ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARTIN COSTABEL
ERNST STEPHAN
WOLFGANG L. WENDLAND

On boundary integral equations of the first kind for the bi-laplacian in a polygonal plane domain

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 10, nº 2 (1983), p. 197-241
<http://www.numdam.org/item?id=ASNSP_1983_4_10_2_197_0>

On Boundary Integral Equations of the First Kind for the bi-Laplacian in a Polygonal Plane Domain.

MARTIN COSTABEL - ERNST STEPHAN
WOLFGANG L. WENDLAND

Introduction.

Here we analyze a system of Fredholm integral equations of the first kind with logarithmic principal part on a polygonal boundary curve. This is the system from Fichera's single layer approach for the Dirichlet problem of the bi-Laplacian. We show for the integral equations continuity in Sobolev spaces, a Gårding inequality and regularity results including a-priori estimates, where the solution is decomposed into corner singularities and smooth remainders. The Gårding inequality holds on $H^{-\frac{1}{2}}(\Gamma)$, the trace space corresponding to the energy norm. We also prove the unique solvability of the integral equations and their equivalence with the variational formulation of the corresponding boundary value problem. These results are obtained by local application of the Mellin transformation.

The boundary integral method for elliptic interior and exterior boundary value problems is one of the main tools for their constructive and also numerical solution. For higher order elliptic equations the method of single layer potentials goes back to Fichera [10] and has been worked out for some equations with constant coefficients by Hsiao and MacCamy in [15]. This method as well as its analysis in Sobolev spaces [16] was developped for closed smooth boundaries only whereas in practical problems one is frequently confronted with piecewise smooth curves having corner points. As the first approximation to this general case we consider here a polygonal boundary curve. The general case of a curved polygon will be treated with similar methods in a forthcoming paper.

Pervenuto alla Redazione il 22 Maggio 1982.

The boundary integral method for the plane mixed Dirichlet-Neumann problem of the Laplacian on a polygon has already been analyzed by Costabel and Stephan in [5], [6]. Here we extend their approach to the bi-Laplacian and the corresponding system of integral equations for the Dirichlet problem and present the method of Mellin transformation in detail which allows the further extension to different and also mixed boundary conditions as well as to other differential equations (as e.g. plane elasticity) and also other boundary integral methods resulting from approaches different from Fichera's (e.g. the direct method [17]).

As representative examples of boundary value problems yielding our boundary integral equations, we present the second fundamental problem of plane elasticity, the clamped plate and exterior (and interior) Stokes flows.

Since boundary integral methods in connection with finite element approximations play a significant role among numerical procedures we have focused our interest on the following principles:

1) The mapping properties of the integral operators in appropriate Sobolev spaces and the relations to the variational solution of the original boundary value problem.

2) Strong ellipticity of the integral equations and a Gårding inequality.

3) Regularity of the solution and its decomposition into singular functions near the corners and regular remainders and corresponding a-priori estimates.

These three principles form the basis of the asymptotic error analysis of Galerkin type boundary element methods (see [5], [6], [34], [29], [30], [16], [33]). For smooth boundaries these properties are obtained by the Fourier transformation and the calculus of pseudodifferential operators. For the polygon, however, we have to use the Mellin transformation and the calculus of Mellin symbols. This technique, which goes back to Kondratiev [18] and Shamir [27] has been used more recently by several authors [8], [9], [19], [24], [7]. Here we present in detail the complete analysis with the Mellin transformation leading to the above three principles. The above mentioned error analysis, however, will not be presented here.

The paper is organized as follows:

In § 1 we show the reduction of boundary value problems to the integral equations on the boundary, formulate the corresponding variational principle and their equivalence. The system on the boundary curve Γ consists of two Fredholm integral equations of the first kind with the logarithmic kernel as principal part and three additional constraints for two boundary densities and three real parameters.

In § 2-4 we present the local analysis of the integral equations near the corner points t_j, $j = 1, \dots, J$. (The indices will be used cyclically mod J, e.g. $t_0 = t_J$.) The polygon Γ with the corners t_j is composed of the open straight line segments Γ^j connecting t_{j-1} with t_j, respectively ($j = 1, \dots, J$). We assume that Γ is the boundary of a simply connected bounded domain Ω. By ω_j we denote the interior angle between Γ^j and Γ^{j+1}. For the local analysis we choose respective local Euclidean coordinates with the origin at the corner point under consideration and identify the complex plane with $\mathbb{R}^2$. Moreover, the two segments joining at this corner point are supposed to be incident with the half lines $\Gamma_- = \exp[i\omega]\mathbb{R}_+$ and $\Gamma_+ = \mathbb{R}_+$ spanning the reference angle $\Gamma^\omega = \Gamma_- \cup \{0\} \cup \Gamma_+$. A function g on Γ^ω will be identified with the pair (g_-, g_+) of functions on $\mathbb{R}^+$ viz. $g_-(x) = g(x\exp[i\omega])$ and $g_+(x) = g(x)$ for $x > 0$. This induces an identification of any scalar integral operator on Γ^ω, which corresponds locally to an integral operator on Γ near the corner, to a 2×2 matrix of integral operators on $\mathbb{R}_+$. Thus, locally our 2×2 system of integral equations on Γ becomes a 4×4 system on $\mathbb{R}_+$.

In § 2 we apply the Mellin transformation to this localized system and calculate the corresponding 4×4 Mellin symbols. Up to finite dimensional operators, the integral operators are of such a form that the Mellin transformation converts them into multiplications with these symbols (up to a shift). Analogously to (Fourier) symbols of pseudodifferential operators on $\mathbb{R}$, we use the Mellin symbols to prove boundedness in weighted Sobolev spaces on $\mathbb{R}_+$. The final results then are formulated in the usual Sobolev spaces without weight, since we are interested in the trace spaces of the variational solution.

§ 3 is devoted to strong ellipticity and a Gårding inequality. We are able to show local positive definiteness of our system of integral operators on a subspace with codimension 1 of $H^{-\frac{1}{2}}(\Gamma^\omega)$. Here we make use of the explicit symbols in order to show that the logarithmic principal part dominates the remainders with respect to the spectral norm. The remainders here are *not* compact as in the case of a smooth curve Γ. The principal part is positive definite due to Costabel and Stephan [5].

In § 4 we perform Kondratiev's technique for our integral equations, *i.e.* the application of the Cauchy residue theorem in the complex plane of the Mellin transformed variable. This gives the decomposition of the solution into singular and regular parts as well as a-priori estimates and regularity in a whole scale of Sobolev spaces. The exponents of the singularity functions turn out to be the roots of the same transcendental equation as derived by the usual Kondratiev technique for the bi-Laplacian in a plane sector [18], [26]. In addition to the singularity functions for the interior angle ω here also appear those belonging to the exterior angle $2\pi - \omega$, like in the potential case [5].

In § 5 we patch up our local results to obtain a global Gårding inequality on $H^{-\frac{1}{2}}(\Gamma)$ as well as continuity and regularity of the integral operators on Γ in the whole scale of Sobolev spaces. These results then are used to prove the equivalence theorem 1.1.

1. – Boundary value problems and boundary integral equations.

In this section we present three representative problems which can all be reduced to the interior or exterior Dirichlet problem for the bi-Laplacian and its variational solution on one hand.

On the other hand, by Fichera's method [10] we reduce these problems to boundary integral equations of the first kind together with appropriate side conditions following [15] and [14]. Theorem 1.1 gives the equivalence of both formulations.

1.1. *The second fundamental problem of plane elasticity.*

Introducing the Airy stress function $U(x, y)$ in plane elasticity, the second fundamental problem reads as follows [14]: Find the weak solution $U(x, y)$ of the bi-Laplacian

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0 \tag{1.1}$$

satisfying the boundary condition

$$\operatorname{grad} U = \boldsymbol{f} + \boldsymbol{a}\,, \tag{1.2}$$

where $\boldsymbol{f}$ corresponds to the given boundary forces and $\boldsymbol{a}$ denotes a yet unknown constant vector to be specified later on by the equilibrium state condition. The specific boundary condition (1.2) requires the compatibility condition for the given $\boldsymbol{f}$

$$\int_\Gamma (f_1\dot{x} + f_2\dot{y})\,ds = 0\,. \tag{1.3}$$

Expressing the stress function U in terms of a single layer potential, *i.e.*

$$U(x, y) = \int_\Gamma \left(\frac{\partial F}{\partial \xi}\,g_1(\zeta) + \frac{\partial F}{\partial \eta}\,g_2(\zeta)\right) ds_\zeta, \tag{1.4}$$

where $z = x + iy \in \Omega$, $\zeta = \xi + i\eta \in \Gamma$ and

$$F(z, \zeta) = -\tfrac{1}{2}|z-\zeta|^2 \log|z-\zeta| \tag{1.5}$$

is the fundamental solution of the bi-Laplacian, we find from (1.2) the integral equation

$$\int_\Gamma \log|z-\zeta| \boldsymbol{g}(\zeta)\, ds_\zeta + \int_\Gamma \mathfrak{L}_\varkappa(z,\zeta)\boldsymbol{g}(\zeta)\, ds_\zeta = \boldsymbol{f}(z) + \boldsymbol{a} + a_3\boldsymbol{k} \tag{1.6}$$

with the matrix function $\mathfrak{L}_\varkappa$ given by

$$\mathfrak{L}_\varkappa(z,\zeta) = \begin{pmatrix} \dfrac{(x-\xi)^2}{|z-\zeta|^2} + \dfrac{1}{2} - 2\varkappa, & \dfrac{(x-\xi)(y-\eta)}{|z-\zeta|^2} \\ \dfrac{(x-\xi)(y-\eta)}{|z-\zeta|^2}, & \dfrac{(y-\eta)^2}{|z-\zeta|^2} + \dfrac{1}{2} - 2\varkappa \end{pmatrix} \tag{1.7}$$

with $\varkappa = 0$. The equilibrium conditions give

$$\int_\Gamma \boldsymbol{g}(\zeta)\, ds_\zeta = \boldsymbol{b} \quad \text{and} \quad \int_\Gamma (g_1\dot{x} + g_2\dot{y})\, ds = b_3 \tag{1.8}$$

with $b_j = 0$, $j = 1, 2, 3$ (see [14]). Here $\boldsymbol{k}$ is a given smooth function on Γ which is fixed arbitrarily such that $\int_\Gamma \boldsymbol{k}\cdot\dot{\boldsymbol{x}}\, ds \neq 0$, where $\dot{\boldsymbol{x}} = (\dot{x}, \dot{y})$, i.e. $\boldsymbol{k}$ does *not* satisfy (1.3). One possible choice is $k(z) = \begin{pmatrix} y \\ -x \end{pmatrix}$. The term $a_3\cdot\boldsymbol{k}$ in (1.6) and the condition in (1.8) are introduced in order to achieve unique solvability of the integral equations (1.6) and (1.8) even if $\boldsymbol{f}$ does not satisfy (1.3). Solutions of the boundary value problem (1.1), (1.2) however, then always correspond to the case $b_3 = 0$ which yields $a_3 = 0$ provided $\boldsymbol{f}$ satisfies (1.3).

1.2. *The clamped plate.*

The problem of the clamped plate can be reduced to finding again a biharmonic function U satisfying (1.2) by setting

$$u(z) = U(z) - a_1x - a_2y + \gamma\,, \tag{1.9}$$

where u corresponds to the normal displacement of the plate. Here the single layer $\boldsymbol{g}$ becomes a fictitious boundary moment and $\boldsymbol{a}, \gamma$ correspond to the yet unknown rigid motion. The boundary conditions of the clamped plate yield again the integral equation (1.6) with side condition (1.8) [14].

1.3. *Viscous flow and Stokes problems.*

Both interior and exterior viscous flow problems with small Reynolds numbers can be reduced to the Stokes problem [31], [2], [15], [14].

The Stokes problem reads as

$$\Delta \boldsymbol{q} - \operatorname{grad} p = 0 \tag{1.10}$$

and

$$\operatorname{div} \boldsymbol{q} = 0 \quad \text{in } \Omega \text{ or } \Omega_c, \tag{1.11}$$

$$\boldsymbol{q}_{|\Gamma} = \boldsymbol{f}, \tag{1.12}$$

where Ω_c denotes the exterior domain $\mathbb{R}^2 \setminus \bar{\Omega}$. For the interior problem $\boldsymbol{f}$ is given, and for the exterior problem we have $\boldsymbol{f} = \boldsymbol{0}$ and the additional condition

$$\boldsymbol{q} = \boldsymbol{A} \log |z| + (-a_2, a_1) + O(|z|^{-1}) \tag{1.13}$$

for $|z| \to \infty$ where $\boldsymbol{A}$ is given and $\boldsymbol{a} = (a_1, a_2)$ to be determined. In both cases we introduce the stream function

$$U(z) = \int_\Gamma \operatorname{grad}_\zeta F(z, \zeta) \cdot \boldsymbol{g}(\zeta)\, ds_\zeta - x a_1 - y a_2 \tag{1.14}$$

which gives

$$\boldsymbol{q} = \left(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}\right). \tag{1.15}$$

The desired single layer $\boldsymbol{g}$ is the yet unknown hydrodynamic stress distribution on the boundary and F denotes the fundamental solution

$$F(z, \zeta) = -\tfrac{1}{2} |z - \zeta|^2 \log |z - \zeta| + \varkappa |z - \zeta|^2 \tag{1.16}$$

with

$$\varkappa = \tfrac{1}{2}\left(\log 4 + \tfrac{2}{3} - (\text{Euler's constant})\right). \tag{1.17}$$

For both boundary value problems we again find the integral equations (1.6) where $\varkappa$ in (1.7) corresponds to (1.17). Here $\boldsymbol{b}$ in (1.8) is either given as $\boldsymbol{0}$ for the interior problem or is given by $\boldsymbol{b} = (-A_2, A_1)$ for the exterior problem [14].

Among the weak solutions the variational solutions $U \in H^2(\Omega)$ respectively $U \in H^2_{\mathrm{loc}}(\Omega_c)$ (W^2_{00} in Giroire's [11] terminology) play a fundamental role in our analysis, since the corresponding variational problems are coercive and uniquely solvable even for our polygonal domains. Moreover, the boundary integral equations (1.6) provide in the corresponding trace spaces a Gårding inequality.

For the formulation of the above boundary value problems and integral equations in the weak and the variational form we need the usual Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$ [20], and the following traces, i.e. corresponding Sobolev spaces on the boundary curve following [12] and [5]:

$H^s(\Gamma)$ is defined as the trace space of $H^{s+\frac{1}{2}}(\mathbb{R}^2)$ for $s > 0$, as $L^2(\Gamma)$ for $s = 0$, and as the dual space of $H^{-s}(\Gamma)$ for $s < 0$.

$H^s(\Gamma^j)$ is defined as the usual trace space of $H^s(\Gamma)$ for $s \geqslant 0$ [20], whereas

$\tilde{H}^s(\Gamma^j)$ denotes the subspace

$$\tilde{H}^s(\Gamma^j) := \{u = \tilde{u}|_{\Gamma_j} | \tilde{u} \in H^s(\Gamma);\ \tilde{u}|_{\Gamma \setminus \Gamma_j} = 0\}$$

equipped with the topology of $H^s(\Gamma)$ for $s > 0$. For $s < 0$ we define $H^s(\Gamma^j) := (\tilde{H}^{-s}(\Gamma^j))'$ and $\tilde{H}^s(\Gamma^j) := (H^{-s}(\Gamma^j))'$ by duality.

With the trace lemma by Grisvard [12] it follows that $H^s(\Gamma)$ is a subspace of $\prod_j H^s(\Gamma^j)$ for $s \geqslant 0$ whose elements satisfy additional compatibility conditions corresponding to the corner points t_j. For $0 \leqslant s < \frac{3}{2}$ these subspaces are characterized by the identification of functions u on the reference angle Γ^ω with pairs (u_-, u_+) of functions on $\mathbb{R}_+$ as follows:

$$(1.18) \qquad \begin{cases} H^s(\Gamma^\omega) = \{(u_-, u_+) \in H^s(\mathbb{R}_+)^2 | u_- - u_+ \in \tilde{H}^s(\mathbb{R}_+)\} \ [12]\,, \\ H^{-s}(\Gamma^\omega) = \{(u_-, u_+) \in H^{-s}(\mathbb{R}_+)^2 | u_- + u_+ \in \tilde{H}^{-s}(\mathbb{R}_+)\} \ [5]\,. \end{cases}$$

For further use it should be noted that we have the following equivalent norms for $0 \leqslant s < \frac{3}{2}$:

$$(1.19) \qquad \|u\|^2_{H^s(\Gamma^\omega)} \sim \|u_+ + u_-\|^2_{H^s(\mathbb{R}_+)} + \|u_- - u_+\|^2_{\tilde{H}^s(\mathbb{R}_+)}$$

$$(1.20) \qquad \|u\|^2_{H^{-s}(\Gamma^\omega)} \sim \|u_+ + u_-\|^2_{\tilde{H}^{-s}(\mathbb{R}_+)} + \|u_- - u_+\|^2_{H^{-s}(\mathbb{R}_+)}\,.$$

In order to formulate the relation between the boundary value problems in the form (1.1), (1.2) and the integral equations (1.6), (1.8) we first formulate the corresponding variational problem.

To any given $\boldsymbol{f} \in H^{\frac{1}{2}}(\Gamma)$ fulfilling (1.3) we find a continuation to $h \in H^2(\Omega)$ satisfying

$$\operatorname{grad} h = \boldsymbol{f} \text{ on } \Gamma \tag{1.21}$$

with a trace theorem by Grisvard [12, Theorem 1.5.2.8]. To this end note that the data

$$\frac{\partial h}{\partial n} = f_1 \frac{dy}{ds} - f_2 \frac{dx}{ds},$$

$$h = \int_0^s f_1\, dx + f_2\, dy \quad \text{ on } \Gamma$$

satisfy the required compatibility conditions [12, Theorem 1.5.2.8]. With h we formulate the variational problem:

Find $w \in \mathring{H}^2(\Omega)$ satisfying

$$\int_\Omega \Delta w\, \Delta v\, dx\, dy = -\int_\Omega \Delta h\, \Delta v\, dx\, dy \tag{1.22}$$

for all test functions $v \in \mathring{H}^2(\Omega)$. As is well known, the above variational problem has a unique solution to any given $h \in H^2(\Omega)$.

THEOREM 1.1. *Let $f \in H^{\frac{1}{2}}(\Gamma)$ be given with* (1.3). *Then we have the following equivalence.*

i) *Any weak* (*distributional*) *solution $U \in H^2(\Omega)$ of* (1.1), (1.2) *is given by*

$$U = h + w + a_1 x + a_2 y + \gamma \tag{1.23}$$

with $h \in H^2(\Omega)$ satisfying (1.21) *and $w \in H^2(\Omega)$ given by* (1.22) *and any $\gamma \in \mathbb{R}$, i.e. U is a variational solution.*

ii) *Any solution w of* (1.22) *with $h \in H^2(\Omega)$ satisfying* (1.21) *defines by* (1.23) *a weak solution of* (1.1), (1.2), *i.e. any variational solution $U \in H^2(\Omega)$ is a weak solution.*

iii) *To any given h with* (1.21) *and w defined by* (1.22) *there exists a density $\boldsymbol{g} \in H^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{a} \in \mathbb{R}^2$ solving the integral equation* (1.6) *with $a_3 = 0$ and a suitable constant γ such that U defined by* (1.4) *is a variational solution of the form* (1.23).

iv) *To any given* $\boldsymbol{f} \in H^{\frac{1}{2}}(\Gamma)$, $(\boldsymbol{b}, b_3) \in \mathbb{R}^3$ *there exists a unique solution* $\boldsymbol{g} \in H^{-\frac{1}{2}}(\Gamma)$, $(\boldsymbol{a}, a_3) \in \mathbb{R}^3$ *of the system* (1.6), (1.8). *The constant part* a_3 *of the solution is such that* $\boldsymbol{f} + a_3\boldsymbol{k}$ *satisfies* (1.3), *in particular* $a_3 = 0$ *for* $\boldsymbol{f}$ *satisfying* (1.3). *In the latter case* U *defined by* (1.4) *is a variational solution of the boundary value problem* (1.1), (1.2).

REMARK 1.2. Note that the equations (1.6), (1.8) differ from the formulations given in [15], [16] by the additional constant a_3 and the last side condition in (1.8). Without these two quantities the system would have the one-dimensional kernel *span* $\{(\dot{x}, \dot{y})\}$ and a one-dimensional cokernel in contrary to Theorem 3 in [15]. Our formulation with three constants (a_1, a_2, a_3) and three constraints (1.8) is in agreement with the rigid motions which appear in corresponding exterior plane elasticity problems.

2. – Mellin symbols and continuity at a corner.

In order to characterize the mapping properties of the operators corresponding to the integral equations (1.6), (1.8) we introduce in equations (1.6)—which will be written in short as

$$\mathcal{A}\boldsymbol{g} = -\boldsymbol{f} - \boldsymbol{a}\,, \tag{2.1}$$

the operators

$$\mathcal{W}\boldsymbol{g}(z) := \int_\Gamma \log|z-\zeta|\,\boldsymbol{g}(\zeta)\,ds_\zeta\,, \tag{2.2}$$

$$\mathfrak{L}_0\boldsymbol{g}(z) := \int_\Gamma \mathfrak{L}_0(z,\zeta)\,\boldsymbol{g}(\zeta)\,ds_\zeta \tag{2.3}$$

with $\mathfrak{L}_0(z,\zeta)$ defined by (1.7) with $\varkappa = \frac{1}{4}$. Then

$$\mathcal{A}\boldsymbol{g} = -(\mathcal{W} + \mathfrak{L}_0)\boldsymbol{g} - (\tfrac{1}{2} - 2\varkappa)\int_\Gamma \boldsymbol{g}\,ds\,. \tag{2.4}$$

In the case of a smooth boundary curve Γ, the operator $\mathcal{W}$ is a pseudodifferential operator of order -1 and $\mathfrak{L}_0$ is a pseudodifferential operator of order -2 and $\mathcal{A}$ satisfies a Gårding inequality [16] in the form

$$(\mathcal{A}\boldsymbol{g}, \boldsymbol{g})_{L_2(\Gamma)} \geqslant \gamma\|\boldsymbol{g}\|^2_{H^{-\frac{1}{2}}(\Gamma)} - c\|\boldsymbol{g}\|^2_{H^{-1}(\Gamma)} \tag{2.5}$$

where $\gamma > 0$ and $c \geqslant 0$ are suitable constants independent of $\boldsymbol{g}$.

In our case of a polygonal domain, however, at the corner points $\mathcal{W}$ and $\mathcal{L}_0$ cannot be considered as pseudodifferential operators anymore. Since continuity of the operators and regularity of the solution are local properties, it suffices to investigate $\mathcal{W}$ and $\mathcal{L}_0$ near each corner point separately following the approach in [5]. To this end we introduce the reference angle Γ^ω and consider $\mathcal{W}, \mathcal{L}_0$ on Γ^ω. Here the identification of functions u on Γ^ω with pairs (u_-, u_+) of functions on $\mathbb{R}_+$ via (1.18) induces an identification of any scalar integral operator forming $\mathcal{W}$ resp. $\mathcal{L}_0$ on Γ^ω with a corresponding (2×2) matrix of integral operators on $\mathbb{R}_+$. Thus $\mathcal{W}$ respectively $\mathcal{L}_0$ will correspond to a matrix of operators whose entries can be ordered as follows:

$$\begin{gathered} \boldsymbol{g} \simeq (\boldsymbol{g}_-, \boldsymbol{g}_+)\,, \qquad \boldsymbol{g}_- = (g_{1-}, g_{2-})\,, \qquad \boldsymbol{g}_+ = (g_{1+}, g_{2+}) \\ \mathcal{W} \simeq \begin{pmatrix} \mathcal{W}_{--} & \mathcal{W}_{+-} \\ \mathcal{W}_{-+} & \mathcal{W}_{++} \end{pmatrix}, \qquad \mathcal{L}_0 \simeq \begin{pmatrix} \mathcal{L}_{0--} & \mathcal{L}_{0+-} \\ \mathcal{L}_{0-+} & \mathcal{L}_{0++} \end{pmatrix}. \end{gathered} \tag{2.6}$$

With local coordinates we write $z = x$ on Γ_+, $z = x \exp[i\omega]$ on Γ_- and correspondingly $\zeta = \xi$ on Γ_+ and $\zeta = \xi \exp[i\omega]$ on Γ_- with $x, \xi \in \mathbb{R}_+$ and obtain the explicit formulas

$$\left\{ \begin{aligned} \mathcal{W}_{--}\boldsymbol{\varphi}(x) &= \mathcal{W}_{++}\boldsymbol{\varphi}(x) = -\pi l\boldsymbol{\varphi} - \pi V_0\boldsymbol{\varphi}(x)\,, \\ \mathcal{W}_{-+}\boldsymbol{\varphi}(x) &= \mathcal{W}_{+-}\boldsymbol{\varphi}(x) = -\pi l\boldsymbol{\varphi} - \pi V_\omega\boldsymbol{\varphi}(x)\,, \end{aligned} \right. \tag{2.7}$$

where

$$\left\{ \begin{aligned} l\boldsymbol{\varphi} \quad &= -\frac{1}{\pi}\int_0^\infty \log \xi\, \boldsymbol{\varphi}(\xi)\, d\xi\,, \\ V_0\boldsymbol{\varphi}(x) &= -\frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x}{\xi}\right| \boldsymbol{\varphi}(\xi)\, d\xi\,, \\ &\text{and} \\ V_\omega\boldsymbol{\varphi}(x) &= -\frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x}{\xi}\exp[i\omega]\right| \boldsymbol{\varphi}(\xi)\, d\xi\,, \end{aligned} \right. \tag{2.8}$$

according to [5, (2.13)]. Note that now all functions as $\boldsymbol{g}_+, \boldsymbol{g}_-, \boldsymbol{\varphi}$ are defined on $\mathbb{R}_+$ as well as their images in (2.8), (2.7). Similarly, we define with

$$\mathcal{L}_{0--}\boldsymbol{\varphi}(x) = \int_0^\infty \mathcal{L}_{0--}(x, \xi)\, \boldsymbol{\varphi}(\xi)\, d\xi$$

and $\mathcal{L}_{0+-}, \mathcal{L}_{0-+}, \mathcal{L}_{0++}$, correspondingly, the kernels of the operators in (2.3), (2.6) by the formulas

$$
(2.9)\quad
\begin{cases}
\mathfrak{L}_{0--}(x,\xi) = \begin{pmatrix} \cos^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega \end{pmatrix}, \\[2ex]
\mathfrak{L}_{0+-}(x,\xi) = \dfrac{1}{|x\exp[i\omega]-\xi|^2} \\
\qquad \cdot\begin{pmatrix} (x\cos\omega-\xi)^2 & (x\cos\omega-\xi)x\sin\omega \\ (x\cos\omega-\xi)x\sin\omega & x^2\sin^2\omega \end{pmatrix} \\
\qquad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathfrak{L}^0_{+-}(x,\xi), \\[2ex]
\mathfrak{L}_{0-+}(x,\xi) = \dfrac{1}{|x-\xi\exp[i\omega]|^2} \\
\qquad \cdot\begin{pmatrix} (x-\xi\cos\omega)^2 & -(x-\xi\cos\omega)\xi\sin\omega \\ -(x-\xi\cos\omega)\xi\sin\omega & \xi^2\sin^2\omega \end{pmatrix} \\
\qquad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathfrak{L}^0_{-+}(x,\xi), \\[2ex]
\mathfrak{L}_{0++}(x,\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\end{cases}
$$

$$
(2.10)\quad
\begin{cases}
\mathfrak{L}^0_{+-}(x,\xi) = \dfrac{1}{|x\exp[i\omega]-\xi|^2} \\
\qquad \cdot\begin{pmatrix} -x^2\sin^2\omega & (x\cos\omega-\xi)x\sin\omega \\ (x\cos\omega-\xi)x\sin\omega & x^2\sin^2\omega \end{pmatrix}, \\[2ex]
\mathfrak{L}^0_{-+}(x,\xi) = \dfrac{1}{|x-\xi\exp[i\omega]|^2} \\
\qquad \cdot\begin{pmatrix} -\xi^2\sin^2\omega & -(x-\xi\cos\omega)\xi\sin\omega \\ -(x-\xi\cos\omega)\xi\sin\omega & \xi^2\sin^2\omega \end{pmatrix}.
\end{cases}
$$

All the above decompositions are constructed with the aim of obtaining simple Mellin multipliers by the use of the Mellin transformation. For the operator $\mathfrak{L}_{0-+}$, however, we shall need another decomposition, namely

$$
\mathfrak{L}_{0-+}(x,\xi) = \mathfrak{L}_{0--}(x,\xi) + \mathfrak{L}^1_{-+}(x,\xi), \tag{2.11}
$$

where $\mathfrak{L}^1_{-+}$ is defined by the relations (2.9)-(2.11).

Since all the desired mapping properties will be obtained by Mellin transformation as in [5], we now list the Mellin symbols of the above operators.

The Mellin transformation is for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ given by

$$\hat{\varphi}(\lambda) = \int_{-\infty}^{+\infty} \exp[-i\lambda t]\varphi(\exp[-t])\,dt = \int_0^\infty x^{i\lambda-1}\varphi(x)\,dx\,.$$

As is well known, the Mellin transformation is the composition of the Euler transformation $x \mapsto \exp[-t]$: $\mathbb{R}_+ \to \mathbb{R}$ with the Fourier tranformation. The inverse transformation is given by

$$\varphi(x) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = \text{const}} x^{-i\lambda}\hat{\varphi}(\lambda)\,d\lambda\,.$$

LEMMA 2.1. *Let $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}_+)$ be any test function. Then the Mellin transformation $\hat{}$ yields the equalities*

$$(2.12)\qquad \left\{\begin{aligned} \widehat{V_0\boldsymbol{\varphi}}(\lambda) &= \hat{V}_0(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i) \\ \widehat{V_\omega\boldsymbol{\varphi}}(\lambda) &= \hat{V}_\omega(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i) \\ \widehat{\mathfrak{L}^0_{+-}\boldsymbol{\varphi}}(\lambda) &= \hat{\mathfrak{L}}^0_{+-}(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i) \\ \widehat{\mathfrak{L}^1_{-+}\boldsymbol{\varphi}}(\lambda) &= \hat{\mathfrak{L}}^1_{-+}(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i) \end{aligned}\right. \qquad \textit{for } \operatorname{Im}\lambda \in (0,1)$$

and

$$(2.13)\qquad \widehat{\mathfrak{L}^0_{-+}\boldsymbol{\varphi}}(\lambda) = \hat{\mathfrak{L}}^0_{-+}(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i) \qquad \textit{for } \operatorname{Im}\lambda \in (-1, 0)\,,$$

where the Mellin symbols are given by

$$(2.14)\qquad \hat{V}_0(\lambda) = \frac{\cosh \pi\lambda}{\lambda \sinh \pi\lambda}\,, \qquad \hat{V}_\omega(\lambda) = \frac{\cosh(\pi - \omega)\lambda}{\lambda \sinh \pi\lambda}\,,$$

$$(2.15)\qquad \hat{\mathfrak{L}}^0_{+-}(\lambda) = -\frac{\pi \sin\omega}{\sinh \pi\lambda}$$

$$\cdot\left\{\sinh(\pi - \omega)\lambda \begin{pmatrix} \cos\omega & \sin\omega \\ \sin\omega & -\cos\omega \end{pmatrix} + i\cosh(\pi - \omega)\lambda \begin{pmatrix} \sin\omega & -\cos\omega \\ -\cos\omega & -\sin\omega \end{pmatrix}\right\},$$

$$(2.16)\qquad \hat{\mathfrak{L}}^1_{-+}(\lambda) = \hat{\mathfrak{L}}^0_{-+}(\lambda) = \hat{\mathfrak{L}}^0_{+-}(-\lambda) = \overline{\hat{\mathfrak{L}}^0_{+-}(\bar{\lambda})}\,.$$

PROOF. The following calculations for $\hat{V}_0(\lambda)$ and $\hat{V}_\omega(\lambda)$ are taken from [5] whereas those for $\hat{\mathfrak{L}}^0_{+-}(\lambda)$ are extensions of the one sided Hilbert transformation on the reference angle Γ^ω in [4] and [12, p. 270]. The latter reads ([5, (2.18)])

$$\widehat{N_\omega\varphi}(\lambda) = -\frac{1}{\sinh \pi\lambda}\exp\left[-(\pi-\omega)\lambda\right]\hat{\varphi}(\lambda) \quad \text{for } 0<\omega<2\pi \text{ and } \operatorname{Im}\lambda \in (-1, 0) \tag{2.17}$$

where

$$N_\omega\varphi(x) := -\frac{1}{i\pi}\int_0^\infty \frac{\varphi(\xi)\,d\xi}{x\exp[i\omega]-\xi} \tag{2.18}$$

and

$$\widehat{N_{2\pi-\omega}\varphi}(\lambda) = -\frac{1}{\sinh \pi\lambda}\exp\left[(\pi-\omega)\lambda\right]\hat{\varphi}(\lambda) \quad \text{for } 0<\omega<2\pi \text{ and } \operatorname{Im}\lambda \in (-1, 0)\,. \tag{2.19}$$

We also need the multiplication operator

$$T\varphi(x) = x\varphi(x) \text{ with } \widehat{T\varphi}(\lambda) = \hat{\varphi}(\lambda - i) \quad \text{for all } \lambda \in \boldsymbol{C}\,. \tag{2.20}$$

i) Partial integration yields for any test function $\varphi \in C_0^\infty[0, \infty)$

$$\left\{\begin{aligned} V_\omega\varphi(x) &= -\frac{1}{\pi}\int_0^\infty \log\left|1-\frac{x}{\xi}\exp[i\omega]\right|\varphi(\xi)\,d\xi \\ &= \frac{1}{\pi}\int_0^\infty \frac{\Xi(\xi)}{\xi}\operatorname{Re}\frac{x\exp[i\omega]}{\xi - x\exp[i\omega]}\,d\xi \\ &= \frac{i}{2}\left\{\exp[i\omega]TN_\omega T^{-1} + \exp[-i\omega]TN_{2\pi-\omega}T^{-1}\right\}\Xi(x)\,, \end{aligned}\right. \tag{2.21}$$

where

$$\Xi(\xi) = \int_0^\xi \varphi(\eta)\,d\eta\,,$$

having the Mellin transform

$$\hat{\Xi}(\lambda) = \frac{1}{i\lambda}\hat{\varphi}(\lambda - i) \qquad \text{for } \operatorname{Im}\lambda \in (0, 1)\,. \tag{2.22}$$

Inserting (2.18)-(2.20) and (2.22) into (2.21), we obtain with a straightforward calculation

$$\widehat{V_\omega \varphi}(\lambda) = \frac{\cosh(\pi-\omega)}{\lambda \sinh \pi\lambda}\,\hat{\varphi}(\lambda - i) \qquad \text{for } \operatorname{Im}\lambda \in (0,1)\,.$$

This is the second formula of (2.12), the first follows with $\omega = 0$.

ii) Because of

$$(x \exp[i\omega] - \xi)^{-1} = |x \exp[i\omega] - \xi|^{-2}(x\cos\omega - \xi - ix\sin\omega)\,,$$

the operator $\mathfrak{L}^0_{+-}$ with kernel (2.10) can be written as

$$\mathfrak{L}^0_{+-} = \frac{1}{\pi}\,T\sin\omega \begin{pmatrix} N_{2\pi-\omega} - N_\omega & -i(N_\omega + N_{2\pi-\omega}) \\ -i(N_\omega + N_{2\pi-\omega}) & N_\omega - N_{2\pi-\omega} \end{pmatrix}. \tag{2.23}$$

For Mellin transformation, we again insert (2.17)-(2.20) and obtain (2.15).

iii) For the proof of (2.16), first note that we have from (2.10)

$$\mathfrak{L}^0_{+-}(x,\xi) = k\left(\frac{x}{\xi}\right) = \mathfrak{L}^0_{-+}(\xi, x)$$

with a matrix function k whose Mellin transform $\hat{k}(\lambda)$ is given by $\hat{\mathfrak{L}}^0_{+-}(\lambda)$. This implies the elementary equality

$$\left|\begin{aligned} \hat{\mathfrak{L}}^0_{-+}(\lambda) &= \int_0^\infty x^{i\lambda} k\left(\frac{1}{x}\right)\frac{dx}{x} = \int_0^\infty t^{-i\lambda}k(t)\frac{dt}{t} = \hat{k}(-\lambda) \\ &= \hat{\mathfrak{L}}^0_{+-}(-\lambda) = \overline{\hat{\mathfrak{L}}^0_{+-}(\bar{\lambda})} \end{aligned}\right. \tag{2.24}$$

for $\operatorname{Im}(-\lambda) \in (0,1)$.

In order to find $\mathfrak{L}^1_{-+}(\lambda)$, we use the analytic continuation of $\widehat{\mathfrak{L}^0_{-+}\boldsymbol{\varphi}}(\lambda)$ from the strip $\operatorname{Im}\lambda \in (-1,0)$ into $\operatorname{Im}\lambda \in (0,1)$. The corresponding operators can be expressed by the use of the inverse Mellin transformation:

$$\mathfrak{L}^0_{-+}\boldsymbol{\varphi}(x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = c\in(-1,0)} x^{-i\lambda}\hat{\mathfrak{L}}^0_{-+}(\lambda)\,\hat{\boldsymbol{\varphi}}(\lambda - i)\,d\lambda$$

and

$$\mathfrak{L}^1_{-+}\boldsymbol{\varphi}(x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = c'\in(0,1)} x^{-i\lambda}\hat{\mathfrak{L}}^0_{-+}(\lambda)\,\hat{\boldsymbol{\varphi}}(\lambda - i)\,d\lambda\,.$$

Both operators are connected by the Cauchy theorem as follows:

$$\mathfrak{L}^0_{-+}\boldsymbol{\varphi}(x) - \mathfrak{L}^1_{-+}\boldsymbol{\varphi}(x) = 2\pi i \operatorname{Res}\left\{\frac{1}{2\pi} x^{-i\lambda}\mathfrak{L}^0_{-+}(\lambda)\hat{\boldsymbol{\varphi}}(\lambda - i)\right\}\bigg|_{\lambda=0}$$

$$= \begin{pmatrix} -\sin^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega \end{pmatrix}\int_0^\infty \boldsymbol{\varphi}(x)\,dx\,.$$

For the corresponding kernels we thus have

$$\begin{aligned}\mathfrak{L}_{0-+}(x,\xi) &= \mathfrak{L}^0_{-+}(x,\xi) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\\ &= \mathfrak{L}^1_{-+}(x,\xi) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\sin^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega \end{pmatrix}\\ &= \mathfrak{L}^1_{-+}(x,\xi) + \mathfrak{L}_{0--}(x,\xi)\,,\end{aligned}$$

the desired relation (2.11).

This completes the proof. □

In order to find the appropriate function spaces to be used for the system (1.6), (1.8) of integral equations, we introduce the spaces $\mathring{W}^s_0 = \mathring{W}^s_0(\mathbb{R}_+)$ as completions of $C^\infty_0(0,\infty)$ with respect to the norms

$$\|\varphi\|^2_{\mathring{W}^s_0} = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = s-\frac{1}{2}} (1+|\lambda|^2)^s|\hat{\varphi}(\lambda)|^2\,d\lambda \tag{2.25}$$

for any $s\in\mathbb{R}$. These spaces are equivalent to weighted Sobolev spaces introduced by Kondratiev [18] and further investigated by Avantaggiati and Troisi [1], and (2.25) corresponds to Perseval's equality.

Now the explicit Mellin symbols in Lemma 2.1 provide with (2.25) the following properties.

LEMMA 2.2. *Let $s\in(\frac{1}{2},\frac{3}{2})$. Then the following mappings are continuous*:

$$V_0,\ V_\omega,\ \mathfrak{L}^0_{+-},\ \mathfrak{L}^1_{-+}: \mathring{W}^{s-1}_0 \to \mathring{W}^s_0 \quad\text{and}\quad \mathfrak{L}^0_{-+}: \mathring{W}^{s-2}_0 \to \mathring{W}^{s-1}_0\,.$$

PROOF. The proof follows directly from (2.25) by taking advantage of the asymptotic behaviour of the Mellin symbols for $|\lambda|\to\infty$ on $\operatorname{Im}\lambda = s-\frac{1}{2}$, i.e.

$$|\hat{V}_0(\lambda)|\leq c(1+|\lambda|^2)^{-\frac{1}{2}}$$

whereas $|\hat{V}_\omega(\lambda)|$, $|\hat{\mathfrak{L}}^0_{+-}(\lambda)|$, $|\hat{\mathfrak{L}}^1_{-+}(\lambda)|$ and $|\hat{\mathfrak{L}}^0_{-+}(\lambda - i)|$ decay exponentially. □

In order to apply these results to our original integral equations note that all functions on Γ^ω will have a common compact support since the original boundary curve is compact. This can be described by a finite and fixed partition of unity on Γ.

To this end let in the following χ be a cut-off function attached to one corner:

(2.26) $\chi \in C_0^\infty(\mathbb{R}^2)$ with $\chi \equiv 1$ in a neighbourhood of the corner point, i.e. the vertex of Γ^ω, and χ depending only on the distance to the corner point.

Now for $v = \chi u$ with fixed χ we make use of the equivalence of norms:

$$\|v\|_{H^s(\mathbb{R}_+)} \simeq \|v\|_{\mathring{W}_0^s(\mathbb{R}_+)} \quad \text{for } s < 0 \tag{2.27}$$

and

$$\|v\|_{\tilde{H}^s(\mathbb{R}_+)} \simeq \|v\|_{\mathring{W}_0^s(\mathbb{R}_+)} \quad \text{for } s \geqslant 0\,. \tag{2.28}$$

This result follows from the interpolation property of weighted Sobolev spaces proved in [32, (4.3.2/7)].

LEMMA 2.3. *Let $s \in (-\frac{1}{2}, \frac{3}{2})$. Then the following mappings are continuous:*

$$\chi \mathcal{W}_{\alpha\beta}\chi,\ \chi \mathcal{L}_{0\alpha\beta}\chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to H^s(\mathbb{R}_+)\,, \tag{2.29}$$

where $\alpha, \beta \in \{+, -\}$ and

$$\chi(\mathcal{L}_{0--} - \mathcal{L}_{0-+})\chi,\ \chi(\mathcal{L}_{0+-} - \mathcal{L}_{0++})\chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)\,, \tag{2.30}$$

$$\begin{aligned} &\chi(\mathcal{W}_{--} - \mathcal{W}_{-+})\chi = \chi(\mathcal{W}_{++} - \mathcal{W}_{+-})\chi\,,\\ &\chi(\mathcal{L}_{0--} - \mathcal{L}_{0-+} - \mathcal{L}_{0+-} + \mathcal{L}_{0++})\chi\colon H^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)\,. \end{aligned} \tag{2.31}$$

PROOF. For the operators $\mathcal{W}_{\alpha\beta}$ the proof was done in [5, Lemma 2.15]. Here we shall repeat these arguments and extend them to the above operators $\mathcal{L}_{0\alpha\beta}$.

i) First observe that l in (2.8), $\mathcal{L}_{0--}$, and $\mathcal{L}_{0++}$ in (2.9) are linear functionals which generate the following continuous mappings:

$$\begin{cases} \chi l\chi,\ \chi\mathcal{L}_{0--}\chi,\ \chi\mathcal{L}_{0++}\chi\colon H^s(\mathbb{R}_+) \to H^t(\mathbb{R}_+) & \text{for } s > -\frac{1}{2},\ t \in \mathbb{R};\\ \chi\mathcal{L}_{0--}\chi,\ \chi\mathcal{L}_{0++}\chi\colon \tilde{H}^s(\mathbb{R}_+) \to H^t(\mathbb{R}_+) & \text{for all } s, t \in \mathbb{R}\,. \end{cases} \tag{2.32}$$

ii) For any of the above operators, the proof of continuity will be executed in three steps. First we find continuity for the spaces with s in one of the intervals

$$I_1 = (-\tfrac{1}{2}, \tfrac{1}{2}), \quad I_2 = (\tfrac{1}{2}, \tfrac{3}{2}), \tag{2.33}$$

then we show continuity in the second interval and thirdly the case $s = \frac{1}{2}$ follows by interpolation.

iii) Let us begin with (2.29). Lemma 2.2 in connection with the equivalence of norms (2.27), (2.28) yields continuity of

$$\chi V_0 \chi,\ \chi V_\infty \chi,\ \chi \mathfrak{L}^0_{+-} \chi,\ \chi \mathfrak{L}^1_{-+} \chi\colon H^{s-1}(\mathbb{R}_+) = \tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+) \quad \text{for } s \in I_2 .$$

In view of (2.32) this is already the proposed continuity of $\chi \mathfrak{W}_{\alpha\beta} \chi$ for $s \in I_2$. The operators $\chi \mathfrak{W}_{\alpha\beta} \chi$ are self adjoint in L_2. This follows from (2.8) by an elementary computation for the corresponding kernels. Hence, by duality they are also continuous in $H^{-s}(\mathbb{R}_+) \to \tilde{H}^{1-s}(\mathbb{R}_+)$ for $s \in I_2$. This means continuity in $H^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)$ for $s \in I_1$. Interpolation completes the proof of (2.29) for $\chi \mathfrak{W}_{\alpha\beta} \chi$.

Besides the above continuity of $\chi \mathfrak{L}^1_{-+} \chi$, Lemma 2.2 also yields the continuity of $\chi \mathfrak{L}^0_{-+} \chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)$ for $s \in I_1$. This implies also continuity of $\chi \mathfrak{L}^0_{-+} \chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)$ since the norm in $\tilde{H}^{s-1}(\mathbb{R}_+)$ dominates that in $H^{s-1}(\mathbb{R}_+)$ for $s \in I_1$. Because of (2.9) and (2.11), $\chi \mathfrak{L}_{0-+} \chi$ differs from both operators $\chi \mathfrak{L}^0_{-+} \chi$ and $\chi \mathfrak{L}^1_{-+} \chi$ only by operators generated by linear functionals of the form (2.32). Thus $\chi \mathfrak{L}_{0-+} \chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$ is continuous for $s \in I_1 \cup I_2$. For $s = \frac{1}{2}$ we find the proposition by interpolation.

The explicit form of the kernels of $\mathfrak{L}_{0+-}$ and $\mathfrak{L}_{0-+}$ in (2.9) shows that

$$(\chi \mathfrak{L}_{0-+} \chi)^* = (\chi \mathfrak{L}_{0+-} \chi) \tag{2.34}$$

holds where * denotes the L_2 adjoint.

Hence, by duality we find continuity

$$\chi \mathfrak{L}_{0+-} \chi\colon \tilde{H}^{-s}(\mathbb{R}_+) \to H^{1-s}(\mathbb{R}_+) \qquad \text{for } s \in (-\tfrac{1}{2}, \tfrac{3}{2}),$$

which coincides with (2.29).

iv) For (2.30) we see from (2.29) the proposed continuity for $s \in I_1$ because of $H^s(\mathbb{R}_+) = \tilde{H}^s(\mathbb{R}_+)$.

For the remaining case $s \in I_2$ note that with (2.11) and (2.9), respectively, we have

$$\chi(\mathfrak{L}_{0--} - \mathfrak{L}_{0-+}) \chi = -\chi \mathfrak{L}^1_{-+} \chi \quad \text{and} \quad \chi(\mathfrak{L}_{0+-} - \mathfrak{L}_{0++}) \chi = \chi \mathfrak{L}^0_{+-} \chi .$$

Both right hand sides are continuous in $\tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)$ for $s \in I_2$ due to Lemma 2.2. Again, interpolation gives (2.30) for all $s \in (-\frac{1}{2}, \frac{3}{2})$.

v) For $s \in I_2$ the proposed continuity (2.31) follows with (2.7) for

$$\chi(\mathcal{W}_{--} - \mathcal{W}_{-+})\chi = \chi(\mathcal{W}_{++} - \mathcal{W}_{+-})\chi = -\pi\chi(V_0 - V_\omega)\chi$$

by Lemma 2.2, that of

$$\chi(\mathcal{L}_{0--} - \mathcal{L}_{0-+} - \mathcal{L}_{0+-} + \mathcal{L}_{0++})\chi$$

by (2.30) because of $\tilde{H}^{s-1}(\mathbb{R}_+) = H^{s-1}(\mathbb{R}_+)$ for $s \in I_2$.

All these operators are self adjoint in L_2 (see (2.34)). Hence, by duality we find continuity for $s \in I_1$ and with interpolation for all $s \in (-\frac{1}{2}, \frac{3}{2})$. □

Now we are in the position to formulate the continuity properties of our integral operators $\mathcal{A}$ in (2.1) on Γ^ω. According to (1.18)-(1.20) we decompose all functions into their even and odd parts. For $\boldsymbol{g} \simeq (\boldsymbol{g}_-, \boldsymbol{g}_+)$ with

$$\boldsymbol{g}_+(x) = \boldsymbol{g}(x) \quad \text{and} \quad \boldsymbol{g}_-(x) = \boldsymbol{g}(x\exp[i\omega]) \qquad \text{for } x \geq 0 \tag{2.35}$$

we define

$$\boldsymbol{g}^e(x) := \boldsymbol{g}_+(x) + \boldsymbol{g}_-(x) \quad \text{and} \quad \boldsymbol{g}^o(x) := \boldsymbol{g}_-(x) - \boldsymbol{g}_+(x)\,. \tag{2.36}$$

This induces a decomposition of $\mathcal{A}$ for $\varkappa = \frac{1}{4}$ as follows:

$$(\mathcal{A}\boldsymbol{g})^e = \mathcal{A}_{ee}\boldsymbol{g}^e + \mathcal{A}_{oe}\boldsymbol{g}^o\,, \quad (\mathcal{A}\boldsymbol{g})^o = \mathcal{A}_{eo}\boldsymbol{g}^e + \mathcal{A}_{oo}\boldsymbol{g}^o; \tag{2.37}$$

where

$$\begin{cases} \mathcal{A}_{ee} = -\frac{1}{2}\{\mathcal{W}_{--} + \mathcal{W}_{-+} + \mathcal{W}_{+-} + \mathcal{W}_{++} + \mathcal{L}_{0--} + \mathcal{L}_{0-+} + \mathcal{L}_{0+-} + \mathcal{L}_{0++}\}, \\ A_{oe} = -\frac{1}{2}\{\mathcal{L}_{0--} + \mathcal{L}_{0-+} - \mathcal{L}_{0+-} - \mathcal{L}_{0++}\}\,, \\ \mathcal{A}_{eo} = -\frac{1}{2}\{\mathcal{L}_{0--} - \mathcal{L}_{0-+} + \mathcal{L}_{0+-} - \mathcal{L}_{0++}\}\,, \\ \mathcal{A}_{oo} = -\frac{1}{2}\{\mathcal{W}_{--} - \mathcal{W}_{-+} - \mathcal{W}_{+-} + \mathcal{W}_{++} + \mathcal{L}_{0--} - \mathcal{L}_{0-+} - \mathcal{L}_{0+-} + \mathcal{L}_{0++}\}. \end{cases} \tag{2.38}$$

THEOREM 2.4. *Let* $s \in (-\frac{1}{2}, \frac{3}{2})$. *Then the operator*

$$\chi\mathcal{A}\chi\colon H^{s-1}(\Gamma^\omega) \to H^s(\Gamma^\omega) \tag{2.39}$$

is continuous.

PROOF. In view of (1.18)-(1.20) we have to show continuity of the following mappings defined by (2.38):

$$\chi\mathcal{A}_{ee}\chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to H^s(\mathbb{R}_+)\,, \tag{2.40}$$

$$\chi\mathcal{A}_{oe}\chi\colon H^{s-1}(\mathbb{R}_+) \to H^s(\mathbb{R}_+)\,, \tag{2.41}$$

$$\chi\mathcal{A}_{eo}\chi\colon \tilde{H}^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)\,, \tag{2.42}$$

$$\chi\mathcal{A}_{oo}\chi\colon H^{s-1}(\mathbb{R}_+) \to \tilde{H}^s(\mathbb{R}_+)\,. \tag{2.43}$$

(2.40) follows from Lemma 2.3, (2.29). (2.42) follows from (2.30) in Lemma 2.3.

Since $\mathfrak{L}_{0--}$ and $\mathfrak{L}_{0++}$ are self adjoint in L_2 and because of (2.34) we see that $\chi\mathcal{A}_{oe}\chi$ is the adjoint to $\chi\mathcal{A}_{eo}\chi$ in L_2. Hence, by duality (2.42) implies (2.41).

Finally, (2.43) follows from (2.31) in Lemma 2.3. □

3. – Local coerciveness at a corner.

The generalization of the Gårding inequality (2.5) for a smooth boundary curve Γ to our polygon creates two significant new difficulties. The first consists of the generalization of (2.5) to $\mathcal{W}$ near a corner point. Here one needs the Mellin transformation and a careful investigation of the mappings (2.7). This generalization was performed in [5], here we shall repeat the proof for our situation which is a special case of that in [5]. The second difficulty arises from $\mathfrak{L}_0$ which is not a compact perturbation anymore as for smooth Γ. Here we shall prove that $\mathfrak{L}_0$ is a small perturbation with respect to a suitable spectral norm of $\mathcal{W}$.

For the analysis we need besides the relations (2.27), (2.28) between the spaces (1.18)-(1.20) and $W_0^s(\mathbb{R}_+)$ with (2.25) also a characterization of $\tilde{H}^{-\frac12}(\mathbb{R}_+)$ by means of the Mellin transformation which is given by the following lemma.

LEMMA 3.1 [5, Corollary 2.4]. *There exists a positive constant c such that for any $u \in C_0^\infty(0, \infty)$*

$$\|u\|^2_{\tilde{H}^{-\frac12}(\mathbb{R}_+)} \leqslant c \int\limits_{\operatorname{Im}\lambda=0} \frac{1+|\lambda|}{|\lambda|^2}\,|\hat{u}(\lambda - i)|^2\,d\lambda\,, \tag{3.1}$$

provided the integral is finite.

LEMMA 3.2. *The set of test functions*

$$M_\chi := \left\{ v \in C_0^\infty(\Gamma^\omega \setminus \{0\}) \,|\, \widehat{(\chi v^e)}(-i) = \frac{d}{d\lambda} \widehat{(\chi v^e)}(-i) = 0 \right\} \tag{3.2}$$

is dense in the subspace

$$H_+^{-\frac{1}{2}}(\Gamma^\omega) := \{ g \in H^{-\frac{1}{2}}(\Gamma^\omega) \,|\, (g, \chi)_{L_2(\Gamma^\omega)} = 0 \} \,. \tag{3.3}$$

PROOF. The first linear functional in (3.2) satisfies

$$\begin{aligned} (v, \chi)_{L_2(\Gamma^\omega)} &= (v_+, \chi)_{L_2(\mathbf{R}_+)} + (v_-, \chi)_{L_2(\mathbf{R}_+)} \\ &= (v^e, \chi)_{L_2(\mathbf{R}_+)} = \widehat{(v^e \chi)}(-i) \end{aligned} \tag{3.4}$$

and, hence is a bounded linear functional on $H^{-\frac{1}{2}}(\Gamma^\omega)$ since $\chi \in H^{\frac{1}{2}}(\Gamma^\omega)$. The second functional, however, is given by

$$\begin{aligned} \frac{d}{d\lambda} \widehat{(\chi v^e)}(-i) &= -i \int_0^\infty \log x \chi(x) v^e(x)\, dx = i\pi l(\chi v^e) \\ &= -i(v^e, \chi \log |\cdot|)_{L_2(\mathbf{R}_+)} \\ &= -i(v, \chi \log |\cdot|)_{L_2(\Gamma^\omega)} \end{aligned} \tag{3.5}$$

and, hence is *not* bounded on $H^{-\frac{1}{2}}(\Gamma^\omega)$ since $\chi \log |\cdot| \notin H^{\frac{1}{2}}(\Gamma^\omega)$. Therefore the kernel of the functional (3.5) is dense in $H_+^{-\frac{1}{2}}(\Gamma^\omega)$. □

The generalization of (2.5) to $\mathcal{W}$ reads as

THEOREM 3.3 (see also [5, Theorem 2.19]). *There exists a positive constant γ depending on χ and ω such that*

$$-(\mathcal{W}\chi \mathbf{g}, \chi \mathbf{g})_{L_2(\Gamma^\omega)} \geq \gamma \|\chi \mathbf{g}\|^2_{H^{-\frac{1}{2}}(\Gamma^\omega)} \tag{3.6}$$

for all $\mathbf{g} \in H_+^{-\frac{1}{2}}(\Gamma^\omega)$.

PROOF. Since $\mathcal{W}$ is defined by a diagonal matrix (cf. (1.6)), it suffices to show (3.6) for the scalar case. Moreover it suffices to prove (3.6) for $v \in M_\chi$ instead of g since M_χ is dense in $H_+^{-\frac{1}{2}}(\Gamma^\omega)$ and for $v \to g$ in $H^{-\frac{1}{2}}$ we have $\|\chi v\|_{H^{-\frac{1}{2}}(\Gamma^\omega)} \to \|\chi g\|_{H^{-\frac{1}{2}}(\Gamma^\omega)}$ and also

$$\begin{aligned} (\mathcal{W}\chi v, \chi v)_{L_2(\Gamma^\omega)} &= (\chi \mathcal{W} \chi v, v)_{L_2(\Gamma^\omega)} \\ &\to (\mathcal{W}\chi g, \chi g)_{L_2(\Gamma^\omega)} \end{aligned}$$

because of the continuity of $\chi \mathcal{W} \chi \colon H^{-\frac{1}{2}}(\Gamma^\omega) \to H^{\frac{1}{2}}(\Gamma^\omega)$ due to Theorem 2.4.

For the following we write

$$\chi v = \varphi \simeq (\varphi_-, \varphi_+) = \tfrac{1}{2}(\varphi^e + \varphi^o, \varphi^e - \varphi^o)\,,$$

and decompose also $\mathcal{W}$ accordingly as in (2.36). The decomposition in (2.38) corresponding to $\mathcal{W}$ in (3.6) gives with (2.7)

$$\begin{aligned}(3.7)\qquad -(\mathcal{W}\varphi, \varphi)_{L_2(\Gamma^\omega)} &= -\tfrac{1}{2}(\mathcal{W}_{ee}\varphi^e + \mathcal{W}_{oe}\varphi^o, \varphi^e) - \tfrac{1}{2}(\mathcal{W}_{eo}\varphi^e + \mathcal{W}_{oo}\varphi^o, \varphi^o)\\ &= -\tfrac{1}{2}((\mathcal{W}_{--} + \mathcal{W}_{-+})\varphi^e, \varphi^e)_{L_2(\mathbb{R}_+)} - \tfrac{1}{2}((\mathcal{W}_{--} - \mathcal{W}_{-+})\varphi^o, \varphi^o)_{L_2(\mathbb{R}_+)}\\ &= \frac{\pi}{2}((V_0 + V_\omega)\varphi^e, \varphi^e)_{L_2(\mathbb{R}_+)} + \frac{\pi}{2}((V_0 - V_\omega)\varphi^o, \varphi^o)_{L_2(\mathbb{R}^+)}\,.\end{aligned}$$

The right hand sides we rewrite with Parseval's equality for the Mellin transformed functions given in (2.14) obtaining

$$\begin{aligned}(3.8)\qquad -(\mathcal{W}\varphi, \varphi)_{L_2(\Gamma^\omega)} &= \frac{1}{4}\int\limits_{\operatorname{Im}\lambda = -\frac{1}{2}} (\hat{V}_0(\lambda) + \hat{V}_\omega(\lambda))\hat{\varphi}^e(\lambda - i)\overline{\hat{\varphi}^e(\lambda)}\,d\lambda\\ &\qquad + \frac{1}{4}\int\limits_{\operatorname{Im}\lambda = -\frac{1}{2}} (\hat{V}_0(\lambda) - \hat{V}_\omega(\lambda))\hat{\varphi}^o(\lambda - i)\overline{\hat{\varphi}^o(\lambda)}\,d\lambda\\ &= \frac{1}{4}\int\limits_{\lambda\in\mathbb{R}} (\hat{V}_0(\lambda) + \hat{V}_\omega(\lambda))|\hat{\varphi}^e(\lambda - i)|^2\,d\lambda\\ &\qquad + \frac{1}{4}\int\limits_{\lambda\in\mathbb{R}} (\hat{V}_0(\lambda) - \hat{V}_\omega(\lambda))|\hat{\varphi}^o(\lambda - i)|^2\,d\lambda\,.\end{aligned}$$

For the last equality we have deformed the path of integration and used the analyticity and exponential decay of the integrand at infinity. To this end we used the holomorphic function

$$\overline{\hat{\varphi}(\bar{\lambda} - i)} \text{ satisfying } \overline{\hat{\varphi}(\bar{\lambda} - i)}|_{\operatorname{Im}\lambda = -\frac{1}{2}} = \overline{\hat{\varphi}(\lambda)}|_{\operatorname{Im}\lambda = -\frac{1}{2}}\,.$$

In order to estimate (3.8) we use the inequalities

$$(3.9)\qquad \begin{cases} c_1 \dfrac{1 + |\lambda|}{\lambda^2} \leqslant \hat{V}_0(\lambda) + \hat{V}_\omega(\lambda) \leqslant c_2 \dfrac{1 + |\lambda|}{\lambda^2}\,,\\[2ex] c_1 \dfrac{1}{1 + |\lambda|} \leqslant \hat{V}_0(\lambda) - \hat{V}_\omega(\lambda) \leqslant c_2 \dfrac{1}{1 + |\lambda|} \quad \text{for all } \lambda \in \mathbb{R}, \end{cases}$$

where c_1 and c_2 are two suitable positive constants. Hence (3.8) yields

$$-(\mathcal{W}\varphi, \varphi)_{L_2(\Gamma^\omega)} \geqslant \gamma\left\{\int\limits_{\lambda\in\mathbb{R}} \frac{1 + |\lambda|}{\lambda^2}\,|\hat{\varphi}^e(\lambda - i)|^2\,d\lambda + \int\limits_{\lambda\in\mathbb{R}} \frac{1}{1 + |\lambda|}\,|\hat{\varphi}^o(\lambda - i)|^2\,d\lambda\right\}.$$

For the first integral on the right hand side we use (3.1) and for the second (2.25) together with (2.27) obtaining

$$-(\mathcal{W}\varphi,\varphi)_{L_2(\Gamma^\omega)} \geqslant \gamma'\{\|\varphi^e\|^2_{\tilde{H}^{-\frac{1}{2}}(\mathbf{R}_+)} + \|\varphi^o\|^2_{H^{-\frac{1}{2}}(\mathbf{R}_+)}\}.$$

With (1.20) this is the desired estimate (3.6). □

LEMMA 3.4. *For $g \in M_\chi$ the bilinear form*

$$(3.10)\qquad -(\mathcal{W}\chi g,\chi g)_{L_2(\Gamma^\omega)} = \frac{1}{4}\int_{\lambda\in\mathbf{R}} (\hat{V}_0(\lambda) + \hat{V}_\omega(\lambda))|\widehat{\chi g^e}(\lambda - i)|^2\,d\lambda + \frac{1}{4}\int_{\lambda\in\mathbf{R}} (\hat{V}_0(\lambda) - \hat{V}_\omega(\lambda))|\widehat{\chi g^o}(\lambda - i)|^2\,d\lambda$$

is equivalent to $\|\chi g\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma^\omega)}$.

PROOF. The relation (3.10) was shown in the foregoing proof of Theorem 3.3, viz. (3.8). The equivalence follows from (3.6) together with the continuity of $\chi\mathcal{W}\chi$ which results from Theorem 2.4. □

The following theorem gives the main result of this section.

THEOREM 3.5. *For $g \in H_+^{-\frac{1}{2}}(\Gamma^\omega)$ there holds*

$$(3.11)\qquad |(\mathcal{L}\chi g,\chi g)_{L_2(\Gamma^\omega)}| \leqslant \max\left\{\left|\frac{\sin\omega}{\omega}\right|, \left|\frac{\sin(2\pi-\omega)}{2\pi-\omega}\right|\right\} |(\mathcal{W}\chi g,\chi g)_{L_2(\Gamma^\omega)}|\,.$$

This implies for $\mathcal{A} = -(\mathcal{W} + \mathcal{L})$ and $0 < \omega < 2\pi$ the coerciveness

$$(3.12)\qquad (\mathcal{A}\chi g,\chi g)_{L_2(\Gamma^\omega)} \geqslant \gamma\|\chi g\|^2_{\tilde{H}^{-\frac{1}{2}}(\Gamma^\omega)}$$

for all $g \in H_+^{-\frac{1}{2}}(\Gamma^\omega)$, where γ is a suitable positive constant depending only on ω and χ.

PROOF. i) The coerciveness (3.12) follows from

$$\left|\frac{\sin\omega}{\omega}\right| < 1 \quad\text{and}\quad \left|\frac{\sin(2\pi-\omega)}{2\pi-\omega}\right| < 1\,,$$

Theorem 3.3, and (3.11) by the triangle inequality.

ii) Because of the continuity of $\chi\mathcal{L}\chi$ and $\chi\mathcal{W}\chi$ (Theorem 2.4), it suffices to show (3.11) for $g \in M_\chi$. In this case all the linear functionals in $\mathcal{L}$ and $\mathcal{W}$ vanish on g. Hence here $\mathcal{L}\chi g = \mathcal{L}_0\chi g$ and $\mathcal{A}$ can be decomposed as in (2.37), (2.38). Furthermore $\mathfrak{L}^0_{++}\chi g = \mathfrak{L}^0_{--}\chi g = 0$. Thus we obtain with $\varphi = \chi g$

$$(3.13)\qquad \begin{aligned}(\mathfrak{L}\varphi, \varphi)_{L_2(\Gamma^\omega)} &= \tfrac{1}{4}\big((\mathfrak{L}^0_{-+} + \mathfrak{L}^0_{+-})\varphi^e + (\mathfrak{L}^0_{-+} - \mathfrak{L}^0_{+-})\varphi^o, \varphi^e\big)_{L_2(\mathbf{R}_+)} \\ &\quad + \tfrac{1}{4}\big((\mathfrak{L}^0_{+-} - \mathfrak{L}^0_{-+})\varphi^e - (\mathfrak{L}^0_{-+} + \mathfrak{L}^0_{+-})\varphi^o, \varphi^o\big)_{L_2(\mathbf{R}_+)} \\ &= \frac{1}{2\pi}\int\limits_{\operatorname{Im}\lambda = -\frac{1}{2}} \{\overline{\hat{\varphi}^{eT}(\lambda)}(\hat{\mathfrak{R}}(\lambda)\hat{\varphi}^e(\lambda - i) + i\hat{\mathfrak{J}}(\lambda)\hat{\varphi}^o(\lambda - i)) \\ &\qquad - \overline{\hat{\varphi}^{oT}(\lambda)}(\hat{\mathfrak{R}}(\lambda)\hat{\varphi}^o(\lambda - i) + i\hat{\mathfrak{J}}(\lambda)\hat{\varphi}^e(\lambda - i))\}\, d\lambda \\ &= \frac{1}{2\pi}\int\limits_{\lambda\in\mathbf{R}} \{\overline{\hat{\varphi}^{eT}(\lambda - i)}(\hat{\mathfrak{R}}(\lambda)\hat{\varphi}^e(\lambda - i) + i\hat{\mathfrak{J}}(\lambda)\hat{\varphi}^o(\lambda - i)) \\ &\qquad - \overline{\hat{\varphi}^{oT}(\lambda - i)}(\hat{\mathfrak{R}}(\lambda)\hat{\varphi}^o(\lambda - i) + i\hat{\mathfrak{J}}(\lambda)\hat{\varphi}^e(\lambda - i))\}\, d\lambda\end{aligned}$$

where

$$\hat{\mathfrak{R}} = \frac{1}{4}(\hat{\mathfrak{L}}^0_{-+} + \hat{\mathfrak{L}}^0_{+-}), \qquad \hat{\mathfrak{J}} = \frac{1}{4i}(\hat{\mathfrak{L}}^0_{-+} - \hat{\mathfrak{L}}^0_{+-}).$$

In view of (3.10) we introduce the new Mellin transformed vector function

$$(3.14)\qquad \Phi(\lambda)^T := \Big((\hat{V}_0(\lambda) + \hat{V}_\omega(\lambda))^{\frac{1}{2}}\hat{\varphi}^{eT}(\lambda - i), (\hat{V}_0(\lambda) - \hat{V}_\omega(\lambda))^{\frac{1}{2}}\hat{\varphi}^{oT}(\lambda - i)\Big).$$

With this notation we write (3.10) as

$$(3.15)\qquad -(\mathfrak{W}\varphi, \varphi)_{L_2(\Gamma^\omega)} = \tfrac{1}{4}\int\limits_{\lambda\in\mathbf{R}} |\Phi(\lambda)|^2\, d\lambda$$

and (3.13) as

$$(3.16)\qquad (\mathfrak{L}\varphi, \varphi)_{L_2(\Gamma^\omega)} = \int\limits_{\lambda\in\mathbf{R}} \overline{\Phi^T(\lambda)}\,\tilde{\mathfrak{L}}(\lambda)\Phi(\lambda)\, d\lambda$$

where $\tilde{\mathfrak{L}}(\lambda)$ is given by

$$(3.17)\qquad \tilde{\mathfrak{L}}(\lambda) = \begin{pmatrix} \alpha\begin{pmatrix}\cos\omega & \sin\omega \\ \sin\omega & -\cos\omega\end{pmatrix} & \beta\begin{pmatrix}\sin\omega & -\cos\omega \\ -\cos\omega & -\sin\omega\end{pmatrix} \\ -\beta\begin{pmatrix}\sin\omega & -\cos\omega \\ -\cos\omega & -\sin\omega\end{pmatrix} & \delta\begin{pmatrix}\cos\omega & \sin\omega \\ \sin\omega & -\cos\omega\end{pmatrix}\end{pmatrix}$$

with

$$(3.18)\qquad \begin{cases} \alpha = -\dfrac{\lambda}{4}\sin\omega\sinh(\pi - \omega)\lambda(\cosh(\pi - \omega)\lambda + \cosh\pi\lambda)^{-1} \\[2ex] \beta = i\,\dfrac{\lambda}{4}\sin\omega\cosh(\pi - \omega)\lambda(\cosh^2\pi\lambda - \cosh^2(\pi - \omega)\lambda)^{-\frac{1}{2}}, \\[2ex] \delta = \dfrac{\lambda}{4}\sin\omega\sinh(\pi - \omega)\lambda(\cosh\pi\lambda - \cosh(\pi - \omega)\lambda)^{-1}. \end{cases}$$

For the comparison of (3.15) with (3.16) in view of the assertion (3.11) we have to show the pointwise estimate

$$|\overline{\Phi^T}\tilde{\mathfrak{L}}\Phi| \leqslant \max_{j=1,\dots,4}|\mu_j||\Phi|^2 \leqslant \frac{1}{4}\max\left\{\left|\frac{\sin(2\pi-\omega)}{(2\pi-\omega)}\right|, \left|\frac{\sin\omega}{\omega}\right|\right\}|\Phi|^2 \tag{3.19}$$

where μ_j denote the eigenvalues of the Hermitian matrix $\tilde{\mathfrak{L}}$ (3.17). (3.19) means « strong ellipticity » of the system on Γ^ω, i.e. a generalization of the corresponding concept for pseudo-differential operators which is defined via Fourier transformation [29], [7, § 26].

For finding μ_j we first compute the determinant

$$\det(\tilde{\mathfrak{L}}-\mu I) = \left(\frac{\lambda}{4}\right)^4 \frac{\sin^4\omega}{\Delta^2} \tag{3.20}$$

$$\cdot\left\{\left(\sinh^2(\pi-\omega)\lambda\left(1-\frac{\mu^2}{\alpha\delta}\right)-\cosh^2(\pi-\omega)\lambda\right)^2-\sinh^4(\pi-\omega)\lambda\left(\frac{1}{\alpha}-\frac{1}{\delta}\right)^2\mu^2\right\}$$

$$=\frac{1}{4^4\Delta^2}\left(\lambda^2\sin^2\omega-(4\mu)^2\sinh^2\omega\lambda\right)\left(\lambda^2\sin^2\omega-(4\mu)^2\sinh^2(2\pi-\omega)\lambda\right)$$

where

$$\Delta = \cosh^2\pi\lambda-\cosh^2(\pi-\omega)\lambda = \sinh(2\pi-\omega)\lambda\cdot\sinh\omega\lambda\,. \tag{3.21}$$

Then for $\lambda\neq 0$ the four eigenvalues are given by

$$\left|\begin{aligned}\mu &= \pm\frac{\lambda\sin\omega}{4\Delta}\{\sinh(\pi-\omega)\lambda\cosh\pi\lambda\pm\cosh(\pi-\omega)\lambda\sinh\pi\lambda\}\\ &= \pm\frac{\lambda}{4}\sin\omega\frac{\sinh(\pi-\omega\pm\pi)\lambda}{\sinh(2\pi-\omega)\lambda\sinh\omega\lambda}.\end{aligned}\right. \tag{3.22}$$

Hence

$$|\mu_j| \leqslant \frac{1}{4}\max\left\{\left|\frac{\lambda\sin\omega}{\sinh(2\pi-\omega)\lambda}\right|, \left|\frac{\lambda\sin\omega}{\sinh\omega\lambda}\right|\right\} \quad \text{for } j=1,\dots,4\,. \tag{3.23}$$

Since

$$\frac{x}{\sinh x}\leqslant 1 \qquad \text{for all } x\in\mathbb{R}$$

and $|\sin\omega| = |\sin(2\pi-\omega)|$ we obtain (3.19) from (3.23), i.e. (3.11). □

4. – Local regularity at a corner.

In order to obtain a local expansion of the solution of our integral equations in terms of singularity functions near a corner we proceed along the lines of Kondratiev [18]. For this expansion we shall need the regularity of the solution on the smooth parts of the boundary which can be characterized by standard a-priori estimates with pseudodifferential equations as follows.

LEMMA 4.1. *Let $\chi \in C_0^\infty$ be a cut-off function having its support on the interior of some segment Γ^j. Let $\boldsymbol{f} \in H^s(\Gamma)$ with $s \geqslant \frac{1}{2}$ and let $\boldsymbol{g} \in H^{-\frac{1}{2}}(\Gamma)$ be a solution of the integral equation* (1.6) *with $a_3 = 0$. Then there holds the a-priori estimate*

$$\|\chi \boldsymbol{g}\|_{H^{s-1}(\Gamma)} \leqslant c\{\|\boldsymbol{f}\|_{H^s(\Gamma)} + \|\boldsymbol{g}\|_{H^{-\frac{1}{2}}(\Gamma)} + |\boldsymbol{a}|\}. \tag{4.1}$$

PROOF. Multiplication of (1.6) by χ gives

$$\chi \mathfrak{W} \boldsymbol{g} = \chi(\boldsymbol{f} + \boldsymbol{a} - \mathfrak{L}\boldsymbol{g}) .$$

Now we introduce another cut-off function $\chi_1 \subset C_0^\infty(\Gamma^j)$ with $\chi = \chi\chi_1$ and obtain

$$\chi \mathfrak{W} \chi_1 \boldsymbol{g} = - \chi \mathfrak{W}(1 - \chi_1)\boldsymbol{g} + \chi(\boldsymbol{f} + \boldsymbol{a} - \mathfrak{L}\boldsymbol{g}) = \chi \boldsymbol{h} . \tag{4.2}$$

Since $\chi(1 - \chi_1) \equiv 0$, the kernel of $\chi\mathfrak{W}(1 - \chi_1)$ as well as $\chi(z)\mathfrak{L}(z, \zeta)$ are C^∞-functions. Hence the right hand side of (4.2) can be estimated as

$$\|\chi \boldsymbol{h}\|_{H^s(\Gamma)} \leqslant c\{\|\boldsymbol{f}\|_{H^s(\Gamma)} + \|\boldsymbol{g}\|_{H^{-\frac{1}{2}}(\Gamma)} + |\boldsymbol{a}|\}. \tag{4.3}$$

(4.2) can be understood as an equation on a simple closed C^∞ curve $\tilde{\Gamma}$ containing Γ^j and having conformal radius $\neq 1$. Then on $\tilde{\Gamma}$ consider V^{-1} which is a pseudodifferential operator of order 1 which gives

$$V^{-1}\chi \mathfrak{W} \chi_1 \boldsymbol{g} = - \chi\pi\chi_1 \boldsymbol{g} + (V^{-1}\chi - \chi V^{-1}) \mathfrak{W} \chi_1 \boldsymbol{g} .$$

Hence (4.2) yields

$$\pi\chi\boldsymbol{g} = (V^{-1}\chi - \chi V^{-1}) \mathfrak{W} \chi_1 \boldsymbol{g} - V^{-1}\chi \boldsymbol{h} .$$

The commutator $V^{-1}\chi - \chi V^{-1}$ is a pseudodifferential operator of order zero and therefore we find the estimate

$$\|\chi g\|_{H^{s-1}(\Gamma)} \leq c\{\|\chi_1 g\|_{H^{s-2}(\Gamma)} + \|\chi h\|_{H^s(\Gamma)}\}.$$

Repeating the above arguments we find an estimate

$$\|\chi_1 g\|_{H^{s-2}(\Gamma)} \leq c'\{\|\chi_2 g\|_{H^{s-3}(\Gamma)} + \|\chi_1 h\|_{H^{s-1}(\Gamma)}\},$$

where the cut-off function $\chi_2 \in C_0^\infty(\Gamma^j)$ satisfies $\chi_1 = \chi_1\chi_2$. After a finite number of repeated estimates we finally find

$$\|\chi g\|_{H^{s-1}(\Gamma)} \leq c\{\|\chi_n g\|_{H^{-\frac{1}{2}}(\Gamma)} + \|\chi_n h\|_{H^s(\Gamma)}\}$$

and with (4.3) the desired estimate (4.1). □

For the regularity at the corner points we use again the Mellin transformed equations and the Cauchy residual theorem in the complex plane of the Mellin transformed variable. As we shall see, the singularity functions are given by

$$v \simeq (v_-, v_+) = (c_-, c_+)x^{-i\lambda-1}\chi(x) \tag{4.4}$$

where λ is a solution of the transcendental equation

$$(\lambda^2 \sin^2 \omega - \sinh^2 \omega\lambda)(\lambda^2 \sin^2 (2\pi - \omega) - \sinh^2 (2\pi - \omega)\lambda) = 0 \tag{4.5}$$

and $(c_-, c_+) \in \mathbb{C}^4$ is the corresponding eigenvector of the Mellin symbol at the above λ. In the case of λ being a multiple root of (4.5) the singularity functions (4.4) may have the more general form

$$v \simeq \sum_{l=0}^{3} (c_{l-}, c_{l+})x^{-i\lambda-1}(\log x)^l \chi(x) \tag{4.6}$$

where (c_{l-}, c_{l+}) are appropriate chains of generalized eigenvectors of the Mellin symbol.

It should be emphasized that (4.5) is exactly the same equation as for the singularity functions of the Dirichlet problem for U with the bi-Laplacian in the interior as well as the exterior domain of Γ near the corner. For the latter we refer to Williams [35], Kondratiev [18] and Seif [26], for further investigations see e.g. [22], [21], [28]. Seif [26] has shown, the roots of each factor in (4.5) have at most multiplicity 2, being in this case purely imaginary.

Furthermore, for $\omega > 0.812\pi$ or $(2\pi - \omega) > 0.812\pi$, respectively, all roots of the corresponding factor are purely imaginary. For the characterization of local regularity of the solution we need the following Lemma.

LEMMA 4.2. *Let* $\chi \in C_0^\infty[0, \infty)$ *with* $\chi_{|[0,\alpha]} \equiv 1$,

$$\chi_{|[\beta,\infty)} \equiv 0 \quad \text{and } 0 \leqslant \chi \leqslant 1 \quad \text{and } u \in \mathring{W}_0^{h+\frac{1}{2}}, \qquad h \in \mathbb{R}.$$

i) *The Mellin transformed function* $\widehat{\chi u}(\lambda)$ *exists for* $\operatorname{Im}\lambda \leqslant h$ *and is holomorphic for* $\operatorname{Im}\lambda < h$. *Moreover, there exist real positive constants* c, N *and* σ *such that*

$$|\widehat{\chi u}(\lambda)| \leqslant c(1 + |\lambda|^2)^\sigma (h - \operatorname{Im}\lambda)^{-N} \beta^{-\operatorname{Im}\lambda} \quad \text{for } \operatorname{Im}\lambda < h \tag{4.7}$$

and

$$\int_{\operatorname{Im}\lambda = k} (1 + |\lambda|^2|)^{h+\frac{1}{2}} |\widehat{\chi u}(\lambda)|^2 d\lambda \leqslant c \cdot \beta^{2(h-k)} \|u\|^2_{\mathring{W}_0^s} \tag{4.8}$$

for any $k \leqslant h$.

ii) *The Mellin transform* $\widehat{(1 - \chi)}(\lambda)$ *exists for* $\operatorname{Im}\lambda > 0$ *and is of the form*

$$\widehat{(1 - \chi)}(\lambda) = \frac{\hat{\varphi}(\lambda)}{\lambda} \tag{4.9}$$

where $\varphi \in C_0^\infty[\alpha, \beta]$ *and, hence,* $\hat{\varphi}(\lambda)$ *is entire analytic satisfying*

$$|\hat{\varphi}(\lambda)| \leqslant c_N (1 + |\lambda|^2)^{-N} \max\{\alpha^{-\operatorname{Im}\lambda}, \beta^{-\operatorname{Im}\lambda}\} \tag{4.10}$$

for any $N \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$.

iii) $\widehat{(1 - \chi)u}(\lambda)$ *exists for* $\operatorname{Im}\lambda \geqslant h$ *and is holomorphic for* $\operatorname{Im}\lambda > h$. *It can be expressed by the convolution*

$$\widehat{(1 - \chi)u}(\lambda) = \frac{1}{2\pi} \int_{\operatorname{Im}\mu = h_0} \hat{u}(\mu) \widehat{(1 - \chi)}(\lambda - \mu)\, d\mu \tag{4.11}$$

for $\operatorname{Im}\lambda > h_0$.

PROOF. i) Since the Mellin transform of χu on the line $\operatorname{Im}\lambda = h$ is defined by the Fourier transform of $\psi(t) = \chi u(\exp[-t]) \cdot \exp[th]$, we have

$$\|\chi u\|^2_{\mathring{W}_0^{h+\frac{1}{2}}} = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = h} (1 + |\lambda|^2)^{h+\frac{1}{2}} |\widehat{\chi u}(\lambda)|^2 d\lambda = \|\psi\|^2_{H^{h+\frac{1}{2}}(\mathbb{R})}.$$

This implies $\psi \in H^{h+\frac{1}{2}}(\mathbb{R}) \subset \mathcal{S}'$ (the space of tempered distributions). Since, furthermore, $\operatorname{supp}(\psi) \subset [-\ln\beta, \infty)$, the desired estimate (4.7) as well as the proposed holomorphy are direct consequences of the Paley-Wiener-Schwartz theorem [25, § IX.3].

In order to verify (4.8) we first introduce the modified Bessel potential operator $\Lambda^{h+\frac{1}{2}}$ by

$$\widehat{\Lambda^{h+\frac{1}{2}}\chi u}(\mu) = (\mu - i)^{h+\frac{1}{2}}\,\widehat{\chi u}(\mu + ih) \quad \text{for } \mu \in \mathbb{R}\,. \tag{4.12}$$

From (4.12) we have obviously the equivalence of norms

$$\|\Lambda^{h+\frac{1}{2}}\chi u(\exp[-t])\|_{L_2(\mathbb{R})} \simeq \|\chi u\|_{\overset{\circ}{W}{}^{h+\frac{1}{2}}_0}\,.$$

Inserting (4.12) into (4.7) yields an estimate of the same type for $\widehat{\Lambda^{h+\frac{1}{2}}\chi u}$, i.e.

$$|\widehat{\Lambda^{h+\frac{1}{2}}\chi u}(\mu)| \leqslant c'(1+|\mu|^2)^{\sigma+\frac{1}{4}+h/2}\beta^{-\operatorname{Im}\mu}\,,$$

implying $\operatorname{supp}(\Lambda^{h+\frac{1}{2}}\chi u) \subset [0, \beta]$ by the Paley-Wiener-Schwartz theorem. Thus the Euler transformed function $\Lambda^{h+\frac{1}{2}}\chi u(\exp[-t])$ belongs to $L_2(\mathbb{R})$ having its support in $[-\ln\beta, \infty)$ and the classical Paley-Wiener Theorem can be applied yielding

$$\int\limits_{\operatorname{Im}\mu = k-h} |\widehat{\Lambda^{h+\frac{1}{2}}\chi u}(\mu)|^2\, d\mu \leqslant \beta^{2(h-k)}\|\Lambda^{h+\frac{1}{2}}\chi u(\exp[-t])\|^2_{L_2(\mathbb{R})}\,.$$

This is equivalent to (4.8).

ii) With $\varphi(x) = -ix\chi'(x)$, $\varphi \in C_0^\infty[\alpha, \beta]$ we find (4.9) by integration by parts for $\operatorname{Im}\lambda > 0$. Then the Euler transformed $\varphi(\exp[-t])$ is in $C_0^\infty[-\ln\beta, -\ln\alpha]$ and (4.10) is the consequence of the Paley-Wiener-Schwartz theorem.

iii) The desired holomorphy follows from the Paley-Wiener-Schwartz theorem applied to

$$\eta(t) := \big((1-\chi)u\big)(\exp[t])\cdot\exp[-th] \quad \text{with } \eta \in H^{-h+\frac{1}{2}}(\mathbb{R}) \subset \mathcal{S}'$$

and $\operatorname{supp}\eta \subset (-\infty, \ln\beta]$.

The convolution property (4.8) follows by direct verification from the convolution theorem for the Fourier transformation. □

In the following theorem we summarize the local regularity properties of the solution of the integral equations.

THEOREM 4.3. *Let* $\boldsymbol{f} \in H^s(\Gamma)$ *with* $s > \frac{1}{2}$ *and* $s \neq \frac{1}{2} + \operatorname{Im}\lambda$ *for all roots* λ *of* (4.5). *Let* $\boldsymbol{g} \in H^{-\frac{1}{2}}(\Gamma)$ *be a solution of the integral equations* (1.6), (1.8). *Then* $\boldsymbol{g}$ *provides the local expansion*

$$\chi \boldsymbol{g} = \chi \boldsymbol{g}^{(s)} + \sum_{0<\operatorname{Im}\lambda_k<s-\frac{1}{2}} c_k \boldsymbol{v}_k\,, \qquad c_k \in \mathbb{R}\,, \tag{4.13}$$

where $\boldsymbol{v}_k$ *are the functions of the form* (4.4) *or* (4.6), *respectively, and where* λ_k *are the roots of* (4.5) *or* $\lambda_k = im$, $m = 1, 2, \ldots$, *and* $\chi \boldsymbol{g}^{(s)} \in H^{s-1}(\Gamma^\infty)$. *Furthermore there holds the a-priori estimate*

$$\|\chi \boldsymbol{g}^{(s)}\|_{H^{s-1}(\Gamma^\infty)} + \sum_{0<\operatorname{Im}\lambda_k<s-\frac{1}{2}} |c_k| \leqslant C\{\|\boldsymbol{f}\|_{H^s(\Gamma)} + |\boldsymbol{a}| + \|\boldsymbol{g}\|_{H^{-\frac{1}{2}}(\Gamma)}\}\,. \tag{4.14}$$

PROOF. In order to use the Kondratiev technique for the system of integral equations we first bring it into such a form that the Mellin transformation can be applied. Therefore we introduce cut-off functions $\chi_1, \chi_2, \chi_3 \in C_0^\infty(\mathbb{C})$ depending only on $|z|$ with $0 \leqslant \chi_j \leqslant 1$, $\chi_j \equiv 1$ for $|z| \leqslant \alpha_j$ and $\chi_j \equiv 0$ for $|z| \geqslant \beta_j$ $(j = 1, 2, 3)$. $\alpha_j < \beta_j$ are chosen such that $\beta_1 < \alpha_2$, $\beta_2 < \alpha_3$ and furthermore the support of χ_j contains only the corner at the origin. Then (1.6) near the corner can be written as

$$\chi_2 \mathcal{A} \chi_1 \boldsymbol{g} = \chi_2(-\boldsymbol{f} - \boldsymbol{a} - \mathcal{A}(1 - \chi_1)\boldsymbol{g}) =: \chi_2 \boldsymbol{F} \text{ on } \Gamma\,. \tag{4.15}$$

Obviously, (4.15) can also be considered as an equation on Γ^∞. The right hand side satisfies the estimate

$$\|\chi_2 \boldsymbol{F}\|_{H^s(\Gamma^\infty)} \leqslant c\{\|\boldsymbol{f}\|_{H^s(\Gamma)} + |\boldsymbol{a}| + \|\boldsymbol{g}\|_{H^{-\frac{1}{2}}(\Gamma)}\}\,. \tag{4.16}$$

This follows, since

$$\chi_2 \mathcal{A}(1 - \chi_1)\boldsymbol{g} = \chi_2 \mathcal{A}(1 - \chi_1)\chi_3 \boldsymbol{g} + \chi_2 \mathcal{A}(1 - \chi_1)(1 - \chi_3)\boldsymbol{g}$$

where on one hand

$$\begin{aligned}\|\chi_2 \mathcal{A}(1 - \chi_1)\chi_3 \boldsymbol{g}\|_{H^s(\Gamma^\infty)} &\leqslant c\|(1 - \chi_1)\chi_3 \boldsymbol{g}\|_{H^{s-1}(\Gamma)}\\ &\leqslant c\{\|\boldsymbol{f}\|_{H^s(\Gamma)} + |\boldsymbol{a}| + \|\boldsymbol{g}\|_{H^{-\frac{1}{2}}(\Gamma)}\}\end{aligned}$$

due to Lemma 4.1 with $\chi = (1-\chi_1)\chi_3$ and on the other hand

$$\|\chi_2 \mathcal{A}(1-\chi_1)(1-\chi_3) g\|_{H^s(\Gamma^\omega)} \leq c \|g\|_{H^{-\frac{1}{2}}(\Gamma)}$$

since the operator has a C^∞-kernel. As we shall see below, we will need the Mellin transformed equation for $\operatorname{Im}\lambda \in (-1, 0)$ whereas (2.12) gives the Mellin transformed equation for $\operatorname{Im}\lambda \in (0,1)$. Therefore we need a different decomposition

$$\mathcal{A} = \mathcal{A}_0^0 + \mathcal{A}_1^0 \tag{4.17}$$

given by

$$\left\{\begin{aligned} \mathcal{A}_0^0 \boldsymbol{\varphi} = & \begin{pmatrix} \pi(V_0^* + V_\omega^*) & \pi(V_0^* - V_\omega^*) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi}^e \\ \boldsymbol{\varphi}^o \end{pmatrix} \\ & -2 \begin{pmatrix} \mathfrak{R} & i\mathfrak{J} + \frac{1}{2}(\mathfrak{L}_{0--} - \mathfrak{L}_{0++}) \\ -i\mathfrak{J} - \frac{1}{2}(\mathfrak{L}_{0--} - \mathfrak{L}_{0++}) & -\mathfrak{R} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi}^e \\ \boldsymbol{\varphi}^o \end{pmatrix} \end{aligned}\right. \tag{4.18}$$

and

$$\left|\begin{aligned} \mathcal{A}_1^0 \boldsymbol{\varphi} = & \begin{pmatrix} 2\pi l^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi}^e \\ \boldsymbol{\varphi}^o \end{pmatrix} \\ & - \begin{pmatrix} \mathfrak{L}_{0--} + \mathfrak{L}_{0++} & 0 \\ \mathfrak{L}_{0--} - \mathfrak{L}_{0++} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi}^e \\ \boldsymbol{\varphi}^o \end{pmatrix} \end{aligned}\right. \tag{4.19}$$

where $\mathfrak{R}$ and $\mathfrak{J}$ correspond to (3.13), $\mathfrak{L}_{0--}$ and $\mathfrak{L}_{0++}$ are given by (2.9),

$$\left\{\begin{aligned} l^* \boldsymbol{\varphi} &= -\frac{1}{\pi} \log x \int_0^\infty \boldsymbol{\varphi}(\xi)\, d\xi\,, \\ V_\omega^* \boldsymbol{\varphi}(x) &= -\frac{1}{\pi} \int_0^\infty \log \left| 1 - \frac{\xi}{x} \exp[i\omega] \right| \boldsymbol{\varphi}(\xi)\, d\xi \end{aligned}\right. \tag{4.20}$$

and V_0 with $\omega = 0$. Now the Mellin transform $\widehat{\mathcal{A}_0^0 \boldsymbol{\varphi}}(\lambda)$ exists for $\operatorname{Im}\lambda \in$ $\in (-1, 0)$ provided $\boldsymbol{\varphi} \in C_0^\infty$ is any test function and has the form

$$\widehat{\mathcal{A}_0^0 \boldsymbol{\varphi}}(\lambda) = \hat{\mathcal{A}}(\lambda)\, \hat{\boldsymbol{\varphi}}(\lambda - i) \tag{4.21}$$

with

$$\hat{\mathcal{A}}(\lambda) = -\hat{\mathfrak{W}}(\lambda) - \hat{\mathfrak{L}}(\lambda) \tag{4.22}$$

and

$$(4.23)\qquad -\hat{\mathfrak{W}}(\lambda) = \begin{pmatrix} \pi(\hat{\mathcal{V}}_0 + \hat{\mathcal{V}}_\omega) & 0 \\ 0 & \pi(\hat{\mathcal{V}}_0 - \hat{\mathcal{V}}_\omega) \end{pmatrix},$$

$$(4.24)\qquad \hat{\mathfrak{L}}(\lambda) = 2\begin{pmatrix} \hat{\mathfrak{R}} & i\hat{\mathfrak{J}} \\ -i\hat{\mathfrak{J}} & -\hat{\mathfrak{R}} \end{pmatrix},$$

(cf. (2.14), (3.13)). Here the transition from $\operatorname{Im}\lambda \in (0, 1)$ to $\operatorname{Im}\lambda \in (-1, 0)$ can be performed in the same manner as the transition from $\mathfrak{L}^1_{-+}$ to $\mathfrak{L}^0_{-+}$ in the proof of Lemma 2.1.

Now approximate $\chi_1 \boldsymbol{g}$ in $H^{-\frac{1}{2}}(\Gamma^\omega)$ by $\boldsymbol{\varphi} \in C_0^\infty(\Gamma^\omega)$. Then $\hat{\boldsymbol{\varphi}}(\lambda - i)$ converges to $\widehat{\chi_1 \boldsymbol{g}}(\lambda - i)$ in the sense of Lemma 4.2, (4.7) for $\operatorname{Im}\lambda < 0$. Hence there holds for $\operatorname{Im}\lambda \in (-1, 0)$

$$(4.25)\qquad \widehat{\mathcal{A}_0^0 \chi_1 \boldsymbol{g}}(\lambda) = \hat{\mathcal{A}}(\lambda)\, \widehat{\chi_1 \boldsymbol{g}}(\lambda - i)\,.$$

Now we write the local integral equation (4.15) with the decomposition (4.17) as

$$\chi_2 \mathcal{A}_0^0 \chi_1 \boldsymbol{g} = \chi_2 \boldsymbol{F} - \chi_2 \mathcal{A}_1^0 \chi_1 \boldsymbol{g} =: \chi_2 \boldsymbol{G}$$

or

$$(4.26)\qquad \mathcal{A}_0^0 \chi_1 \boldsymbol{g} \quad = (1 - \chi_2)\mathcal{A}_0^0 \chi_1 \boldsymbol{g} + \chi_2 \boldsymbol{G} =: \boldsymbol{H}\,.$$

The term $\chi_2 \mathcal{A}_1^0 \chi_1 \boldsymbol{g}$ consists of constant, respectively logarithmic functions (cf. (4.19), (4.20)) multiplied by χ_2. Therefore its Mellin transform exists and is holomorphic for $\operatorname{Im}\lambda < 0$. Moreover it has the form

$$(4.27)\qquad \widehat{\chi_2 \mathcal{A}_1^0 \chi_1 \boldsymbol{g}}(\lambda) = \frac{\hat{\varphi}(\lambda)}{\lambda^2}$$

where $\hat{\varphi}(\lambda)$ is an entire analytic function satisfying (4.10). Note that the functionals in $\mathcal{A}_1^0$ are continuous in $H^{-\frac{1}{2}}(\Gamma^\omega)$. The Mellin transform $\widehat{\chi_2 \boldsymbol{F}}(\lambda)$ exists and is holomorphic for $\operatorname{Im}\lambda < 0$ and has a meromorphic extension to $\operatorname{Im}\lambda < s - \frac{1}{2}$ possessing at most simple poles at $\lambda = ki$, $k \in \mathbb{N}_0$, since $\chi_2 \boldsymbol{F}(x) = \chi_2 \boldsymbol{F}_0(x) + \chi_2 \boldsymbol{P}(x)$ where $\boldsymbol{P}(x)$ is the Taylor polynomial of $\boldsymbol{F}$ about $x = 0$ of degree $\geqslant s$ and $\chi_2 \boldsymbol{F}_0 \in \tilde{H}^\sigma(\mathbb{R}_+)$ for all $\sigma < s$. The Mellin transform $\widehat{\chi_2 \boldsymbol{F}_0}(\lambda)$ is holomorphic for $\operatorname{Im}\lambda < s - \frac{1}{2}$ due to Lemma 4.2 and $\widehat{\chi_2 \boldsymbol{P}}(\lambda)$ is meromorphic in $\mathbb{C}$.

If (4.26) is Mellin transformed for $\operatorname{Im}\lambda \in (-1, 0)$ then the above arguments yield there the existence of $\widehat{(1 - \chi_2)\mathcal{A}_0^0 \chi_1 \boldsymbol{g}}(\lambda)$, too. On the other hand $\mathcal{A}_0^0\chi_1 g \in \mathring{W}_0^\sigma$ for any $\sigma \in (-\frac{1}{2}, \frac{1}{2})$ and Lemma 4.2 implies that $\widehat{(1-\chi_2)\mathcal{A}_0^0\chi_1 g}(\lambda)$ is holomorphic for all λ with $\mathrm{Im}\,\lambda > -1$.

Thus the Mellin transformed equation (4.26),

$$\widehat{\mathcal{A}_0^0\chi_1 g}(\lambda) = \hat{H}(\lambda)$$

which is valid for $\mathrm{Im}\,\lambda \in (-1, 0)$ can be extended meromorphically to all λ with $\mathrm{Im}\,\lambda \in (-1, s-\frac{1}{2})$. Inserting (4.25) gives the equation

$$\hat{\mathcal{A}}(\lambda)\,\hat{U}(\lambda - i) = \hat{H}(\lambda) \tag{4.28}$$

where $\hat{U}(\lambda - i)$ denotes the meromorphic extension of $\widehat{\chi_1 g}(\lambda - i)$ from $\mathrm{Im}\,\lambda \in (-1, 0)$ to $\mathrm{Im}\,\lambda \in (-1, s-\frac{1}{2})$. The function

$$U^h(x) = \frac{1}{2\pi}\int_{\mathrm{Im}\,\lambda = h} \hat{U}(\lambda - i)x^{-i\lambda-1}\,d\lambda \tag{4.29}$$

exists for all $h \in (-1, s-\frac{1}{2})$ with h not equal to $\mathrm{Im}\,\lambda_p$ where λ_p denotes the poles of $\hat{\mathcal{A}}^{-1}(\lambda)\hat{H}(\lambda)$ and for $h \in (-1, 0)$ we have

$$U^h(x) = \chi_1 g(x)\,.$$

For the other h the Cauchy integral theorem gives the decomposition

$$U^h(x) = \chi_1 g(x) - i \sum_{\mathrm{Im}\,\lambda \in (-1,h)} \mathrm{Res}\left(\hat{U}(\lambda - i)x^{-i\lambda-1}\right). \tag{4.30}$$

Because of (4.28) the residues appear at the poles of $\hat{H}(\lambda)$ and $\hat{\mathcal{A}}(\lambda)^{-1}$, which can be classified as follows:

1) $\lambda = 0$, as a pole of $\hat{H}(\lambda)$,

2) poles of $\hat{H}(\lambda)$ at $\lambda = ik$ with $k \in \mathbb{N}$ and

3) zeros of (4.5) defining poles of $\hat{\mathcal{A}}(\lambda)^{-1}$.

1) In view of (4.27), $\hat{H}(\lambda)$ may have a pole of order one or two at $\lambda = 0$. The meromorphic function $\hat{U}(\lambda - i)$, however, is regular at the origin, since Lemma 4.2 with $h = -1$ provides the estimate

$$\int_{\mathrm{Im}\,\lambda = k+1} (1 + |\lambda|^2)^{-\frac{1}{2}}|\hat{U}(\lambda - i)|^2\,d\lambda \leqslant c\|\chi g\|_{\mathring{W}_0^{-\frac{1}{2}}} < \infty\,,$$

where $k \in (-2, -1]$, thus excluding poles of $\hat{U}(\lambda - i)$ at $\lambda = 0$. Hence $\lambda = 0$ gives no contribution to (4.30).

2) The equation (4.28) can be written as

$$\hat{B}(\lambda)\,\hat{U}(\lambda - i) = \lambda \sinh \pi\lambda\,\hat{H}(\lambda) \tag{4.31}$$

where

$$\hat{B}(\lambda) = \lambda \sinh \pi\lambda\,\hat{\mathcal{A}}(\lambda) \tag{4.32}$$

is holomorphic for all λ (cf. (2.14), (2.15), (3.13)). Since $\hat{H}(\lambda)$ has at most simple poles at $\lambda = ik$, $k \in \mathbb{N}$, the right hand side of (4.31) is there holomorphic. Therefore these poles do not contribute to (4.30) unless $\det \hat{B}(\lambda) = 0$ which is equivalent to (4.5) as we shall see below. Hence all poles in (4.30) belong to case 3).

3) An elementary computation yields the equation

$$\hat{\mathcal{A}}(\lambda) = 4\pi\left(-\frac{1}{\pi}\,\hat{\mathfrak{W}}(\lambda)\right)^{\frac{1}{2}}\left(\frac{1}{4}\,I - \tilde{\mathfrak{L}}(\lambda)\right)\left(-\frac{1}{\pi}\,\hat{\mathfrak{W}}(\lambda)\right)^{\frac{1}{2}} \tag{4.33}$$

where $\tilde{\mathfrak{L}}(\lambda)$ is given by (3.17) and I denotes the 4×4 identity matrix. Hence

$$\det \hat{B}(\lambda) = \lambda^4 \sinh^4 \pi\lambda \cdot (4\pi)^4 \det\left(-\frac{1}{\pi}\,\hat{\mathfrak{W}}(\lambda)\right) \det\left(\frac{1}{4}\,I - \tilde{\mathfrak{L}}(\lambda)\right).$$

From (4.23) and (2.14) one finds

$$\det\left(-\frac{1}{\pi}\,\hat{\mathfrak{W}}(\lambda)\right) = \Delta^2(\lambda \sinh \pi\lambda)^{-4}$$

where Δ is given by (3.21). Furthermore, $\det\left(\frac{1}{4}I - \tilde{\mathfrak{L}}(\lambda)\right)$ is given by (3.20) with $\mu = \frac{1}{4}$. Thus we obtain

$$\det \hat{B}(\lambda) = \pi^4(\lambda^2 \sin^2 \omega - \sinh^2 \omega\lambda)\big(\lambda^2 \sin^2 (2\pi - \omega) - \sinh^2 (2\pi - \omega)\,\lambda\big) \tag{4.34}$$

which corresponds to (4.5). Note that (4.34) has zeros of at most 4-th order and therefore the poles of $\hat{U}(\lambda - i)$ are also at most of order 4.

It remains to give the connection between the decompositions (4.13) and (4.30) and to show (4.14). To this end we note that there is an interval $I = (s - \frac{1}{2} - \varepsilon,\ s - \frac{1}{2})$ such that $\hat{U}(\lambda - i)$ is holomorphic for $\operatorname{Im} \lambda \in I$. Therefore $U^h(x)$ (cf. (4.29)) is independent of $h \in I$ due to (4.30).

Now we define

$$\chi g^{(s)}(x) := \chi U^h(x) \tag{4.35}$$

for $h \in I$. This defines with (4.30) the decomposition (4.13) since the residues are of the form

$$\sum_{l=0}^{3} c_l \left(\frac{d}{d\lambda}\right)^l (x^{-i\lambda-1}),$$

where the constants c_l correspond to Laurent coefficients of $\hat{U}(\lambda - i)$ and since $\chi\chi_1 = \chi$. In this way we are led to the explicit form (4.6) of the singularity functions.

It remains to show that $U^h(\cdot) \in \mathring{W}_0^{s-1}$. Then the a-priori estimate (4.14) is a consequence of the closed graph theorem.

For brevity let us consider first the case $s - \frac{1}{2} \notin \mathbb{Z}$. We note first that (4.29) is valid (in the distributional sense) for $h = s - \frac{1}{2}$, too.

As above we have

$$\hat{H}(\lambda) = \widehat{\chi_2 F_0}(\lambda) + \widehat{\chi_2 P}(\lambda) - \widehat{\chi_2 \mathcal{A}_1^0 \chi_1 g}(\lambda) + \widehat{(1-\chi_2)\mathcal{A}_0^0 \chi_1 g}(\lambda) \tag{4.36}$$

where $\chi_2 F_0 \in \mathring{W}_0^s$ and therefore with Lemma 4.2, (4.8)

$$\int_{\operatorname{Im}\lambda = k} (1 + |\lambda|^2)^s |\widehat{\chi_2 F_0}(\lambda)|^2 \, d\lambda \leqslant c \tag{4.37}$$

for $k \in [s - \frac{1}{2} - \varepsilon, s - \frac{1}{2}] =: \bar{I}$. Next note that $P(x)$ is a polynomial and $\widehat{\chi_2 P}(\lambda)$ decays rapidly for $\operatorname{Im}\lambda \in \bar{I}$, $|\lambda| \to \infty$. The same holds for $\widehat{\chi_2 \mathcal{A}_1^0 \chi_1 g}(\lambda)$ in view of (4.27). The inequality corresponding to (4.37) for the last term of (4.36), namely

$$\int_{\operatorname{Im}\lambda = k} (1 + |\lambda|^2)^s |\widehat{(1-\chi_2)\mathcal{A}_0^0 \chi_1 g}(\lambda)|^2 \, d\lambda \leqslant c \tag{4.38}$$

will be obtained from Lemma 4.2 in connection with the Cauchy integral theorem. In particular, (4.11) yields

$$\widehat{(1-\chi_2)\mathcal{A}_0^0 \chi_1 g}(\lambda) = \int_{\operatorname{Im}\mu = p} \hat{\mathcal{A}}(\mu)\, \widehat{\chi_1 g}(\mu - i)\, \widehat{(1-\chi_2)}(\lambda - \mu)\, d\mu \tag{4.39}$$

for $p \in (-1, 0)$ and $\operatorname{Im}\lambda > p$. Furthermore (4.7) and (4.10) yield the estimates

$$(4.40)\qquad |\widehat{\chi_1 g}(\mu - i)| \leqslant c(1 + |\mu|^2)^\sigma \beta_1^{-\operatorname{Im}\mu} \qquad \text{for all } \mu \text{ with } \operatorname{Im}\mu \leqslant p\,,$$

$$(4.41)\qquad |\widehat{(1 - \chi_2)}(\lambda - \mu)| \leqslant c(1 + |\lambda - \mu|^2)^{-N} \alpha_2^{\operatorname{Im}\mu}$$

for all μ with $\operatorname{Im}\mu \leqslant p$ and $\operatorname{Im}\lambda \in \bar{I}$.

For $|\operatorname{Re}\mu| \geqslant \eta > 0$ we also have that $\hat{\mathcal{A}}(\mu)$ is bounded. Therefore we can deform the path of integration in (4.39) into $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ where

$$\mathcal{C}_1 = \{-\eta + i\tau | -\infty < \tau \leqslant p\}\,,$$

$$\mathcal{C}_2 = \{\xi + ip | -\eta \leqslant \xi \leqslant \eta\}$$

and

$$\mathcal{C}_3 = \{\eta + i\tau | p \geqslant \tau > -\infty\}\,.$$

In this way we obtain

$$(4.42)\qquad \widehat{(1 - \chi_2)\mathcal{A}_0^0 \chi_1 g}(\lambda) = J_1(\lambda) + J_2(\lambda) + J_3(\lambda)$$

with

$$J_k(\lambda) = \int_{\mathcal{C}_k} \hat{\mathcal{A}}(\mu) \widehat{\chi_1 g}(\mu - i) \widehat{(1 - \chi_2)}(\lambda - \mu)\, d\mu \qquad (k = 1, 2, 3)\,,$$

i.e.

$$J_1(\lambda) = i\int_{-\infty}^{p} w(\lambda, -\eta + i\tau)\, d\tau\,,$$

$$J_2(\lambda) = \int_{-\eta}^{\eta} w(\lambda, \xi + ip)\, d\xi\,,$$

$$J_3(\lambda) = i\int_{p}^{-\infty} w(\lambda, \eta + i\tau)\, d\tau\,,$$

where

$$w(\lambda, \mu) := \hat{\mathcal{A}}(\mu) \widehat{\chi_1 g}(\mu - i) \widehat{(1 - \chi_2)}(\lambda - \mu)\,.$$

From (4.40), (4.41) we get the estimates (for any $N > 0$)

for J_1: $|w(\lambda, -\eta + i\tau)| \leqslant C(1 + |\lambda|^2)^{-N}(1 + \tau^2)^\sigma \left(\dfrac{\alpha_2}{\beta_1}\right)^\tau$,

for J_2: $|w(\lambda, \xi + ip)| \leqslant C(1 + |\lambda|^2)^{-N}$

and

$$\text{for } J_3\colon |w(\lambda, \eta + i\tau)| \leqslant C(1 + |\lambda|^2)^{-N}(1 + \tau^2)^{\sigma}\left(\frac{\alpha_2}{\beta_1}\right)^{\tau}.$$

Now we note that $\alpha_2 > \beta_1$ which implies the convergence of the integrals J_1 and J_3 as well as the estimates $|J_k(\lambda)| \leqslant C(1 + |\lambda|^2)^{-N}$ for $k = 1, 2, 3$. In this way, (4.38) follows from (4.42).

Returning to (4.36), we see now that all terms on the right hand side satisfy an inequality corresponding to (4.37). Therefore

$$\int\limits_{\operatorname{Im}\lambda = k} |\hat{H}(\lambda)|^2(1 + |\lambda|^2)^s d\lambda \leqslant c < \infty\,, \qquad \text{for } k \in \bar{I}\,. \tag{4.43}$$

Now $\hat{\mathcal{A}}(\lambda)^{-1}$ has no poles in $\operatorname{Im}\lambda \in \bar{I}$. Therefore $|\hat{\mathcal{A}}(\lambda)^{-1}| \leqslant C(1 + |\lambda|^2)^{\frac{1}{2}}$ (Note that the diagonal terms of $\hat{\mathcal{A}}(\lambda)$ in (4.22)-(4.24) behave like $\frac{1}{2}\coth \pi\lambda$, all other terms decay exponentially for $|\operatorname{Re}\lambda| \to \infty$). Therefore (4.28) gives with $k = s - \frac{1}{2}$

$$\int\limits_{\operatorname{Im}\lambda = s - \frac{1}{2}} |\hat{U}(\lambda - i)|^2(1 + |\lambda|^2)^{s-1} d\lambda \leqslant c \int\limits_{\operatorname{Im}\lambda = s - \frac{1}{2}} |\hat{H}(\lambda)|^2(1 + |\lambda|^2)^s d\lambda \leqslant c < \infty\,.$$

By (4.29) this gives $U^h \in \mathring{W}_0^{s-1}$, and from the definition (4.35) of $\chi \boldsymbol{g}^{(s)}$ and the equivalence of norms (2.27), (2.28) we finally obtain $\chi \boldsymbol{g}^{(s)} \in \boldsymbol{H}^{s-1}(\Gamma^\omega)$.

The remaining case $s - \frac{1}{2} \in \mathbb{Z}$ is treated by interpolation which is possible if there is no zero of the transcendental equation (4.5) on the line $\operatorname{Im}\lambda = s - \frac{1}{2}$, i.e. there is no change in the decomposition (4.13) across this value of s. □

REMARK 4.4. It is not only possible to determine the exponents of the singular functions $\boldsymbol{v}_k$ in (4.13), but also to calculate the constant vectors $\boldsymbol{c}_{l-}$, $\boldsymbol{c}_{l+}$ in (4.6). By the decomposition (4.30) according to the Cauchy integral theorem, this is done by computing the Laurent expansion of $\hat{U}(\lambda - i)$ at a pole. In view of (4.28) this means to compute the generalized eigenvectors (generalized Jordan expansion) of the 4×4-matrix valued Mellin symbol $\hat{\mathcal{A}}(\lambda)$ which is explicitly given by (4.22)-(4.24) together with (2.14)-(2.16). In [5], where the symbol was a 2-2-matrix, this was done in detail.

5. – Mapping properties of the integral equations on the whole boundary and the proof of the equivalence Theorem 1.1.

In this chapter we combine the foregoing local results to obtain the mapping properties of $\mathcal{A}$ on Γ, i.e global continuity, Gårding inequality and regularity. These results then easily yield also the equivalence Theorem 1.1.

THEOREM 5.1. *The operator $\mathcal{A}$ defined by* (1.6) *maps $H^{s-1}(\Gamma)$ into $H^s(\Gamma)$ continuously for $s \in (-\frac{1}{2}, \frac{3}{2})$.*

PROOF. As in [5, (2.8)] we use a partition of unity $\{\chi_1, \ldots, \chi_J\}$ with the properties

$$\chi_j \text{ is the restriction of a } C_0^\infty(\mathbb{R}^2)\text{-function to } \Gamma, \tag{5.1}$$

$$\chi_j \equiv 1 \text{ in a neighbourhood of the vertex } t_j, \text{ and}$$
$$\operatorname{supp} \chi_j \subset \Gamma^j \cup \{t_j\} \cup \Gamma^{j+1}.$$

Then

$$\mathcal{A}g = \sum_{j,k=1}^{J} \chi_j \mathcal{A} \chi_k g \tag{5.2}$$

splits into the operators $x_j \mathcal{A} \chi_k$ whose continuity has to be shown for the three cases $|j-k| = 0, 1, > 1$.

For $j = k$ use the local continuity result of Theorem 2.4. For $|j-k| > 1$ note that the kernels of the integral operators in $\chi_j \mathcal{A} \chi_k$ are C^∞-functions providing continuity from $H^{s-1}(\Gamma)$ into any Sobolev space $H^t(\Gamma)$, $t \in \mathbb{R}$.

For $|j-k| = 1$ one can use a combination of the two preceding cases: Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be chosen with $\chi|_{\operatorname{supp} \chi_k} \equiv 1$ and $\Gamma \cap \operatorname{supp} \chi \subset \Gamma^k \cup \Gamma^{k+1} \cup \{t_k\}$. Then

$$\chi_j \mathcal{A} \chi_k = \chi_j \chi \mathcal{A} \chi \chi_k + (1-\chi) \chi_j \mathcal{A} \chi_k$$

where the first operator is continuous from $H^{s-1}(\Gamma)$ into $H^s(\Gamma)$ due to the continuity of $\chi \mathcal{A} \chi$ and the second one has a C^∞-kernel. □

THEOREM 5.2. *$\mathcal{A}$ satisfies the Gårding inequality*

$$(\mathcal{A}g, g)_{L_2(\Gamma)} \geq \gamma \|g\|^2_{H^{-\frac{1}{2}}(\Gamma)} - k[g, g] \tag{5.3}$$

for all $g \in H^{-\frac{1}{2}}(\Gamma)$ where $\gamma > 0$ and $k[g, g]$ denotes a compact bilinear form on $H^{-\frac{1}{2}}(\Gamma)$.

PROOF. Note that by setting

$$g = g^0 + \alpha \chi$$

with

$$\alpha = (\chi, g)_{L_2(\Gamma^\infty)} / \|\chi\|^2_{L_2(\Gamma^\infty)},$$

$g^0 \in H_+^{-\frac{1}{2}}(\Gamma^\infty)$ and Theorem 3.5 yields with (3.12) a local Gårding inequality (5.3). Then as in [5, Lemma 2.20] the global Gårding inequality (5.3) follows from the following lemma.

LEMMA 5.3. *Let* $A: H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ *be a continuous operator having the following properties:*

a) A satisfies the local Gårding inequality

$$(A\varphi,\varphi)_{L_2(\Gamma)} \geqslant \gamma_j \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} - k_j[\varphi,\varphi] \tag{5.4}$$

for all $\varphi \in C_0^\infty(S_j)$ *with* S_j *any compact subset of* $\Gamma^j \cup \{t_j\} \cup \Gamma^{j+1}$, *where* $\gamma_j > 0$ *and* $k_j[\varphi,\varphi]$ *is a compact bilinear form on* $H^{-\frac{1}{2}}(\Gamma)$ *depending only on* S_j, $j = 1, \ldots, J$.

b) For any $\Phi_1, \Phi_2 \in C_0^\infty(\mathbb{R}^2)$ *with* $\operatorname{supp}\Phi_1 \cap \operatorname{supp}\Phi_2 = \emptyset$ *the operator* $\Phi_1 A \Phi_2$ *from* $H^{-\frac{1}{2}}(\Gamma)$ *into* $H^{\frac{1}{2}}(\Gamma)$ *is compact. For any* $\Phi, \Psi \in C_0^\infty(\mathbb{R}^2)$ *with* $\Gamma \cap \operatorname{supp}\Psi \subset \Gamma^j$ *for some* j *the operator* $\Psi(A\Phi - \Phi A)\Psi$ *is also compact.*

Then A satisfies a global Gårding inequality (5.3).

The proof follows as in [5, Lemma 2.20].

This completes the proof of Theorem 5.2 since $\mathcal{A}$ has the properties *a*) and *b*).

REMARK 5.4. If we denote the operator belonging to the system (1.6), (1.8) by $\mathcal{C}$:

$$\mathcal{C}\begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{a} \\ a_3 \end{pmatrix} = \begin{pmatrix} \mathcal{A}\boldsymbol{g} + \boldsymbol{a} + a_3\boldsymbol{k} \\ -\int_\Gamma \boldsymbol{g}\, ds \\ -\int_\Gamma \boldsymbol{g}\cdot\dot{\boldsymbol{x}}\, ds \end{pmatrix} \tag{5.5}$$

then $\mathcal{C}: H^{-\frac{1}{2}}(\Gamma)\times\mathbb{R}^3 \to H^{\frac{1}{2}}(\Gamma)\times\mathbb{R}^3$ satisfies also a Gårding inequality in the form

$$(\mathcal{A}\boldsymbol{g} + \boldsymbol{a} + a_3\boldsymbol{k}, \boldsymbol{g})_{L_2(\Gamma)} - \int_\Gamma \boldsymbol{g}\, ds \cdot \boldsymbol{a} - \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{x}\, ds\, a_3 \geqslant \gamma\{\|\boldsymbol{g}\|^2_{H^{-\frac{1}{2}}(\Gamma)} + |\boldsymbol{a}|^2 + a_3^2\} - k_1\left[\begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{a} \\ a_3 \end{pmatrix}, \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{a} \\ a_3 \end{pmatrix}\right] \tag{5.6}$$

which is obvious from (5.3).

THEOREM 5.5. *Let* $\boldsymbol{f} \in H^s(\Gamma)$ *with* $s > \frac{1}{2}$ *and* $s \notin \{\lambda_{jk} + \frac{1}{2}\}$ *where* λ_{jk} *are the roots of* (4.5) *with* $\omega = \omega_j$. *Let* $\boldsymbol{g} \in H^{-\frac{1}{2}}(\Gamma)$ *be a solution of the integral* *equations* (1.6), (1.8). *Then* g *is of the form*

$$g = g^{(s)} + \sum_{j=1}^{J} \sum_{0<\operatorname{Im}\lambda_{jk}<s-\frac{1}{2}} d_{jk} v_{jk} \tag{5.7}$$

where $d_{jk} \in \mathbb{R}$ *and* v_{jk} *are the singularity functions given by*

$$v_{jk}(z) = \sum_{l=0}^{3} c_{jl}(z)|z - z_j|^{-i\lambda_{jk}-1}(\log|z - z_j|)^l \chi_j(z)\,. \tag{5.8}$$

Here c_{jl} *are vector functions taking constant but possibly different values on* Γ^j *and* Γ^{j+1}, *respectively, which correspond to the generalized eigenvectors* (c_{l-}, c_{l+}) *of the local Mellin symbol* $\hat{\mathcal{A}}(\lambda)$ *at the corner* z_j, *and* $g^{(s)} \in H^{s-1}(\Gamma)$. *Furthermore there holds the a-priori estimate*

$$\|g^{(s)}\|_{H^{s-1}(\Gamma)} + \sum_{j=1}^{J} \sum_{0<\operatorname{Im}\lambda_{jk}<s-\frac{1}{2}} |d_{jk}| \\ \leqslant C_1\{\|f\|_{H^s(\Gamma)} + |b| + |b_3|\} + C_2\{\|g\|_{H^{-\frac{1}{2}}(\Gamma)} + |a| + |a_3|\}\,. \tag{5.9}$$

The proof is obvious in view of Theorem 4.3.

REMARK 5.6. Due to the uniqueness of the solution of the integral equations (1.6), (1.8), the a-priori estimate (5.9) holds even with $C_2 = 0$.

In order to prove the unique solvability of the integral equations (1.6), (1.8) as well as the equivalence with the boundary value problem (1.1), (1.2) we need the following lemma.

LEMMA 5.7. *Let* $\varepsilon \in (0, \frac{1}{2})$, *let* $g \in H^{-\frac{1}{2}+\varepsilon}(\Gamma)$ *be a solution of the integral equations* (1.6), *and let* U *be given by the representation formula* (1.4), *i.e.*

$$U(z) = Bg(z) := \int_{\Gamma} (\operatorname{grad}_\zeta F(z, \zeta)) \cdot g(\zeta)\, ds_\zeta\,. \tag{5.10}$$

Then $U \in H^{2+\varepsilon}(\Omega)$ *and* U *satisfies the boundary value problem* (1.1), (1.2). *The mapping* $B\colon H^{-\frac{1}{2}+\varepsilon}(\Gamma) \to H^{2+\varepsilon}(\Omega)$ *is continuous for* $\varepsilon \in [0, \frac{1}{2})$.

PROOF. Let us consider Ω as an intersection of half-planes with boundaries which are straight lines containing the segments Γ^j. We can decompose the potential U (5.10) into the sum of integrals over the individual segments Γ^j. These potentials can then be viewed as potentials on respective half-planes generated by densities on the respective lines which are extensions by zero of $g|_{\Gamma^j}$. These extensions belong to $H^{-\frac{1}{2}+\varepsilon}(\mathbb{R})$. Now it suffices to show the above continuity of B for one of these potentials where Γ^j is an interval on the real axis. Here

$$Bg(z) = \boldsymbol{\Phi} * (g \otimes \delta) \tag{5.11}$$

with $*$ the two-dimensional convolution and where

$$\boldsymbol{\Phi}(z-\zeta) = \operatorname{grad}_\zeta F(z,\zeta) \quad \text{and} \quad g \otimes \delta(z) = g(x)\cdot\delta(y)$$

with $\delta(y)$ the Dirac distribution. Then $g \otimes \delta \in H^{-1+\varepsilon}(\mathbb{R}^2)$ having compact support and the convolution by $\boldsymbol{\Phi}$ is a pseudodifferential operator of order -3 since F is the fundamental solution to Δ^2 in $\mathbb{R}^2$. Thus

$$\boldsymbol{\Phi} * : H^s_{\text{compact supp}}(\mathbb{R}^2) \to H^{s+3}_{\text{loc}}(\mathbb{R}^2)$$

is continuous providing the proposed continuity of B. (See Eskin [7, p. 106 ff.]).

For $\varepsilon > 0$ we therefore have $U \in H^{2+\varepsilon}(\Omega) \subset C^1(\bar{\Omega})$ and grad $U \in C^0(\bar{\Omega})$ being given by

$$\operatorname{grad} U(z) = \int_\Gamma \operatorname{grad}_z\big(\operatorname{grad}_\zeta F(z,\zeta)\cdot g(\zeta)\big)\, ds_\zeta . \tag{5.12}$$

For $z \in \Gamma$ this gives with (1.6) the boundary values (1.2). The differential equation (1.1) on $\mathbb{R}^2\setminus\Gamma$ is obvious. □

THEOREM 5.8. *The operator* $\mathfrak{C}$, *i.e. the integral equations* (1.6), (1.8), *define an isomorphism*

$$\mathfrak{C}\colon H^{-\frac{1}{2}}(\Gamma)\times\mathbb{R}^3 \to H^{\frac{1}{2}}(\Gamma)\times\mathbb{R}^3 .$$

PROOF. From the Gårding inequality (5.6) we see that $\mathfrak{C}$ is a Fredholm mapping of index zero. Hence in view of Fredholm's alternative we have only to show injectivity. Let $(g, a, a_3) \in H^{-\frac{1}{2}}(\Gamma)\times\mathbb{R}^3$ be a solution of the homogeneous equations (1.6), (1.8),

$$\begin{pmatrix} \mathcal{A}g \\ a \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then Theorem 5.5, viz. a-priori estimate (5.9) assures $g \in H^{-\frac{1}{2}+\varepsilon}$ with some $\varepsilon > 0$. Moreover the special form (5.8) of the singularity functions implies

$g \in L^1(\Gamma)$ and $g|_{\Gamma_j} \in C^{1+\varepsilon'}$ with some $\varepsilon' > 0$ if s in Theorem 5.5 is chosen large enough. The potential

$$U_0(z) = Bg(z) - xa_1 - ya_2 - \gamma$$

now satisfies the differential equation

$$\Delta^2 U_0 = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{5.13}$$

and, by Lemma 5.7, the boundary condition (1.2), i.e.

$$\operatorname{grad} U_0 = a_3 \boldsymbol{k} \qquad \text{on } \Gamma$$

which yield explicitly

$$a_3 = \Big(\int_\Gamma \boldsymbol{k} \cdot \dot{\boldsymbol{x}}\, ds\Big)^{-1} \int_\Gamma \operatorname{grad} U_0 \cdot \dot{\boldsymbol{x}}\, ds = 0$$

due to the continuity of grad U_0. This implies

$$U_{0|\Gamma} = 0 \qquad \text{and } \left.\frac{\partial U_0}{\partial n}\right|_\Gamma = 0 \tag{5.14}$$

if $\gamma \in \mathbb{R}$ is chosen appropriately. Now we want to use the uniqueness theorem by Giroire [11, p. 68] for the bi-Laplacian in $W^2_{0,0}(\mathbb{R}^2)$. It implies $U_0 \equiv 0$ if we can show that the second derivatives of U_0 belong to $L_2(\mathbb{R}^2)$ [11, Lemma II.3].

Since g satisfies (1.8) with $\boldsymbol{b} = \boldsymbol{0}$ we have

$$\begin{aligned} Bg(z) &= \int_\Gamma \big(\boldsymbol{\phi}(z-\zeta) - \boldsymbol{\phi}(z)\big)\cdot \boldsymbol{g}(\zeta)\, ds_\zeta \\ &= -\int_{\zeta\in\Gamma} \boldsymbol{g}(\zeta) \int_{\tau=0}^{1} \left\{\frac{\partial \boldsymbol{\phi}}{\partial \xi}(z-\tau\zeta)\,\xi + \frac{\partial \boldsymbol{\phi}}{\partial \eta}(z-\tau\zeta)\,\eta\right\} d\tau\, ds_\zeta . \end{aligned} \tag{5.15}$$

Moreover, $g \in L_1(\Gamma)$ and (5.15) can be differentiated with respect to x, y for large $|z|$ under the integral sign. The second derivatives $D_z^2 U_0$ are then of the form

$$\int_{\zeta\in\Gamma} \boldsymbol{g}(\zeta)\cdot \int_{\tau=0}^{1} \boldsymbol{\psi}(z-\tau\zeta, \zeta)\, d\tau\, ds_\zeta$$

where $\boldsymbol{\psi}$ is composed of 4-th derivatives of $F(z, \tau\zeta)$ which are $O\big(|z|^{-2}\big)$ for

$|z| \to \infty$ uniformly for $\zeta \in \Gamma$, $\tau \in [0,1]$. Hence $D_z^2 U_0 \in L_2(\mathbb{R}^2)$ and U_0 vanishes everywhere.

On Γ^j, i.e. away from the corner points, with $\boldsymbol{g}|_{\Gamma^j} \in C^{1+\varepsilon'}$ we can use the standard jump relations as given e.g. by Hsiao and MacCamy [15, A.2, A.3]:

$$g_1 \frac{dy}{ds} - g_2 \frac{dx}{ds} = \boldsymbol{n}\cdot\boldsymbol{g} = \frac{1}{2\pi}[\Delta U_0]_{|\Gamma} = 0$$

(5.16) and

$$\frac{d}{ds}\left(g_1 \frac{dx}{ds} + g_2 \frac{dy}{ds}\right) = \frac{d}{ds}(\boldsymbol{g}\cdot\dot{\boldsymbol{x}}) = \frac{1}{2\pi}\left[\frac{\partial}{\partial n} U_0\right]_{|\Gamma} = 0,$$

where $[\cdot]$ denotes the jump across Γ from the outside to the inside of Ω. This implies

$$\boldsymbol{g}\cdot\dot{\boldsymbol{x}}|_{\Gamma^j} = c^j, \tag{5.17}$$

a constant on each Γ^j. Inserting (5.16), (5.17) into (5.10) gives

$$0 = U_0(z) = \int_\Gamma \left(\left(\frac{\partial F}{\partial n_\zeta}(z,\zeta)\right)(\boldsymbol{n}\cdot\boldsymbol{g}) + \left(\frac{\partial F}{\partial s_\zeta}(z,\zeta)\right)(\boldsymbol{x}\cdot\boldsymbol{g})\right) ds - xa_1 - ya_2 - \gamma$$

$$= \sum_{j=1}^{J} c^j \int_{\Gamma^j} \frac{\partial F}{\partial s_\zeta}(z,\zeta)\, ds_\zeta - xa_1 - ya_2 - \gamma,$$

$$0 = \sum_{j=1}^{J} F(z, t_j)(c^j - c^{j+1}) - xa_1 - ya_2 - \gamma \tag{5.18}$$

for all $z \in \mathbb{R}^2$. This implies $c^j = c^{j+1} = c^1$ for $j = 1, \ldots, y$ which can be seen by applying Δ^2 to (5.18). Now (5.16) yields $\boldsymbol{g} = c^1 \dot{\boldsymbol{x}}$.

From the second equation in (1.8) we find

$$0 = b_3 = \int_\Gamma \boldsymbol{g}\cdot\dot{\boldsymbol{x}}\, ds = c^1 \int_\Gamma |\dot{\boldsymbol{x}}|^2 ds,$$

hence $c_1 = 0$. Thus $(\boldsymbol{g}, \boldsymbol{a}, a_3) = \boldsymbol{0}$ which completes the proof. □

PROOF OF THEOREM 1.1. The properties i) and ii) are well known results from the variational calculus, see e.g. Necas [23].

Now let us prove iv). The first assertion corresponds to Theorem 5.8.

As we have seen in the proof of Theorem 5.8, the self-adjoint integral operator $\mathcal{A}$ has the kernel span $\{\dot{\boldsymbol{x}}\}$. Hence its range is characterized by the orthogonality relation (1.3). Now, if $\boldsymbol{f}$ also satisfies (1.3) then there follows $a_3 = 0$.

Thus, it remains to show that a solution $(\boldsymbol{g}, \boldsymbol{a}, 0) \in H^{-\frac{1}{2}}(\Gamma) \times \mathbb{R}^3$ of (1.6), (1.8) generates by (5.10) a weak solution (1.23) of the boundary value problem (1.1), (1.2). To this end we approximate $\boldsymbol{f}$ in $H^{\frac{1}{2}}(\Gamma)$ by a sequence $\boldsymbol{f}_n \in H^{\frac{1}{2}+\varepsilon}(\Gamma)$ with $\varepsilon > 0$ and $\boldsymbol{f}_n$ satisfying also (1.3). Then Theorem 5.8 implies the existence of $(\boldsymbol{g}_n, \boldsymbol{a}_n, 0) \in H^{-\frac{1}{2}+\varepsilon}(\Gamma) \times \mathbb{R}^3$ satisfying (1.6), (1.8) and $\boldsymbol{g}_n \to \boldsymbol{g}$ in $H^{-\frac{1}{2}}(\Gamma)$, $|\boldsymbol{a}_n - \boldsymbol{a}| \to 0$. If we define $U_n := B\boldsymbol{g}_n$ we have by Lemma 5.7 $U_n \in H^{2+\varepsilon}(\Omega)$ for any n, U_n satisfying (1.1), (1.2) with

$$\text{grad}\, U_n = \boldsymbol{f}_n + \boldsymbol{a}_n \quad \text{on } \Gamma . \tag{5.19}$$

Moreover $U_n \to U = B\boldsymbol{g}$ in $H^2(\Omega)$. On the other hand we find a sequence of variational solutions $\tilde{U}_n \in H^2(\Omega)$ satisfying also (5.19) and converging also in $H^2(\Omega)$ to a variational solution $\tilde{U}$ of (1.1), (1.2). By uniqueness of the variational solution we have

$$U_n = \tilde{U}_n + \gamma_n \quad \text{with } \gamma_n \in \mathbb{R}$$

and from $U_n \to U$ and $\tilde{U}_n \to \tilde{U}$ in $H^2(\Omega)$ it follows with the Sobolev imbedding theorem that the convergence also takes place pointwise implying $\gamma_n \to \gamma \in \mathbb{R}$. Therefore $U = \tilde{U} + \gamma$ is also a variational solution which proves iv).

iii) For given h with (1.21) and w defined by (1.22) we have that $\boldsymbol{f} \in H^{\frac{1}{2}}(\Gamma)$ satisfies (1.3). Therefore (1.6), (1.8) has a solution $(\boldsymbol{g}, \boldsymbol{a}, 0)$ to $\boldsymbol{f}$ and $\boldsymbol{b} = \boldsymbol{0}$, $b_3 = 0$. By iv) $U = B\boldsymbol{g}$ defines a variational solution of the form (1.23). □

REFERENCES

[1] A. Avantaggiati - M. Troisi, *Spazi di Sobolev con peso e problemi ellittici in un angolo, I, II, III*, Ann. Mat. Pura Appl., **95** (1973), pp. 361-408; Ann. Mat. Pura Appl., **97** (1973), pp. 207-252; Ann. Mat. Pura Appl., **99** (1974), pp. 1-51.

[2] J. D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Comp., Waltham, Mass. (1968).

[3] H. O. Cordes - E. Herman, *Singular integrals on the half-line*, Proc. Nat. Acad. Sci., **56** (1966), pp. 1668-1673.

[4] M. Costabel, *Singular integral operators on curves with corners*, Integral Equations Oper. Theory, **3** (1980), pp. 323-349.

[5] M. COSTABEL - E. STEPHAN, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, THD-preprint Nr. 593, Darmstadt (1981), to appear in Banach Center Publications.

[6] M. COSTABEL - E. STEPHAN, *The boundary integral method for a mixed boundary value problem in a polygonal domain*, Advances in Computer Methods for Partial Diff. Equations IV, IMACS (1981), pp. 300-304.

[7] G. I. ESKIN, *Boundary value problems for elliptic pseudo-differential equations*, A.M.S. (Transl. of Math. Mon., **52**), Providence, Rhode Island (1981).

[8] E. B. FABES - M. JODEIT - J. E. LEWIS, *On the spectra of a Hardy Kernel*, J. Functional Analysis, **21** (1976), pp. 187-194.

[9] E. B. FABES - M. JODEIT - J. E. LEWIS, *Double layer potentials for domains with corners and edges*, Indiana Univ. Math. J., **26** (1977), pp. 95-114.

[10] G. FICHERA, *Linear elliptic equations of higher order in two independent variables and singular integral equations*, Proc. Conf. partial diff. equations and cont. mechanics, Univ. Wisconsin Press, (1961), pp. 55-80.

[11] J. GIROIRE, *Formulation variationelle par équations intégrales de problèmes aux limites extérieurs*, Rapport Interne École Polytechnique 6 (1976).

[12] I. GOHBERG - N. KRUPNIK, *Einführung in die Theorie der eindimensionalen singulären Integraloperatoren*, Birkhäuser Verlag, Basel (1979).

[13] P. GRISVARD, *Boundary value problems in non-smooth domains*, Univ. Maryland, Lecture Notes **19** (1980).

[14] G. C. HSIAO - P. KOPP - W. L. WENDLAND, *Some applications of a Galerkin-collocation method for integral equations of the first kind*, THD-preprint Nr. 768, Darmstadt (1983), to appear in Math. Meth. in the Appl. Sci..

[15] G. C. HSIAO - R. MACCAMY, *Solution of boundary value problems by integral equations of the first kind*, SIAM Rev., **15** (1973), pp. 687-705.

[16] G. C. HSIAO - W. L. WENDLAND, *A finite element method for some integral equations of the first kind*, J. Math. Anal. Appl., **58** (1977), pp. 449-481.

[17] M. A. JASWON - G. T. SYMM, *Integral Equation Methods in Potential Theory and Elastostatics*, Academic Press, New York - San Francisco - London (1977).

[18] V. A. KONDRATIEV, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., **16** (1967), pp. 227-313.

[19] J. E. LEWIS - C. PARENTI, *Pseudodifferential operators and Hardy Kernels on $L^p(\mathbf{R}^+)$*, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **7** (1980), pp. 481-503.

[20] J. L. LIONS - E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, Berlin - Heidelberg - New York (1972).

[21] R. LOZI, *Résultats numériques de régularité du problème de Stokes et du Laplacien itéré dans un polygon*, preprint Univ. Nice (1978).

[22] H. MELZER - R. RANNACHER, *Spannungskonzentrationen in Eckpunkten der Kirschhoffschen Platte*, Bauingenieur, **55** (1980), pp. 181-184.

[23] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris (1967).

[24] B. A. PLAMENEVSKII, *On an algebra of pseudodifferential operators in spaces with weighted norms*, Math. USSR-Sb., **34** (1978), pp. 841-865.

[25] M. REED - B. SIMON, *Methods of Modern Mathematical Physics II*, Academic Press, New York - San Francisco - London (1975).

[26] T. B. SEIF, *On the Green's function for the biharmonic equation in an infinite wedge*, Trans. Amer. Math. Soc., **182** (1973), pp. 241-260.

[27] E. SHAMIR, *Mixed boundary value problems for elliptic equations in the plane —the L^p-theory*, Ann. Scuola Norm. Sup. Pisa, Ser. III, **17** (1963), pp. 117-139.
[28] E. STEPHAN, *Conform and mixed finite element schemes for the Dirichlet problem for the bilaplacian in plane domains with corners*, Math. Meth. in the Appl. Sci., **1** (1979), pp. 354-382.
[29] E. STEPHAN - W. L. WENDLAND, *Remarks to Galerkin and least squares methods with finite elements for general elliptic problems*, Manuscripta geodaetica, **1** (1976), pp. 93-123.
[30] E. STEPHAN - W. L. WENDLAND, *Boundary element method for membrane and torsion crack problems*, Comput. Methods Appl. Mech. Engrg, **36** (1983), pp. 331-358.
[31] R. TEMAM, *Navier-Stokes Equations*, North-Holland, Amsterdam (1977).
[32] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, North-Holland Publishing Company, Amsterdam - New York - Oxford (1978).
[33] W. L. WENDLAND, *On the asymptotic convergence of some boundary element methods*, Mathematics of Finite Elements and Appl. IV, MAFELAP 1981, Academic Press, London (1982), pp. 281-312.
[34] W. L. WENDLAND - E. STEPHAN - G. C. HSIAO, *On the integral equation method for the plane mixed boundary value problem of the Laplacian*, Math. Meth. in the Appl. Sci., **1** (1979), pp. 265-321.
[35] M. L. WILLIAMS, *Stress singularities resulting from various boundary conditions in angular corners of plates in extensions*, J. Appl. Mech., **19** (1952), pp. 526-528.

Technische Hochschule
Darmstadt
Schlossgartenstr. 7
6100 Darmstadt - W. Germany