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Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time

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Well-Posedness in the Gevrey Classes of the Cauchy Problem for a Non-Strictly Hyperbolic Equation with Coefficients Depending on Time.

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1. – Introduction.

We shall consider here the Cauchy problem

$$\text{(1)} \qquad \begin{cases} u_{tt} - \sum_{i,j}^{1,n} a_{ij}(t)\, u_{x_i x_j} = 0 \\ u(x, 0) = \varphi(x)\,, \qquad u_t(x, 0) = \psi(x) \end{cases}$$

on $\mathbb{R}^n \times [0, T]$, under the *non-strict hyperbolicity* condition

$$\text{(2)} \qquad \sum a_{ij}(t)\xi_i\xi_j \geqslant 0\,, \qquad \forall \xi \in \mathbb{R}^n\,.$$

It is known (see [1]) that (1) is *well-posed* [1] in the space $\mathcal{A}$ of analytic functions on $\mathbb{R}^n$, whenever the coefficients belong to $L^1([0, T])$. On the other side (1) can fail to be well posed in the class $\mathcal{E}$ of the C^∞ functions, even if the coefficients are C^∞ (see [2]).

The aim of this paper is to prove the well-posedness of (1) in some Gevrey class $\mathcal{E}^s$, assuming only the minimum of regularity on the coefficients.

Going into detail, we shall prove (see th. 1 and Remark 2 below) that:

If the coefficients $a_{ij}(t)$ *belong to* $C^{k,\alpha}([0, T])$, *with* k *integer* $\geqslant 0$ *and* $0 \leqslant \alpha \leqslant 1$, *then problem* (1) *is well posed in the Gevrey class* $\mathcal{E}^s$ *provided that*

$$\text{(3)} \qquad 1 \leqslant s < 1 + \frac{k + \alpha}{2}\,.$$

If the coefficients are analytic on $[0, T]$, *then* (1) *is well posed in* $\mathcal{E}$.

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(1) We shall say that problem (1) is well-posed in some space $\mathcal{F}$ of functions on functionals on $\mathbb{R}^n_x$ if for any φ, ψ in $\mathcal{F}$ it admits one and only one solution u in $C^1([0, T], \mathcal{F})$.

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Such a result is *optimal*, in the sense that *there exist* $a_{ij}(t)$ *of class* $C^{k,\alpha}$ *and* $\varphi(x)$, $\psi(x)$, *belonging to* $\mathcal{E}^s$ *for every* $s > 1 + (k + \alpha)/2$, *for which problem* (1) *is not solvable in the space of distributions* (see § 4 below).

It can be expected that similar results also hold for the more general hyperbolic equation

$$u_{tt} - \sum \left(a_{ij}(x, t) u_{x_i}\right)_{x_j} = 0 .$$

For instance, we can conjecture that the Cauchy problem for such an equation is well-posed in $\mathcal{E}^s$ when the coefficients $a_{ij}(x, t)$ belong to $C^{k,\alpha}([0, T], \mathcal{E}^s)$ while k, α, s satisfy (3) (see [6] for the case $k = \alpha = 0$, $s = 1$), and that it is well-posed in $\mathcal{E}$ when the coefficients are analytic in t and C^∞ in x (cf. OLEINIK [8] and NISHITANI [7]).

A consequence of th. 1 is that (1) is well posed in every Gevrey class when the coefficients $a_{ij}(t)$ are C^∞. In this connexion we can observe that such a conclusion can become false if we replace the equation in (1) by a non homogeneous equation as

$$u_{tt} - \sum a_{ij}(t) u_{x_i x_j} + \sum b_i(t) u_{x_i} = 0 . \tag{4}$$

(For instance, if we consider the equation $u_{tt} - u_x = 0$ the corresponding Cauchy problem is well-posed in $\mathcal{E}^s$ only if $1 \leqslant s < 2$).

Here (Remark 2 below) we get also some result for an equation like (4). For instance we prove that the Cauchy problem for (4), with $a_{ij}(t)$ in $C^{k,\alpha}([0, T])$ and $b_i(t)$ in $L^1([0, T])$, is well-posed in $\mathcal{E}^s$ for

$$1 \leqslant s < 1 + \operatorname{Min}\left\{1, \frac{k + \alpha}{2}\right\}.$$

As a special case we have the well-posedness in every $\mathcal{E}^s$ with $1 \leqslant s < 2$ as soon as the a_{ij} have first derivatives Lipschitz-continuous and the b_i are integrable on $[0, T]$.

An extensive study of the necessary Levi conditions for the well-posedness in the Gevrey classes has been made by Ivrii and Petkov in [5].

Finally we remark that the present paper can be considered an extension of [1], where problem (1) was extensively studied under the *strict hyperbolicity* condition

$$\sum a_{ij}(t) \xi_i \xi_j \geqslant \lambda_0 |\xi|^2 \qquad (\lambda_0 > 0) . \tag{5}$$

In this case, to get the well-posedness in $\mathcal{E}$ of the Cauchy problem (1) it is sufficient that the coefficients $a_{ij}(t)$ are Lipschitz-continuous, while a very little regularity on the a_{ij} insures the well-posedness in some Gevrey class. More precisely (see [1]) *if the* $a_{ij}(t)$ *belong to* $C^{0,\alpha}([0, T])$, *the Cauchy problem* $\{(1), (5)\}$ *is well posed in* $\mathcal{E}^s$ *for*

$$1 \leqslant s < 1 + \frac{\alpha}{1-\alpha}.$$

The techniques used in the present paper are fundamentally the same of [1], namely the Fourier-Laplace transform and the *approximate energy* estimate. Besides this, we shall use the following result of real analysis (Lemma 1 below): *if* $f(t)$ *is a function* $\geqslant 0$ *of class* $C^{k,\alpha}$ *on* $[0, T]$, *then* $f^{1/(k+\alpha)}$ *is absolutely continuous on* $[0, T]$. We have not been able to find this result in the literature, but for the case $k = 2$, $\alpha = 0$ (Gleaser [4], see also Dieudonné [3]). For this reason we shall exhibit a proof (see § 2) of it. Such a proof has been essentially suggested to us by F. Conti, whom we thank warmly.

NOTATIONS:

$\mathcal{H}$ is the topological vector space of entire functions on $\mathbb{R}^n$.

$\mathcal{A}$ is the t.v.s. of analytic functions on $\mathbb{R}^n$.

$\mathcal{E}^s$, for s real $\geqslant 1$, is the t.v.s. of Gevrey functions on $\mathbb{R}^n$, i.e. the C^∞ functions φ verifying

$$|D^r\varphi(x)| \leqslant C_K A_K^{|r|} |r|^{s|r|}, \qquad \forall x \in K,\ \forall r,$$

for any compact subset $K \subset \mathbb{R}^n$.

When $s = 1$, we have the coincidence $\mathcal{E}^1 = \mathcal{A}$.

$\mathcal{E}$ is the t.v.s. of C^∞ functions on $\mathbb{R}^n$.

$\mathcal{D}$ is the t.v.s. of C^∞ functions with compact support in $\mathbb{R}^n$

$\mathcal{D}^s = \mathcal{E}^s \cap \mathcal{D}$.

$\mathcal{H}'$, $\mathcal{A}'$, $\mathcal{D}'$, $(\mathcal{D}^s)'$ are the dual spaces of $\mathcal{H}$, $\mathcal{A}$, $\mathcal{D}$, $\mathcal{D}^s$.

All these spaces are endowed by the usual topologies.

$C^k([0, T], \mathcal{F})$, with $\mathcal{F}$ equal to one of the t.v.s. introduced above. is the t.v.s. of functions $u\colon [0, T] \to \mathcal{F}$ having k continuous derivatives on $[0, T]$. The elements u of $C^k([0, T], \mathcal{F})$ shall be treated, as usual, as functions or functionals on $\mathbb{R}^n \times]0, T[$. In this sense we shall write $u(x, t)$, $\partial u/\partial x_j$, $\partial u/\partial t$.

$C^{k,\alpha}([0, T])$, with k integer $\geqslant 0$ and $0 \leqslant \alpha \leqslant 1$, is the Banach space of the functions having k derivatives continuous on $[0, T]$, and the k-th derivative Hölder-continuous with exponent α when $\alpha > 0$.

The norm in this space is

$$\|u\|_{C^{k,\alpha}} = \sum_{h=0}^{k} \sup_{[0,T]} |u^{(h)}| + \sup_{t \neq s} |u^{(k)}(t) - u^{(k)}(s)| |t - s|^{-\alpha}.$$

2. - A lemma of real analysis.

LEMMA 1. *Let $f(t)$ be a real function of class $C^{k,\alpha}$ on some compact interval $I \subset \mathbb{R}$, with k integer $\geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, and assume that*

$$f(t) \geqslant 0 \quad \text{on } I.$$

Then the function $f^{1/(k+\alpha)}$ is absolutely continuous on I. Moreover

$$\|(f^{1/(k+\alpha)})'\|_{L^1(I)}^{k+\alpha} \leqslant C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}. \tag{6}$$

PROOF. The conclusion of the Lemma is obvious when $k = 1$, $\alpha = 0$. Moreover the case $k = \nu \geqslant 2$, $\alpha = 0$, can be reduced to the case $k = \nu - 1$, $\alpha = 1$. Thus we shall consider only the case $\alpha > 0$.

Let us firstly assume that $f(t) > 0$ on I. In such a case the function $f^{1/(k+\alpha)}$ is C^1 as well as f, and we must only prove that

$$\Big(\int_I f^{(1/(k+\alpha))-1} |f'|\, dt\Big)^{k+\alpha} \leqslant C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}. \tag{7}$$

In order to treat the general case $\big(f(t) \geqslant 0\big)$ we must only approximate $f(t)$ by $f(t) + \varepsilon$, $\varepsilon \to 0$. Since $(f + \varepsilon)^{1/(k+\alpha)-1} |f'|$ is increasing for ε decreasing to zero, then, by Beppo Levi's theorem and inequality (7) for $f + \varepsilon$, we get that the functions $(f + \varepsilon)^{1/(k+\alpha)-1} |f'|$ are equi-integrable on I. This gives the conclusion of Lemma 1.

Hence we assume that $f(t) > 0$ on I and we are aiming at inequality (7). We shall also can assume, without a real loss of generality, that f is C^∞ on I.

Now let $\mathcal{T} \equiv \{x_0, x_1, \ldots, x_N\}$, with $a = x_0 < x_1 < \ldots < x_N = b$, be a *partition* of $I \equiv [a, b]$. We define, for every real function g on I,

$$V_s(\mathcal{T}, g) = \sum_{j=0}^{N-1} |g(x_{j+1}) - g(x_j)|^{1/s}, \qquad s > 0, \tag{8}$$

and

$$V_s^*(g) = \operatorname*{Sup}_{\mathcal{T}\in P(g)} V_s(\mathcal{T}, g)\,,$$

where $P(g)$ is the class of partitions $\mathcal{T}$ of I such that

$$(9) \qquad g'(x_j) = 0 \quad \text{for } j = 1, \dots, N-1\,.$$

We claim that the following inequalities hold:

$$(10) \qquad \operatorname{Var}(g) \leqslant V_1^*(g)\,,$$

$$(11) \qquad V_1^*(|g|^{1/s}) \leqslant V_s^*(g)\,, \quad \text{for } s \geqslant 1\,,$$

$$(12) \qquad V_s^*(g) \leqslant \|g\|_{C^{0,s}(I)} \cdot |I|\,, \quad \text{for } 0 < s \leqslant 1\,,$$

$$(13) \qquad V_s^*(g) \leqslant [V_{s-1}^*(g')^{(s-1)/s} + \|g'\|_{C^0(I)}^{1/s}]|I|^{1/s}\,, \quad \text{for } s > 1\,,$$

where $|I|$ denotes the length of I and $\operatorname{Var}(g)$ the variation on I of g, i.e. the supremum of $V_1(\mathcal{T}, g)$ as $\mathcal{T}$ runs in the class of *all* partitions of I.

From these inequalities it is easy to derive (7), i.e. the conclusion of the Lemma.

Indeed, by applying successively (13) with $g = f$ and $s = k + \alpha$; $g = f'$ and $s = k + \alpha - 1; \dots; g = f^{(k-1)}$ and $s = \alpha + 1$; and finally using (12) with $g = f^{(k)}$ and $s = \alpha$, we get

$$(14) \qquad V_{k+\alpha}^*(f) \leqslant C_0(k, \alpha, |I|)\|f\|_{C^{k,\alpha}(I)}^{1/(k+\alpha)} \quad (k \geqslant 1;\ 0 < \alpha \leqslant 1)\,.$$

Now from (10), (11) and (14) it follows

$$\operatorname{Var}(f^{1/(k+\alpha)}) \leqslant V_1^*(f^{1/(k+\alpha)}) \leqslant V_{k+\alpha}^*(f) \leqslant C_0(k, \alpha, |I|)\|f\|_{C^{k,\alpha}(I)}^{1/(k+\alpha)}$$

and hence (7).

Let us then prove (10), (11), (12) and (13).

In order to prove (10) we show that for every partition $\mathcal{T}$ on I, there exists another partition $\tilde{\mathcal{T}}$ verifying (9) and such that

$$(15) \qquad V_1(\mathcal{T}, g) \leqslant V_1(\tilde{\mathcal{T}}, g)\,.$$

To this end, if $\mathcal{T} = \{x_0, \dots, x_N\}$ we consider these values of j such that g' has some zero on $[x_j, x_{j+1}]$ and correspondingly we denote by y_j and z_j respectively the first and the last of these zeros. Then the partition $\tilde{\mathcal{T}}$ whose endpoints are a, b, y_j, z_j belongs to $P(g)$ and verifies (15).

Inequalities (11) and (12) are obvious.

In order to get inequality (13) it is sufficient to prove that for any partition $\mathcal{T}$ belonging to $P(g)$, i.e. verifying (9), there exists a partition $\tilde{\mathcal{T}} \in P(g')$ in such a way that

$$V_s(\mathcal{T}, g) \leqslant \left(V_{s-1}(\tilde{\mathcal{T}}, g')^{(s-1)/s} + \|g'\|^{1/s}_{C^0(I)} \right) |I|^{1/s} \tag{16}$$

for $s > 1$.

To this end, if $\mathcal{T} = \{x_0, x_1, \ldots, x_N\}$, we denote by y_j the first point of maximum of $|g'|$ on the interval $[x_j, x_{j+1}]$, for $j = 0, 1, \ldots, N-1$. Afterwards we denote by z_j the first point of minimum (resp. of maximum) of g' on the interval $[y_j, y_{j+1}]$ if $g'(y_j) \geqslant 0$ (resp. $g'(y_j) \leqslant 0$), for $j = 0, 1, \ldots, N-2$. In particular, taking into account that $g'(x_{j+1}) = 0$ and x_{j+1} belongs to $[y_j, y_{j+1}]$, we have

$$g'(y_j) \cdot g'(z_j) \leqslant 0 \; . \tag{17}$$

Now let $\tilde{\mathcal{T}}$ be the partition having as endpoints a, b and y_j, z_j. We shall verify that $\tilde{\mathcal{T}}$ belongs to $P(g')$, i.e. g'' vanishes at every endpoint different from a and b, and that (16) holds.

Let y_j be different from a and b. Two cases are then possible: either y_j lies at the interior of $[x_j, x_{j+1}]$, or it coincides with x_j or with x_{j+1}. In the first case we get immediately that $g''(y_j) = 0$; in the second case we know that $g'(y_j) = 0$ since $\mathcal{T}$ verifies (9), and by consequence g' must be identically zero on $[x_j, x_{j+1}]$. In both cases we have $g''(y_j) = 0$.

Let now z_j be different from a and b. Since $z_j \in [y_j, y_{j+1}]$, if z_j is equal to y_j or to y_{j+1} we have just seen that $g''(z_j) = 0$, while if z_j is internal to $[y_j, y_{j+1}]$ we get obviously $g''(z_j) = 0$.

Thus $\tilde{\mathcal{T}}$ belongs to $P(g')$.

It remains only to verify (16). Now, remembering the definition of y_j and using (17) and the Hölder inequality, we get, for $s > 1$,

$$\begin{aligned}
\sum_{j=0}^{N-1} |g(x_{j+1}) - g(x_j)|^{1/s} &\leqslant \sum_{j=0}^{N-1} |g'(y_j)|^{1/s} |x_{j+1} - x_j|^{1/s} \\
&\leqslant \sum_{j=0}^{N-2} |g'(y_j) - g'(z_j)|^{1/s} |x_{j+1} - x_j|^{1/s} + |g'(y_{N-1})|^{1/s} |x_N - x_{N-1}|^{1/s} \\
&\leqslant \left[\sum_{j=0}^{N-2} |g'(y_j) - g'(z_j)|^{1/(s-1)} \right]^{(s-1)/s} |I|^{1/s} + \|g'\|^{1/s}_{C^0(I)} |I|^{1/s},
\end{aligned}$$

whence (16) follows.

This completes the proof of Lemma 1. //

3. – The existence theorem.

THEOREM 1. *Let us consider the problem*

$$\left\{\begin{array}{l} u_{tt}-\sum_{i,j}^{1,n} a_{ij}(t)u_{x_ix_j}=0 \\ u(x,0)=\varphi(x)\,, \qquad u_t(x,0)=\psi(x) \end{array}\right. \tag{18}$$

on $\mathbb{R}^n\times[0,T]$, *assuming that*

$$\sum a_{ij}(t)\xi_i\xi_j\geqslant 0\,, \qquad \forall\xi\in\mathbb{R}^n\,, \tag{19}$$

and

$$a_{ij}\in C^{k,\alpha}([0,T])\,, \qquad k \text{ integer } > 0,\ 0\leqslant\alpha\leqslant 1\,. \tag{20}$$

Then for every φ *and* ψ *in* $\mathcal{E}^s$, *the problem admits one and only one solution* $u\in C^2([0,T],\mathcal{E}^s)$, *provided that*

$$1\leqslant s<1+\frac{k+\alpha}{2}\,. \tag{21}$$

REMARK 1. When $k=\alpha=0$, (21) does not make sense However in [1], § 8, has been proved that problem (18) is well posed in $\mathcal{E}^1$ $(=\mathcal{A})$ whenever the coefficients a_{ij} belong to $C^0([0,T])$, or even to $L^1([0,T])$.

PROOF OF TH. 1. We can devote ourselves to the case $s>1$ (see Remark 1 here above).

The coefficients $a_{ij}(t)$ are taken continuous on $[0,T]$, thus we can assume that, for some $\Lambda>0$,

$$\sum a_{ij}(t)\xi_i\xi_j\leqslant\Lambda|\xi|^2\,, \qquad \forall\xi\in\mathbb{R}^n,\ \forall t\,. \tag{22}$$

By Holmgren's theorem we know that every solution $u(x,t)$ of (18), whose initial data are identically zero on some ball $\{|x-x_0|\leqslant r\}$, is zero on the cone $\{|x-x_0|\leqslant r-(1+\Lambda)t\}$ (more precisely, $u\equiv 0$ on the cone $\{|x-x_0|\leqslant r-\sqrt{\Lambda}t$; cf. [1], formula (90)).

This fact gives the uniqueness of solutions to (18), and moreover allows us to reduce ourselves to the case of initial data having a compact support in $\mathbb{R}^n$

Hence we assume, from now on, that $\varphi(x)$ and $\psi(x)$ belong to $\mathcal{D}^s$

Now $\mathcal{D}^s$ is a subspace of the space $\mathcal{H}'$ of holomorphic functionals on $\mathbb{C}^n$ and the Ovciannikov theorem ensures the well-posedness of (18) in $\mathcal{H}'$ (even without the hyperbolicity assumption (19)). Hence (18) admits a solution $u \in C^2([0, T], \mathcal{H}')$: our task is to prove that u belongs to $C^2([0, T], \mathcal{D}^s)$ when (19) and (21) are satisfied. To this purpose, denoting by

$$v(\xi, t) = \langle u(x, t), \exp[-i(\xi, x)]\rangle , \qquad \xi \in \mathbb{R}^n ,$$

the Fourier transform of u with respect to x, it will be sufficient to prove that

$$|v(\xi, t)| \leqslant M \exp[-\delta|\xi|^{1/s}] \tag{23}$$

for every $\xi \in \mathbb{R}^n$ and $t \in [0, T]$, and some $M, \delta > 0$.

Indeed from (23) it follows, in virtue of Paley-Wiener theorem, that $u(\cdot, t)$ belongs to $\mathcal{D}^s$ or rather that $\{u(\cdot, t) | t \in [0, T]\}$ is bounded in $\mathcal{D}^s$. Thus, taking into account that u is a solution of (18), (23) gives that $u \in C^2([0, T], \mathcal{D}^s)$.

Let us hence prove inequality (23), assuming that $\hat{\varphi}(\xi)$ and $\hat{\psi}(\xi)$, i.e. the Fourier transforms of the initial data, verify an analogous inequality and that $1 < s < 1 + (k + \alpha)/2$.

By Fourier transform, (18) becomes

$$v'' + (a(t)\xi, \xi)v = 0 , \qquad t \in [0, T] , \tag{24}$$

where we have put

$$(a(t)\xi, \xi) = \sum a_{ij}(t)\xi_i\xi_j .$$

Now we approximate $a(t)$, in a suitable way, by a family $\{a_\varepsilon(t)\}_{\varepsilon>0}$ of C^1 stricly positive quadratic forms, and we introduce, for any $\varepsilon > 0$, the ε-approximate energy of u

$$E_\varepsilon(\xi, t) = (a_\varepsilon(t)\xi, \xi)|v|^2 + |v'|^2 . \tag{25}$$

Our goal will be to get a good estimate of the growth of E_ε as $|\xi| \to \infty$.

By differentiating in t, we have

$$E'_\varepsilon(\xi, t) = (a'_\varepsilon\xi, \xi)|v|^2 + 2(a_\varepsilon\xi, \xi)\operatorname{Re}(v\bar{v}') + 2\operatorname{Re}(\bar{v}'v'') ,$$

whence, taking (24) into account,

$$E'_\varepsilon \leqslant |(a'_\varepsilon\xi, \xi)||v|^2 + 2|((a_\varepsilon - a)\xi, \xi)||v||v'|$$

i.e.

$$E'_\varepsilon \leqslant \frac{|(a'_\varepsilon \xi, \xi)|}{(a_\varepsilon \xi, \xi)} E_\varepsilon + \frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} E_\varepsilon .$$

By Gronwall lemma we then derive, $\forall t \in [0, T]$,

$$E_\varepsilon(\xi, t) \leqslant E_\varepsilon(\xi, 0) \exp\left[\int_0^T \frac{|(a'_\varepsilon \xi, \xi)|}{(a_\varepsilon \xi, \xi)} ds + \int_0^T \frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} ds\right]. \tag{26}$$

Let us now define the approximating coefficients $a_\varepsilon(t)$, by considering separately the case in which $a(t)$ belongs to $C^{k,\alpha}$ with $k \geqslant 1$ and the case in which $a(t)$ belongs to $C^{0,\alpha}$.

In the first case we take

$$a_\varepsilon(t) = a(t) + \varepsilon I ,$$

where I denotes the identity matrix.

We have then obviously

$$(a_\varepsilon \xi, \xi) \geqslant (a_\varepsilon \xi, \xi)^{1-1/(k+\alpha)} \big(\varepsilon |\xi|^2\big)^{1/(k+\alpha)} \tag{27}$$

and

$$\frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} \leqslant \sqrt{\varepsilon}\, |\xi| . \tag{28}$$

On the other hand, using Lemma 1 with $f(t) = \big(a(t)\xi, \xi\big)$ and remarking that $\mathrm{Var}_{[0,T]}(a_\varepsilon \xi, \xi) = \mathrm{Var}_{[0,T]}(a\xi, \xi)$, we get

$$\int_0^T \frac{(a'_\varepsilon \xi, \xi)}{(a_\varepsilon \xi, \xi)^{1-1/(k+\alpha)}} ds \leqslant C(k, \alpha, T) \|a\|_{C^{k,\alpha}}^{1/(k+\alpha)} |\xi|^{2/(k+\alpha)}. \tag{29}$$

Introducing (27), (28) and (29) in (26), we obtain then the estimate

$$E_\varepsilon(\xi, t) \leqslant E_\varepsilon(\xi, 0) \exp\left[C_1\big(\varepsilon^{-1/(k+\alpha)} + \sqrt{\varepsilon}|\xi|\big)\right], \tag{30}$$

where $C_1, \ldots, C_i, \ldots$ denote constants depending only on $\|a\|_{C^{k,\alpha}([0,T])}$:

Now let us compare the ε-energy E_ε with the functional E defined as

$$E(\xi, t) = |\xi|^2 |v(\xi, t)|^2 + |v'(\xi, t)|^2 .$$

We see immediately that

$$\varepsilon E(\xi, t) \leqslant E_\varepsilon(\xi, t) \leqslant (1 + \Lambda) E(\xi, t)$$

for $\varepsilon < 1$, Λ being defined by (22).

By consequence, (30) with $\varepsilon = (1 + |\xi|)^{-2(k+\alpha)/(2+k+\alpha)}$ gives

$$(31) \qquad E(\xi, t) \leqslant C_2 (1 + |\xi|)^{2(k+\alpha)/(2+ \quad +\alpha)} E(\xi, 0) \exp [C_3 |\xi|^{2/(2+k+\alpha)}] .$$

But the initial data φ, ψ of (18) belong to $\mathfrak{D}^s$, thus their transforms $\hat{\varphi}(\xi)$, $\hat{\psi}(\xi)$, and consequently $E(\xi, 0)$, can be estimated by $M_0 \cdot \exp(-\delta_0 |\xi|^{1/s})$ for some M_0, $\delta_0 > 0$.

Therefore by (31) we get

$$E(\xi, t) \leqslant M_0 C_4 \exp\left(-\frac{\delta_0}{2} |\xi|^{1/s} + C_3 |\xi|\right)$$

and hence (23), as $1/s > 2/(2 + k + \alpha)$.

Let us now pass to examine the case $k = 0$, in which $a(t)$ belongs to $C^{0,\alpha}([0, T])$. In this case we must not only make $a(t)$ strictly positive but also regularise it.

We then take

$$a_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^{+\infty} \tilde{a}(t + s)\, \varrho\left(\frac{s}{\varepsilon}\right) ds + \varepsilon^\alpha I ,$$

where $\tilde{a}(t)$ is the continuation of $a(t)$ on $[0, +\infty[$ such that $\tilde{a} \equiv a(T)$ on $[T, +\infty[$, and $\varrho(t)$ is a non negative C^∞ function such that $\varrho \equiv 0$ on $]-\infty, 0]$ and on $[1, +\infty[$, and $\int_{-\infty}^{+\infty} \varrho\, ds = 1$.

The α-Hölder continuity of $a(t)$ gives

$$\int_0^T |(a'_\varepsilon \xi, \xi)|\, ds \leqslant C_5 \varepsilon^{\alpha - 1} |\xi|^2$$

and

$$\int_0^T |((a_\varepsilon - a)\xi, \xi)|\, ds \leqslant C_6 \varepsilon^\alpha |\xi|^2 ,$$

while by definition

$$(a_\varepsilon \xi, \xi) \geqslant \varepsilon^\alpha |\xi|^2 .$$

Introducing these estimates in (26) we get

$$E_\varepsilon(\xi, t) \leqslant E_\varepsilon(\xi, 0) \exp [C_7(\varepsilon^{-1} + \varepsilon^{\alpha/2}|\xi|)] .$$

From now on, we proceed in the same manner that in the case $k \geqslant 1$. The only difference is the choice of ε, now taken equal to $(1 + |\xi|)^{-2/(2+\alpha)}$
In both cases, (23) is obtained and the theorem is proved. //

REMARK 2. As a corollary of th. 1, we have that problem (18) is well posed in $\mathcal{E}^s$ for every $s \geqslant 1$, when the coefficients $a_{ij}(t)$ are C^∞ on $[0, T]$.

Concerning the well-posedness in $\mathcal{E}$, we must assume further regularity on the a_{ij} (see the example of [2]).

For instance, when the $a_{ij}(t)$ are analytic on $[0, T]$ it is easy to prove that (18) is well posed in $\mathcal{E}$. Indeed, in virtue of the analyticity, one can prove that $(a'(t)\xi, \xi)$ admits at most N isolated zeros for every $\xi \in \mathbb{R}^n$, with N independent on ξ. Therefore

$$\int_0^T \frac{(a'(t)\xi, \xi)}{(a(t)\xi, \xi) + \varepsilon|\xi|^2}\, dt \leqslant (N+1) \log \frac{\Lambda + \varepsilon}{\varepsilon} ,$$

where Λ is defined by (22). Thus, going back to the proof of th. 1, we see that (26) becomes

$$E_\varepsilon(\xi, t) \leqslant E_\varepsilon(\xi, 0) \exp\left((N+1) \log \frac{\Lambda + \varepsilon}{\varepsilon} + \sqrt{\varepsilon}\, |\xi| T\right).$$

Hence, taking $\varepsilon = |\xi|^{-2}$, we obtain that $E(\xi, t)/E(\xi, 0)$ has a polynomial growth for $|\xi| \to \infty$, so that (18) is well posed in $\mathcal{E}$.

REMARK 3. Let us consider the more general equation

$$u_{tt} - \sum a_{ij}(t) u_{x_i x_j} + \sum b_i(t) u_{x_i} + c(t) u + d(t) u_t = 0 \tag{32}$$

where the a_{ij} are in $C^{k,\alpha}([0, T])$, k integer > 0 and $0 < \alpha \leqslant 1$, and satisfy (2), while b_i, c and d belong to $L^1([0, T])$.

Moreover let us assume the following sort of Levi's condition:

$$\left| \sum b_i(t)\xi_i \right| \leqslant \lambda(t, \xi)\left(\sum a_{ij}(t) \xi_i \xi_j \right)^\beta$$

for some $\beta \in [0, \frac{1}{2}]$ and some λ such that

$$\operatorname*{Sup}_{|\xi|=1} \int_0^T \lambda(t, \xi)\, dt < +\infty .$$

Therefore, using the same technique of th. 1, we can prove that the Cauchy problem for the equation (32) is well posed in $\mathcal{E}^s$ for every s verifying

$$1 \leqslant s < 1 + \operatorname{Min}\left\{\frac{k+\alpha}{2}, \frac{1}{1-2\beta}\right\}.$$

For $\beta = \frac{1}{2}$ we get in particular the same conclusion as in the homogeneous equation (th. 1).

Finally let us observe that if $\beta = 0$, i.e. if no condition is imposed on the coefficients $b_i(t)$, we cannot have in general the well-posedness in $\mathcal{E}^s$ for $s \geqslant 2$.

REMARK 4. Under the same assumptions of th. 1, we can prove, in a similar way, that problem (18) is well posed in $(\mathcal{D}^s)'$, space of the Gevrey ultradistributions with order $s < 1 + (k+\alpha)/2$.

4. – Counter-examples.

In this section we put ourselves in the case $n = 1$, considering the problem

$$u_{tt} - a(t) u_{xx} = 0 \tag{33}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \tag{34}$$

for $x \in \mathbb{R}$, $t \geqslant 0$, with $a(t) \geqslant 0$.

Our purpose is to show that th. 1 cannot be improved, by constructing for any (k, α) a coefficient $a(t)$ of class $C^{k,\alpha}$ in such a way that $\{(33), (34)\}$ is not well-posed in $\mathcal{E}^s$ if $s > 1 + (k+\alpha)/2$.

More precisely we shall prove the following result.

THEOREM 2. *For every $T_* > 0$ and every (k, α) (k integer $\geqslant 0$, $0 \leqslant \alpha < 1$) it is possible to construct a function $a(t)$, C^∞ and strictly positive on $[0, T_*[$, identically zero on $[T_*, +\infty[$, and a solution $u(x, t)$ of (33) in such a way that*

$$a(t) \quad \textit{belongs to} \quad C^{k,\alpha}([0, +\infty[) \tag{35}$$

and

$$u \textit{ belongs to } C^1([0, T_*[, \mathcal{E}^s), \quad \forall s > 1 + \frac{k+\alpha}{2}, \tag{36}$$

whereas

$$\{u(\cdot, t)\} \quad \textit{is not bounded in } \mathcal{D}', \textit{ as } \quad t \to T_*^- . \tag{37}$$

REMARK 5. From (36) it follows in particular that $u(\cdot, 0)$ and $u_t(\cdot, 0)$ belong to $\mathcal{E}^s$, $\forall s > 1 + (k + \alpha)/2$. Hence $u(x, t)$ is a solution (in fact the *unique* solution) of problem $\{(33), (34)\}$ with $\varphi(x) = u(x, 0)$ and $\psi(x) = u_t(x, 0)$.

Thus, th. 2 says that this problem is not well-posed in the Gevrey space $\mathcal{E}^s$ if $s > 1 + (k + \alpha)/2$.

PROOF OF TH. 2. The construction of $a(t)$ and $u(x, t)$ will be very similar to the one made in [2], where it was given an example of $a(t)$ of class C^∞ such that the Cauchy problem $\{(33), (34)\}$ is not well-posed in C^∞ (the example of [2] can be in some sense considered as the limit case of th. 2 as $k + \alpha \to \infty$).

However we shall give here for sake of completness a self-consistent exposition, referring to [2] for some technical step.

Fixed $T_* > 0$, let us introduce the following *parameters*, whose values will be chosen at the end of the proof:

a sequence $\{\varrho_j\}$ of positive numbers, decreasing to zero and verifying

$$\sum_{j=1}^{\infty} \varrho_j = T_*; \tag{38}$$

a sequence $\{\delta_j\}$ of positive numbers, decreasing to zero;

a sequence $\{\nu_j\}$ of integers $\geqslant 1$, increasing to ∞.

Correspondingly let us consider the points of $[0, T_*[$

$$t_j = \varrho_1 + \dots + \varrho_{j-1} + \frac{\varrho_j}{2},$$

and the intervals

$$J_j = \left[t_j - \frac{\varrho_j}{2}, \, t_j + \frac{\varrho_j}{2}\right[.$$

We have then

$$[0, T_*[= \bigcup_{j=1}^{\infty} J_j .$$

Finally let us consider, inside J_j, the points

$$t_j' = \left(t_j - \frac{\varrho_j}{2}\right) + \frac{\varrho_j}{8\nu_j}, \qquad t_j'' = \left(t_j + \frac{\varrho_j}{2}\right) - \frac{\varrho_j}{8\nu_j},$$

and denote by

$$\tilde{I}_j = \left[t_j - \frac{\varrho_j}{2}, t'_j\right[\qquad \text{and} \qquad I_j = \left[t'_j, t_j + \frac{\varrho_j}{2}\right[$$

the intervals into which J_j is divided by t'_j.

The definition of $a(t)$ will be given piece by piece on each J_j and it will be based on two auxiliary functions, $\alpha(\tau)$ and $\beta(\tau)$.

As $\beta(\tau)$ we take any C^∞ function on $\mathbb{R}$, strictly increasing on $[0, 1]$, equal to zero on $]-\infty, 0]$ and equal to 1 on $[1, +\infty[$.

As $\alpha(\tau)$ we take the function

$$\alpha(\tau) = 1 - \frac{4}{10}\sin 2\tau - \frac{1}{100}(1 - \cos 2\tau)^2 . \tag{39}$$

Observe that $\alpha(\tau)$ is π-periodic and valued in $[\frac{1}{2}, 2]$.

Now let us define $a(t)$ by taking

$$\begin{cases} a = a_j b_j + a_{j-1}(1 - b_j) & \text{on } J_j \ (j \geq 1) , \\ a \equiv 0 \quad \text{on } [T_*, +\infty[, \end{cases} \tag{40}$$

where a_j, b_j are defined by

$$\begin{cases} a_j(t) = \delta_j \cdot \alpha\left(2\nu_j \pi \dfrac{t - t_j}{\varrho_j}\right), & j \geq 1 , \\[2ex] b_j(t) = \beta\left(8\nu_j \dfrac{t - (t_j - \varrho_j/2)}{\varrho_j}\right), & j \geq 1 , \\[2ex] a_0(t) = 2\delta_1 . \end{cases} \tag{41}$$

Observe that $a(t) \equiv a_j(t)$ on I_j and that $a(t)$ is C^∞ on $[0, T_*[$.

Now let us define the solution $u(x, t)$ as

$$u(x, t) = \sum_{j=1}^{\infty} v_j(t) \sin(h_j x) , \tag{42}$$

with

$$h_j = 2\pi \frac{\nu_j}{\varrho_j} \frac{1}{\sqrt{\delta_j}} , \tag{43}$$

and $v_j(t)$ equal to the solution of

$$\begin{cases} v'' + h_j^2 \cdot a(t) v = 0 , & t \geq 0 , \\ v(t_j) = 0 , \quad v'(t_j) = 1 . \end{cases} \tag{44}$$

Clearly (42) defines a solution, in some weak sense, of equation (33). Hence the problem is to find ϱ_j, δ_j, ν_j in such a way that (35), (36) and (37) are satisfied.

To get (35), let us differentiate k-times (40). We then obtain

$$a^{(k)}|_{J_j} = \sum_{r=0}^{k} \binom{k}{r} b_j^{(k-r)} \cdot (a_j^{(r)} - a_{j-1}^{(r)}) + a_{j-1}^{(k)},$$

whence, using the monotonicity of $\{\delta_j\}$ and $\{\varrho_j/\nu_j\}$, we derive the estimate

$$\|a\|_{C^{k,\alpha}(J_j)} \leqslant C(k, \alpha)\, \delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^{k+\alpha}. \tag{45}$$

Hence a sufficient condition for the C^k-regularity of $a(t)$ on $[0, +\infty]$ is that

$$\delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^{k} \to 0 \qquad \text{as } j \to \infty. \tag{46}$$

As the C^α-regularity of $a^{(k)}(t)$ on $[0, +\infty[$, we can see that a sufficient condition is

$$\delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^{k+\alpha} \leqslant M, \qquad \forall_j. \tag{47}$$

Indeed from (47) we derive, using (45) with $\alpha = 0$,

$$\|a\|_{C^k(J_j)} \leqslant C(k, 0)\, M \frac{\varrho_j^\alpha}{\nu_j^\alpha} \leqslant C(k, 0)\, M \varrho_j^\alpha,$$

and this inequality, together with (45), enables us to get

$$|a^{(k)}(t'') - a^{(k)}(t')| \leqslant 2M\big(C(k, \alpha) + C(k, 0)\big)|t'' - t'|^\alpha.$$

Let us now look for a sufficient condition on the parameters which ensures (36). To this end we must estimate the growth for $j \to \infty$ of the coefficients $v_j(t)$ of Fourier expansion (42).

Since $a(t) \equiv \delta_j \cdot \alpha\big(2\pi(\nu_j/\varrho_j)(t - t_j)\big)$ on I_j, we can calculate $v_j(t)$ on I_j. In fact we have

$$v_j(t) = \frac{\varrho_j}{2\pi\nu_j}\, w\left(2\pi \frac{\nu_j}{\varrho_j}(t - t_j)\right), \qquad \text{on } I_j, \tag{48}$$

having denoted by $w(\tau)$ the solution of

$$\begin{cases} w'' + \alpha(\tau)w = 0\,, & \text{on } \mathbb{R}\,, \\ w(0) = 0\,, & w'(0) = 1\,. \end{cases} \tag{49}$$

But we defined $\alpha(\tau)$ in such a way that (49) admits a solution of the form $w(\tau) = p(\tau)\cdot\exp(\gamma\tau)$, with $p(\tau)$ periodic and $\gamma > 0$. More precisely the solution of (49) is

$$w(\tau) = \sin\tau\cdot\exp\left[\frac{1}{10}\left(\tau - \frac{1}{2}\sin 2\tau\right)\right]. \tag{50}$$

Thus (48) and (50) give an explicit expression of $v_j(t)$ on I_j, and in particular

$$\begin{cases} |v_j(t_j')| = \tilde{c}_1 \dfrac{\varrho_j}{\nu_j} \exp\left(-\dfrac{\pi}{10}\nu_j\right) \\ |v_j'(t_j')| = \tilde{c}_2 \exp\left(-\dfrac{\pi}{10}\nu_j\right) \end{cases} \tag{51}$$

and

$$\begin{cases} |v_j(t_j'')| = \tilde{c}_3 \dfrac{\varrho_j}{\nu_j} \exp\left(\dfrac{\pi}{10}\nu_j\right) \\ |v_j'(t_j'')| = \tilde{c}_4 \exp\left(\dfrac{\pi}{10}\nu_j\right) \end{cases} \tag{52}$$

with $\tilde{c}_j > 0$.

If we introduce the *energy* of $v_j(t)$ as

$$E_j(t) = h_j^2 a(t) v_j^2 + v_j'^2\,, \tag{53}$$

we get by (51)

$$E_j(t_j') = C_0 \exp\left(-\frac{\pi}{5}\nu_j\right). \tag{54}$$

Now, starting from (54), we estimate $E_j(t)$ for $t < t_j'$.

To this end we use the energy estimate

$$E_j(t) \leqslant E_j(s)\exp\left[\int_t^s \frac{|a'(\xi)|}{a(\xi)}\,d\xi\right], \qquad t < s\,, \tag{55}$$

which can be easily derived from equation (44).

We use (55) with $s = t_j'$ and $t < t_j'$, thus we must estimate the integral $\int^{t'} |a'|a^{-1}\,d\xi$. For this purpose we take into account the behaviour of $a(t)$

on the interval

$$[0, t'_j] \equiv \tilde{I}_1 \cup I_1 \cup \dots \cup \tilde{I}_{j-1} \cup I_{j-1} \cup \tilde{J}_j ,$$

and, more precisely, the following facts:

- $a(t)$ is decreasing near the points $t = 0$, $t = t'_t$, and $a(0) = 2\delta_1$, $a(t'_t) = c_1 \cdot \delta_1$ $(c_1 = \alpha(\pi/4))$;
- $a(t)$ has exactly $2\nu_h$ points of minimum and $2\nu_h$ points of maximum on I_h, where $\delta_h/2 \leqslant a(t) \leqslant 2\delta_h$;
- $a(t)$ is decreasing in a neighborhood of $\tilde{I}_h$.

The first two of these facts are direct consequences of definition itself of $a(t)$, whereas to have the third we must impose a supplementary assumption on the parameters, namely that

$$2\delta_j \leqslant \frac{\delta_{j-1}}{2}, \qquad \forall j. \tag{56}$$

Using the properties of $a(t)$ enumerated above, we derive from (55)

$$E_j(t) \leqslant E_j(t'_t) \exp\left[2(\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg\left(\frac{2}{c_1} \cdot \frac{\delta_1}{\delta_j}\right)\right] \tag{57}$$

for any $t \leqslant t'_j$.

Finally, observing that $(h_j^2 a(t))^{-1} \leqslant c_2$ for $t \leqslant t'_j$, we derive from (57), (54) and (53) that

$$\sup_{[0,t'_j]} |v_j| \leqslant c_3 \exp\left[-\frac{\pi}{10}\nu_j + (\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg \frac{\delta_1}{\delta_j}\right]. \tag{58}$$

On the other side, Paley-Wiener theorem ensures that the series (42) is converging near some $u(x, t)$ in $C([0, T_* - \varepsilon], \mathcal{E}^s)$ for some $\varepsilon > 0$ and $s \geqslant 1$, if and only if

$$\sup_{[0, T_* - \varepsilon]} |v_j| \leqslant M_\varepsilon \cdot \exp(-\mu_\varepsilon h_j^{1/s})$$

with M_ε and $\mu_\varepsilon > 0$.

Thus, taking into account that $t'_j \to T_*$ as $j \to \infty$, we get from (58) the following sufficient condition for $u(x, t)$ belong to $C([0, T_*[, \mathcal{E}^s)$:

$$-\frac{\pi}{10}\nu_j + (\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg \frac{\delta_1}{\delta_j} \leqslant -\mu h_j^{1/s} + \lg M \tag{59}$$

for some $M, \mu > 0$.

Remembering that $h_j = 2\pi\nu_j \varrho_j^{-1}\delta_j^{-1/2}$, we see that (59) is true in particular when

$$(\nu_1 + \dots + \nu_{j-1}) \lg 4 < \frac{\pi}{11}\nu_j \tag{60}$$

and

$$\operatorname*{Sup}_j \nu_j^{1-s}\varrho_j^{-1}\delta_j^{-1/2} < \infty . \tag{61}$$

Let us moreover observe that, if the series in (42) converges in $C([0, T_*[, \mathcal{E}^s)$, then $u(x,t)$ is a weak solution of equation (33); so that, by the regularity of $a(t)$ on $[0, T_*[$, we also get that $u \in C^\infty([0, T_*[, \mathcal{E}^s)$.

In conclusion, in order that (36) holds, it is sufficient that (60) and (61), with $s > 1 + (k + \alpha)/2$, are satisfied.

It remains to find conditions ensuring (37). To this purpose let us go back to (52) and observe that if (59) holds for some $s \geqslant 1$, then (52) gives

$$|v_j(t_j'')| \geqslant \frac{1}{c_4}\exp\left(\frac{\mu}{2}h_j\right), \tag{62}$$

where $\mu > 0$.

Inequality (62) gives the unboundedness of $\{u(\cdot, t_j'')\}$ in $\mathcal{D}'$. Hence no further assumption on the parameters is needed, in order to have (37).

Summarizing, in order to have (35), (36) and (37), we must only exhibit a choice of the parameters ϱ_i, δ_i, ν_j verifying conditions (38), (46), (47), (56), (60) and (61) for $s > 1 + (k + \alpha)/2$. Incidentally, let us observe that it is impossible to satisfy simultanously (46) and (61) if $s < 1 + (k + \alpha)/2$.

A good choice is the following

$$\begin{cases} \varrho_j = 2^{-j}T_* \\ \nu_j = 2^{j^2} \\ \delta_j = 2^{-(k+\alpha)(j+1)(j+2)-2j} \end{cases} \tag{63}$$

which gives in particular

$$h_j = \frac{2\pi}{T_*}2^{j^2+2j+(k+\alpha)(j+1)(j+2)/2}. \qquad //$$

REMARK 6. In th. 2 we have constructed on $\mathbb{R}\times[0, T_*[$ a solution of (33), $u(x,t)$, which cannot be continued on the *closed* interval $[0, T_*]$ as an element of the space $C([0, T_*], \mathcal{D}')$.

Moreover, as it is easily seen, such a solution cannot be continued as a distribution on $\mathbb{R}\times[0, T_* + \varepsilon[$, for any $\varepsilon > 0$.

On the other side we know that u can be continued to some $\tilde{u} \in C^1([0, +\infty[, (\mathfrak{D}^s)')$, with $s < 1 + (k + \alpha)/2$. Indeed, (36) gives in particular that $u(x, 0)$ and $u_t(x, 0)$ belong to $(\mathfrak{D}^r)'$ for every $r \geqslant 1$, and problem $\{(33), (34)\}$ is well-posed in $(\mathfrak{D}^s)'$ for $s < 1 + (k + \alpha)/2$ (see Rem. 4).
Now one can ask if the ultradistributions $\tilde{u}(\cdot, T_*)$ and $\tilde{u}_t(\cdot, T_*)$ are belonging to $\mathfrak{D}'$.

The answer to this question is that they cannot *both* belong to $\mathfrak{D}'$.

To prove this fact, let us introduce the λ-energy of $v_j(t)$ as

$$E_{j,\lambda}(t) = \lambda h_j^2 |v_j(t)|^2 + |v_j'(t)|^2 \qquad (\lambda > 0)\,,$$

with h_j, $v_j(t)$ as in the proof of th. 2.

It is then easy to prove, in a similar way that (26), the following energy estimate:

$$E_{j,\lambda}(s) \leqslant E_{j,\lambda}(t) \cdot \exp\left(\frac{h_j}{\sqrt{\lambda}} \left| \int_s^t |a(\xi) - \lambda|\, d\xi \right| \right), \qquad \forall s, t\,. \tag{64}$$

Let us take $\lambda = \delta_{j+1}$, $s = t_j''$, $t = T_*$, and observe that

$$\int_{t_j''}^{T_*} |a(\xi) - \delta_{j+1}|\, d\xi = \int_{t_j''}^{t_{j+1}'} |a(\xi) - \delta_{j+1}|\, d\xi + \int_{t_{j+1}'}^{T} |a(\xi) - \delta_{j+1}|\, d\xi$$
$$\leqslant (t_{j+1}' - t_j'')\, 2\delta_j + (T_* - t_{j+1}')\, \delta_{j+1} \leqslant C\left(\frac{\varrho_j}{\nu_j}\, \delta_j + \left(\sum_{j+1}^{\infty} \varrho_h\right) \delta_{j+1}\right)$$

and that (see (52))

$$E_{j,\lambda}(t_j'') \geqslant \frac{1}{C} \exp\left(\frac{\pi}{5}\, \nu_j\right).$$

Then, in virtue of our choice of ϱ_j, δ_j, ν_j (see (63)), we get by (64) the estimate from below

$$|v_j(T_*)| + |v_j'(T_*)| \geqslant \frac{1}{C'} \exp(\mu \nu_j)\,,$$

for some C' and $\mu > 0$ and j large enough, which shows that $\{|v_j(T_*)| + |v_j'(T_*)|\}$ has an exponential growth with respect to h_j for $j \to \infty$ and hence that $u(\cdot, T_*)$ and $u_t(\cdot, T_*)$ cannot be both distributions. //

The solution $u(x, t) \equiv \sum v_j(t) \sin(h_j x)$ constructed in th. 2 has the property to be very regular for $t < T_*$ and to become irregular at $t = T_*$. In fact $|v_j(t)|$ decreases to zero as $\exp(-\mu_1 h_j^{1/s})$ for $t < T_*$, whereas $|v_j(T_*)| + |v_j'(T_*)|$ grows as $\exp(\mu_2 h_j^{1/s})$, with $\mu_j > 0$, $s > 1 + (k + a)/2$ and $j \to \infty$.

Now, in view of Th. 3 below, we shall indicate how to construct a solution $\tilde{u}(x, t)$ of an equation of type (33), say

$$\tilde{u}_{tt} - \tilde{a}(t)\tilde{u}_{xx} = 0 , \quad t \geqslant 0 , \tag{65}$$

which has just the opposite property that u. Namely we look for some solution $\tilde{u}$ of (65) which is very irregular if $t < T_*$ but becomes regular when $t = T_*$.

To construct $\tilde{u}(x, t)$, we proceed as in the proof of th. 2, using in addition the techniques of Rem. 6. The main difference is actually that, to define $\tilde{a}(t)$, we use this time the function

$$\tilde{\alpha}(\tau) = 1 + \frac{4}{10} \sin 2\tau - \frac{1}{100} (1 - \cos 2\tau)^2$$

in place of the function $\alpha(\tau)$ defined by (39).

The solution of

$$\begin{cases} \tilde{w}'' + \tilde{\alpha}(\tau)\tilde{w} = 0 \\ \tilde{w}(0) = 0 , \quad \tilde{w}'(0) = 1 \end{cases}$$

is given by

$$\tilde{w}(\tau) = -\sin \tau \exp\left[-\frac{1}{10}\left(\tau - \frac{1}{2} \sin 2\tau\right)\right],$$

so that

$$|\tilde{w}(\tau)| \leqslant C \exp\left(-\frac{\tau}{10}\right).$$

By means of $\tilde{\alpha}(\tau)$ we then construct the coefficient $\tilde{a}(t)$ of equation (65) in the same manner that $a(t)$ in the proof of th. 2 (see (40), (41)).

Let us now construct the wished solution $\tilde{u}$, belonging to $C^1([0, +\infty[, (\mathfrak{D}^s)')$ for some $s > 1$, by taking again

$$\tilde{u}(x, t) = \sum \tilde{v}_j(t) \sin (h_j x)$$

with $\tilde{v}_j(t)$ such that

$$\begin{cases} \tilde{v}_j'' + h_j \tilde{a}(t)\tilde{v}_j = 0 \\ \tilde{v}_j(t_j) = 0, \ \tilde{v}_j'(t_j) = 1 . \end{cases}$$

We have then (cf. (51), (52))

$$|v_j(t_j')| = \tilde{\tilde{c}}_1 \frac{\varrho_j}{\nu_j} \exp\left(\frac{\pi}{10} \nu_j\right); \qquad |v_j'(t_j')| = \tilde{\tilde{c}}_2 \exp\left(\frac{\pi}{10} \nu_j\right) \tag{66}$$

and

$$(67) \qquad |v_j(t_j'')| = \tilde{\tilde{c}}_3 \frac{\varrho_j}{\nu_j} \exp\left(-\frac{\pi}{10}\nu_j\right); \qquad |v_j'(t_j'')| = \tilde{\tilde{c}}_4 \exp\left(-\frac{\pi}{10}\nu_j\right),$$

with $\tilde{\tilde{c}}_j > 0$.

Now, using the energy estimate (64) with $\lambda = \delta_{j+1}$, $s = T_*$ and $t = t_j''$, we derive from (67) that $|v_j(T_*)|$ and $|v_j'(T_*)|$ are less than $C \cdot \exp(-\mu h_j^{1/s})$ for some $\mu > 0$ and every $s > 1 + (k + a)/2$. Thus $u(\cdot, T_*)$ and $u_t(\cdot, T_*)$ are belonging to $\mathcal{E}^s$ for $s > 1 + (k + \alpha)/2$.

Finally we derive from (66) that u and u_t are not two distributions on any strip $\mathbb{R} \times]T_* - \varepsilon, T_*[$ for $\varepsilon > 0$.

In conclusion, if we effect the change of variable $t \mapsto T_* - t$, we get the following result.

THEOREM 3. *For every k, α, k integer $\geqslant 0$ and $0 \leqslant \alpha < 1$, it is possible to construct a function $a(t)$, vanishing at $t = 0$ and strictly positive for $t > 0$, and two initial data $\varphi(x)$, $\psi(x)$ which belong to $\mathcal{E}^s$ for any $s > 1 + (k + \alpha)/2$, in such a way that:*

i) *$a(t)$ belongs to $C^{k,\alpha}([0, +\infty[)$;*

ii) *the problem $\{(33), (34)\}$ does not admit solutions in the space $\mathcal{D}'(\mathbb{R} \times \times [0, \varepsilon[)$, $\forall \varepsilon > 0$.*

ADDED IN PROOF. After the drawing up of the present paper, T. Nishitani sent us a manuscript containing the extension of th. 1, when $k + \alpha \leqslant 2$, to the more general case of an equation whose coefficients $a_{ij}(x, t)$ are $C^{k,\alpha}$ with respect to t and Gevrey functions of order s with respect to x, and (k, α, s) satisfies condition (3).

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