - Theta functions on the Tate space.

In this section and in the next one the ground field k is assumed to be of characteristic p , and l is a rational prime different from p . We denote by G the group of points of A such that $l^n P = e_A$ for some non-negative integer n ; and we denote by $G(i)$ the kernel of the endomorphism $P \mapsto l^i P$ of G .

Let X be a divisor on A whose support doesn't meet the points of $G(2n)$. We denote by F a co-cycle of X normalized by the condition $F(e_A, P, Q) = 1$ for each P, Q in $G(2n)$. F is regular and non-zero at each point of $G(2n)^3$, hence we may define a function on $G(2n) \times G(2n)$ with values in k^* , by putting

$$(4.1) \quad \psi_{2n}(P, Q) = \prod_{j=1}^{l^n-1} \frac{F(l^n Q, P, jP)}{F(P, l^n Q, jl^n Q)}.$$

As one can see from the following proposition, the function ψ_{2n} is a first step in the construction of our theta functions.

(4.2) PROPOSITION. *For each $(P, Q, R) \in G(2n)^3$ one has*

$$(4.3) \quad \frac{\psi_{2n}(P, Q + R)}{\psi_{2n}(P, Q) \psi_{2n}(P, R)} = F(l^n P, l^n Q, l^n R) = \frac{\psi_{2n}(P + Q, R)}{\psi_{2n}(P, R) \psi_{2n}(Q, R)}.$$

Moreover, the map defined by

$$\chi_{2n}(P, Q) = \frac{\psi_{2n}(P, Q)}{\psi_{2n}(Q, P)},$$

is a bi-homomorphism from $G(2n)$ to the multiplicative group k^* .

PROOF. In the proof of the first equality of (4.3), we use only the properties (1.1) and (1.2) of F , *i.e.* its co-cycle properties.

$$\begin{aligned}
 & \frac{\psi_{2n}(P, Q + R)}{\psi_{2n}(P, Q) \psi_{2n}(P, R)} \\
 &= \prod_{j=1}^{2n-1} \frac{F(\iota^n Q + \iota^n R, P, jP) F(P, \iota^n Q, j\iota^n Q) F(P, \iota^n R, j\iota^n R)}{F(P, \iota^n Q + \iota^n R, j\iota^n Q + j\iota^n R) F(\iota^n Q, P, jP) F(\iota^n R, P, jP)} \\
 &= \prod_{j=1}^{2n-1} \frac{F(\iota^n Q, \iota^n R + P, jP) F(\iota^n R, P, jP) F(P, \iota^n Q, j\iota^n Q) F(P, \iota^n R, j\iota^n R)}{F(\iota^n Q, \iota^n R, jP) F(P, \iota^n Q, j\iota^n Q + (j+1)\iota^n R) F(P, \iota^n R, j\iota^n Q + j\iota^n R)} \\
 & \qquad \qquad \qquad \cdot \frac{F(P, \iota^n Q, \iota^n R)}{F(\iota^n Q, P, jP) F(\iota^n R, P, jP)} \\
 &= \prod_{j=1}^{2n-1} \frac{F(\iota^n Q, \iota^n R, (j+1)P) F(\iota^n Q, P, jP) F(P, \iota^n Q, j\iota^n Q) F(P, \iota^n R, j\iota^n R)}{F(\iota^n Q, \iota^n R, P) F(\iota^n Q, \iota^n R, jP) F(P, (j+1)\iota^n Q, (j+1)\iota^n R)} \\
 & \qquad \qquad \qquad \cdot \frac{F(P, \iota^n Q, \iota^n R) F(P, j\iota^n Q, (j+1)\iota^n R) F(P, j\iota^n Q, j\iota^n R)}{F(P, \iota^n Q, j\iota^n Q) F(P, (j+1)\iota^n R, j\iota^n Q) F(P, \iota^n R, j\iota^n R) F(\iota^n Q, P, jP)} \\
 &= \prod_{j=1}^{2n-1} \frac{F(\iota^n Q, \iota^n R, (j+1)P) F(P, j\iota^n Q, j\iota^n R)}{F(\iota^n Q, \iota^n R, jP) F(P, (j+1)\iota^n Q, (j+1)\iota^n R)} \\
 & \qquad \qquad \qquad = \frac{F(\iota^n Q, \iota^n R, \iota^n P) F(P, \iota^n Q, \iota^n R)}{F(\iota^n Q, \iota^n R, P) F(P, \iota^n Q, \iota^n R)} = F(\iota^n Q, \iota^n R, \iota^n P).
 \end{aligned}$$

The proof of the second equality is a perfect analogous and is left to the reader.

At this point, from the symmetry of F and from (4.3), one deduces that

$$\frac{\psi_{2n}(P, Q + R)}{\psi_{2n}(P, Q) \psi_{2n}(P, R)} = \frac{\psi_{2n}(Q + R, P)}{\psi_{2n}(Q, P) \psi_{2n}(R, P)}.$$

Finally, from this and from the skew-symmetry of χ_{2n} , we can conclude that

$$(4.4) \quad \frac{\chi_{2n}(P, Q + R)}{\chi_{2n}(P, Q) \chi_{2n}(P, R)} = 1 = \frac{\chi_{2n}(Q + R, P)}{\chi_{2n}(Q, P) \chi_{2n}(R, P)}, \quad \text{Q.E.D.}$$

From (4.3) we deduce that

$$(4.5) \quad \psi_{2n}(P, Q + R) \psi_{2n}(Q, R) = \psi_{2n}(P + Q, R) \psi_{2n}(P, Q),$$

which shows that ψ_{2n} is a *non-commutative* (in general) factor set from $G(2n)^2$ to k^* . In this way one has a non-commutative extension of $G(2n)$

by k^* , which we denote by $\mathfrak{G}(2n)$ and which we call the *Mumford Group of level $2n$ of X* (see also p. 444). Later on we'll give a condition for the commutativity of the Mumford group.

In the rest of this section we suppose $l \neq 2$. The case $l = 2$ presents some diversity, therefore it will be treated separately in the next section. With this hypothesis, the map $P \mapsto 2P$ is an isomorphism of $G(2n)$ and hence the application of $G(2n)^2$ in k^* defined by

$$(P, Q) \mapsto \chi_{2n}(1/2P, Q)$$

is a bi-homomorphism.

If we add to this (4.2), we have the following

(4.6) PROPOSITION. *The equation (4.3) has a symmetric solution; that is, there exists a function*

$$\varphi_{2n}: G(2n) \times G(2n) \rightarrow k^*$$

subject to the conditions:

- i) $\varphi_{2n}(P, Q) = \varphi_{2n}(Q, P)$
- ii) $\frac{\varphi_{2n}(P, Q + R)}{\varphi_{2n}(P, Q)\varphi_{2n}(P, R)} = F(l^n P, l^n Q, l^n R)$
- iii) $\varphi_{2n}(P, Q + R)\varphi_{2n}(Q, R) = \varphi_{2n}(P + Q, R)\varphi_{2n}(P, Q)$

PROOF. We define:

$$\varphi_{2n}(P, Q) = \psi_{2n}(P, Q)\chi_{2n}(1/2Q, P);$$

then, ii) follows from (4.3), and (4.4); iii) follows from (4.5); and i) is an easy consequence of the relations $\chi_{2n}(1/2P, Q) = \chi_{2n}(1/2P, 1/2Q)^2 = \chi_{2n}(P, 1/2Q)$, Q.E.D.

From this we deduce the following

(4.7) COROLLARY. *There exist functions*

$$\sigma_{2n}: G(2n) \rightarrow k^*$$

such that

$$(4.8) \quad F(l^n P, l^n Q, l^n R) = \frac{\sigma_{2n}(P + Q + R)\sigma_{2n}(P)\sigma_{2n}(Q)\sigma_{2n}(R)}{\sigma_{2n}(P + Q)\sigma_{2n}(P + R)\sigma_{2n}(Q + R)}$$

for each P, Q, R in $G(2n)$.

PROOF. According to ii) of (4.6) it is sufficient to choose a function σ_{2n} such that

$$(4.9) \quad \frac{\sigma_{2n}(P + Q)}{\sigma_{2n}(P)\sigma_{2n}(Q)} = \varphi_{2n}(P, Q).$$

The existence of functions of this kind depends only on the fact that φ_{2n} is a symmetric factor set, which is associated to an extension of $G(2n)$ by k^* , and that any such extension is trivial, Q.E.D.

We state here a remark, that will be useful later

REMARK. Any function $\sigma_{2n}: G(2n) \rightarrow k^*$ which satisfies (4.9) is completely determined by its values on a set of generators $P_1, P_2, \dots, P_{2\sigma}$ of $G(2n)$, and these values must satisfy the equations

$$(4.10) \quad \sigma_{2n}(P_i)^{i^{2n}} = \left[\prod_{j=1}^{i^{2n}-1} \varphi_{2n}(P_i, jP_i) \right]^{-1}.$$

In fact, from (4.9), we get

$$(4.11) \quad \sigma_{2n}(mP) = \sigma_{2n}(P)^m \prod_{j=1}^{m-1} \varphi_{2n}(P, jP),$$

which shows the necessity of (4.10). Then by (4.11) and (4.9) we can extend any function defined on $\{P_1, \dots, P_{2\sigma}\}$ in a unique way to $G(2n)$. Finally one may verify directly the sufficiency of (4.10).

In order to use the σ_{2n} in the construction of theta functions, we must compare σ^{2n} with σ_{2m} when $m > n$. In this comparison will be useful the following

(4.12) PROPOSITION. Let F be a normalized co-cycle of a divisor X , whose support doesn't meet $G(2n + 2r)$. Then if P and Q belong to $G(2n + r)$, with the notation of (4.1), one has

$$\psi_{2(n+r)}(P, Q) = \psi_{2n}(t^r P, t^r Q).$$

PROOF. From (4.1) we get

$$\psi_{2(n+r)}(P, Q) = \prod_{j=1}^{i^{n+r}-1} \frac{F(t^{n+r}Q, P, jP)}{F(P, t^{n+r}Q, jt^{n+r}Q)}.$$

Hence we must prove the relation

$$(4.13) \quad \prod_{j=1}^{i^{n+r}-1} \frac{F(t^{n+r}Q, P, jP)}{F(P, t^{n+r}Q, jt^{n+r}Q)} = \prod_{j=1}^{i^n-1} \frac{F(t^{n+r}Q, t^r P, jt^r P)}{F(t^r P, t^{n+r}Q, jt^{n+r}Q)}.$$

In the course of the proof of (4.13), we will use repeatedly the following relations:

$$\begin{aligned}
 \text{i)} \quad & F(P, mQ, mR) = F(P, Q, R)^m \prod_{j=1}^{m-1} \frac{F(P, Q + R, j(Q + R))}{F(P, Q, jQ)F(P, R, jR)} \\
 \text{ii)} \quad & F(mP, Q, R) = F(P, Q, R)^m \prod_{j=1}^{m-1} \frac{F(P, jP, Q + R)}{F(P, jP, Q)F(P, jP, R)},
 \end{aligned}$$

where P, Q and R are in $G(2n + 2r)$ and m is a positive integer. Here we prove only i); the proof of ii) is similar and is left to the reader.

Using only the co-cycle properties of F , we get

$$\begin{aligned}
 \prod_{j=1}^{m-1} \frac{F(P, Q + R, jQ + jR)}{F(P, Q, jQ)F(P, R, jR)} &= \prod_{j=1}^{m-1} \frac{F(P, Q, jQ + (j + 1)R)F(P, R, jQ + jR)}{F(P, Q, R)F(P, Q, jQ)F(P, R, jR)} \\
 &= \prod_{j=1}^{m-1} \frac{F(P, (j + 1)Q, (j + 1)R)F(P, Q, jQ)F(P, (j + 1)R, jQ)F(P, R, jR)}{F(P, jQ, (j + 1)R)F(P, jQ, jR)F(P, Q, R)F(P, Q, jQ)F(P, R, jR)} \\
 &= \frac{F(P, mQ, mR)}{F(P, Q, R)^m};
 \end{aligned}$$

this proves i). Now we begin the proof of (4.13). We put $R = t^{n+r}Q$ and we state some equalities. At the end we'll give an explanation of each one. The right side of (4.13) is equal to:

$$\begin{aligned}
 \text{a)} \quad & \prod_{j=1}^{t^n-1} \frac{F(R, t^r P, jt^r P)}{F(t^r P, R, jR)} = \\
 \text{b)} \quad & \prod_{j=1}^{t^n-1} \frac{F(R, P, jP)^{t^r}}{F(P, R, jR)^{t^r}} \\
 & \cdot \prod_{j=1}^{t^n-1} \prod_{i=1}^{t^r-1} \frac{F(R, (j + 1)P, i(j + 1)P)F(P, iP, R)F(P, iP, jR)}{F(R, P, iP)F(R, jP, iP)F(P, iP, (j + 1)R)} = \\
 \text{c)} \quad & \prod_{j=1}^{t^n-1} \frac{F(R, P, jP)^{t^r}}{F(P, R, jR)^{t^r}} \prod_{i=1}^{t^r-1} \frac{F(R, t^n P, it^n P)F(P, iP, R)}{F(R, P, iP)F(P, iP, t^n R)} = \\
 \text{d)} \quad & \prod_{j=1}^{t^n-1} \frac{F(R, P, jP)^{t^r}}{F(P, R, jR)^{t^r}} \prod_{i=1}^{t^r-1} F(R, t^n P, it^n P) = \\
 \text{e)} \quad & \prod_{j=1}^{t^n-1} \frac{F(R, P, jP)^{t^r}}{F(P, R, jR)^{t^r}} \prod_{i=1}^{t^r-1} F(R, P, it^n P)^{t^n} \prod_{i=1}^{t^r-1} \prod_{j=1}^{t^n-1} \frac{F(P, jP, R + it^n P)}{F(P, jP, R)F(P, jP, it^n P)} = \\
 \text{f)} \quad & \frac{\prod_{j=1}^{t^n-1} F(R, P, jP)^{t^r} \prod_{i=1}^{t^r-1} F(R, P, it^n P)^{t^n}}{\prod_{j=1}^{t^n-1} F(P, R, jR)^{t^r} \prod_{j=1}^{t^n-1} F(P, jP, R)^{t^r-1}} \\
 & \cdot \prod_{i=1}^{t^r-1} \prod_{j=1}^{t^n-1} \frac{F(P, (it^n + j)P, R)F(P, jP, it^n P)}{F(P, it^n P, R)F(P, jP, it^n P)} =
 \end{aligned}$$

$$\begin{aligned}
 g) \quad & \frac{\prod_{j=1}^{t^n-1} F(R, P, jP) \prod_{i=1}^{t-1} F(R, P, i t^n P)^{t^n} \prod_{i=1}^{t-1} \prod_{j=1}^{t^n-1} F(R, P, (i t^n + j)P)}{\prod_{j=1}^{t^n-1} F(P, R, jR)^{t^r} \prod_{i=1}^{t-1} F(P, R, i t^n P)^{t^n-1}} = \\
 h) \quad & \prod_{j=1}^{t^{n+r}-1} \frac{F(R, P, jP)}{F(P, R, jR)},
 \end{aligned}$$

and h) is the left side of (4.13). Now, we explain the steps one by one. We go from a) to b) by applying i) to the numerator and ii) to the denominator; from b) we obtain c) by developing the product, and then we arrive at d) if we recall that $t^n R = e_A$ and that F is normalized. Now we obtain e) by applying ii) to the second factor of d). From e) we arrive at f) and then g) by the simple remark that $F(P, jP, R)$ doesn't depend on i and that $F(P, R, i t^n P)$ doesn't depend on j . Finally we obtain h) because

$$\begin{aligned}
 & \{1, 2, \dots, t^n - 1\} \cup \{t^n, 2t^n, \dots, (t-1)t^n\} \\
 & \cup \{t^n + 1, t^n + 2, \dots, (t-1)t^n + 1, \dots, t^{n+r} - 1\} = \{1, \dots, t^{n+r} - 1\}
 \end{aligned}$$

and $sR = (t^n + s)R$ for any positive integer s .

This concludes the proof.

Now we can compare σ_{2n} with σ_{2m} . In fact we have the following

(4.14) COROLLARY. *Let F be a normalized co-cycle of X , where X is a divisor whose support doesn't meet $G(2(n+r))$. Then if σ_{2n} satisfies (4.8) we may choose $\sigma_{2(n+r)}$ such that*

$$(4.15) \quad \sigma_{2(n+r)}(P) = \sigma_{2n}(tP)$$

for each P in $G(2n+r)$.

PROOF. Because of the definitions in (4.2) and (4.6), it follows from (4.12) that

$$\chi_{2n}(tP, tQ) = \chi_{2(n+r)}(P, Q),$$

and so

$$(4.16) \quad \varphi_{2n}(tP, tQ) = \varphi_{2(n+r)}(P, Q),$$

for each P, Q in $G(2n+r)$.

If we recall the remark after the corollary (4.7), we see that $\sigma_{2(n+r)}$ is determined by its values at $\{P_1, \dots, P_{2\theta}\}$, and these values must satisfy the

following conditions:

$$(4.17) \quad \sigma_{2(n+r)}(P_i)^{\ell^{2(n+r)}} = \prod_{j=1}^{\ell^{2(n+r)}-1} \varphi_{2(n+r)}(P_i, jP_i)^{-1}.$$

It follows that, if we fix the function $\sigma_{2(n+r)}$ by means of its values $\sigma_{2(n+r)}(P_i)$, any other solutions of (4.17) is obtained by multiplying $\sigma_{2(n+r)}(P_i)$ by a $\ell^{2(n+r)}$ -th root of unity.

Now, from (4.16) we deduce that

$$(4.18) \quad \frac{\sigma_{2n}(\ell P + \ell Q)}{\sigma_{2n}(\ell P)\sigma_{2n}(\ell Q)} = \frac{\sigma_{2(n+r)}(P + Q)}{\sigma_{2(n+r)}(P)\sigma_{2(n+r)}(Q)}$$

for each P, Q in $G(2n + r)$; thus the map which sends P to $\sigma_{2n}(\ell P)/\sigma_{2(n+r)}(P)$ is a homomorphism from $G(2n + r)$ to k^* .

Hence $\zeta_i = \sigma_{2n}(\ell P_i)/\sigma_{2(n+r)}(P_i)$ is a ℓ^{2n+r} -th root of 1 in k^* . Now, if we fix ξ_1, \dots, ξ_{2g} , $\ell^{2(n+r)}$ -th roots of 1, such that $\xi_i^{\ell^r} = \zeta_i$, then the function $\sigma'_{2(n+r)}$ defined by putting

$$\sigma'_{2(n+r)}(P_i) = \xi_i \sigma_{2(n+r)}(P_i),$$

satisfies the following equalities:

$$(4.19) \quad \sigma_{2n}(\ell^{2r} P_i) = \sigma'_{2(n+r)}(\ell^r P_i), \quad i = 1, \dots, 2g.$$

In fact, from (4.11), it follows that $\sigma'_{2(n+r)}(\ell^r P_i) = \zeta_i \sigma_{2(n+r)}(\ell^r P_i)$, that is $\sigma_{2n}(\ell^{2r} P_i)/\sigma'_{2(n+r)}(\ell^r P_i) = 1$.

Finally, because $\{\ell P_1, \dots, \ell P_{2g}\}$ spans $G(2n + r)$, and $\sigma'_{2(n+r)}$ satisfies (4.18), (4.19), it also satisfies (4.15), Q.E.D.

At this point we are able to construct the theta functions associated to a divisor X . We begin with a positive divisor X .

First of all we fix some notations: we denote by $V = V_A$ the ℓ -adic Tate space of A :

$$V = \{p = (P_0, P_1, \dots) \mid P_i \in G, \ell P_i = P_{i-1}, (i = 1, 2, \dots)\},$$

i.e. $V = \varprojlim (G \xleftarrow{\ell} G \xleftarrow{\ell} \dots)$. We denote by π_A the natural projection of V in G defined by $p \rightarrow P_0$; and then we denote by A its kernel. Finally, if f is a function defined on a subset U of G , we denote by $\pi_A^* f$ the map defined on $\pi_A^{-1} U$ by putting $(\pi_A^* f)(p) = f(\pi_A p)$.

(4.20) THEOREM. *Let X be a positive divisor on A and F one of its co-cycles. Then, there exists a k -valued function $\Theta = \Theta_x$, defined at all points of V , such that*

$$(4.21) \quad F(p, q, r) = \frac{\Theta(p + q + r)\Theta(p)\Theta(q)\Theta(r)}{\Theta(p + q)\Theta(p + r)\Theta(q + r)}$$

for each (p, q, r) of $V \times V \times V$, at which F is defined. The divisor X determines Θ_x up to a factor aq where $a \in k^*$ and $q: V \rightarrow k^*$ is a second degree character of V .

We remark here that F in (4.21) is in fact $\pi_{A \times A \times A}^* F$ and it is defined at the points of $V^3 \setminus \pi_{A \times A \times A}^{-1}((G \times G \times G) \cap \text{supp}((p_1 + p_2)^{-1}X + (p_1 + p_3)^{-1}X + (p_2 + p_3)^{-1}X))$

PROOF. Let m be a positive integer much greater than 0. Then, let X' be a divisor linearly equivalent to X , whose support doesn't meet $G(m)$. Thus one has $X = X' + (f)$, where f is an element of $k(A)$, regular at the points of $G(m)$ and determined up to a multiplicative constant.

We denote by F' the normalized co-cycle of X' and we choose f in such a way that

$$(4.22) \quad F(x_1, x_2, x_3) = F'(x_1, x_2, x_3) \frac{f(x_1 + x_2 + x_3)f(x_1)f(x_2)f(x_3)}{f(x_1 + x_2)f(x_1 + x_3)f(x_2 + x_3)}.$$

If $2n \leq m$, from (4.22) it follows that we obtain a solution of (4.8) by setting $\sigma_{2n}(P) = \sigma'_{2n}(P)f(\ell^n P)$, where σ'_{2n} is linked to F' by (4.8). In other words, for each (P, Q, R) in $G(2n)^3$ at which F' is defined, we have

$$(4.23) \quad F(\ell^n P, \ell^n Q, \ell^n R) = \frac{\sigma_{2n}(P + Q + R)\sigma_{2n}(P)\sigma_{2n}(Q)\sigma_{2n}(R)}{\sigma_{2n}(P + Q)\sigma_{2n}(P + R)\sigma_{2n}(Q + R)}.$$

If $2(n + r)$ is also less than m and if we choose $\sigma'_{2(r+n)}$ which satisfies (4.15), then we have

$$\sigma_{2(n+r)}(P) = \sigma'_{2(n+r)}(P)f(\ell^{n+r} P) = \sigma'_{2n}(\ell^r P)f(\ell^n(\ell^r P)) = \sigma_{2n}(\ell^r P),$$

for each P in $G(2n + r)$.

It's an easy remark to see that σ_{2n} doesn't depend on the choice of the divisor X' used in its construction. In fact if we use another divisor X'' with the same properties as X' and $X = X'' + (f')$, then from the fact that $X' - X''$ doesn't meet $G(2n)$ it follows that we may choose σ''_{2n} such that

$$\sigma'_{2n}(P)f(\ell^n P) = \sigma''_{2n}(P)f'(\ell^n P).$$

Thus we may also choose σ_{2n} which satisfies (4.23) and (4.15). If we have made this choice, then for each $p = (P_0, P_1, \dots)$ in V we put

$$(4.24) \quad \Theta(P) = \sigma_{2n}(P_n),$$

where $P_0 = \pi p$ belongs to $G(n)$. In this way Θ is well defined. In fact, if P_0 is in $G(n)$, then for each positive integer in r , P_{n+r} belongs to $G(2n + r)$, and hence, from (4.15), we obtain

$$\sigma_{2(n+r)}(P_{n+r}) = \sigma_{2n}(P_n),$$

as desired. This proves the first statement of (4.20). In order to prove the second, we observe that X determines F up to a multiplicative constant and that F determines σ_{2n} up to a second degree character (quadratic exponential) $q_n: G(2n) \rightarrow k^*$. By using (4.15) we build up from these q_n 's a second degree character from V to k^* , Q.E.D.

Any k -valued function, defined on V , which solves (4.21) is said to be a *theta function of X* .

In order to extend the constructions of a theta to any divisor (not necessarily positive), we must define the place where these theta functions lie. The group G , as a subset of A , is dense in A ; thus, it is an irreducible space when endowed with induced topology. Now we put on V the weakest topology when endowed with induced topology. Now we put on V the weakest topology \mathfrak{B} for which the projection $\pi_A: V \rightarrow G$ is continuous. We denote by \mathcal{F}_A the set of the functions defined on some open subset of V , and (as in section 3) we impose an equivalence relation \sim defined as follows: f_1 , defined on U_1 , is equivalent to f_2 , defined on U_2 , and we write $f_1 \sim f_2$, if and only if they coincide on some open set contained in $U_1 \cap U_2$. Once again the set \mathcal{F}_A/\sim has a natural $k(A)$ -algebra structure, and we denote it by \mathcal{C}_A .

As in section 3, we are interested in the multiplicative group \mathcal{U}_A of the invertible elements of \mathcal{C}_A (we can represent any such element by a function different from zero on an open set of V).

It is immediate to see that \mathcal{U}_A contains the subgroup Q spanned by the second degree characters from V to k^* . In fact, if two characters coincide on an open set, they coincide everywhere. And then a multiple of a second degree character is contained in $k(A)^*$, if and only if it is a constant.

Hence there is a natural embedding of $k(A)^*/k^*$ into \mathcal{U}_A/Q . From this and from (4.20), we deduce the following

(4.25) THEOREM. *Let $\bar{\Theta}_X$ be an element of \mathcal{U}_A/Q determined by a theta of X*

as in (4.20). Then the map which sends X to $\bar{\Theta}_X$, defined on the set of the positive divisors, extends in a unique way to a homomorphism

$$\tilde{j}_A: \mathcal{D}_A \rightarrow \mathcal{U}_A/Q,$$

where \mathcal{D}_A is the group of all the divisor of A . Moreover the restriction j_A of \tilde{j}_A to the subgroup of principal divisors coincides with the natural embedding of $k(A)^*/k^*$ in \mathcal{U}_A/Q .

PROOF. First of all we observe that if X_1 and X_2 are positive divisors, then $\Theta_{X_1}\Theta_{X_2} \equiv \Theta_{X_1+X_2} \pmod Q$, where the Θ_{X_i} 's are as in (4.20). From this, it follows that the map $X \rightarrow \bar{\Theta}_X$, defined on the semigroup of positive divisors, is a homomorphism.

Thus if $X = X_0 - X_1$ is a divisor on A , where X_0 and X_1 are positive, we put $\tilde{j}_A X = \bar{\Theta}_{X_0}\bar{\Theta}_{X_1}^{-1}$.

Now we have to prove only the second statement. But if $X = (f)$ and $X = X_0 - X_1$ where X_0, X_1 are positive and without common prime components, then the functions $p \mapsto \pi^*f(p)$ and $p \mapsto \Theta_{X_0}(p)\Theta_{X_1}(p)^{-1}$, both defined on the open set $V \setminus \pi^{-1}(G \cap \text{supp } X_1)$, differ only by a multiple of a second degree character, and this is what we need, Q.E.D.

5. - Theta functions on the Tate space; continuation.

Our main interest in this section is the Severi group of A : we'll show how one may determine the class of a divisor with respect to algebraic equivalence by examining its theta functions.

As we will see, the most important object in this topic is the function $\psi_{2n}: \mathcal{G}(2n) \times \mathcal{G}(2n) \rightarrow k^*$ defined in (4.1); or equivalently, the corresponding Mumford group of level $2n$. In view of our application, it is convenient to substitute the single Mumford groups of finite level $\mathcal{G}(2n)$, by the extension \mathcal{G} of V by k^* which they produce when we pass to the limit.

In order to describe the group \mathcal{G} , we begin by observing that the functions of $k(A \times A)$

$$(5.1) \quad \psi_{2n}(x_1, x_2) = \prod_{j=1}^{2n-1} \frac{F(\ell^j \omega_2, x_1, j \omega_1)}{F(x_1, \ell^j \omega_2, j \ell^j \omega_2)}$$

associated to the divisor X , are determined by the linear system \mathcal{L}_X of X , up to a symmetric factor. In other words, if we replace X with a linearly equivalent divisor X' , the corresponding function $\psi'_{2n}(x_1, x_2)$ differs from (5.1) by a symmetric factor $\nu(x_1, x_2)$ on $\mathcal{G}(2n)$. That is a element of $k(A \times A)$

such that the rational function $\nu(x_1, x_2)/\nu(x_2, x_1)$ is regular and equal to 1 at each point of $G(2n) \times G(2n)$.

If we recall that any commutative extension of $G(2n)$ by k^* is trivial, from the previous arguments we deduce that $\mathfrak{G}(2n)$ depends only on \mathfrak{L}_x .

As in section 4, we denote by Λ the kernel of the natural projection $\pi: V \rightarrow G$, that is the points of V of the form $(e_\Lambda, P_1, P_2, \dots)$. Clearly $V = \varinjlim (\Lambda \hookrightarrow 1/t\Lambda \hookrightarrow 1/t^2\Lambda \hookrightarrow \dots)$.

Now the linear system \mathfrak{L}_x gives, for each n , an extension \mathfrak{G}_n of $1/t^n\Lambda$ by k^* , defined by the co-cycle (factor set)

$$\psi_n: 1/t^n\Lambda \times 1/t^n\Lambda \rightarrow k^*,$$

which sends (p, q) to $\psi_{2n}(P_n, Q_n)$.

Proposition (4.12) shows us that $\psi_{2(n+r)}(P_{n+r}, P_{n+r}) = \psi_{2n}(P_n, Q_n)$ for each $p = (P_0, P_1, \dots)$ and $q = (Q_0, Q_1, \dots)$ in $1/t^n\Lambda$. Therefore if $n \leq m$, the restriction of ψ_m to $1/t^n\Lambda$ coincides with ψ_n . Hence there exists the extension \mathfrak{G} of V by k^* defined by putting:

$$\mathfrak{G} = \varinjlim_{n \in \mathbb{N}} \mathfrak{G}_n.$$

We call such extension \mathfrak{G} the *Mumford group* of X (or of \mathfrak{L}_x).

If $\psi: V \times V \rightarrow k^*$ is a factor set which determines \mathfrak{G} , then the map

$$e_x: V \times V \rightarrow k^*,$$

defined by putting

$$e_x(p, q) = \frac{\psi(p, q)}{\psi(q, p)},$$

is \mathbb{Q}_t -bimultiplicative and skew-symmetric, and depends only on \mathfrak{G} . The last assertion follows from the fact that $e_x(p, q)$ is the commutator in \mathfrak{G} of any two elements \tilde{p} and \tilde{q} whose projections in V are respectively p and q .

Thus, we may calculate the restriction of $e = e_x$ to $1/t^n\Lambda \times 1/t^n\Lambda$ by using ψ_{2n} , and we obtain

$$e(p, q) = \frac{\psi_{2n}(P_n, Q_n)}{\psi_{2n}(Q_n, P_n)} = \chi_{2n}(P_n, Q_n),$$

because of the definition (4.2) of χ_{2n} .

We call the pairing e , the *Riemann form of X with values in k^** . Given e_x we may construct a \mathbb{Q}_t -bilinear, skew-symmetric pairing. Let us consider

the subgroup U of k^* , defined in the following way:

$$U = \{x \mid x \in k^*, x^{l^n} = 1 \text{ for some } n \in \mathbb{N}\} .$$

Now we denote by U^∞ the group $\varprojlim (U \leftarrow U \leftarrow \dots)$ where the arrows denote « raising to the l -th power ». Because k is algebraically closed and of characteristic $p \neq l$, one has

$$U \simeq \varinjlim_{n \in \mathbb{N}} \mathbb{Z}/l^n \mathbb{Z} ,$$

and hence there is a (non-canonical) homomorphism j from U^∞ to the additive group of \mathbb{Q}_l . Thus we define $\chi = \chi_x$ by setting,

$$\chi(p, q) = j(a_0, a_1, a_2, \dots) ,$$

where $a_i = e(1/l^i p, q) = e(p, 1/l_i q)$. Now the bilinearity and the skew-symmetry of χ follows from the analogous properties of e . We call χ_x the *l-adic Riemann form of X* .

Before we state the main result of this chapter, we show some properties of the factor sets ψ associated to \mathcal{G} :

$$(5.2) \quad \psi(p, q + \lambda) = \psi(p, q) ,$$

$$(5.3) \quad \psi(p + \lambda, q) = \psi(p, q) e(\lambda, q) ,$$

where p, q are in V and λ in \mathcal{A} . In order to prove (5.2) we observe that, if $\psi_{2n}(x_1, x_2)$ (cfr. (5.1)) is associated to a divisor of \mathcal{L}_X , which doesn't meet $G(2n)$, then the map $\nu: 1/l^n \mathcal{A} \times 1/l^n \mathcal{A} \rightarrow k^*$ defined by

$$\nu(p, q) = \frac{\psi(p, q)}{\psi_{2n}(P_n, Q_n)} ,$$

is symmetric and periodic with periods $\mathcal{A} \times \mathcal{A}$.

Since, by its definition, the map which sends (p, q) to $\psi_{2n}(P_n, Q_n)$ (is periodic with periods \mathcal{A} in its 2nd argument, this proves (5.2). By putting $q = 0 = (e_{\mathcal{A}}, e_{\mathcal{A}}, \dots)$ in (5.2), we find that $\psi(p, \lambda) = 1$ and hence that $\psi(\lambda, p) = e(\lambda, p)$. Finally, if we recall that ψ is a factor set we obtain (5.3). In fact:

$$\psi(p + \lambda, q) = \frac{\psi(\lambda, p + q) \psi(p, q)}{\psi(\lambda, p)} = \frac{e(\lambda, p + q)}{e(\lambda, p)} \psi(p, q) = e(\lambda, q) \psi(p, q) .$$

(5.4) THEOREM. *The map which associates the pairing χ_x to a divisor X*

on A is a group-homomorphism. Its kernel is the subgroup of divisors algebraically equivalent to zero. In fact the following statements are equivalent:

- i) X is algebraically equivalent to 0;
- ii) e_x is identically equal to 1;
- iii) ψ is symmetric (that is $\psi(p, q) = \psi(q, p)$);
- iv) \mathfrak{G} is isomorphic to $V \times k^*$;
- v) ψ is periodic with the elements of $A \times A$ as periods;
- vi) χ_x is identically equal to 0.

PROOF. Because the map which sends a divisor X to the set of its co-cycles is a homomorphism from \mathfrak{D}_A to $k(A \times A \times A)/k^*$, the first statement of the theorem follows from the construction of χ_x .

In order to prove the rest of the theorem, we establish the following chain of implications ii) \Rightarrow iii), iii) \Leftrightarrow iv), iii) \Rightarrow v), v) \Rightarrow vi), vi) \Rightarrow ii), i) \Rightarrow iii) and finally v) \Rightarrow i).

ii) \Rightarrow iii): follows directly from the definition of e_x .

iii) \Leftrightarrow iv): it's a well-known fact that any commutative extension of V by k^* is trivial; and, clearly, if \mathfrak{G} is a commutative group any factor set associated to it is symmetric.

iii) \Rightarrow v): because of (5.2), in any case ψ is periodic with periods $\{0\} \times A$. If we add the symmetry of ψ , it follows that ψ admits also the periods $A \times \{0\}$, and hence $A \times A$.

v) \Rightarrow vi): ψ is periodic, from (5.3) it follows that $e(\lambda, q) = 1$, for each λ in A and q in V . This implies $e(p, q) = 1$ for each p, q in V ; in fact, if p belongs to $1/t^n A$, we have $e(p, q) = e(p, 1/t^n q)^{t^n} = e(t^n p, 1/t^n q) = 1$.

vi) \Rightarrow ii): recalling the construction of χ_x , one sees that e_x is the element a_0 .

i) \Rightarrow iii): if Z is a divisor of \mathfrak{L}_x whose support doesn't meet $G(2n)$, and if $\varphi(x_1, x_2)$ is the normalized equation of $(p_1 + p_2)^{-1}Z - p_1^{-1}Z - p_2^{-1}Z$, then

$$F(x_1, x_2, x_3) = \frac{\varphi(x_1, x_2 + x_3)}{\varphi(x_1, x_2)\varphi(x_1, x_3)},$$

where F is the normalized co-cycle of Z .

From this and from the symmetry of φ , by (5.1), we have

$$\varphi(t^n P, t^n Q) = \psi_{2n}(P, Q),$$

for each P, Q in $G(2n)$. Hence ψ_{2n} is also symmetric and this implies the symmetry of ψ .

v) \Rightarrow i): because the points of G are dense in A , in order to show that $X \equiv 0$, it is sufficient to prove that $X_P - X$ is principal for each P in G . But, from v), it follows that if we choose Q such that $t^n Q = P$, then

$$\prod_{j=1}^{n-1} \frac{F(x, Q, jQ)}{F(Q, x, jx)}$$

in an equation of a divisor linearly equivalent to $X_P - X$, Q.E.D.

We state here explicitly a corollary, which is included in the last arguments of the preceding proof.

(5.5) COROLLARY. *Let X be a divisor on A and set $X = X_0 - X_1$, where X_0 and X_1 are positive divisors without common prime components. Consider the function Θ_X , defined on $V \setminus \pi^{-1}(\text{supp } X_1 \cap G)$, by putting $\Theta_X(p) = \Theta_{X_0}(p)/\Theta_{X_1}(p)$; then Θ_X is A -periodic if and only if $X \equiv 0$.*

PROOF. The periodicity of Θ_X is equivalent to the periodicity of ψ , hence the statement follows from (5.4), Q.E.D.

This last result points out the deep difference between the present situation and the one examined in section 3.

At this point it seems natural to ask what are the relations between the Riemann form e_x (or χ_x) which appears in (5.4) and the Riemann form introduced by Weil. We prove here that they coincide.

First of all we recall the Weil definition. If P and Q are in $G(n)$, the divisor $Y = (t^n)^{-1}(X - X_P)$ is principal. We suppose that its support doesn't meet either e_A or $-Q$, and we denote by $g_P(x)$ the normalized ($g_P(e_A) = 1$) equation of such divisor. Because $g_P(x + Q)$ is also an equation of Y , it follows that $g_P(x)/g_P(x + Q) = g_P(-Q)$ is a constant.

Finally, if we set

$$(5.6) \quad e_n(P, Q) = g_P(-Q),$$

the map from $G(n) \times G(n)$ to k^* defined by $(P, Q) \rightarrow e_n(P, Q)$ is bilinear and

skew-symmetric, the compatibility conditions (cfr. [9] p. 185) show that

$$e_{n+r}(P, Q)^v = e_n(vP, vQ),$$

for each P, Q in $G(n+r)$, and therefore, if in particular P, Q are in $G(2n+r)$, one has

$$(5.7) \quad e_{2(n+r)}(P, Q) = e_{2n}(vP, vQ).$$

The formula (5.7) is the compatibility condition satisfied by the χ_{2n} 's, and Weil's Riemann form e' , is obtained from the e_i as e_X is obtained from the χ_{2i} (cfr. 5.2). Thus we have reduced our problem to proving the following

(5.8) LEMMA. *Notation as above. For each P, Q in $G(2n)$ one has*

$$e_{2n}(P, Q) = \chi_{2n}(P, Q).$$

PROOF. Let R be a point of $G(n)$ and $g_R(x)$ an equation of $(t^n)^{-1}(X - X_R)$. We want to write g_R by means of the (normalized) co-cycle F of X . Thus, if we note that $F(x, -R, -jR)$ is an equation of $X_{(j+1)R} + X - X_R - X_{jR}$, and hence that $\prod_{j=1}^{v-1} F(x, -R, -jR)$ is an equation of $t^n(X - X_R)$, we can write

$$g_R(x)^v = \prod_{j=1}^{v-1} F(t^n x, -R, -jR).$$

But, using the co-cycle properties of F , and also recalling that $t^n R = e_A$, we obtain

$$\prod_{j=1}^{v-1} F(t^n x, -R, jR) = \prod_{j=1}^{v-1} \left(\frac{F(x, -R, -jR)}{F(x, jx, -R)} \right)^v;$$

therefore

$$g_R(x) = \prod_{j=1}^{v-1} \frac{F(x, -R, -jR)}{F(x, jx, -R)}.$$

In this way we obtain

$$e_n(R, S) = g_R(-S) = \prod_{j=1}^{v-1} \frac{F(-S, -R, -jR)}{F(-S, -jS, -R)},$$

for each R and S in $G(n)$.

Now it suffices to verify that

$$(5.9) \quad \chi_{2n}(P, Q) = \prod_{j=1}^{v-1} \frac{F(-Q, -P, -jP)}{F(-P, -Q, -jQ)}.$$

This is easy; in fact from (4.1) and the definition of χ_{2n} , in (4.2), we get

$$\begin{aligned} \prod_{j=1}^{2^n-1} \frac{F(-Q, -P, -jP)}{F(-P, -Q, -jQ)} &= \psi_{4^n}(-P, -1/t^{2^n}Q) = \chi_{4^n}(P, 1/t^{2^n}Q) \\ &= \chi_{4^n}(1/t^n P, 1/t^{2^n}Q)^{2^n} = \chi_{4^n}(1/t^n P, 1/t^n Q) = \chi_{2n}(P, Q), \end{aligned}$$

where $1/t^i R$ denote some point R' such that $t^i R' = R$, Q.E.D.

To conclude this section we want to show the behaviour of theta functions with respect to the action of \mathcal{A} ; that is, we want to calculate the automorphy factors. This is related to the anomaly of the case $t = 2$. Therefore we give here a general construction of Θ_x which also includes this case.

We have seen that the map $e_x: V \times V \rightarrow k^*$ is bimultiplicative, as a consequence

$$\varphi(p, q) = \psi(p, q)e(1/2q, p)$$

is a symmetric factor set on V ; hence φ is trivial. Thus if X is, for instance, a positive divisor, the equation

$$(5.9) \quad \Theta(p + q) = \Theta(p)\Theta(q)\psi(p, q)e(1/2q, p)$$

has a solution $\Theta: V \rightarrow k$. The solution of (5.9) are precisely the theta functions of X . In particular, if we restrict Θ to $t^{-n}\mathcal{A}$, it satisfies the equation

$$(5.10) \quad \Theta(p + q) = \Theta(p)\Theta(q)\psi_{2n}(P_n, Q_n)e(1/2q, p),$$

where $p = (P_0, P_1, \dots)$ and $q = (Q_0, Q_1, \dots)$. Now, if one recalls the definition of ψ_{2n} in (4.1), one sees immediately that the restriction of ψ_{2n} to $G(2n) \times G(n)$ is identically equal to 1. From this, it follows that

$$(5.11) \quad \Theta(p + \lambda) = \Theta(p)\Theta(\lambda)e(1/2\lambda, p),$$

for each p in V and each λ in \mathcal{A} . From the definition of e_x it follows that $e(\lambda, \mu) = 1$ for each λ, μ and \mathcal{A} . Hence, if $t \neq 2$, by (5.11) we find that the map $\lambda \mapsto \Theta(\lambda)$ is a character of \mathcal{A} . Otherwise, if $t = 2$ such a map is only a second degree character.

As we may extend any character of \mathcal{A} to a character of V , if $t \neq 2$, we may suppose $\Theta(\lambda) = 1$ for each λ in \mathcal{A} ; on the contrary, i.e. in the case $t = 2$, if we want to write explicitly a solution of

$$(5.12) \quad \Theta(\lambda + \mu) = \Theta(\lambda)\Theta(\mu)e(1/2\lambda, \mu),$$

we must consider the decompositions

$$A = A_1 \oplus A_2 \quad \text{and} \quad V = V_1 \oplus V_2,$$

where $V_i = A_i \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$ ($i = 1, 2$) are maximal isotropic subspaces of V with respect to e_x . If we denote by λ_i the projection of λ to A_i ($i = 1, 2$), then the map $h: A \rightarrow k^*$ defined by

$$h(\lambda) = e(1/2\lambda_1, \lambda_2),$$

is a solution of (5.12). In fact, $e(1/2\lambda_1, \lambda_2) = e(1/2\lambda_2, \lambda_1)$ and therefore

$$\frac{h(\lambda + \mu)}{h(\lambda)h(\mu)} = e(1/2\lambda_1, \mu_2)e(1/2\mu_1, \lambda_2) = e(1/2\lambda_1, \mu_2)e(1/2\lambda_2, \mu_1) = e(1/2\lambda, \mu).$$

Because any other solution of (5.11) differs from h by a character of A in k^* , we obtain that, if $l = 2$, there exists a theta of X such that

$$\Theta(p + \lambda) = \Theta(p)e(1/2\lambda_1, \lambda_2)e(1/2\lambda, \mu).$$

The above arguments establish the first part of the following

(5.13) THEOREM. *Let X be a positive divisor on A . Then there exists a theta function of X which satisfies the following functional equations:*

$$(5.14) \quad \Theta(p + \lambda) = \Theta(p)e(1/2\lambda, p), \quad \text{if } l \neq 2$$

$$(5.15) \quad \Theta(p + \lambda) = \Theta(p)e(1/2\lambda_1, \lambda_2)e(1/2\lambda, p), \quad \text{if } l = 2.$$

relations (5.14), (5.15) determine Θ up to a homomorphism $q: V \rightarrow k^*$. In particular, if X is totally symmetric (that is $(- \iota)^{-1}X = X$), (5.14) and (5.15) have a unique solution Θ such that $\Theta(p) = \Theta(-p)$.

PROOF. A theta function of X is determined (in general) up to a second degree character, $\tilde{q}: V \rightarrow k^*$, and if it satisfies (5.14), or (5.15), \tilde{q} is periodic with periods A . Because non-trivial bihomomorphisms from $G \times G$ to k^* don't exist, it follows that the map

$$(p, q) \mapsto \frac{\tilde{q}(p + q)}{\tilde{q}(p)\tilde{q}(q)}$$

is identically equal to 1. Hence \tilde{q} is a character of V in k^* .

In order to prove the last statement of (5.13), it suffices to remark that V (as a 2-divisible group) doesn't admit a non-trivial character \tilde{q} such that $\tilde{q}(p) = \tilde{q}(-p)$, Q.E.D.

If we compare these results with § 8 of [11] (with special attention to p. 116), we see that, if $l = 2$ and X is totally symmetric, the theta which satisfies (5.15) is the function that Mumford denotes with $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The symmetric bi-extensions and some of their relations with theta function are studied in a recent paper by L. Breen (cfr. [3]).

6. – Theta functions on the Barsotti-Tate group of A ; the global case.

This section welds together the results of paragraphs 2 and 3; however it doesn't follow immediately after these because the arguments used here, which are self-consistent, acquire a greater transparency if one looks first at sections 4 and 5.

Let S be the affine algebra of the Barsotti-Tate group of A . It is well known (cfr., for instance, [MA]) that $S = \prod_{P \in G_{\acute{e}t}} R_P$, where R_P is the completion (in the local topology) of the local ring at P . Using the extensions of the translations σ_P^* of $k(A)$, we identify R_P with R_e , and we denote the last one by R . In this way S coincides with $R^{G_{\acute{e}t}}$. With this identification the natural embedding of $k(A)$ into $\prod_{P \in G_{\acute{e}t}} \text{Frac } R_P$, becomes the map i_A of $k(A)$ into $(\text{Frac } R)^{G_{\acute{e}t}}$ defined by $i_A(f) = (f_P)_{P \in G_{\acute{e}t}}$ where $f_P = \sigma_P^* f$.

But, the algebra $(\text{Frac } R)^{G_{\acute{e}t}}$ doesn't suffice. In general, in order to find a solution for the equation (*), we must use the algebra $C_A = (\text{Frac } \mathcal{R})^{V_A}$, where V_A is the p -adic Tate space of A . For this reason in the following we denote by i_A the embedding of $k(A)$ in C_A . This is the map obtained by the composition of the previous i_A with the natural embedding of $(\text{Frac } \mathcal{R})^{G_{\acute{e}t}}$ in C_A (that is, the map induced by the natural projection $\tau: V_A \rightarrow G_{\acute{e}t}$; $\tau(P_0, P_1, \dots) = P_0$). Thus, for instance, if F is a co-cycle of a divisor X of A , $i_A F = (F_{(P,Q,R)})_{(P,Q,R) \in V_A^2}$, where $F_{(P,Q,R)} = \sigma_{(\tau P, \tau Q, \tau R)}^* F$.

Now, if we recall that for each element $y = (y_P(t))_{P \in V_A}$, of $(\text{Frac } \mathcal{R})^{V_A}$ we have

$$(p_1 + p_2)^* y = (y_{P_1 + P_2}(t_1 + t_2))_{(P_1, P_2) \in V_A^2};$$

then it follows immediatly that (*) has a solution $\Theta = (\vartheta_P)_{P \in V_A}$, if and only if the $\vartheta_P = \vartheta_P(t)$'s, are solutions for the system Σ formed by the

equations

$$(6.1) \quad F_{(P_1, P_2, P_3)}(t_1, t_2, t_3) = \frac{\vartheta_{P_1+P_2+P_3}(t_1 + t_2 + t_3) \vartheta_{P_1}(t_1) \vartheta_{P_2}(t_2) \vartheta_{P_3}(t_3)}{\vartheta_{P_1+P_2}(t_1 + t_2) \vartheta_{P_1+P_3}(t_1 + t_3) \vartheta_{P_2+P_3}(t_2 + t_3)},$$

when (P_1, P_2, P_3) varies in V_A^3 . Our next goal is to solve Σ .

In order to do this, as in chap. 4, we proceed by approximations. More precisely: first of all we suppose X positive and we denote by $R(n)$ the algebra $R/(p^n \iota R^+)R$ and by $S(n)$ the algebra $R(n)^{G_{\text{ét}}(2n)} = S/(p^n \iota S^+)S$. With this assumption we show that the system induced by Σ over $S(2n)$ has solutions $\Theta^{(2n)}$ and that such solutions may be chosen in a coherent way so that they determine an element Θ_X of \mathcal{R}^{V_A} . Because for any such Θ_X , we have $\vartheta_P \neq 0$ for each P , we extend the definition of Θ_X to any divisor in the natural way.

Before to begin the construction of Θ_X , we write explicitly the compatibility conditions which allow us to pass from $\Theta^{(2n)}$ to Θ .

(6.2) COMPATIBILITY CONDITIONS. *If we denote by p_- the canonical projection from $S(n)$ to $S(n - 1)$, and by p_+ the map from $S(n)$ to $S(n + 1)$ induced by the endomorphism pi of S , then the sequence $(y^{(2n)})_{n \in \mathbb{N}}$, where $y^{(2n)}$ belongs to $S(2n)$, determines a unique element of \mathcal{R}^{V_A} if and only if $p_- y^{(2(n+1))} = p_+ y^{(2n)}$.*

One may verify this claim directly observing that if

$$\tilde{S}_i = \varinjlim (S(i) \xrightarrow{p_+} S(i + 1) \xrightarrow{p_+} \dots),$$

then $\mathcal{R}^{V_A} = \varprojlim (\tilde{S}_0 \xleftarrow{p_-} \tilde{S}_1 \xleftarrow{p_-} \dots)$ (cfr., for instance, ch. 4 of [MA]).

Now, let X be a divisor on A whose support doesn't meet $G_{\text{ét}}(2n)$ we denote by F the normalized co-cycle of X , and by

$$F^{(2n)} = (F_{(P_1, P_2, P_3)}^{(2n)}(t_1, t_2, t_3))_{(P_1, P_2, P_3) \in G_{\text{ét}}^3(2n)}$$

the image of F in $S(2n)$; then we consider the element $\psi^{(2n)}$ of $S(2n) \otimes S(2n)$ defined by

$$(6.3) \quad \psi_{(P_1, P_2)}^{(2n)}(t_1, t_2) = \prod_{j=1}^{v^n-1} \frac{F_{(p^n P_2, P_1, iP_1)}^{(2n)}(p^n \iota t_2, t_1, jt_1)}{F_{(P_1, p^n P_2, ip^n P_2)}^{(2n)}(t_1, p^n \iota t_2, jp^n \iota t_2)}.$$

Because in the proof (4.2) and (4.12) we use only the co-cycle properties of F and the fact that $G(2n)$ is a torsion group of finite exponent, we may repeat these arguments (replacing ι with p). In this way we obtain:

(6.4) PROPOSITION. *The element $\psi^{(2n)}$, defined by (6.3), satisfies the fol-*

lowing equations:

$$\begin{aligned}
 \text{i) } p_-^r \psi^{(2(n+r))} &= p_+^r \psi^{(2n)}; \\
 \text{ii) } \frac{(p_1, (p_2 + p_3))^* \psi^{(2n)}}{((p_1, p_2)^* \psi^{(2n)})(p_1, p_3)^* \psi^{(2n)}} &= p_+^n F^{(n)} \\
 &= \frac{((p_1 + p_2), p_3)^* \psi^{(2n)}}{((p_1, p_3)^* \psi^{(2n)})(p_2, p_3)^* \psi^{(2n)}}.
 \end{aligned}$$

From (6.4), looking at (6.2), we obtain the following

(6.5) COROLLARY. *If $sc: S(2n) \otimes S(2n) \rightarrow S(2n) \otimes S(2n)$ is the map defined by $t_1 \otimes t_2 \mapsto t_2 \otimes t_1$, then the element*

$$\chi^{(2n)} = \frac{\psi^{(2n)}}{sc\psi^{(2n)}}$$

of $S(2n) \otimes S(2n)$ is bimultiplicative. Moreover there exists a unique bimultiplicative element $\chi = \chi_x$ in $\mathbb{R}^{V_A} \widehat{\otimes} \mathbb{R}^{V_A}$ whose projection on \tilde{S}_i coincides with $\chi^{(2i)}$.

The element χ_x defined above is the Riemann form of X .

We remark here that, as in analogous situations, if we replace X with a linearly equivalent divisor X' , $\psi^{(2n)}$ is modified by a symmetric factor and hence for the definition of χ_x is unnecessary to suppose that X doesn't meet $G_{\text{ét}}(2n)$.

Once again, the arguments which lead to the symmetrization of ψ_{2n} in section 4, may be repeated also in this case and we find a symmetric factor set

$$(6.6) \quad \varphi^{(2n)} = \psi^{(2n)}(1/2\iota \otimes \iota)^* \chi^{(2n)}.$$

Such a factor set determines a commutative extension of a finite group by the multiplicative group, so it is trivial. That is there exists an element $\Theta^{(2n)}$ of $S(2n)$ which satisfies the equation

$$(6.7) \quad \frac{(p_1 + p_2)^* \Theta^{(2n)}}{(p_1^* \Theta^{(2n)})(p_2^* \Theta^{(2n)})} = \varphi^{(2n)}.$$

Instead of proving directly that we may choose the $\Theta^{(2n)}$'s which satisfy the compatibility condition (6.2), we show here the relations between $\Theta^{(2n)}$ and the theta functions of sections 2 and 3, and we deduce from these the existence of compatible solutions.

First of all, we remark a general fact, used also in lemma (2.11): the factor set $\varphi^{(2n)}$ of $R(2n)^{G_{\text{ét}}(2n)} \otimes R(2n)^{G_{\text{ét}}(2n)}$ determines two factor sets, in $R(2n) \otimes R(2n)$, and in $k^{G_{\text{ét}}(2n) \times G_{\text{ét}}(2n)}$ respectively (we suppose that $\text{supp } X$ doesn't meet $G_{\text{ét}}(2n)$). Using this fact we may decompose $\varphi^{(2n)}$ in the following way:

$$(6.8) \quad (\varphi_{(P_1, P_2)}^{(2n)}(t_1, t_2))_{P_1, P_2 \in G_{\text{ét}}(2n)} = (\varphi_{(e_A, e_A)}^{(2n)}(t_1, t_2)) (\varphi_{(P_1, P_2)}^{(2n)}(0, 0))_{P_1, P_2 \in G_{\text{ét}}(2n)} \cdot \left(\frac{\varphi_{(P_1 + P_2, e_A)}^{(2n)}(0, t_1 + t_2)}{\varphi_{(P_1, e_A)}^{(2n)}(0, t_1) \varphi_{(P_2, e_A)}^{(2n)}(0, t_2)} \right)_{P_1, P_2 \in G_{\text{ét}}(2n)},$$

where we identify $k^{G_{\text{ét}}(2n)}$ and $R(2n)$ with their images in $S(2n)$. Thus, we may solve separately the equations, obtained from (6.7) by substituting $\varphi^{(2n)}$ with the factors of (6.8). Now, if $\vartheta(t)$ is a theta of X on the local component of the Barsotti-Tate group of A (cfr. section 2), we denote by $\vartheta_{\text{loc}}^{(2n)}(t)$ the image of $\vartheta(p^n t)$ in $R(2n)$. It follows easily from the previous arguments that

$$\frac{\vartheta_{\text{loc}}^{(2n)}(t_1 + t_2)}{\vartheta_{\text{loc}}^{(2n)}(t_1) \vartheta_{\text{loc}}^{(2n)}(t_2)} = \varphi_{(e_A, e_A)}^{(2n)}(t_1, t_2).$$

Because there exists no non-trivial bi-homomorphisms from $G_{\text{ét}}(2n)^2$ to k^* , one sees that

$$(6.9) \quad \varphi_{(P_1, P_2)}^{(2n)}(0, 0) = \psi_{(P_1, P_2)}^{(2n)}(0, 0) = \prod_{j=1}^{n-1} \frac{F^{(2n)}(p^n P_2, P_1, jP_1)}{F^{(2n)}(P_1, p^n P_2, jp^n P_2)}.$$

Moreover, the factor set $\varphi_{(P_1, P_2)}^{(2n)}(0, 0)$ is associated to a commutative extension of $G_{\text{ét}}(2n)$ by k^* , hence it is trivial; so there exists a function $\Theta_{\text{ét}}^{(2n)}$ on $G_{\text{ét}}(2n)$ which satisfies the following equation

$$(6.10) \quad \frac{\Theta_{\text{ét}}^{(2n)}(P_1 + P_2)}{\Theta_{\text{ét}}^{(2n)}(P_1) \Theta_{\text{ét}}^{(2n)}(P_2)} = \varphi_{(P_1, P_2)}^{(2n)}(0, 0).$$

Because there are no non-trivial homomorphism from $G_{\text{ét}}(2n)$ to k^* , the solution of (6.10) is unique and hence

$$(6.11) \quad \Theta_{\text{ét}}^{(2(n+r))}(P) = \Theta_{\text{ét}}^{(2n)}(p^r P)$$

for each P in $G_{\text{ét}}(2n + r)$ (cfr. (4.14)).

At this point, if we change X with a $X' = X + (f)$, which doesn't meet $G_{\text{ét}}(2n)$, then we obtain

$$f^{-1}(p^n P) \Theta^{(2n)'}(P) = \Theta_{\text{ét}}^{(2n)}(P),$$

(cfr. (4.20)). Thus, from now on, X will be a positive divisor (without restrictions).

From (6.11) one has that there exists a map $\Theta_{\text{ét}}$ from V_A to k , defined by putting

$$(6.12) \quad \Theta_{\text{ét}}(P_0, P_1, \dots) = \Theta_{\text{ét}}^{(2n)}(P_n),$$

where $p^n P_0 = e_A$. It follows immediately from (6.9) that $\varphi_{(P_1, P_2)}^{(2n)}(0, 0)$ admits the elements $G_{\text{ét}}(n)^2$ as periods, thus

$$(6.13) \quad \Theta_{\text{ét}}^{(2n)}(P + Q) = \Theta_{\text{ét}}^{(2n)}(P),$$

for each P in $G_{\text{ét}}(2n)$ and Q in $G_{\text{ét}}(n)$. Therefore the $\Theta_{\text{ét}}$ of (6.12) has the elements of $T_A = \{(P_0, P_1, \dots) \in V \mid P_0 = e_A\}$ as periods; hence it defines a function (which we call $\Theta_{\text{ét}}$ again) from $G_{\text{ét}} = V_A/T_A$ to k . Such a function coincides with Θ_X of section 3, that is

$$\Theta_{\text{ét}}^{(2n)}(P) = \Theta_X(p^n P),$$

for each $P \in G_{\text{ét}}(2n)$. As one can see, these facts show that in section 3 the hypothesis that A is ordinary is irrelevant in the construction of theta.

Finally, if we look at the third factor of (6.8), and if we remark that $\varphi^{(2n)}$, as $\psi^{(2n)}$, satisfies the i) of (6.4), we conclude that the sequence

$$(\varphi_{(P, e_A)}^{(2n)}(0, t))_{n \in \mathbb{N}},$$

as P varies in $G_{\text{ét}}(2n)$, satisfies (6.2).

In this way we have proved the following

(6.4) THEOREM. *Let X be a divisor on A . Then the equation (*) of the introduction has solutions in $(\text{Frac } \mathcal{R})^{V_A}$. If, in particular, X is positive, each solution Θ of (*) may split in the following way:*

$$(6.15) \quad \Theta = \vartheta_{\text{loc}} \Theta_{\text{ét}} \tau$$

where ϑ_{loc} is a theta of X on the local component of the Barsotti-Tate group of A , $\Theta_{\text{ét}}$ is a k -valued function on V_A which admits T_A as set of periods; and finally τ is an element of \mathcal{R}^{V_A} which determines a section of the extension of $G_{\text{ét}} \times G_I$ by G_m related to X .

If A is an ordinary abelian variety, then $\Theta_{\text{ét}}$ is the theta of X on the étale component of the Barsotti-Tate group of A described in section 3.

The splitting (6.15) and the arguments of section 2, show that the information contained in the Θ of (6.15) are redundant; in fact we may reconstruct Θ from ϑ_{loc} .

Now, if one looks at the symmetry in the structure of Barsotti-Tate groups associated to abelian varieties (cfr. for instance [MA]), it's natural to ask whether there exists a decomposition of Θ analogous to (6.15), where this symmetry appears. The construction of ϑ_{loc} performed in section 2 shows that the answer is «yes». If $R = R_\pi \widehat{\otimes} R_r \widehat{\otimes} R_{\text{ét}}$ is the decomposition of the hyperalgebra of G corresponding to the splitting of $G = G_\pi \times G_r \times G_{\text{ét}}$; then we may split Θ into the product:

$$(6.16) \quad \Theta = \vartheta_\pi \vartheta_r \Theta_{\text{ét}} \tau',$$

where $\vartheta_\pi \in R_\pi$, $\vartheta_r \in R_r$, $\Theta_{\text{ét}} \in R_{\text{ét}}$ and τ' is a factor of cohomological meaning, whose projection either on $R_r \widehat{\otimes} R_{\text{ét}}$ or on $R_\pi \widehat{\otimes} R_{\text{ét}}$ is equal to 1 (we identify $\mathcal{R}/\mathcal{R}R_\pi^+$ and $\mathcal{R}_r \widehat{\otimes} \mathcal{R}_{\text{ét}}$, etc ...).

At this point the following is nearly immediate

(6.17) THEOREM. *Let ${}^\pi\mathcal{R}$ (resp. ${}^t\mathcal{R}$) be the algebra which one obtain from \mathcal{R} by reduction mod $\mathcal{R}R_{\text{ét}}^+$ (resp. $\mathcal{R}R_\pi^+$). With the notations of (6.14), Θ is completely determined either by its canonical image ${}^\pi\vartheta = {}^\pi\mathcal{R}$, or by its canonical image ${}^t\vartheta$ in ${}^t\mathcal{R}$.*

PROOF. Because one has ${}^\pi\vartheta = \vartheta_\pi \vartheta_r = \vartheta_{\text{loc}}$; we have only to show that ${}^t\vartheta$ determines Θ . But this follows immediately from the natural embedding of the field $k(A)$ into $(\text{Frac } \mathcal{R}_r)^{V_A}$, and from the fact that ${}^t\vartheta$ is a solution in ${}^t\mathcal{R} = \mathcal{R}_r^{V_A}$ of the equation (*) if the introduction, Q.E.D.

At the end of this section we observe that from the decomposition (6.15) and from (8.2) and (8.16) of [6] it follows that the Riemann form χ_x defined in (6.5) is the element of $\mathcal{R}^{V_A} \widehat{\otimes} \mathcal{R}^{V_A}$, denoted by ϑ_x in chapter 7 of [MA]. Therefore also in this case the map: $X \mapsto \chi_x$ is a group-homomorphism whose kernel is the group of divisors algebraically equivalent to zero.

7. - Abstract theta function.

This section, where an algebraically closed field k of characteristic p , and a prime l different from p are fixed, is devoted to the characterization of l -adic theta functions as particular maps, defined on (subsets of) $V = \mathbf{Q}_l^{2g}$ and with values in k . Moreover we show how one may construct from such a function a (abelian) variety, and then one may reconstruct the divisor

to which it is attached. The techniques are inspired from [1], [6] and by some results of [7].

First of all we make some general remarks. Let \mathfrak{C} be a topology on V ; we denote by $\mathfrak{C}^{(2)}$ the smallest topology on V^2 for which the projections p_1 and p_2 and the map $p_1 + p_2$ are continuous; analogously we denote by $\mathfrak{C}^{(3)}$ the smallest topology on V^3 for which p_1, p_2, p_3 and the maps $p_1 + p_2, p_1 + p_3, p_2 + p_3, p_1 + p_2 + p_3$ are continuous. It's a straightforward remark that, if \mathfrak{C} is irreducible (that is two non-empty open sets of \mathfrak{C} have non-empty intersection), then $\mathfrak{C}^{(2)}$ and $\mathfrak{C}^{(3)}$ are also irreducible.

If \mathfrak{U} is an irreducible topology on a set S , for each U in \mathfrak{U} , we denote by $\Gamma(U)$ the k -algebra of the k -valued functions defined on U . If $U' \subseteq U$ we denote by $\varrho_{U'}^U: \Gamma(U) \rightarrow \Gamma(U')$ the natural restriction map. Finally we set

$$R_S = R_S(\mathfrak{U}) = \lim_{\substack{\longrightarrow \\ U \in \mathfrak{U}}} (\Gamma(U), \varrho).$$

Thus the subring of R_S , which contains 0 and the elements represented by functions different from zero on (each point of) a non-empty open set of \mathfrak{U} is a field which we denote by $\mathfrak{K}_S = \mathfrak{K}_S(\mathfrak{U})$.

We remark that \mathfrak{K}_V is functorial with respect to the maps $p_1, p_2, p_3, p_1 + p_2, p_1 + p_3, p_2 + p_3, p_1 + p_2 + p_3$; that is any such a map gives an embedding of $\mathfrak{K}_V(\mathfrak{C})$ into $\mathfrak{K}_{V^3}(\mathfrak{C}^{(3)})$. For instance, if \bar{f} belongs to \mathfrak{K}_V and is represented by a map f , defined on the open set U of \mathfrak{C} , $(p_1 + p_2)^* \bar{f}$ is represented by a map g defined on $(p_1 + p_2)^{-1}U$ by putting $g(P_1, P_2, P_3) = f(P_1 + P_2)$. Moreover, there is a natural embedding of $\text{Frac}(\mathfrak{K}_V \otimes_k \mathfrak{K}_V \otimes_k \mathfrak{K}_V)$ into \mathfrak{K}_{V^3} ; by this we identify $\text{Frac}(\mathfrak{K}_V \otimes \mathfrak{K}_V \otimes \mathfrak{K}_V)$ with its image in \mathfrak{K}_{V^3} .

We say that a subset C of V is *negligible*, if we cannot obtain V as a finite join of C and its traslates. This means that the smallest topology in which C and its translates are closed sets is irreducible. In order to state that some statement is true in the complement of a negligible set of V we say that is true *almost everywhere (a.e.) on V* . Therefore we say that the elements of \mathfrak{K}_V are represented by functions defined a.e. on V , and a.e. different from zero.

It follows from this that a function f , defined almost everywhere on V , determines by means of its domain an irreducible topology on V and hence a field \mathfrak{R} . We observe here that V acts naturally on \mathfrak{K}_V : if $\bar{\theta}$ belongs to \mathfrak{K}_V and is represented by a $\theta \in \Gamma(U)$, and P is a point of V , we denote by θ_P the element of \mathfrak{K}_V , represented by the function which maps Q to $\theta(P + Q)$: θ_P belongs to $\Gamma(U - P)$. We denote by K_V the subfield of \mathfrak{K}_V of the

elements which are stable with respect to the action of the subgroup $T = \mathbb{Z}_i^{2^a}$ of V .

(7.1) DEFINITION. A function Θ , defined almost everywhere on V , is of type theta on V , if

$$(7.2) \quad \frac{((p_1 + p_2 + p_3)^* \bar{\Theta})(p_1^* \bar{\Theta})(p_2^* \bar{\Theta})(p_3^* \bar{\Theta})}{((p_1 + p_2)^* \bar{\Theta})(p_1 + p_3)^* \bar{\Theta})(p_2 + p_3)^* \bar{\Theta}} \in \text{Frac}(K_V \otimes K_V \otimes K_V)$$

where $\bar{\Theta}$ denotes the image of Θ in the field K_V determined by Θ as above.

We may associate to a type theta function a group-variety; in fact, we prove the following

(7.3.) THEOREM. Let Θ be a function of type theta on V . Then there exists a minimal field containing k and contained in K_V , which we denote by C_V and, which satisfies (7.2) in place of K_V . Such a field is a finitely generated k -hyperfield; it is spanned over k by the elements $(\bar{\Theta} \bar{\Theta}_{P+Q}) / (\bar{\Theta}_P \bar{\Theta}_Q)$ where P and Q vary in V .

PROOF. Let Θ be of type theta; if we denote by $\bar{F} = \bar{F}_\Theta$ the left hand side of (7.2), we may write:

$$(7.4) \quad \bar{F} = \frac{\sum_{i=1}^r \bar{f}_i \otimes \bar{g}_i \otimes \bar{h}_i}{\sum_{j=1}^s \bar{f}'_j \otimes \bar{g}'_j \otimes \bar{h}'_j}$$

where $\bar{f}_1, \bar{g}_1, \bar{h}_1, \dots, \bar{f}'_s, \bar{g}'_s, \bar{h}'_s$ are elements of K_V .

Therefore there exist subfields of K_V , finitely generated over k , which satisfy (7.2); for instance $k(\bar{f}_1, \dots, \bar{h}'_s)$ is one of them. In order to show that there exists a minimal one, let us consider two open sets U and W in V such that, for each (P, Q) in $U \times W$, the element (notations as in 7.4)

$$(7.5) \quad \bar{h}_{(P,Q)} = \frac{\sum_{i=1}^r \bar{f}_i g_i(P) h_i(Q)}{\sum_{j=1}^s \bar{f}'_j g'_j(P) h'_j(Q)}$$

is defined and belongs to K_V .

It is easy to see that the field K , generated over k by the $\bar{h}_{(P,Q)}$'s, depends only on \bar{F} (and hence from $\bar{\Theta}$) and not on U, W or on the representative of \bar{F} in (7.4).

From (7.5) it follows that K is contained in $k(\bar{f}'_1, \dots, \bar{f}'_s)$, and also that the \bar{f}'_i and the \bar{f}'_j , as solutions of the linear system (7.5), with coefficients in K , may be chosen in K . From this and from the symmetry of \bar{F} one has that $K = C_\Theta$.

In order to show that C_Θ is generated by the element $(\bar{\Theta}\bar{\Theta}_{P+Q})/(\bar{\Theta}_P\bar{\Theta}_Q)$, as P and Q vary in V , it suffices to observe that we may choose a representative Θ of $\bar{\Theta}$ such that, for each P, Q in V , there exists

$$\bar{h}_{(P,Q)} = \frac{\bar{\Theta}_{P+Q}\bar{\Theta}\Theta(P)\Theta(Q)}{\bar{\Theta}_P\bar{\Theta}_Q\Theta(P+Q)}.$$

Now we have only to show that C_Θ is a hyperfield. We use the definition of hyperfield of [7]; almost all the checks are *routine* and they are essentially similar to the ones given in the proof of (7.4) in [6]. For this reason we limit ourselves to prove only that

$$(7.6) \quad (p_1 + p_2)^* C_\Theta \subseteq \text{Frac}(C_\Theta \otimes_k C_\Theta).$$

If we denote by p_i ($i = 1, \dots, 4$) the i -th projection from V^4 to V , then for the co-cycle properties of \bar{F} , one has

$$(7.7) \quad (p_1 + p_2, p_3, p_4)^* \bar{F} = \frac{((p_1, p_2, p_3 + p_4)^* \bar{F})((p_1, p_3, p_4)^* \bar{F})((p_2, p_3, p_4)^* \bar{F})}{((p_1, p_2, p_3)^* \bar{F})((p_1, p_2, p_4)^* \bar{F})};$$

and from this one deduce that

$$(p_1 + p_2)^* \bar{h}_{(P,Q)} \in \text{Frac}(C_\Theta \otimes_k C_\Theta)$$

and this implies (7.6), Q.E.D.

We return now to the action of V on \mathcal{K}_V , defined above. From (7.1) and (7.3) we obtain the following

(7.8) COROLLARY. *The hyperfield C_Θ is stable with respect to the action of V . Moreover, if we denote by φ the homomorphism of V into $\text{Aut } C_\Theta$, defined by the action of V , then $\mathbb{Z}_i^{2\sigma} = T$ is contained in $\ker \varphi$. Finally, the automorphisms of C_Θ which belong to the image of φ are invariant.*

PROOF. The stability of C_Θ follows from the observation that for each P, Q, R one has

$$\frac{\bar{\Theta}_{P+Q+R}\bar{\Theta}_R}{\bar{\Theta}_{P+R}\bar{\Theta}_{Q+R}} = \frac{\bar{\Theta}_{P+Q+R}\bar{\Theta}}{\bar{\Theta}_{P+R}\bar{\Theta}_Q} \cdot \frac{\bar{\Theta}_Q\bar{\Theta}_R}{\bar{\Theta}\bar{\Theta}_{Q+R}}.$$

Then it follows from (7.1) that $T \subseteq \ker \varphi$.

Finally an automorphism $\varphi(P)$ in $\text{Im } \varphi$ is invariant if and only if it satisfies the equation

$$(\varphi(P) \otimes \iota)[(p_1 + p_2)^* \bar{f}] = (p_1 + p_2)^* [\varphi(P) \bar{f}];$$

and this is immediatly verified for each P in V and \bar{f} in C_θ , Q.E.D.

From (7.8) we obtain that the group of \mathfrak{l} -torsion points of the group-variety A_θ , determined by C_θ , contains a subgroup G_θ isomorphic to $V/\ker \varphi$.

We say that a function Θ of type theta on V is a *theta function* if G_θ is the group of \mathfrak{l} -torsion points of A_θ . If Θ is a theta function on V the *dimension of Θ* , $\dim \Theta$, is the dimension over $\mathbb{Q}_\mathfrak{l}$ of the vector space $V_\theta = \varprojlim (G_\theta \xleftarrow{\mathfrak{l}} G_\theta \xleftarrow{\mathfrak{l}} \dots)$.

If $V_\theta = V$, the theta function Θ is said to be *non-degenerate*.

The group of \mathfrak{l}^n -torsion points of a group variety A of dimension g is isomorphic to some subgroup of $(\mathbb{Z}/\mathfrak{l}^n \mathbb{Z})^{2g}$ and coincides with the last one if and only if A is an abelian variety. Thus, from the previous arguments one has the following

(7.9) PROPOSITION. *Let Θ be a non-degenerate theta function on $\mathbb{Q}_\mathfrak{l}^{2g}$; then $\dim A_\theta \geq g$. In particular, A_θ is an abelian variety if and only if $\dim A = g$.*

We remark here that, for a theta function Θ , the natural homomorphism from $(\mathbb{Q}_\mathfrak{l}/\mathbb{Z}_\mathfrak{l})^{2g}$ to G_θ , induced by the φ of (7.8), gives a homomorphism σ of V onto V_θ . Such a map is an isomorphism only if Θ is non-degenerate.

On the contrary, if σ isn't an isomorphism, there exists a second degree character $\lambda: V \rightarrow k^*$ such that $(\overline{\Theta \lambda})_P = \overline{\Theta} \lambda$ for each P in $\ker \sigma$. We may describe also these facts by the bimultiplicative pairing e_θ associated to Θ .

In fact, if we, replace Θ with another representative Θ' of $\overline{\Theta}$, when it is necessary, from (5.11) we see that this pairing is

$$(7.10) \quad e_\theta(1/2p, q) = \frac{\Theta(\mathfrak{l}^n p + 1/\mathfrak{l}^n q)}{\Theta(\mathfrak{l}^n p) \Theta(1/\mathfrak{l}^n q)},$$

for each p in $(1/\mathfrak{l} \mathbb{Z}_\mathfrak{l})^{2g}$ and q in V .

Because we may multiply Θ by a second degree character and obtain that $\Theta^2(\lambda) = 1$ for each λ in $(\mathbb{Z}_\mathfrak{l})^{2g}$, we obtain from (7.10)

$$e_\theta(p, q) = \frac{\Theta(\mathfrak{l}^n p + 1/\mathfrak{l}^n q)^2}{\Theta(1/\mathfrak{l}^n q)^2};$$

hence $e_\theta(p, q) = 1$ for each q in V and for each p in $\ker \sigma$.

Thus if Θ degenerates, so also does e_Θ : *Vice versa*, if e_Θ degenerates i.e. if there exists $W \subseteq V$ such that $e_\Theta(p, q) = 1$ for each p in W and each q in V ; we have (if $t \neq 2$)

$$1 = \Theta(t^n p + t^{-n} q) / \Theta(t^{-n} q),$$

that is $\bar{\Theta}_{t^n p} = \bar{\Theta}$. Thus $\ker \sigma \subseteq W \cap (\mathbb{Z}_t)^{2g}$ and so $W \subseteq \ker \sigma$. This implies that Θ degenerates.

Our results are summarized in the following

(7.11) PROPOSITION. *A theta function Θ is non-degenerate if and only if the bimultiplicative pairing e_Θ associated to it is non-degenerate.*

At the end of this section we observe that one may characterize the divisor X associated to Θ as the unique divisor X of A_Θ such that

$$(p_1 + p_2 + p_3)^{-1} X + p_1^{-1} X + p_2^{-1} X + p_3^{-1} X - (p_1 + p_2)^{-1} X - (p_1 + p_3)^{-1} X - (p_2 + p_3)^{-1} X$$

coincides with the divisor of \bar{F}_Θ on $A_\Theta \times A_\Theta \times A_\Theta$. But we may also recover X directly from Θ : this is shown in section 5. For instance by those arguments we have the following

(7.12) PROPOSITION. *Let F be the function defined by \bar{F} on the open set U in G_Θ^3 , where U is the complement of the polar variety of \bar{F} . Let $\pi: V \rightarrow G$ be the canonical projection. If there exists a representative Θ of $\bar{\Theta}$ defined on V , such that*

$$\frac{\Theta(p + q + r) \Theta(p) \Theta(q) \Theta(r)}{\Theta(p + q) \Theta(p + r) \Theta(q + r)} = F(\pi p, \pi q, \pi r)$$

whenever $(\pi p, \pi q, \pi r)$ belongs to U ; then X_Θ is a positive divisor and it is the closure in A_Θ of the π -image of the zeros of Θ .

We don't give here other tests to recognize the entire theta functions. On the (quartic?) relations satisfied by these we will return another time.

8. - Abstract theta functions; continuation.

Let k be an algebraically closed field of characteristic $p \neq 0$; we suppose to have given a bidomain \mathbb{C} over k and, by analogy with the previous section, we want to pick out the theta functions among the elements of \mathbb{C} .

If \mathbb{C} is a local bidomain, the problem is completely solved in [6]; therefore we are interested in the global case. This means that we suppose $\mathbb{C} = (\text{Frac } \mathcal{R})^V$, where $V = \mathbb{Q}_p^f$ and $\mathcal{R} = \mathcal{R}_\pi \widehat{\otimes} \mathcal{R}_r$ is a local bidomain with a fixed hyperdomain R inside of it, $R = R_\pi \widehat{\otimes} R_r$. In other words, the candidates for the rôle of theta functions are maps from V and with values in $\text{Frac } \mathcal{R}$. As usual if Θ is an element of \mathbb{C} and P a point of V , we denote by Θ_P the translate of Θ : that is, if $\Theta = (\vartheta_Q)_{Q \in V}$, then $\Theta_P = (\vartheta_{P+Q})_{Q \in V}$.

An element Θ of $(\text{Frac } \mathcal{R})^V$ is said to be *periodic* if $\Theta_P = \Theta$, for each P in $T = \mathbb{Z}_p^f$ and ϑ_Q belongs to $\text{Frac } R$ for each Q in V .

An element Θ of $(\text{Frac } \mathcal{R})^V$ is said to be *admissible* if the k -algebra R_Θ , spanned by Θ and its translates, is an integral domain. If Θ is admissible, we denote by \mathcal{K}_Θ the field $\text{Frac } R_\Theta$.

It's clear that the subset of the periodic elements in \mathcal{K}_Θ is a subfield; we will denote it by K_Θ . At this point we may give a definition corresponding to (7.1):

(8.1) DEFINITION. *A function of type theta is an admissible element, Θ , of $(\text{Frac } \mathcal{R})^V$ which satisfies the following relation*

$$(8.2) \quad \frac{((\iota \widehat{\otimes} \mathbf{P})(\mathbf{P}\Theta))(\Theta \widehat{\otimes} \Theta \widehat{\otimes} \Theta)}{(\mathbf{P}\Theta \widehat{\otimes} 1)(sc_{23}(\mathbf{P}\Theta \widehat{\otimes} 1))(1 \widehat{\otimes} \mathbf{P}\Theta)} \in \text{Frac}(K_\Theta \otimes K_\Theta \otimes K_\Theta),$$

where \mathbf{P} is the coproduct of $(\text{Frac } \mathcal{R})^V$.

Because of the periodicity of the elements of K_Θ , we may represent them as functions defined on $G = (\mathbb{Q}_p/\mathbb{Z}_p)^f$ and with values in R ; thus the left hand of (8.2) will be denoted by

$$F = (F_{(P,Q,R)})_{(P,Q,R) \in G^3}.$$

Such element is the co-cycle of a bi-extension, hence, reasoning as in the Proof of (7.3), we obtain the following

(8.3) THEOREM. *Let Θ be a function of type theta in $(\text{Frac } \mathcal{R})^V$; then there exists a minimal field, containing k and contained in K_Θ , (and therefore in $(\text{Frac } \mathcal{R})^{V/T}$), which we denote by C_Θ , such that F belongs to $\text{Frac}(C_\Theta \otimes C_\Theta \otimes C_\Theta)$. Moreover, C_Θ is a finitely generated k -hyperfield.*

By analogy with section 7, we may show that C_Θ is stable under the action of V , that the kernel of the homomorphism $\varrho: V \rightarrow \text{Aut } C_\Theta$ contains T , and, finally, that ϱV is contained in the group of invariant automorphisms of C_Θ (that is in the group of points of the group-variety A_Θ associated to C_Θ).

We are interested in functions of type theta with the property that ϱV is exactly the group of p -torsion points in A_Θ ; we call such functions *theta functions*.

If Θ is a theta function; the map j , defined on C_Θ , which sends each function $f = (f_P)_{P \in G}$ to its value at 0, f_0 , is an embedding of C_Θ in $\text{Frac } \mathcal{R}$; thus, from (8.3) follows the

(8.4) COROLLARY. *Let $\Theta = (\vartheta_P)_{P \in V}$ be a theta function; then ϑ_0 is a theta element in the sense on n. 7 of [6]. Moreover, the map j , defined above, is an isomorphism between C_Θ and the hyperfield C_{ϑ_0} of theorem (7.1) of [6].*

From this fact it follows that a theta function, $\Theta = (\vartheta_P)_{P \in V}$, is determined by its component ϑ_0 . For this reason, in order to construct the variety and the divisor related to a theta function, we may use the theory of n. 7 of [6].

Before ending this section, we want to mention another situation, in some way dual to the one given in corollary (8.4), related to the fact that the information contained in a theta function is redundant.

We denote by ϱ either the canonical projection (with kernel $\mathcal{R}_+^x \mathcal{R}$) of \mathcal{R} onto \mathcal{R}_r , or the homomorphism from \mathcal{R}^V to \mathcal{R}_r^V induced by such projection. Now we suppose, in order to make the things easier, that Θ is a theta function in \mathcal{R}^V and we denote by R_Θ the k -algebra spanned by Θ and its translates:

$$R_\Theta = k[\{\Theta_P \mid P \in V\}].$$

The case to which we are referring happens when the restriction of ϱ to R_Θ is injective; if this is the case, ϱR_Θ , as R_Θ , is an integral domain and, in this way, we may embed C_Θ into $\text{Frac}(\varrho R_\Theta)$. For instance, if the radical part is missing ($\mathcal{R}_r = k$), $\varrho \Theta$ is a function from V to k . So, in order to characterize such theta functions, we may use the same methods used in section 7 for the l -adic theta functions.

Appendix

Comparison between the classical theta functions and the l -adic ones.

If the ground field is the complex field \mathbf{C} , to each divisor X on A one may associate both the classical theta functions and the l -adic ones, as defined in Section 4. Now we wish to explain the relation between them.

It's suitable to begin with some general remark about the co-cycles of bi-extensions. Thus, let k be a field of characteristic zero and let $R = k[[t]]$

be the ring of formal power series where $(t) = (t^{(1)}, \dots, t^{(g)})$ are indeterminates over k . We will denote the elements of R by series of the kind: $\sum_{\nu \in \mathbb{N}^g} a_\nu t^\nu$, where t^ν means $(t^{(1)})^{\nu_1} \dots (t^{(g)})^{\nu_g}$; then $\partial t^{(i)}$ will denote the differential of $t^{(i)}$ and (d_1, \dots, d_g) will be the dual basis of $(\partial t^{(1)}, \dots, \partial t^{(g)})$. Finally the differential operators will be subject to the same convention which works for monomials; that is d^ν is the operator $d_1^{\nu_1} \dots d_g^{\nu_g}$.

Because over a ground field of characteristic 0 any local formal group is additive, we give the following

(A.1) DEFINITION. An element $F = F(t_1, t_2, t_3)$ of $R \widehat{\otimes} R \widehat{\otimes} R$ is said to be a normalized co-cycle of a bi-extension of R if:

$$(A.2) \quad \begin{cases} F(t_1, t_2, t_3) = F(t_{\sigma_1}, t_{\sigma_2}, t_{\sigma_3}) & \text{for each permutation } \sigma, \\ F(0, t_2, t_3) = 1 \\ F(t_1 + t_2, t_3, t_4) F(t_1, t_2, t_4) = F(t_1, t_2 + t_3, t_4) F(t_2, t_3, t_4). \end{cases}$$

Let F be a normalized co-cycle of a bi-extension of R , that is:

$$F(t_1, t_2, t_3) = 1 + \sum_{\mu, \nu \in \mathbb{N}^g - \{0\}} B_{\mu\nu}(t_1) t_2^\mu t_3^\nu,$$

where the $B_{\mu\nu}(t)$'s are elements of R uniquely determined by F . Moreover, because $\varepsilon B_{\mu\nu} = 0$ (ε is the k -homomorphism defined by setting $\varepsilon t = 0$), $\log F$ exists in $R \widehat{\otimes} R \widehat{\otimes} R$ and equals

$$(A.3) \quad \log F(t_1, t_2, t_3) = \sum_{\mu, \nu \in \mathbb{N}^g - \{0\}} A_{\mu\nu}(t_1) t_2^\mu t_3^\nu,$$

where the $A_{\mu\nu}$'s are in R and $\varepsilon A_{\mu\nu} = 0$.

Now we may state the following

(A.4) THEOREM. Let F be a normalized co-cycle of a bi-extension of R . We denote by e_l ($l = 1, \dots, g$) the element $(\delta_{l1}, \dots, \delta_{lg})$ of \mathbb{N}^g where the δ_{li} 's are the Kronecker deltas. Then, with the notations of (A.3), the differentials

$$\omega_i = \sum_{j=1}^g A_{e_i e_j}(t) \partial t^{(j)}$$

are closed; therefore they are exact in R .

If we denote by $\eta_i = \eta_i(t)$ the integral of ω_i , normalized by putting $\varepsilon\eta_i = 0$, the differential

$$\zeta = \sum_{j=1}^g \eta_j(t) \partial t^{(j)}$$

is also closed.

If we denote by γ the integral of ζ , normalized by putting $\varepsilon\gamma = 0$, and we set $\vartheta = \exp \gamma$, we obtain

$$(A.5) \quad F(t_1, t_2, t_3) = \frac{\vartheta(t_1 + t_2 + t_3) \vartheta(t_1) \vartheta(t_2) \vartheta(t_3)}{\vartheta(t_1 + t_2) \vartheta(t_1 + t_3) \vartheta(t_2 + t_3)}.$$

Finally, if Θ is a solution of (A.5) all the other solutions are of the kind $\vartheta \exp q$, where $q = q(t)$ is a polynomial of second degree and $\varepsilon q = 0$.

PROOF. First of all we want to show that

$$(A.6) \quad d_i A_{e_j e_l} = \bar{d}_j A_{e_l e_i},$$

for each $i, j, l = 1, \dots, g$. Now, from (A.2) and (A.3) we deduce that:

$$(A.7) \quad \sum A_{\mu\nu}(t_1 + t_2) t_3^\mu t_4^\nu + \sum A_{\mu\nu}(t_1) t_2^\mu t_3^\nu = \sum A_{\mu\nu}(t_2 + t_3) t_1^\mu t_4^\nu + \sum A_{\mu\nu}(t_2) t_3^\mu t_4^\nu$$

and now, in order to obtain (A.6) it suffices to apply to both sides of (A.7) the operator

$$\varepsilon \bar{d}_l \widehat{\otimes} \iota \widehat{\otimes} \varepsilon \bar{d}_j \widehat{\otimes} \varepsilon \bar{d}_i,$$

where ι denotes, as usual, the identity operator.

Now, we must show that ζ is closed, that is

$$d_i(\zeta \bar{d}_j) = \bar{d}_j(\zeta \bar{d}_i).$$

It is a straightforward verification because $\bar{d}_i(\zeta \bar{d}_j) = d_i \eta_j = A_{e_j e_i}$ and $\bar{d}_j(\zeta \bar{d}_i) = A_{e_i e_j}$ and from (A.2) it follows that $A_{\mu\nu} = A_{\nu\mu}$ for each μ, ν .

In order to show that ϑ satisfies (A.5), we observe that from the first row of (A.2), it follows that

$$\sum A_{\mu\nu}(t_2 + t_3) t_1^\mu t_4^\nu = \sum A_{\mu\nu}(t_1) (t_2 + t_3)^\mu t_4^\nu.$$

Using this fact, we may write (A.7) in the form

$$(A.8) \quad \begin{aligned} \sum A_{\mu\nu}(t_1 + t_2) t_3^\mu t_4^\nu + \sum A_{\mu\nu}(t_1) t_2^\mu t_4^\nu \\ = \sum A_{\mu\nu}(t_1) (t_2 + t_3)^\mu t_4^\nu + \sum A_{\mu\nu}(t_2) t_3^\mu t_4^\nu; \end{aligned}$$

now, if we apply to both side of (A.8) the operator

$$\iota \otimes \varepsilon d^\mu \otimes \varepsilon d^\nu \otimes \varepsilon d^\lambda,$$

we get

$$(A.9) \quad \nu! \lambda! d^\mu A_{\nu\lambda}(t) = (\mu + \nu)! \lambda! A_{\mu+\nu,\lambda}(t) + \nu! \lambda! \varepsilon(d^\mu A_{\nu\lambda}(t)).$$

In view of the symmetry of $A_{\mu\nu}$, with respect to (μ, ν) , we may rewrite (A.9) in the form:

$$(A.10) \quad A_{\nu+\mu,\lambda+\eta}(t) = \frac{\nu! \lambda!}{(\nu + \mu)! (\lambda + \eta)!} (d^{\mu+\eta} A_{\nu,\lambda} - \varepsilon(d^{\mu+\eta} A_{\nu,\lambda})).$$

This last formula is really expressive; in fact it shows that $\log F$ is completely determined by the differentials ω_i in the statement. Moreover it gives an explicit formula by which to construct the $A_{\mu\nu}$'s from the $A_{e_i e_j}$:

$$(A.11) \quad A_{\mu\nu} = \frac{1}{\mu! \nu!} (d^{\mu+\nu-e_i-e_j} A_{e_i e_j} - \varepsilon(d^{\mu+\nu-e_i-e_j} A_{e_i e_j})),$$

where $\mu \geq e_i, \nu \geq e_j$:

Now, if we recall that $A_{e_i e_j} = \bar{d}_j(\zeta \bar{d}_i) = \bar{d}_j \bar{d}_i \gamma$, we may write (A.11) in the form:

$$A_{\mu\nu} = \frac{1}{\mu! \nu!} (d^{\mu+\nu} \gamma - \varepsilon(d^{\mu+\nu} \gamma)).$$

With such a formula for the $A_{\mu\nu}$'s, we may rewrite (A.3) in the form

$$\begin{aligned} \log F(t_1, t_2, t_3) &= \sum_{\mu, \nu \in \mathbb{N}^g - \{0\}} \frac{1}{\mu! \nu!} d^{\mu+\nu} \gamma(t_1) t_2^\mu t_3^\nu - \sum_{\mu, \nu \in \mathbb{N}^g - \{0\}} \frac{1}{\mu! \nu!} \varepsilon(d^{\mu+\nu} \gamma(t_1)) t_2^\mu t_3^\nu \\ &= \gamma(t_1 + t_2 + t_3) - \gamma(t_1 + t_2) - \gamma(t_1 + t_3) - \gamma(t_2 + t_3) + \gamma(t_1) + \gamma(t_2) + \gamma(t_3), \end{aligned}$$

where the last equality follows from the Taylor's formula. At this point it is clear that one has (A.5) by applying \exp to the end sides of the last equalities. Thus, the direct relation between F and ϑ is

$$d^{e_i+e_j} \log \vartheta = (\iota \widehat{\otimes} \varepsilon \bar{d}_i \widehat{\otimes} \varepsilon \bar{d}_j) \log F.$$

Finally, the statements about the quadratic exponentials are immediate and the proof is complete.

Now, let A be an abelian variety of dimension g over k ; if we denote by R the affine algebra of the completion of A at the identity point e_A ,

it is well known that one may choose a set of regular parameters of R , $(t^{(1)}, \dots, t^{(g)})$, such that $(p_1 + p_2)^* t^{(j)} = t_1^{(j)} + t_2^{(j)}$: $(t^{(1)}, \dots, t^{(g)})$ is a basis of the normalized ($\epsilon t^{(j)} = 0$) integrals of the first kind. By such a choice of parameters, to each divisor X on A , which doesn't meet e_A , one may associate a unique normalized co-cycle of a bi-extension of R . By Theorem (A.4) we get an elementary method of construction of theta functions associated to X .

Theta functions of this kind were constructed for the first time in [1], where the structure of the De Rham cohomology was used as a crucial tool.

If, in particular, $k = \mathbf{C}$ is the field of complex numbers, and if we denote by U the universal covering space of A , by $\pi: U \rightarrow A$ the canonical projection and by $\Omega = \ker \pi$ the lattice, then we may interpret R as the affine algebra of the completion of U at the identity point e_U (a basis of integrals of the first kind of A is, in natural way, a set of coordinate on U). In this case, if X is a positive divisor, the solutions of (A.5) are germs of entire functions on U ; we identify such germs and the functions determined by them: these are the *classical entire theta functions*.

We observe here that an entire functions on U is clearly determined by its values on the subgroup $U_l = \varinjlim (\Omega \hookrightarrow 1/l\Omega \hookrightarrow 1/l^2\Omega \dots)$ of U , where l is a rational prime. What allows us to perform the comparison is the embedding j of U_l into the l -adic Tate space V , defined by

$$j(z) = (\pi z, \pi(1/lz), \dots),$$

We have, in fact, the following

(A.12) THEOREM. *Let X be a positive divisor on A and $\Theta: V \rightarrow \mathbf{C}$ one of the l -adic theta functions of X . Then the function defined on U_l by sending z to $\Theta(j(z))$, is the restriction to U_l of a classical theta of X .*

PROOF. If Θ is a l -adic theta function of X , it results

$$(A.13) \quad \frac{\Theta(j(z_1 + z_2 + z_3))\Theta(j(z_1))\Theta(j(z_2))\Theta(j(z_3))}{\Theta(j(z_1 + z_2))\Theta(j(z_1 + z_3))\Theta(j(z_2 + z_3))} = F(\pi z_1, \pi z_2, \pi z_3),$$

for each (z_1, z_2, z_3) in U_l^3 such that $\pi z_i + \pi z_j \notin (p_i + p_j)^{-1}X$ (cfr. section 4). But U_l is dense in U and the map $z \rightarrow \Theta(j(z))$ is continuous in U_l ; therefore its extension to U , which we denote by ϑ , satisfies (A.5) and hence it is a classical theta, Q.E.D.

In this way we have proved that, given an l -adic theta function Θ and a classical theta function ϑ , both associated to the same divisor X , there

exists a quadratic exponential q on U_i with values in \mathbf{C}^* , such that

$$\Theta(j(z)) = \vartheta(z)q(z),$$

for each z in U_i .

Now we want determine q in some special case. It's well known that one of the solutions of (A.5) is a ϑ such that

$$(A.14) \quad \vartheta(z + \omega) = \vartheta(t)\chi(\omega) \exp 2\pi i \left(-\frac{i}{2} H(z, \omega) - \frac{i}{4} H(\omega, \omega) \right),$$

where H is a hermitian pairing on $U \times U$, defined by the \mathbf{R} -bilinear skew-symmetric real-valued pairing E by putting

$$H(z, z') = E(iz, z') + iE(z, z'),$$

and where χ is a second degree character of Ω , which satisfies the equation

$$(A.15) \quad \frac{\chi(\omega + \omega')}{\chi(\omega)\chi(\omega')} = \exp i\pi E(\omega, \omega')$$

for each ω, ω' in Ω ; thus it is determined up to a character. But, for any decomposition of U as a sum of maximal isotropic subspaces with respect to E : $U = U_1 \oplus U_2$, where $U_i = \Omega_i \otimes_{\mathbf{Z}} \mathbf{R}$ and $\Omega = \Omega_1 \oplus \Omega_2$, one may find a solution of (A.15) by putting:

$$\chi(\omega) = \exp i\pi E(\omega_1, \omega_2),$$

where ω_i is the projection of ω to Ω_i . By such a choice of χ , if ϑ is normalized by the assumption $\vartheta(0) = 1$, from (A.14) we get

$$(A.16) \quad \vartheta(\omega) = \exp i\pi E(\omega_1, \omega_2) \exp \frac{\pi}{2} H(\omega, \omega).$$

Now we state the last

(A.17) PROPOSITION. *Let X be positive and totally symmetric; then if Θ denotes the unique l -adic theta of X of theorem (5.12), and ϑ denotes the unique classical theta of X which satisfies (A.14) and which assume on Ω the values*

determined by (A.16), there results:

$$\text{i) } \vartheta(z) = \Theta(j(z)) \exp \pi i E(z_1, z_2) \exp \frac{\pi}{2} H(z, z), \quad \text{if } \mathfrak{l} \neq 2$$

$$\text{ii) } \vartheta(z) = \Theta(j(z)) \exp \frac{\pi}{2} H(z, z), \quad \text{if } \mathfrak{l} = 2,$$

for each z in $U_{\mathfrak{l}}$.

PROOF. First of all, we observe that from [9] (cfr. also [5]) one has that the pairing e of section 5, which gives the automorphy factors of the \mathfrak{l} -adic theta functions by (5.13), (5.14), is related to E in the following way:

$$(A.18) \quad \exp(-2\pi i E(z, z')) = e(j(z), j(z')).$$

Now, if $\mathfrak{l} \neq 2$, from i), (5.13) and (A.18), one gets

$$a) \quad \frac{\vartheta(\omega)}{\Theta(j(\omega))} = \exp i\pi E(\omega_1, \omega_2) \exp \frac{\pi}{2} H(\omega, \omega),$$

and analogously, from ii), and (5.14):

$$b) \quad \frac{\vartheta(\omega)}{\Theta(j(\omega))} = \frac{\exp i\pi E(\omega_1, \omega_2) \exp(\pi/2) H(\omega, \omega)}{\exp(-i\pi E(\omega_1, \omega_2))} = \exp \frac{\pi}{2} H(\omega, \omega),$$

for each ω in Ω .

At this point, in order to obtain i) and ii) of the statement, it suffices to remark that a second degree character q from $U_{\mathfrak{l}}$ to \mathbf{C}^* , which is Ω -periodic, is a character; but there are no non-trivial characters q such that $q(z) = q(-z)$, Q.E.D.

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