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A bifurcation theory for periodic solutions of nonlinear dissipative hyperbolic equations

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A Bifurcation Theory for Periodic Solutions of Nonlinear Dissipative Hyperbolic Equations (*).

WALTER CRAIG

1. – Introduction.

In this paper I demonstrate the existence of nontrivial branches of periodic solutions for a class of nonlinear hyperbolic equations. Denoting the quantities

$$Du = (u_t, u_{x_j}) , \qquad D^2 u = (u_{tt}, u_{tx_j}, u_{x_i x_j})$$

and considering $x \in \Omega$ a smooth bounded domain in $\mathbb{R}^n$, $t \in \mathbb{R}$, problems of the following form will be considered;

$$\begin{aligned} (1.1) \qquad & 0 = u_{tt} - \Delta u + \alpha u_t - mu + F(x; t; u, Du, D^2 u) \equiv \mathcal{F}(u, m) \\ & u(x, t + \tau) = u(x, t) \\ & u(x, t)|_{x \in \partial\Omega} = 0 \end{aligned}$$

F is smooth in its arguments, with $F(x, t + \tau; ...) = F(x, t; ...)$. If it is supposed that $F(x, t; 0, 0, 0) = 0$, then $u = 0$ represents a trivial solution for all values of the parameter m. If also $dF(x, t; 0, 0, 0) = 0$, then for particular discrete values of m there exists a nontrivial kernel of the linearized operator $d\mathcal{F}(0, m)$. In particular when $m = \lambda_1$, where $-\lambda_1$ is the first eigenvalue of the Dirichlet problem for the domain Ω, $d\mathcal{F}(0, \lambda_1)$; $H^2 \to L^2$ has a one dimensional kernel and corange, both spanned by the eigenfunction $\varphi_1(x)$ associated with $-\lambda_1$. Excepting the fact that the linearized operator is hyperbolic, this suggests a bifurcation theoretic approach to an existence

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theorem. However, being hyperbolic, the linearized operator also loses derivatives, a phenomenon similar to the small divisor problem of celestial mechanics. One main interest in this paper is that the Nash-Moser technique may be used in conjunction with a Lyapounov-Schmidt decomposition to overcome the existence of a kernel and this loss of derivatives, proving the following result.

THEOREM. *There exists a branch of nontrivial solutions*

$$\big(u(x, t), m\big) \in H_0^\varrho \times \mathbb{R}$$

of the equation (1.1), *for* $\varrho > [(n+1)/2] + 4$. *These solutions can be parametrized by* $\sigma \in (-\delta, \delta)$; *where*

$$u(\sigma) = \sigma\varphi_1 + \sigma w(\sigma)$$

and

$$\int_0^t \int_\Omega w\varphi_1 \, dx\, dt = 0 .$$

Moreover $u(\sigma)$ *is a Lipschitz curve in* $H_0^{\varrho-1}$

$$u(0) = 0 , \qquad w(0) = 0$$

$$m(0) = \lambda_1 , \qquad \frac{\partial}{\partial\sigma} u\Big|_{\sigma=0} = \varphi_1 . \qquad \square$$

In other words, there is a branch of nontrivial solutions which intersects transversely the branch of trivial solutions $u = 0$, $m \in \mathbb{R}$ at the point $m = \lambda_1$.

Speaking abstractly, two things may hamper the invertibility of a nonlinear operator $\mathcal{F}$ in a neighborhood of a solution u. The range of $d\mathcal{F}(u)$ may not be closed, or the range may have nontrivial codimension. Classically the latter difficulty has been treated using the Lyapounov-Schmidt procedure. More recently a wide variety of problems which have dense, but not closed range have been solved using the Nash-Moser technique. Typically a rapidly convergent iteration is alternated with smoothing operations to overcome loss of differentiability or loss of decay properties of solutions of the associated linear equations. In the above problem both difficulties occur, and a combination of the two techniques is used to obtain solutions of (1.1).

I will briefly describe the method of Lyapounov-Schmidt. Consider the functional $\mathcal{F}(u, m)$ mapping a neighborhood of a Banach space $X \times \mathbb{R}$ into a Banach space Y. Suppose $\mathcal{F}(0, m) = 0$, so $u = 0$, $m \in \mathbb{R}$ is a trivial branch of solutions. For certain critical values of the parameter m, let the linearized operator $d\mathcal{F}(0, m)$ have a nontrivial kernel X_1, and nontrivial corange Y_1, where we are able to write

$$X = X_1 \oplus X_2 \qquad Y = Y_1 \oplus Y_2 .$$

Denoting P a projection onto Y_2, the decomposition suggests to solve first the equation

$$P\mathcal{F}(u_1 + u_2, m) = 0 \tag{1.2}$$

for $u_1 + u_2 \in X_1 \oplus X_2$. This is the first bifurcation equation. Taking the Frechet derivative of (1.2) with respect to u_2 at the point $u_1 + u_2 = 0$, we find that

$$Pd_{u_2}\mathcal{F}(0, m_{\text{critical}}); \qquad X \to Y_2$$

has dense range. In many cases, for example if $d\mathcal{F}(0, m)$ were elliptic, the range is also closed, and the « soft » implicit function theorem can be applied to find solutions

$$(u_1 + u_2(u_1, m), m)$$

of (1.2) for $|u_1|_x + |m - m_{\text{critical}}|$ small. It then remains to solve the second bifurcation equation

$$[I - P]\mathcal{F}(u, m) = 0 . \tag{1.3}$$

Often this is a finite dimensional problem, whose solution gives a characterization of solutions of the full nonlinear equation in a neighborhood of the point $(0, m_{\text{critical}}) \in X \times \mathbb{R}$.

In the problem studied in this paper the linearized operator $d\mathcal{F}$ is hyperbolic. Best estimates on its inverse are of the form

$$|v|^2_{H^2} - \text{const}\,|v|^2_{L^2} \leq \text{const}\left[|d\mathcal{F}\cdot v|^2_{L^2} + \left|\frac{\partial}{\partial t}(d\mathcal{F}\cdot v)\right|^2_{L^2}\right]. \tag{1.4}$$

The extra time derivative appearing on the right hand side represents a loss of derivatives, i.e. the range of the linearized operator is not closed, its inverse is unbounded. Since the nonlinear function F contains second derivatives, these estimates are not sufficient for the application of the usual implicit function theorem to the first bifurcation equation, requiring the use of a more rapidly convergent Newton iteration scheme.

The Nash-Moser technique, based on Newton iteration, requires the invertibility of the linearized operator in a full neighborhood of the solution

$u = 0$. However $d\mathcal{F}(u)$ is a perturbation of $d\mathcal{F}(0)$, and in general their kernels will not coincide. It is important to the iteration procedure that we are able to invert $P d\mathcal{F}(u)$, where the projection P is kept fixed.

The existence of solutions of the first bifurcation equation will be shown by satisfying the hypotheses of a theorem of Moser [9]. This is the subject of section 2. It requires a linear existence theory, and rather careful control of the regularity of solutions of the linear equations. This work, which is found in sections 4 and 5, makes up the bulk of the paper. Once the first bifurcation equation is solved, regularity of the solution with respect to parameters is demonstrated, and the existence question is reduced to the second bifurcation equation. In this case it is a finite dimensional problem with a particularly simple solution. In section 6 are some results on the stability of the above solutions. Finally in section 7 the linear estimates are used to prove perturbation results about the kernel of $d\mathcal{F}(u)$ as u varies.

The idea that the Nash-Moser technique can be applied to the genuinely nonlinear periodic dissipative wave equation comes from Rabinowitz [12]. Methods for obtaining higher regularity also come from this paper, with help from some ideas of Kohn-Nirenberg [6], [7]. With minor modifications the estimates presented here can be used to generalize the results of Rabinowitz [12] to any spatial dimension. That is, solutions exist to problems of the following forms

$$\begin{aligned} u_{tt} - \Delta u + \alpha u_t + \varepsilon F(x, t; u, Du, D^2 u) &= 0 \\ u|_{\partial\Omega} = 0 \qquad u(x, t+\tau) &= u(x, t) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} u_t &= \varkappa \Delta u + \varepsilon F(x, t; u, Du, D^2 u) \\ u|_{\partial\Omega} = 0 \qquad u(x, t+\tau) &= u(x, t) \,. \end{aligned} \tag{1.6}$$

It is worth noting that in finding nontrivial solutions of (1.6), the nonlinearity may be such that when linearized about the solution, small variable coefficients occur on highest order terms, so that the equation is of undetermined type.

To agree upon notation, let C^∞ denote the infinitely differentiable functions $\varphi(x, t)$ on $\Omega \times \mathbb{R}$ which satisfy

$$\varphi(x, t+\tau) = \varphi(x, t) \,.$$

Let C_0^∞ denote those functions $\varphi \in C^\infty$ which vanish on $\partial\Omega$ for all t

$$\varphi(t, \partial\Omega) = 0 \,.$$

Denote by H^r the completion of C^∞ with respect to the r-th Sobolev norm

$$|\varphi|^2_{H^r} = \sum_{r \leqslant |\alpha|} |D^\alpha \varphi|^2_{L^2} \quad 0 \leqslant r \in \mathbb{Z}^+$$

where α is a multiindex

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \qquad D^\alpha = D_t^{\alpha_0} D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$$

H_0^r is the completion of C_0^∞ is the same norm. Double bars denote the supremum norm

$$\|\varphi\|_r = \sum_{|\alpha| \leqslant r} \sup_{(x,t) \in \Omega \times [0,\tau]} |D^\alpha \varphi(x,t)| .$$

Often for convenience $|\varphi|_{H^0} = |\varphi|_{L^2}$ will just be written $|\varphi|$, when it will be clear by context that a function space norm is intended.

This paper consists principally of work done for my doctoral dissertation at the Courant Institute. I would like to thank my advisor, Professor Louis Nirenberg, for his suggestions, and Professors Jürgen Moser and Paul Rabinowitz for their influential work.

2. – The first bifurcation equation.

I will proceed to describe the existence theory for the first bifurcation equation of the nonlinear periodic dissipative wave equation. Solutions $u(x, t)$ of (1.1) will be classical solutions, where we have assumed that

$$F(x, t; 0, 0, 0) = 0 = dF(x, t; 0, 0, 0) .$$

Solutions will be periodic in time of period τ, where τ is determined by the period of $F(x, t; \ldots)$ with respect to time. Denote

$$\mathcal{F}(u, m) = u_{tt} - \Delta u + \alpha u_t - mu + F(x, t; u, Du, D^2 u) . \tag{2.1}$$

The conditions on the function F imply that $u = 0$ is a solution of $\mathcal{F}(u, m) = 0$ for all values of the parameter m, and the Frechet derivative at the zero solution is the constant coefficient dissipative wave operator

$$d\mathcal{F}(0, m)\cdot\varphi = \varphi_{tt} - \Delta\varphi + \alpha\varphi_t - m\varphi .$$

When the operator $d\mathcal{F}(0, 0)$ is restricted to the domain $H_0^2 \leqslant L^2$, its eigenvalues are easily seen to be

$$\lambda_k - \left(\frac{2\pi}{\tau} j\right)^2 + i\alpha \frac{2\pi}{\tau} j \qquad \forall k \in \mathbb{Z}^+, \quad j \in \mathbb{Z}$$

where $-\lambda_k$ is the k-th eigenvalue of the Dirichlet Laplacian for the domain Ω. In particular the only real eigenvalues are exactly the λ_k's. This is due to the dissipative character of the equation, in particular to the $\alpha(\partial/\partial t)$ term, which is the «friction» or «heat loss» term. It is well known that λ_1 is simple. We will find a branch of nontrivial solutions of the above hyperbolic problem bifurcating from $m = \lambda_1$.

Following the procedure of Lyapounov-Schmidt, we reduce the existence question to a problem in finitely many variables. This is done by first solving the nonlinear equation projected onto the closure of the range of $d\mathcal{F}(0, \lambda_1)$. Since the operator is hyperbolic the range is not a closed subspace. The representation of P that is used is

$$P\varphi(x, t) = \varphi(x, t) - \varphi_1(x)\int_0^\tau\int_\Omega \varphi(y, s)\varphi_1(y)\,dy\,ds$$

where φ_1 is the normalized eigenfunction. This projection commutes with time differentiation and with $d\mathcal{F}(0, m)$, facts which simplify some of the regularity computations. However P does not in general commute with $d\mathcal{F}(u, m)$.

In the following we write $\psi \perp \varphi$ to mean that $\int_0^\tau\int_\Omega \psi(x, t)\varphi(x, t)\,dx\,dt = 0$.

Main Theorems.

To obtain the following results an ansatz is made. We look for solutions of the form

$$u = \sigma(\varphi_1 + w) \qquad w \perp \varphi_1 \qquad \sigma \in \mathbb{R}\,.$$

A solution $w = w(\sigma, m)$ is found for the first bifurcation equation. This is the content of the following theorem.

THEOREM 1. *There exists* $\delta = \delta(\Omega, \tau, \varrho, F)$ *such that if* $|\sigma| + |m - \lambda_1| < \delta$ *and* $\varrho > \max\{[(n+1)/2] + 4, n+1\}$ *there is a unique solution*

$$w = w(\sigma, m) \in H_0^\varrho$$

$$w \perp \varphi_1$$

$$|w|_{H_\varrho} < \text{const}\,\delta$$

of the equation

$$\begin{aligned} 0 &= P\frac{1}{\sigma}\mathcal{F}(\sigma(\varphi_1 + w), m) \\ &= w_{tt} - \Delta w + \alpha w_t - mw + \frac{1}{\sigma}PF(x, t; \sigma(\varphi_1 + w), \sigma D(\varphi_1 + w), \sigma D^2(\varphi_1 + w))\,. \end{aligned}$$

Furthermore

$$w(0, \lambda_1) = 0 . \quad \square$$

Theorem 1 is the heart of this paper. It is of course here that the Nash-Moser technique is used. Its proof involves rather careful control of solutions of the inhomogeneous linear problems in terms of the coefficients, the right hand side, and their derivatives. Once solutions $w(\sigma, m)$ are obtained we have achieved a reduction to finitely many dimensions. It must be shown that these solutions depend smoothly on the parameters. This is the content of Theorem 2. Denote $u(\sigma, m) = \sigma(\varphi_1 + w(\sigma, m))$.

THEOREM 2. *Let* $N = \{(\sigma, m);\ |\sigma| + |m - \lambda_1| < \delta\}$, *and suppose* $|u|_{H^\varrho} < \infty$, $\|u\|_4 < \delta$.
For $s < \varrho$

$$u;\ N \to H^s \textit{ is Lipschitz}.$$

For $\varrho > |p| + [(n+1)/2] + 4$, *where* $p = (p_1, p_2)$

$$D_\sigma^{p_1} D_m^{p_2} u;\ N \to H^{\varrho - |p| - 1} \textit{ is Lipschitz}. \quad \square$$

REMARK. – When the «soft» implicit function theorem is applicable, and when the parameters enter the equation analytically, one expects analyticity of the solution in those parameters. This is not necessarily the case here, due to the unboundedness of the inverse of the linearized operator.

In this case the finite dimensional problem is particularly simple, and the following result guarantees the existence of a branch of nontrivial solutions bifurcating from the trivial branch $u = 0$, $m \in \mathbb{R}$ at the value of the parameter $m = \lambda_1$.

THEOREM 3. (i) *There exists a branch of nontrivial solutions* $(u(x, t), m)$ *of the equation* (1.1).

(ii) *Solutions along this branch are parametrized by* σ, $(u(\sigma), m(\sigma))$ *such that*

$$u(0) = 0 \qquad m(0) = \lambda_1$$

$$\left.\frac{\partial u}{\partial \sigma}\right|_{\sigma=0} = \varphi_1$$

$$u(\sigma);\ (-\delta, \delta) \to H_0^{\varrho-1} \textit{ is continuous}$$

$$u \in C^p((-\delta, \delta); H_0^{\varrho-p-1}) \qquad \textit{if } \varrho - p > \left[\frac{n+1}{2}\right] + 4$$

(iii) $u(\sigma)$ *is of the form*

$$u(\sigma) = \sigma\varphi_1 + \sigma w(\sigma)$$

where

$$w(\sigma) \perp \varphi_1 \quad \textit{and } w(0) = 0 . \quad \square$$

The branch of solutions of Theorem 3 is a mapping of $(-\delta, \delta)$ into a Hilbert space of functions which depend upon both x and t. However the eigenvalue λ_1 is associated with an eigenfunction which is constant in time. If $F(x, t; \dots)$ were independent of t, solutions along the whole branch would be time independent as well. When however $F(x, t; \dots)$ is τ-periodic in time, solutions along the branch are forced oscillations, and inherit the τ-periodic behavior.

The First Bifurcation Equation.

Projecting the nonlinear equation (1.1) onto the range of $d\mathcal{F}(0, \lambda_1)$, the first bifurcation equation is written

$$P\mathcal{F}(u, m) = P[u_{tt} - \Delta u + \alpha u_t - mu + F(x, t; u, Du, D^2u)] = 0 .$$
$$u(t, \partial\Omega) = 0 \qquad u(x, t+\tau) = u(x, t) .$$

Making the substitution $u(x, t) = \sigma(\varphi_1 + w(x, t))$, $w \perp \varphi_1$ and $|\sigma|$ small, we find upon dividing by σ:

$$(2.2) \qquad P\frac{1}{\sigma}\mathcal{F}\big(\sigma(\varphi_1 + w), m\big)$$
$$= w_{tt} - \Delta w + \alpha w_t - mw + P\frac{1}{\sigma}F\big(x, t; \sigma(\varphi_1 + w), \sigma D(\varphi_1 + w), \sigma D^2(\varphi_1 + w)\big) .$$

Considering this as a mapping

$$\mathbb{R}^2 \times [H_0^\varrho \ominus \{\varphi_1\}] \to H^{\varrho-2} \ominus \{\varphi_1\}$$

we wish to find solutions $w = w(\sigma, m)$ for all (σ, m) in a neighborhood N of $(0, \lambda_1) \in \mathbb{R}^2$. In contrast to the case usually encountered, vis Crandall-Rabinowitz [2], the « soft » implicit function theorem is not applicable due to the unboundedness of the inverse of the linear operator. Since we are employing a Newton scheme it is important that the linearized operator be invertible not only at the point $(0, \lambda_1, 0)$, but for all (σ, m, w) in a neighborhood. The linearized equation takes the following form:

For $v \perp \varphi_1$, $v \in H_0^\varrho$, $g \perp \varphi_1$, $g \in H^{\varrho-2}$

$$(2.3) \qquad d_w P\frac{1}{\sigma}\mathcal{F}\big(\sigma(\varphi_1 + w), m\big)\cdot v \equiv Lv$$
$$= v_{tt} - \Delta v + \alpha v_t - mv + P\Big[\sum_{|i|+|j|\le 2} a_{ij}(x, t) v_{t^i x^j}\Big] = g$$

where the coefficients are

$$a_{ij}(x, t) = F_{u_{x^i t^j}}(x, t; \sigma(\varphi_1 + w), \sigma D(\varphi_1 + w), \sigma D^2(\varphi_1 + w)) .$$

It is important to control the smoothness of $(1/\sigma)F(x, t; \sigma(\varphi_1 + w) \ldots)$ and of the coefficients $a_{ij}(x, t)$ in terms of the function w.

LEMMA 2.1. *If $F(x, t; u, Du, D^2u)$ is sufficiently differentiable with respect to all variables, and both*

$$F(x, t; 0, 0, 0) = dF(x, t; 0, 0, 0) = 0 .$$

Then for $\|u\|_2 \leqslant 1$

(2.4) (i) $|F(x, t; u, Du, D^2u)|_{H^p} \leqslant \text{const } \|u\|_2 |u|_{H^{p+2}}$

(ii) $\|a_{ij}(x, t)\|_p \leqslant \text{const } \|u\|_{p+2}$

(iii) $|a_{ij}(x, t)|_{H^p} \leqslant \text{const } |u|_{H^{p+2}}$. □

The proof of this lemma is standard, and is relegated to the appendix.

Using the lemma, and setting $u = \sigma(\varphi_1 + w)$, we have that if $w \in H_0^\varrho$ and σ is taken small

$$\|a_{ij}(x, t)\|_2 \leqslant \sigma \text{ const } \|\varphi_1 + w\|_4 .$$

In section 4 it will be shown that if $\|a_{ij}\|_2$ are small, one may solve the projected linear equation (2.3) for v. Then in section 5 global bounds on the regularity of the solution are obtained in terms of derivatives of the right hand side and of the coefficients. These results are stated in the following two theorems.

THEOREM 4 [linear existence]. *There is a δ such that if $\|a_{ij}\|_2 < \delta$ and $m - \lambda_1 < \delta$, then given g;*

$$g \perp \varphi_1$$
$$|g|_{L^2}^2 + \left|\frac{\partial}{\partial t} g\right|_{L^2}^2 < \infty$$

there exists a v solving $Lv = g$

$$v \perp \varphi_1$$
$$v \in H_0^1$$

$$|v|_{H^2}^2 \leqslant \text{const}\left[|g|_{L^2}^2 + \left|\frac{\partial}{\partial t} g\right|_{L^2}^2\right]. \qquad (2.5)$$ □

THEOREM 5 [higher regularity]. *There is a* δ *such that if* $\|a_{ij}\|_2 < \delta$, *solutions to the equation*

$$Lv = g \qquad v \perp \varphi_1$$

satisfy the estimate

$$(2.6) \qquad |v|^2_{H^r} \leqslant \text{const}\left[|g|^2_{H^{r-2}} + \left|\frac{\partial}{\partial t} g\right|^2_{H^{r-2}} + \|a_{ij}\|^2_{r-2} + \left\|\frac{\partial}{\partial t} a_{ij}\right\|^2_{r-2}\right].$$

If additionally $r \geqslant [(n+1)/2] + 1$, *and*

$$\|D^2 u\|_1 < 1$$

$$\|a_{ij}\|_{[(n+1)/2]-1} + \left\|\frac{\partial}{\partial t} a_{ij}\right\|_{[(n+1)/2]-1} + |a_{ij}|_{H^{[(n+1)/2]+1}} < \delta .$$

Then

$$(2.7) \qquad |v|^2_{H^r} \leqslant \text{const}\left[|g|^2_{H^{r-2}} + \left|\frac{\partial}{\partial t} g\right|^2_{H^{r-2}} + |a_{ij}|^2_{H^{r-2}} + \left|\frac{\partial}{\partial t} a_{ij}\right|^2_{H^{r-2}}\right]. \qquad \square$$

The presence of the extra time derivative on the right hand side is the assertion of the fact that solutions of the linearized equations lose derivatives. It is noteworthy that with dissipation the inverse gains back all but one time derivative. If it were not for this loss a standard Picard iteration would suffice. In fact, if nonlinearity of highest order did not appear in F, again a Picard type method would work. In this paper we are concerned with the case in which $F = F(x, t; u, Du, D^2 u)$ is fully nonlinear.

In the iteration smoothing operators are used to improve the regularity of the coefficients and inhomogeneous terms of the linear equations. The ones appearing here have been used by Moser [9]; they are Galerkin type projections. If $\varphi \in H^r$, denote by P_S the projection of L^2 to the finite dimensional subspace corresponding to that part of the spectrum of $\Delta + \partial^2/\partial t^2$ for which $|\lambda| < S$. S is taken larger than 1. The following smoothing estimates hold.

$$(2.8) \qquad \text{(i)} \quad |P_S\varphi|_{H^r} \leqslant S^r |P_S \varphi|_{H^0} \leqslant S^r |\varphi|_{H^0}$$

$$\text{(ii)} \quad |[I - P_S]\varphi|_{H^0} \leqslant S^{-r} |[I - P_S]\varphi|_{H^r} \leqslant S^{-r}|\varphi|_{H^r}$$

$$\text{(iii)} \quad \|[I - P_S]\varphi\|_0 \leqslant |[I - P_S]\varphi|_{H^{[(n+1)/2]}} \leqslant S^{-r+[(n+1)/2]}|\varphi|_{H^r} .$$

We now prove Theorem 1 by demonstrating that the first bifurcation equation satisfies the hypotheses of a version of the Nash-Moser implicit function theorem. The theorem is taken from Moser [9]; for completeness we state it on page 15. The necessary hypotheses are numbered (2.9) (1) through (6), and (2.10) (1) through (4).

(2.9) (1) The domain $\mathcal{U}$ of the nonlinear operator

$$\begin{aligned}\mathcal{G}(w) &= P\frac{1}{\sigma}\mathcal{F}(\sigma(\varphi_1+w), m)\\ &= w_{tt}-\Delta w+\alpha w_t - mw + P\frac{1}{\sigma}F(x, t;\, \sigma(\varphi_1+w), \dots)\end{aligned}$$

consists of those functions $w \in H_0^r$ such that

$$\begin{aligned}&w \perp \varphi_1\\ &|w|_{H^0}^2+|w|_{H^\varrho}^2<1 \qquad 0<\varrho<r\,.\end{aligned}$$

For any $w \in \mathcal{U}$, if a constant M is picked large enough, we have

$$\text{(2.9) (2)}\qquad \begin{aligned}&|\mathcal{G}(w)|_{H^0}=\left|P\frac{1}{\sigma}\mathcal{F}(\sigma(\varphi_1+w), m)\right|_{H^0}\leqslant \text{const}\, |w|_{H^2}<M\\ &\left|P\frac{1}{\sigma}\mathcal{F}(\sigma(\varphi_1+w), m)\right|_{H^{r-2}}\leqslant \text{const}\, |w|_{H^r}<\infty\,.\end{aligned}$$

Given that $w \in \mathcal{U}$ is such that $|w|_{H^r}<K$ for some $K>1$

$$\text{(2.9) (3)}\qquad \begin{aligned}\left|P\frac{1}{\sigma}\mathcal{F}(\sigma(\varphi_1+w), m)\right|_{H^{r-2}} &\leqslant |w_{tt}-\Delta w+\alpha w_t-mw|_{H^{r-2}}\\ &\qquad + \left|P\frac{1}{\sigma}F(x, t;\dots)\right|_{H^{r-2}}\\ &\leqslant M|w|_{H^r}\leqslant MK\end{aligned}$$

where M is chosen perhaps larger.

Now suppose that $g \perp \varphi_1$ is a function such that $|g|_{H^0}<K^{-\lambda}$, $|g|_{H^{r-2}}<K$ for some fixed constant $\lambda>0$, and suppose that $|w|_{H^r}<K$ for $w \in \mathcal{U}$. We will find a smooth approximate solution $v \perp \varphi_1$ of the linearized first bifurcation equation. Referring to the linear existence and regularity theorems, Theorem 4 and Theorem 5, we solve the equation;

$$\begin{aligned}&v_{tt}-\Delta v+\alpha v_t-mv+P\sum_{|i|+|j|\leqslant 2} a_{ij}^S(x, t)\, v_{t^i x^j}=g^S\\ &v(t, \cdot)|_{\partial\Omega}=0\end{aligned}$$

where $a_{ij}^S(x, t)$ and g^S denote respectively the smooth functions $P_S a_{ij}(x, t)$ and $P_S g$. Notice that if $g \perp \varphi_1$, then $P_S g$ is also, since both the projections P and P_S are defined relative to the eigenfunction expansion of the Dirichlet Laplacian of Ω.

The existence theorem requires that $\|a_{ij}(x, t)\|_2$ be small. We use the Sobolev lemma and the composition of functions Lemma 2.1 to estimate

$$\begin{aligned}\|a_{ij}(x, t)\|_2 &\leqslant \text{const}\, |a_{ij}|_{H^{[(n+1)/2]+2}} \\ &\leqslant \text{const}\, \sigma|\varphi_1 + w|_{H^{[(n+1)/2]+4}} .\end{aligned}$$

If $\varrho \geqslant \max\{[(n+1)/2]+4, n+1\}$, and σ is sufficiently small, the coefficients will be small enough to apply the existence and regularity results to obtain a solution v of the smoothed linearized equation. Comparing the smoothed equation to the non-smoothed one;

$$\begin{aligned}&\left| \mathrm{d}P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m)\cdot v - g \right|_{H^0} \\ &\qquad \leqslant \Big| \sum_{|i|+|j|\leqslant 2} \big(a_{ij}^S(x, t) - a_{ij}(x, t)\big) v_{t^i x^j}\Big|_{H^0} + |g^S - g|_{H^0} \\ &\qquad \leqslant \Big\| \sum \big(a_{ij}^S(x, t) - a_{ij}(x, t)\big)\Big\|_0 |v|_{H^2} + |g^S - g|_{H^0} \\ &\qquad \leqslant \text{const}\, S^{-r+2+[(n+1)/2]} \Big|\sum a_{ij}(x, t)\Big|_{H^{r-2}} |v|_{H^2} + S^{-r-2}|g|_{H^{r-2}} .\end{aligned}$$

It remains to bound $|v|_{H^2}$. Since

$$\begin{aligned}|g^S|_{H^1} &\leqslant \text{const}\, |g^S|_{H^0}^{1-1/(r-2)} |g^S|_{H^{r-2}}^{1/(r-2)} \\ &\leqslant \text{const}\, K^{(\lambda+1)/(r-2)-\lambda}\end{aligned}$$

we find from Theorem 4 that

$$\begin{aligned}|v|_{H^2}^2 &\leqslant \text{const}\left(|g^S|_{L^2}^2 + \left| \frac{\partial}{\partial t} g^S \right|_{L^2}^2 \right) \\ &\leqslant \text{const}\, K^{-2(\lambda-(\lambda+1)/(r-2))} .\end{aligned}$$

If r is sufficiently large we have $(\lambda+1)/(r-2)-\lambda \leqslant 0$, so for $K > 1$, $|v|_{H^2}$ is bounded independently of K. We have shown that if $g \perp \varphi_1$, $|g|_{H^0} < K^{-\lambda}$ $|g|_{H^{r-2}} < K$ and $|w|_{H^r} < K$ then there exists a solution v of the smoothed linearized equation such that

$$\begin{aligned}(2.9)\ (4)\quad (\mathrm{i})\ &\left| dP \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m)\cdot v - g \right|_{H^0} \\ &\qquad \leqslant \text{const}\, S^{-r+2+[(n+1)/2]} \Big|\sum a_{ij}(x, t)\Big|_{H^{r-2}} + S^{-r-2}|g|_{H^{r-2}} \\ &\qquad \leqslant \text{const}\, K S^{-r+2+[(n+1)/2]} = \text{const}\, K S^{-\mu}\end{aligned}$$

where we defined $\mu = r - 2 - [(n+1)/2]$ and have used again the composition of functions Lemma 2.1.

To estimate $|v|_{H^r}$ we refer to Theorem 5, which provides the regularity estimates for the linear equation. With $r > \varrho > \max\{[(n+1)/2]+4, n+1\}$ and δ perhaps chosen smaller;

$$|v|^2_{H^r} \leqslant \text{const}\left(|g^S|^2_{H^{r-2}} + \left|\frac{\partial}{\partial t} g^S\right|^2_{H^{r-2}} + |a^S_{ij}|_{H^{r-2}} + \left|\frac{\partial}{\partial t} a^S_{ij}\right|_{H^{r-2}}\right)$$
$$\leqslant \text{const}\,(1+S^2)\big(|g|^2_{H^{r-2}} + |a_{ij}|^2_{H^{r-2}}\big)\,.$$

Using again the composition of functions Lemma 2.1, and the fact that $S > 1$

$$|v|^2_{H^r} \leqslant \text{const}\, S^2(1+|w|^2_{H^r})\,.$$

For $|w|_{H^r} < K$ we have shown that

(2.9) (4) (ii) $|v|_{H^r} \leqslant \text{const}\, SK$.

The next hypothesis of Moser's theorem to be verified is the estimate

$$\text{(2.9) (5)} \qquad |v|_{H^0} \leqslant \left| dp\,\frac{1}{\sigma}\,\mathcal{F}(\sigma(\varphi_1+w), m)\cdot v\right|_{H^0}.$$

Defining $\Lambda v = (\partial/\partial t)v + (\alpha/2)v$, and $Lv = dP(1/\sigma)\mathcal{F}(\sigma(\varphi_1+w), m)\cdot v$, we take the L^2 inner product;

$$(\Lambda v, Lv) = \alpha|v_t|^2 + (v_t, Pa_{ij}(x,t)v_{t^ix^j}) - \frac{\alpha}{2}|v_t|^2 + \frac{\alpha}{2}|\nabla v|^2 - \frac{\alpha m}{2}|v|^2 + \frac{\alpha}{2}(v, Pa_{ij}(x,t)v_{t^ix^j})\,.$$

To estimate the terms involving variable coefficients we integrate by parts as follows;

$$(v_t, a_{20}(x,t)v_{tt}) = \iint a_{20}(x,t)\left(\frac{1}{2}|v_t|^2\right)_t dx\,dt \leqslant \frac{1}{2}\left\|\frac{\partial}{\partial t}a_{20}\right\|_0 |v_t|^2_{H^0}$$

$$(v_t, a^l_{11}(x,t)v_{tx_l}) = \iint a^l_{11}(x,t)\left(\frac{1}{2}|v_t|^2\right)_{x_l} dx\,dt \leqslant \frac{1}{2}\left\|\frac{\partial}{\partial x_l}a^l_{11}\right\|_0 |v_t|^2$$

$$\begin{aligned}(v_t, a^{lm}_{02}(x,t)v_{x_lx_m}) &= \iint a^{lm}_{02}(x,t)v_{x_lx_m}v_t\,dx\,dt \\ &= \iint \frac{\partial}{\partial x_l}a^{lm}_{02}(x,t)v_{x_m}v_t\,dx\,dt + \iint a^{lm}_{02}(x,t)v_{x_l}v_{x_mt}\,dx\,dt \\ &\leqslant \left\|\frac{\partial}{\partial x_l}a^{lm}_{02}\right\|_0 |v_{x_m}||v_t| + \left\|\frac{\partial}{\partial t}a^{lm}_{02}\right\|_0 |v_{x_l}||v_{x_m}|\end{aligned}$$

where it has been used that $a_{02}^{lm}(x, t) = a_{02}^{ml}(x, t)$. Terms of lower order are estimated similarly, for example

$$|(v_t, a_{10}(x, t) v_t)| \leq \|a_{10}\|_0 |v|_{H^1}^2 .$$

We find that

$$\frac{\alpha}{2} |v|_{H^1}^2 - \text{const} \|a_{ij}\|_1 |v|_{H^2}^2 \leq (\Lambda v, Lv) + \frac{\alpha}{2} m |v|_{L^2}^2 \leq \text{const} |v|_{H^1} |Lv|_{L^2} + \frac{\alpha}{2} m |v|_{L^2}^2 .$$

If all $\|a_{ij}\|_1$ are sufficiently small we apply the generalized Poincaré inequality, Lemma 3.4, to find that if $m < \lambda_2$ and $v \perp \varphi_1$

$$|v|_{H^1} \leq \text{const} \left| dP \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m) \cdot v \right|_{L^2} .$$

This implies inequality (2.9) (5).

There remains one more condition to be satisfied in order to apply the nonlinear existence theorem.

Denote

$$Q(w, v) = P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w + v), m) - P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m) - d_w P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m) \cdot v$$

the quadratic part of $P(1/\sigma) \mathcal{F}(\sigma(\varphi_1 + w), m)$. $Q(w, v)$ must satisfy the estimate

(2.9) (6) $|Q(w, v)|_{H^0} \leq \text{const}\, M |v|_{H^0}^{2-\beta} |v|_{H^r}^{\beta}$

for some fixed constant $0 \leq \beta < 1$.

Verification:

$$|Q(w, v)|_{H^0} = \left| P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w + v), m) - P \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m) - dP \frac{1}{\sigma} \mathcal{F}(\sigma(\varphi_1 + w), m) \cdot v \right|_{H^0} \leq \left| \frac{1}{\sigma} P\big(F(x, t; \sigma(\varphi_1 + w + v), \ldots) - F(x, t; \sigma(\varphi_1 + w), \ldots)\big) - P\, dF(x, t; \sigma(\varphi_1 + w), \ldots) \cdot v \right|_{H^0} .$$

Using the mean value theorem

$$\leq \|d^2 F(x, t; \text{ intermediate point})\|_0 \|v\|_2 |v|_{H^2} .$$

The boundedness of $|w|_{H^{[(n+1)/2]+2}}$ implies the boundedness of the first term. Interpolating,

$$\|v\|_2|v|_{H^2}\leqslant \text{const } |v|_{H^0}^{2-\beta}|v|_{H^r}^{\beta}$$

where $\beta = ([n+1]/2] + 4)/r$, and hence (2.9) (6) is satisfied if r is sufficiently large.

One is led to choose λ, the order of the nonlinear approximation, to satisfy the following inequalities;

$$(2.10)\quad (1)\qquad \left[\frac{n+1}{2}\right]+4<\varrho<\frac{\lambda}{\lambda+1}r$$

$$n+1<\varrho$$

so that K large will imply that $\|a_{ij}(x,t)\|_2$ and $\|F(x,t;u,Du,D^2u)\|_2$ are small.

$$(2.10)\quad (2)\qquad r=\mu+2+\left[\frac{n+1}{2}\right]$$

which defines μ the order of linear approximation in terms of the high norm r.

$$(2.10)\quad (3)\qquad 0<\lambda+\frac{1}{2}(\mu+1)\,.$$

$$(2.10)\quad (4)\qquad 0<\beta=\frac{n+9}{2r}<\frac{\lambda}{\lambda+1}\frac{\mu}{\mu+1}\left(1-2\frac{\lambda+1}{\mu+1}\right)$$

to satisfy the hypotheses of Moser's theorem. If $n=1$ for example, these are satisfied by setting $\lambda=2$, $\mu=13$, $r=16$, and $\varrho=6$. For any n we notice that $\beta=O(1/r)$ while the r.h.s. of (4) approaches $\lambda/(\lambda+1)$ as $r\to\infty$, insuring that a choice of r exists.

We now state Moser's implicit function theorem [9].

THEOREM. *Assume that $\mathcal{G}(n)$ satisfies the properties* (2.9) (1) *through* (6), *and that the constants β, μ, λ, r, ϱ satisfy* (2.10) (1) *through* (4). *Then there is a constant $K_0(M,\beta,\mu,\lambda)>1$ such that if u_0 and $\mathcal{G}(u_0)$ satisfy*

$$|\mathcal{G}(u_0)|_{H^0}<K_0^{-\lambda}$$

$$|u_0|_{H^r}<K_0$$

$$|\mathcal{G}(u_0)|_{H^{r-2}}<MK_0$$

then there is a sequence of approximations $u_n\in\mathcal{U}$ and real numbers $K_n\to\infty$

such that

$$|\mathcal{G}(u_n)|_{H^0} < K_n^{-\lambda}$$
$$|u_n|_{H^r} < K_n .$$

The sequence u_n converges to a function $\hat{u}$ in H^ϱ norm; $|u_n - \hat{u}|_{H^\varrho} \to 0$, where $\varrho < r\lambda/(\lambda + 1)$. If $\mathcal{G}(u)$ is continuous from $H^\varrho \to H^0$, it follows $\mathcal{G}(\hat{u}) = 0$. □

In the case at hand we have taken $\mathcal{G}(w) = P(1/\sigma)\mathcal{F}(\sigma(\varphi_1 + w), m)$ for parameters σ, m such that $|\sigma| + |\lambda_1 - m| < \delta$. Setting $w_0 = 0$ and δ small enough so that

$$\left| P\frac{1}{\sigma}\mathcal{F}(\sigma\varphi_1, m)\right|_{H^0} \leqslant \text{const } |\sigma||\varphi_1| < K_0^{-\lambda}$$

the theorem asserts that there exists $w = w(\sigma, m)$ a solution of the first bifurcation equation (2.2).

REMARKS. (i) Convergence is actually very rapid. The successive K_n's are defined by

$$K_{n+1} = K_n^{\varkappa}, \qquad \text{where } 1 < (1 - 3/(\mu + 1))^{-1} < \varkappa < 2 .$$

(ii) Since initially we ask that $[(n + 1)/2] + 4 \leqslant \varrho$, our sequence of approximations actually converges to a function with 4 classical derivatives. Two derivatives suffices to have a classical solution to the equation, but 4 are needed to insure that the second derivatives of the coefficients of the linearized operator remain sufficiently small.

It remains to demonstrate the uniqueness statement of Theorem 1. This is the content of the following Lemma.

LEMMA 2.2. *Assume that $P\mathcal{F}(u, m) = P\mathcal{F}(v, m) = 0$, and that*

$$\|u\|_4 , \qquad \|v\|_4 < \delta$$
$$|u|_{H^\varrho} , \qquad |v|_{H^\varrho} < 1$$
$$|m - \lambda_1| + |\sigma| < \delta .$$

If $(u - v) \perp \varphi_1$, then $u = v$. □

The lemma asserts that for (m, σ) fixed, $w(\sigma, m)$ is the unique solution of $P\mathcal{F}(\sigma(\varphi_1 + w), m) = 0$ such that $\|w\|_4 < \delta$. The lemma also asserts that any other solution u of $P\mathcal{F}(u, m) = 0$, with $u \perp \varphi_1$ and $\|u\|_4 < \delta$, $|u|_{H^\varrho} < 1$ must be identically zero.

PROOF.

$$\begin{aligned}&|dP\mathcal{F}(u, m)\cdot(u - v)|_{H^1}\\ &\leqslant |P\mathcal{F}(u, m) - P\mathcal{F}(v, m) - dP\mathcal{F}(u, m)\cdot(u - v)|_{H^1}\\ &\leqslant \text{const } \|u - v\|_3 |u - v|_{H^2}\,.\end{aligned}$$

Denoting $w = (u - v) \perp \varphi_1$, and referring to estimate (2.5) of the linear problem,

$$\begin{aligned}|w|^2_{H^2} &\leqslant \text{const}\left(|dP\mathcal{F}(u, m)\cdot w|^2_{L^2} + \left|\frac{\partial}{\partial t}(dP\mathcal{F})u, m)\cdot w)\right|^2_{L^2}\right)\\ &\leqslant \text{const } |dP\mathcal{F}(u, m)\cdot w|^2_{L^2}\,.\end{aligned}$$

Therefore

$$|w|_{H^2} \leqslant \text{const } \|w\|_3 |w|_{H^3}\,.$$

Denoting Galerkin truncation operators P_T and P_S, and using the smoothing estimates (2.8)

$$\begin{aligned}|w^S|_{H^2} &\leqslant |w|_{H^2} \leqslant \text{const } \|w\|_3 |w|_{H^3}\\ &\leqslant \text{const } (\|w^T\|_3 + \|w - w^T\|_3)(|w^T|_3 + |w - w^T|_3)\\ &\leqslant \text{const } T^{[(n+1)/2]+2}(|w^T|^2_{H^2} + T^{4-2\varrho})\,.\end{aligned}$$

If $|w^T|_{H^2} < T^{-(\varrho-2)}$, for $\varrho > 3 + \frac{1}{2}[(n + 1)/2]$ define $S > T$ such that

$$|w^S|_{H^2} \leqslant \text{const } T^{6+[(n+1)/2]} < S^{-(\varrho-2)}\,.$$

If initially $|w|_{H^2} < 2^{-(\varrho-2)}$, then a sequence of $S_n \to \infty$ can be defined inductively such that

$$|w^{S_n}|_{H^2} \leqslant S_n^{-(\varrho-2)}\,.$$

Hence $|w|_{H^2} = 0$ and the proof is complete. □

3. – The second bifurcation equation.

Having obtained a solution $u(\sigma, m)$ of the first bifurcation equation, it must be shown that the solution varies smoothly with respect to the parameters (σ, m). That is, the mapping

$$u;\ N = \{(\sigma, m);\ |\sigma| + |m - \lambda_1| < \delta\} \to u(\sigma, m) \in H^2$$

should be at least continuously differentiable, so that the second bifurcation equation may be solved.

LEMMA 3.1. *$|w(\sigma, m)|_{H^s} \to 0$ as $\sigma \to 0$ for $s < \varrho$.* □

PROOF.

$$w_{tt} - \Delta w + \alpha w_t - m w = -P\frac{1}{\sigma}F\big(x, t;\ \sigma(\varphi_1 + w)\ \ldots\big).$$

Applying the linear estimates (2.6) with $a_{ij}(x, t) = 0$

$$|w|^2_{H^{\varrho-1}} \leqslant \text{const}\left[\left|\frac{1}{\sigma}F\big(x, t;\ \sigma(\varphi_1 + w)\ \ldots\big)\right|^2_{H^{\varrho-3}} + \left|\frac{\partial}{\partial t}\left(\frac{1}{\sigma}F(x, t;\ \ldots)\right)\right|^2_{H^{\varrho-3}}\right]$$
$$\leqslant \text{const}\ \|\varphi_1 + w\|_2^2 |\sigma(\varphi_1 + w)|^2_{H^\varrho}.$$

Since $|w|_{H^\varrho}$ and $\|w\|_2$ are bounded independently of σ for $|\sigma| < \delta$, we are done. □

LEMMA 3.2. *Denoting $u(\sigma, m) = \sigma\big(\varphi_1 + w(\sigma, m)\big)$, if*

$$|u|^2_{H^\varrho} < \infty, \qquad \|u\|_4 < \delta$$

then

$$u;\ N \to H^S$$

is Lipschitz for any $0 \leqslant s < \varrho$. □

PROOF. Difference quotients with respect to the parameters are uniformly bounded. To simplify notation consider $F = F(x, t;\ D^2 u)$. Let

$$u^q = \frac{1}{q}[u(\sigma + q, m) - u(\sigma, m)]$$

where q is so small that $|\sigma + q| < \delta$. Taking the difference quotient of the equation (2.2),

$$(\sigma w)^q_{tt} - \Delta(\sigma w)^q + \alpha(\sigma w)^q_t - m(\sigma w)^q$$
$$+ PdF(x, t;\ \text{intermediate point})D^2(\sigma w)^q$$
$$= -PdF(x, t;\ \text{intermediate point})D^2(\sigma\varphi_1)^q\,.$$

Using the estimates for the linear equation

$$|(\sigma w)^q|_{H^{\varrho-1}} \leqslant \text{const}\left[|dF(x, t; \text{int. pt.}) D^2(\sigma\varphi_1)^q|_{H^{\varrho-2}} + |dF(x, t; \text{int. pt.})|^2_{H^{\varrho-2}}\right]$$

and the right hand side may be bounded independently of q. □

A similar estimate holds for

$$u^q = \frac{1}{q}[u(\sigma, m + q) - u(\sigma, m)].$$

Hence $u(\sigma, m)$ is Lipschitz, and derivatives of u exist almost everywhere, and are such that

$$|D_\sigma u|_{H^{\varrho-1}}, \qquad |D_m u|_{H^{\varrho-1}} < \infty .$$

Denote $D^p u = D_\sigma^{p_1} D_m^{p_2} u$. Assume inductively that

i) $|D^\lambda u|_{H^{\varrho-\lambda}} < \infty$ for all $0 \leqslant \lambda < |p|$

ii) $\varrho - |p| - 1 > \left[\dfrac{n+1}{2}\right] + 2$

iii) $|u|_{H^\varrho} < \infty$, $\|u\|_4 < \delta$.

Taking $|p| - 1$ derivatives and one difference quotient of the equation

$$P[u_{tt} - \Delta u + \alpha u_t - mu] + PF(x, t; D^2 u) = 0$$

we find that

$$v = D^{p-1}(\sigma w(\sigma, m))^q \perp \varphi_1$$

satisfies the equation

$$v_{tt} - \Delta v + \alpha v_t - mv + PdF(x, t; \text{int. pt.}) D_x^2 v$$
$$= PD^{p-1}u - P \sum_{\substack{q+s \\ q>1}} \frac{\partial^q}{\partial u^q} F(x, t; \text{int. pt.}) \prod_{\lambda=1}^{s} (D^\lambda D_x^2 u)^{\beta_\lambda} .$$

The right hand side can be bounded in the $H^{\varrho-|p|-1}$ norm independently of q, using the above hypotheses. We have demonstrated the following lemma.

LEMMA 3.3. *Suppose* (i) $|u|_{H^\varrho} < \infty$, $\|u\|_4 < \delta$

$$\text{(ii)}\ \varrho - |p| > \left[\frac{n+1}{2}\right] + 3 .$$

Then $D^{p-1}u;\ N \to H^{\varrho-|p|}$ *is Lipschitz.* □

This is the conclusion of Theorem 2. For $p > 1$, i.e. for $\varrho > [(n+1)/2] + 4$ we may solve the second bifurcation equation.

$$\begin{aligned}(3.1)\qquad 0 &= [I - P]\,\mathcal{F}(\sigma(\varphi_1 + w), m)\\ &= (\lambda_1 - m)\varphi_1 + \varphi_1 \int_0^\tau \int_\Omega \varphi_1(y) F(s, y; \sigma(\varphi_1 + w) \ldots)\, dy\, ds\,.\end{aligned}$$

For $\sigma = 0$, $m \in \mathbb{R}$, (3.1) is satisfied. This is the trivial branch of solutions. Otherwise divide by σ. One computes that

$$\left|\frac{1}{\sigma} F(x, t; \sigma(\varphi + w) \ldots)\right|_{L^2} \leqslant \text{const}\, \sigma \|\varphi_1 + w\|_2 |\varphi_1 + w|_{H^2}$$

$$\left|\frac{\partial}{\partial\sigma}\left[\frac{1}{\sigma} F(x, t; \sigma(\varphi_1 + w)\ \ldots)\right]\right|_{L^2} \leqslant \text{const}\, \|\varphi_1 + w\|_2 |\varphi_1 + w + \sigma D_\sigma w|_{H^2}$$

$$\left|\frac{\partial}{\partial m}\left[\frac{1}{\sigma} F(x, t; \sigma(\varphi_1 + w) \ldots)\right]\right|_{L^2} \leqslant \text{const}\, \|\varphi_1 + w\|_2 |\sigma D_m w|_{H^2}\,.$$

The mapping

$$(\lambda_1 - m) + \frac{1}{\sigma}\int_0^\tau \int_\Omega \varphi_1(y) F(y, s; \sigma(\varphi_1 + w) \ldots)\, dy\, ds$$

is at least C^1 for $(m, \sigma) \in N$.

$$\frac{\partial}{\partial m}\bigg|_{\substack{m=\lambda_1\\ \sigma=0}} \left[(\lambda_1 - m) + \frac{1}{\sigma}\int_0^\tau \int_\Omega \varphi_1(y) F(y, s; \sigma(\varphi_1 + w) \ldots)\, dy\, ds\right] = -1\,.$$

Hence by the implicit function theorem there exists a branch of solutions $(m(\sigma), \sigma)$ of the second bifurcation equation (3.1), which intersects transversely the trivial solutions $\{\sigma \equiv 0\}$. This concludes the proof of Theorem 3.

4. – The linear equation.

In order to use a Newton method to solve a nonlinear equation it is necessary to be able to solve the linear equations at each step of the iteration. Taking the Frechet derivative with respect to w of the projected equation, evaluated at a given $\sigma(\varphi_1 + w) \in H^r$, we are led to solve

$$(4.1) \qquad \begin{aligned} g &= dP\mathcal{F}(\sigma(\varphi_1 + w), m)\cdot v \\ &= v_{tt} - \Delta v + \alpha v_t - mv + P\Big[\sum_{\substack{i,j \\ |i|+|j|\leqslant 2}} a_{ij}^{lm}(x, t) v_{t^i x^j}\Big] \end{aligned}$$

where the coefficients $a_{ij}^{lm}(x, t)$ are;

$$\begin{aligned} a_{20}(x, t) &= F_{u_{tt}}(\sigma(\varphi_1 + w), \ldots) \\ a_{10}(x, t) &= F_{u_t}(\sigma(\varphi_1 + w), \ldots) \\ a_{11}^l(x, t) &= F_{u_{tx_l}}(\sigma(\varphi_1 + w), \ldots) \\ a_{02}^{lm}(x, t) &= F_{u_{x_l x_m}}(\sigma(\varphi_1 + w), \ldots) \\ a_{01}^l(x, t) &= F_{u_{x_l}}(\sigma(\varphi_1 + w), \ldots) \\ a_{00}(x, t) &= F_u(\sigma(\varphi_1 + w), \ldots) . \end{aligned}$$

It must be required that these coefficients and their first and second derivatives be sufficiently small in supremum norm. Since F includes second derivatives of $\sigma(\varphi_1 + w)$, we must be able to control fourth derivatives of w in sup norm. Using the composition of functions inequalities from the appendix,

$$\begin{aligned} \|a_{ij}^{lm}(x, t)\|_2 = \|F_{u_{t^i x^j}}(x, t; \sigma(\varphi_1 + w) \ldots)\|_2 \\ \leqslant \text{const}\ \|\sigma(\varphi_1 + w)\|_4 \\ \leqslant \text{const}\ |\sigma(\varphi_1 + w)|_{H^{[(n+1)/2]+4}} . \end{aligned}$$

Thus in the iteration we must be able to guarantee that $|w|_{H^{[n+1/2]+4}}$ and $|\sigma|$ be sufficiently small. This is achieved by taking r, the order of the high norm, large enough. In the setting of Moser's theorem, r must be so large that $\varrho/r < \lambda/(\lambda + 2)$, where $\varrho > [(n + 1)/2] + 4$.

We will not solve the exact linearized equation, but an approximate one, in which the coefficients and the inhomogeneous right hand side have been smoothed. We also will not approach this directly, but will first solve and derive estimates for the equation with an added artificial viscosity term. These techniques are similar to the method of Rabinowitz [12]. Care is taken to estimate independently of the viscosity coefficient ν. Taking the limit $\nu \to 0$, solutions of the smoothed linearized equation will be obtained.

The modified linear equation is

$$L_\nu v = \nu[v_{t^6} + \Delta v_{ttt}] + v_{tt} - \Delta v + \alpha v_t - mv + P[\sum (a_{ij}^{lm}(x, t))^S v_{t^i x^j}] = g^S$$

where $(a_{ij}^{lm}(x, t))^S$ and g^S are smooth approximations of a_{ij}^{lm} and g. From now on in the linear theory the S will be deleted.

THEOREM 4. *There is a δ such that if $\|a_{ij}^{lm}\|_2 < \delta$, $m - \lambda_1 < \delta$ then, given g*

$$g \perp \varphi_1$$
$$|g|_{L^2} + \left|\frac{\partial}{\partial t} g\right|^2_{L^2} < \infty$$

there exists a unique $v \perp \varphi_1$ such that

$$L_\nu v = g .$$

Furthermore

$$\nu^2 |v_{t^3}|^2_{H^2} + |v|^2_{H^2} \leqslant \text{const}\left[|g|^2_{L^2} + \left|\frac{\partial}{\partial t} g\right|^2_{L^2}\right]. \tag{4.2}$$ □

The proof will use a negative norm argument, but first some estimates are needed.

LEMMA 4.1. *Define the operator $\Lambda_\nu \varphi$ for $\varphi \in C^\infty \cap H_0^1$ to be*

$$\Lambda_\nu \varphi = \nu[\varphi_{t^5} + \Delta\varphi_{t^3}] - \varphi_{ttt} - \frac{2\alpha}{3}\varphi_{tt} - \frac{\alpha}{3}\Delta\varphi + \varphi_t + \frac{\alpha}{3}\varphi .$$

Then if $\|a_{ij}\|_1$ are sufficiently small,

$$\nu^2 |\varphi_{t^3}|^2_{H^2} + \frac{\alpha}{3}|\varphi|^2_{H^2} \leqslant 2(\Lambda_\nu \varphi, L_\nu \varphi) + m\frac{\alpha}{3}|\varphi|^2_{L^2} . \tag{4.3}$$ □

PROOF. One proceeds by taking the inner product $(\Lambda_\nu \varphi, L_\nu \varphi)$ and integrating by parts. Considering each term separately,

$$\text{(4.4)}\quad (1)\quad \left(\nu[\varphi_{t^5} + \Delta\varphi_{t^3}], L_\nu \varphi\right) = \nu^2\Big[|\varphi_{t^5}|^2 + \sum_l |\varphi_{t^4 x_l}|^2 + |\Delta\varphi_{t^3}|^2\Big]$$
$$+ \alpha\nu\Big[|\varphi_{t^3}|^2 + \sum_l |\varphi_{t^2 x_l}|^2\Big] + \nu\left([\varphi_{t^5} + \Delta\varphi_{t^3}], P\sum a_{ij}\varphi_{t^i x^j}\right) .$$

The inner product on the right is controlled by

$$\nu\left([\varphi_{t^5} + \Delta\varphi_{t^3}], P\sum a_{ij}\varphi_{t^i x^j}\right) \leqslant \frac{\nu^2}{2}|\varphi_{t^3}|^2_{H^2} + \frac{1}{2}\|a_{ij}\|^2_0 |\varphi|^2_{H^2} .$$

The next term,

$$\text{(4.4)}\quad (2)\qquad (-\varphi_{tt}, L_\nu\varphi) = -|\varphi_{tt}|^2 + \sum_l |\varphi_{x_l t}|^2 - m|\varphi_t|^2 - \left(\varphi_{tt}, P\sum a_{ij}\varphi_{t^i x^j}\right)$$

where terms such as $(\varphi_{t^5}, \varphi_{tt})$ have vanished because φ is τ-periodic in time. The remaining inner product is estimated by

$$\left(-\varphi_{tt}, P\sum a_{ij}\varphi_{t^i x^j}\right) \leqslant \|a_{ij}\|_0 |\varphi_{tt}| |\varphi_{t^i x^j}| .$$

The following term

$$(4.4)\quad (3)\quad (-\Delta\varphi, L_\nu\varphi) = -\sum_l |\varphi_{x_l t}|^2 + |\Delta\varphi|^2 - m\sum_l |\varphi_{x_l}|^2 + (-\Delta\varphi, P\sum a_{ij}\varphi_{t^i x^j})$$

where again the remaining inner product is controlled by $\|a_{ij}\|_0|\varphi|^2_{H^2}$. Next

$$(4.4)\quad (4)\qquad \left(\varphi_t + \frac{\alpha}{3}\varphi, L_\nu\varphi\right) = \nu\left[|\varphi_{t^3}|^2 + \sum_l |\varphi_{x_l t}|^2\right] + \frac{2\alpha}{3}|\varphi_t|^2$$

$$+\frac{\alpha}{3}\sum|\varphi_{x_l}|^2 - \alpha\frac{m}{3}|\varphi|^2 + \left(\varphi_t + \frac{\alpha}{3}\varphi, P\sum a_{ij}\varphi_{t^i x^j}\right)$$

where the remaining inner product is certainly bounded by $\|a_{ij}\|_0|\varphi|^2_{H^2}$.

The hardest term has been saved for last.

$$(4.4)\quad (5)\ (-\varphi_{ttt}, L_\nu\varphi) = \nu\left[|\varphi_{t^4}|^2 + \sum_l |\varphi_{x_l t^3}|^2\right] + \alpha|\varphi_{tt}|^2 + (-\varphi_{ttt}, P\sum a_{ij}(x,t)\varphi_{t^i x^j}).$$

The remaining inner products are handled by throwing one of the derivatives onto the variable coefficient. The hardest term is

$$\begin{aligned}
&(-\varphi_{ttt}, Pa_{02}^{lm}(x,t)\varphi_{x_l x_m})\\
&\quad = \left(\varphi_{tt}, \frac{\partial}{\partial t}a_{02}^{lm}\varphi_{x_l x_m}\right) + (\varphi_{tt}, a_{02}^{lm}\varphi_{x_l x_m t})\\
&\quad = \left(\varphi_{tt}, \frac{\partial}{\partial t}a_{02}^{lm}\varphi_{x_l x_m}\right) - \left(\varphi_{tt}, \frac{\partial}{\partial x_l}a_{00}^{lm}\varphi_{x_m t}\right)\\
&\quad = \left(\varphi_{tt}, \frac{\partial}{\partial t}a_{02}^{lm}\varphi_{x_l x_m}\right) - \left(\varphi_{tt}, \frac{\partial}{\partial x_l}a_{02}^{lm}\varphi_{x_m t}\right) + \frac{1}{2}\int_0^\tau\int_\Omega \frac{\partial}{\partial t}a_{02}^{lm}(x,t)|\varphi_{x_l t}|^2\,dx\,dt\\
&\quad \leqslant \text{const}\,\|a_{ij}\|_1|\varphi|^2_{H^2}
\end{aligned}$$

where it has been used that $a_{02}^{lm}(x,t) = a_{02}^{ml}(x,t)$.

The remaining terms are similarly handled.

Summing (4.4) (1) through (5) we find

$$\begin{aligned}
&\nu^2[|\varphi_{t^5}|^2 + |\varphi_{x_l t^4}|^2 + |\varphi_{x_l x_m t^3}|^2] + \frac{\alpha}{3}[|\varphi_{tt}|^2 + |\varphi_{x_l t}|^2 + |\varphi_{x_l x_m}|^2]\\
&\qquad\qquad - m\frac{\alpha}{3}|\varphi|^2 - \text{const}\,\|a_{ij}^{lm}\|_1|\varphi|^2_{H^2}\\
&\quad \leqslant 2(\Lambda_\nu\varphi, L_\nu\varphi)\,.
\end{aligned}$$

If $\|a_{ij}\|_1$ is sufficiently small, the lemma follows. □

LEMMA 4.2. *Define the operator $\Lambda\varphi$ for $\varphi \in C^\infty \cap H_0^1$ to be*

$$\Lambda\varphi = -\varphi_t + \frac{\alpha}{2}\varphi\,.$$

Then if $\|a_{ij}(x, t)\|_2$ is small enough, we have the inequality:

$$\nu\Big[|\varphi_{t^3}|^2 + \sum_l |\varphi_{t^2 x_l}|^2\Big] + \frac{\alpha}{2}\Big[|\varphi_t|^2 + \sum_l |\varphi_{x_l}|^2\Big] \tag{4.5}$$
$$\leqslant 2(\Lambda\varphi, L_\nu^*\varphi) + \frac{\alpha}{2} m|\varphi|^2\,. \qquad \square$$

PROOF.

$$L_\nu^*\varphi = -\nu[\varphi_{t^5} + \Delta\varphi_{t^5}] + \varphi_{tt} - \Delta\varphi - \alpha\varphi_t - m\varphi + \sum (-1)^{|i|+|j|} D_x^i D_t^i [a_{ij}(x, t) P\varphi].$$

The result follows by performing computations similar to the proof of the preceeding lemma, and using explicitly that $P\varphi = \varphi - \varphi_1 \int_0^\tau \int_\Omega \varphi_1(y)\varphi(y, s)\,dy\,ds$. Two derivatives of the coefficients a_{ij} must be sufficiently small, since they appear twice differentiated as coefficients of lower order terms in the adjoint. $\square$

LEMMA 4.3 [Poincaré Inequality]. *If $\varphi(x, t) \perp \varphi_1$, then*

$$|\varphi|^2 \leqslant \frac{1}{\lambda_2} \sum_l |\varphi_{x_l}|^2 + \frac{\tau^2}{4\pi^2} |\varphi_t|^2\,. \tag{4.6}$$ $\square$

PROOF. Expanding φ in terms of eigenfunctions

$$\varphi = \sum_{\substack{k>0\\ \xi\in\mathbf{Z}}} \alpha_{k\xi}\varphi_k(x) \exp\left[i\frac{2\pi}{\tau}\xi t\right]$$

$$|\varphi^2 = \sum_{\substack{k>0\\ \xi\in\mathbf{Z}}} |\alpha_{k\xi}|^2$$

$$|\varphi_t|^2 = \sum_{\substack{k>0\\ \xi\in\mathbf{Z}}} \left(\frac{2\pi}{\pi}\xi\right)^2 |\alpha_{k\xi}{}^2|$$

$$\sum_l |\varphi_{x_l}|^2 = (\varphi, -\Delta\varphi) = \sum_{\substack{k>0\\ \xi\in\mathbf{Z}}} \lambda_k |\alpha_{k\xi}|^2\,.$$

Since $\varphi \perp \varphi_1$,

$$\varphi = \sum_{\substack{\xi=0\\ k>1}} \alpha_{k0}\varphi_k(x) + \sum_{\substack{\xi\neq 0\\ k\geqslant 1}} \alpha_{k\xi}\varphi_k(x) \exp\left[i\frac{2\pi}{\tau}\xi t\right],$$

so that

$$\Big|\sum_{\substack{\xi=0\\k>1}}\alpha_{k0}\varphi_k(x)\Big|^2\leqslant\frac{1}{\lambda_2}\sum_{k>1}\lambda_k|\alpha_{k0}|^2$$

$$\Big|\sum_{\substack{\xi\neq0\\k\geqslant1}}\alpha_{k\xi}\varphi_k(x)\exp\left[i\frac{2\pi}{\tau}\xi t\right]\Big|^2\leqslant\frac{\tau^2}{4\pi^2}\sum_{\substack{k\geqslant1\\\xi\neq0}}\left(\frac{2\pi}{\tau}\xi\right)^2|\alpha_{k\xi}|^2$$

and we are done. □

It is now possible to improve lemma 4.1 to be able to obtain an existence theorem for solutions of the projected linear equation. Redefine the operator

$$\Lambda_\nu\varphi=\nu[\varphi_{t^5}+\Delta\varphi_{t^3}]-\varphi_{tt}-\frac{2}{3}\alpha\varphi_{ttt}-\frac{\alpha}{3}\Delta\varphi+q\varphi_t+p\alpha\varphi\,.$$

LEMMA 4.4. *Suppose that $\varphi\in C^\infty\cap H^0_1$, and that $\varphi\perp\varphi_1$. If $m\in(-\infty,\lambda_2)$ and if $\|a_{ij}\|_2$ are sufficiently small, we can find p and q such that*

$$\nu^2|\varphi_{ttt}|^2_{H^2}+\frac{\alpha}{3}\Big[|\varphi_{tt}|^2+\sum_l|\varphi_{tx_l}|^2+|\Delta\varphi|^2\Big]\leqslant2(\Lambda_\nu\varphi,L_\nu\varphi)\,. \tag{4.7}$$

PROOF. Inspect the proof of Lemma 1 more carefully.

$$\begin{aligned}(\Lambda_\nu\varphi,L_\nu\varphi)\geqslant\frac{\nu^2}{2}|\varphi_{t^3}|^2_{H^2}+\frac{\alpha}{3}\big[|\varphi_{tt}|^2+\sum_l|\varphi_{tx_l}|^2+|\Delta\varphi|^2\big]&\\ +\alpha\left(q-p-\frac{2m}{3}\right)|\varphi_t|^2+\alpha\left[p-\frac{m}{3}\right]\sum_l|\varphi_{x_l}|^2&\\ -mp\alpha|\varphi|^2-\text{const}\,\|a_{ij}\|_1|\varphi|^2_{H^2}\,.&\end{aligned}$$

Using the Poincaré inequality to bound the term $-mp\alpha|\varphi|^2$, we must be able to chose p,q such that

$$\frac{q-p-2m/3}{mp}\geqslant\frac{\tau^2}{4\pi^2}\qquad\text{and}\qquad\frac{p-m/3}{mp}\geqslant\frac{1}{\lambda_2}\,.$$

Taking $(\lambda_2-m)p>m/3$ and then $q/mp\geqslant\tau^2/4\pi^2+(p-2m/3)/mp$ and we are done. If m is bounded away from λ_2 then $p(m)$ and $q(m)$ are bounded. □

LEMMA 4.5. *If $\|a_{ij}\|_2$ are sufficiently small, and $|g|^2_{L^2}+|(\partial/\partial t)g|^2_{L^2}<\infty$, then there exists a $v\perp\varphi_1$, $v|_{\partial\Omega}=0$ such that*

$$(L_\nu v,\varphi)=(g,\varphi)\qquad\textit{for all }\varphi\in C^\infty\cap H^1_0\,,\quad\varphi\perp\varphi_1\,.$$

Furthermore v is unique, and

$$\nu^2|v_{ttt}|^2_{H^2}+\frac{\alpha}{3}|v|^2_{H^2}\leq \text{const}\left[|g|^2_{L^2}+\left|\frac{\partial}{\partial t}g\right|^2_{L^2}\right]. \qquad \square$$

COROLLARY [Theorem 4]. *If $g\perp\varphi_1$, then $L_\nu v=g$.* $\square$

PROOF. The proof uses a negative norm argument. Define the norms

$$|\varphi|^2_E=\nu^2|\varphi_{ttt}|^2_{H^2}+\frac{\alpha}{3}|\varphi|^2_{H^2}$$

$$|\varphi|^2_K=|\varphi|^2_{L^2}+\left|\frac{\partial}{\partial t}\varphi\right|^2_{L^2}.$$

Define the space E to be the completion of $\{\varphi\in C^\infty\cap H^1_0;\ \varphi\perp\varphi_1\}$ with respect to the E norm, define K to be the completion of C^∞ with respect to the K norm. These are both Hilbert spaces. Let E^* and K^* be the respective negative norm dual spaces, dual with respect to the L^2 inner-product. For each $\varphi\in C^\infty\cap H^1_0$, $\varphi\perp\varphi_1$ define $\psi=L^*_\nu\varphi$. Of course $\psi\in E^*$, since for any $\theta\in E$

$$|(\psi,\theta)|=|(L^*_\nu\varphi,\theta)|=|(\varphi,L_\nu\theta)|$$

$$\leq[|\varphi|+|\varphi_t|]|\theta|_E\,.$$

We know that ψ is well defined from Lemma 4.2 and the Poincaré inequality.

Now define a linear function $l(\psi)\equiv(\varphi,g)$.

$$|l(\psi)|\leq|g|_K|\varphi|_{K^*}$$

LEMMA 4.6. *For $\varphi\in L^2$ there exists a solution $\theta\in E$ of the equation $\Lambda_\nu\theta=\varphi$. If $\varphi\in C^\infty$ then so is θ.* $\square$

PROOF. θ can be constructed for example by an eigenfunction expansion. If

$$\theta=\sum_{\xi,k}\theta_{\xi k}\varphi_k(x)\exp\left[i\frac{2\pi}{\tau}\xi t\right]$$

$$\begin{aligned}\Lambda_\nu\theta&=\sum_{\xi,k}\left[\frac{\alpha}{3}(2\xi^2+\lambda_k)+\alpha p+i[\nu(\xi^5+\xi^3\lambda_k)+\xi^3+q\xi]\right]\cdot\theta_{\xi k}\varphi_k(x)\exp\left[i\frac{2\pi}{\tau}\xi t\right]\\&=\sum_{\xi,k}\varphi_{\xi k}\varphi_k(x)\exp\left[i\frac{2\pi}{\tau}\xi t\right].\end{aligned}$$

Notice that both

$$\frac{\alpha}{3}(2\xi^2+\lambda_k)+\alpha p \geqslant \text{const}\,[|\xi|^2+|\lambda_k|]$$

and

$$\nu[\xi^5+\xi^3\lambda_k]+\xi^3+q\xi \geqslant \text{const}\,|\xi|^3[|\xi|^2+\lambda_k]$$

so that one may divide. Notice that if $\varphi \perp \varphi_1$ then so is θ. □

Now we have

$$\begin{aligned}|\varphi|_{K^*} &= \sup_{w\in K}\frac{|(\varphi, w)|}{|w|_K}=\sup_{w\in K}\frac{1}{|w|_K}|(\Lambda_\nu\theta, w)|\\ &\leqslant \text{const}\sup_{w\in K}\frac{1}{|w|_K}\left[\nu|\theta_{ttt}|_{H^2}|w|+\frac{\alpha}{3}|\theta|_{H^2}|w|+\frac{\alpha}{3}|\theta_{tt}||w_t|\right]\\ &\leqslant \text{const}\,|\theta|_E.\end{aligned}$$

Recalling that $\theta \in C^\infty \cap H_0^1$, $\theta \perp \varphi_1$ we know

$$\begin{aligned}|\theta|_E &= \nu^2|\theta_{ttt}|_{H^2}^2+\frac{\alpha}{3}|\theta|_{H^2}^2 \leqslant 2(\Lambda_\nu\theta, L_\nu\theta)\\ &= 2(\varphi, L_\nu\theta)=2(\psi,\theta)\leqslant 2|\psi|_{E^*}|\theta|_E\,.\end{aligned}$$

Hence

$$|l(\psi)|\leqslant |g|_K|\varphi|_{K^*}\leqslant \text{const}\,|g|_K|\theta|_E \leqslant \text{const}\,|g|_K|\psi|_{E^*}\,.$$

In other words, l is a continuous linear functional on a subspace of E^*, whose norm is bounded by const $|g|_K$. Extend l by Hahn-Banach to all of E^*. By a representation theorem, there is a $v \in E$ such that for all $\psi \in E^*$

$$l(\psi)=(v, \psi)$$

and $|v|_E \leqslant \text{const}\,|g|_K$. Whenever $\varphi \in C^\infty \cap H_0^1 \subseteq E^*$, $\varphi \perp \varphi_1$,

$$(\varphi, g)=(L_\nu^*\varphi, v)\,.$$

Since $v \in E$ we may integrate by parts to obtain

$$(\varphi, g)=(\varphi, L_\nu v)$$

$$\nu^2|v_{ttt}|_{H^2}^2+\frac{\alpha}{3}|v|_{H^2}^2 \leqslant \text{const}\left[|g|^2+\left|\frac{\partial}{\partial t}g\right|^2\right]$$

and we are done.

If $g \perp \varphi_1(x)$ the corollary follows easily. □

5. – Technical lemmata.

It is necessary to obtain good control in Sobolev norms of the solutions of the inhomogeneous linear equations. This chapter contains three lemmata used to obtain estimates of the higher derivatives of these solutions in terms of the inhomogeneous part and the coefficients. Methods are similar to ones used in elliptic problems; using cutoff functions and limits of difference quotients to obtain bounds on higher derivatives either interior to Ω or in tangential directions near the boundary. Time derivatives are easier to treat, since the condition of time periodicity allows one to integrate by parts freely. As usual, normal derivatives at the boundary are estimated using the equation. Higher regularity is demonstrated not for the original equation, but for one in which an artificial viscosity term has been added. The inviscid limit is our goal, so care is taken to obtain bounds independent of the coefficient of viscosity.

Gaining one more time derivative.

LEMMA 5.1. *If* (i) $L_\nu u = G,\ u \in H_0^1$

(ii) $\nu^2 |u_{t^3}|^2_{H^2} + |u|^2_{H^2} < \text{const}\,[|G|^2 + |G_t|^2]$

(iii) $\|a_{ij}(x, t)\|_2$ *are sufficiently small*

(iv) $|G_{tt}|^2 < \infty$ *as well.*

Then

$$\nu^2 |u_{t^4}|^2_{H^2} + |u_t|^2_{H^2} \leqslant \text{const}\,[|G|^2 + |G_t|^2 + |G_{tt}|^2]\,. \qquad \square \tag{5.1}$$

PROOF. We will take difference quotients in the time direction, using the fact that all functions are time periodic with period τ.

Let

$$u^h = \frac{1}{h}[u(t+h) - u(t)]$$

$$\Lambda_\nu u^h = \nu[u^h_{t^5} + \Delta u^h_{t^3}] - u^h_{ttt} - \frac{2\alpha}{3} u^h_{tt} - \frac{\alpha}{3}\Delta u^h$$

$$(-[\Lambda_\nu u^h]^{-h}, L_\nu u) = (\Lambda_\nu u^h, G^h)\,.$$

Computing the left hand side,

$$\begin{aligned}(\Lambda_\nu u^h, [L_\nu u]^h) = {} & \nu_\varepsilon^2\Big[|u^h_{t^5}|^2 + \sum_l |u^h_{t^4 x_l}|^2 + \Delta u^h_{t^3}|^2\Big] + \nu\Big[|u^h_{t^4}|^2 + \sum_l |u^h_{x_l t^3}|^2\Big] \\ & + \alpha\nu\Big[|u^h_{t^3}|^2 + \sum_l |u^h_{t^2 x_l}|^2\Big] + \alpha/3\,\Big[|u^h_{tt}|^2 + \sum_l |u^h_{t x_l}|^2 + |\Delta u^h|^2\Big] \\ & - \alpha/3\, m\Big[2|u^h_t|^2 + \sum_l |u^h_{x_l}|^2\Big] + (\Lambda_\nu u^h, [P a_{ij}(x, t)\, u_{t^i x^j}]^h)\,.\end{aligned} \tag{5.2}$$

The inner product that remains must be shown to be small. Term by term this can be done in a manner similar to the method of proof of Lemma 4.1, to achieve

$$|(\Lambda_\nu u^h, [Pa_{ij}(x,t)u_{t^ix^j}]^h)| \leqslant \text{const } \|(a_{ij})^h\|_1 |u^h|_{H^2}|u|_{H^2} + \text{const } \|a_{ij}\|_1 |u^h|^2_{H^2}.$$

Assuming that the coefficients $\|a_{ij}\|_2$ are small, and estimating

$$|(\Lambda_\nu u^h, G^h)| \leqslant \frac{\nu^2}{2}\Big[|u^h_{t^5}|^2 + \sum_l |u^h_{t^4x_l}|^2 + |\Delta u^h_{t^3}|^2\Big] + \frac{\alpha}{6}|u^h|^2_{H^2} + \text{const}\,[|G^h|^2 + |G^h_t|^2],$$

we find

$$\nu^2|u^h_{ttt}|^2_{H^2} + \frac{\alpha}{3}|u^h|^2_{H^2} - \text{const } \|(a_{ij})^h\|_1|u^h|_{H^2}|u|_{H^2} - \text{const}\|a_{ij}\|_1|u^h|^2_{H^2}$$
$$\leqslant \text{const}\Big[|u|^2_{H^2} + |u^h_t|^2 + \sum_l |u^h_{x_l}|^2 + |G^h|^2 + |G^h_t|^2\Big].$$

By the usual limiting process we find that if const $\|(a_{ij})^h\|_1 < \alpha/6$

$$\nu^2|u_{t^4}|^2_{H^2} + \frac{\alpha}{3}|u_t|^2_{H^2} - \text{const } \|a_{ij}\|_1|u_t|^2_{H^2} \leqslant \text{const}\,[|G|^2 + |G_t|^2 + |G_{tt}|^2]\,. \tag{5.3}$$

If $\|a_{ij}\|_1$ is now sufficiently small, we have the result. □

More interior and tangential x-derivatives.

Let $\eta(x)$ be a cutoff function for an open set Ω' interior to Ω, or let it isolate a neighborhood of a straightened section of the boundary. In the straightened coordinates, let the new expression for the Laplacian be

$$b_{lm}(x)D_{x_lx_m} + b_l(x)D_{x_l}\,.$$

For simplicity let us still denote our variable coefficients which arise from linearizing $F(u)$ by $a^{lm}_{ij}(x,t)D^i_tD^i_x$.

$$L_\nu u = \nu[u_{t^5} + b_{lm}(x)u_{x_lx_mt^3} + b_l(x)u_{x_lt^3}]$$
$$+ u_{tt} - b_{lm}(x)u_{x_lx_m} - b_l(x)u_{x_l} + \alpha u_t - mu + P[a^{ij}_{lm}(x,t)u_{t^ix^j}]\,. \tag{5.4}$$

Let

$$\Lambda_\nu = \nu[u_{t^5} + b_{lm}(x)u_{x_lx_mt^3} + b_l(x)u_{x_lt^3}] - u_{ttt} - \frac{2\alpha}{3}u_{tt} - \frac{\alpha}{3}[b_{lm}(x)_{x_lx_m} + b_l(x)u_{x_l}]\,.$$

For h so small that $\operatorname{dist}(\Omega', \partial\Omega) > h$, or in the case of derivatives D_{x_k} in coordinate directions e_k tangential to the boundary, for h so small that $\operatorname{supp}(\eta(x + he_k)) \subseteq \Omega$, we form the difference quotients

$$\eta^2(x)u^h = \eta^2(x)\frac{1}{h}[u(x + he_k, t) - u(x, t)].$$

We will obtain higher derivatives by integrating the differential equation against $-[\eta^2(x)\Lambda_\nu u^h]^{-h}$.

LEMMA 5.2. *If $L_\nu u = G$, such that*

(i) $\nu^2|u_{t^3}|^2_{H^2} + \frac{\alpha}{3}|u|^2_{H^2} < \text{const}\,[|G|^2 + |G_t|^2]$

(ii) $\|a_{ij}\|_2$ *are sufficiently small*

(iii) $|G_{x_k}|^2 + |G_{tx_k}|^2 < \infty$

(iv) $\nu^2|u_{t^4}|^2_{H^2} + \frac{\alpha}{3}|u_t|^2_{H^2} \leqslant \text{const}\,[|G|^2 + |G_t|^2 + |G_{tt}|^2]$

(v) *in the tangential case, if* $u|_{\partial\Omega \cap \operatorname{supp}\eta(x)} = 0$.

Then

$$(5.5) \qquad \nu^2|\eta u_{x_k t^3}|^2_{H^2} + \frac{\alpha}{3}|\eta u_{x_k}|^2_{H^2}$$
$$\leqslant \text{const}\,[|G|^2 + |G_t|^2 + |G_{x_k}|^2 + |G_{x_k}|^2 + |G_{tt}|^2]. \qquad \square$$

COMMENTS. The highest order part of L_ν has the form of an elliptic operator in space and time νEu. However, we are working independently of ν; νEu_{t^3} is used as a smoothing, so that the $-u_{ttt}$ term in $\Lambda_\nu u$ can be employed. Roughly, one must be able to take three derivatives in time in order to get an estimate on second derivatives. The major difference between these lemmata and interior estimates for elliptic operators is that this $-u_{ttt}$ term appears, and must be controlled independently of ν. Assumption (iv), which is basically the conclusion of Lemma 5.1 does this for us.

PROOF.

$$(-[\eta^2(x)\Lambda_\nu u^h]^{-h}, G) = (\eta^2(x)\Lambda_\nu u^h, [L_\nu u]^h).$$

Integrating by parts, similarly to the proof of Lemma 5.1,

$$
\begin{aligned}
(5.6)\quad (a) = {} & \nu^2\Big[|\eta u^h_{t^5}|^2 + \Big(\eta^2 \sum_{l,m} b_{lm} u^h_{x_l t^4},\, u^h_{x_m t^4}\Big)\Big] \\
& + \nu\Big[|\eta u^h_{t^4}|^3 + \alpha|\eta u^h_{t^3}|^2 + \Big(\eta^2 \sum_{l,m} b_{lm} u^h_{x_l t^3},\, u^h_{x_m t^3}\Big)\Big] \\
& + \frac{\alpha}{3}\Big[|\eta u^h_{tt}|^2 + \Big|\eta \sum_{l,m} b_{lm} u^h_{x_l x_m}\Big|^2 + \Big(\eta^2 \sum_{l,m} b_{lm} u^h_{x_l t},\, u^h_{x_m t}\Big)\Big] \\
& - \frac{\alpha}{3} m \Big[2|\eta u^h_t|^2 + \Big(\eta^2 \sum_{l,m} b_{lm} u^h_{x_l x_m},\, u^h\Big)\Big]
\end{aligned}
$$

$$
(5.6)\quad (b)\ + (\eta^2 \Lambda_\nu u^h, [P a_{ij} u_{t^t x^j}]^h)
$$

$$
\begin{aligned}
(5.6)\quad (c)\ & + \nu^2[(\eta^2 B u^h_{t^3}, D^2 u_{t^3}) + (\eta C u^h_{t^3}, D^2 u^h_{t^3})] \\
& + \nu\Big[(\eta^2 E u^h_{t^3}, D^2 u^h) + (\eta^2 F u_{t^3}, D^2 u^h) + \Big(\sum_{l,m} [\eta^2 b_{lm}]_{x_m} u^h_{t^3},\, u^h_{t^3 x_m}\Big) \\
& + \Big(\eta^2 \sum_{l,m} b_{lm} u_{t^3 x_l x_m},\, u^h_{t^3}\Big)\Big] + (\eta G u^h_t, D^2 u^h) + (\eta^2 H u, D^2 u^h) \\
& + \Big(\sum_{l,m} [\eta^2 b_{lm}]_{x_l} u^h_{tt},\, u^h_{t x_m}\Big) + \Big(\eta^2 u^h_{tt},\, \sum b^h_{lm} u_{x_l x_m t}\Big)\,.
\end{aligned}
$$

B, F and H are second order operators arising when a difference quotient has fallen on a coefficient, while C, E and G are first order operators arising when a derivative has fallen on a coefficient.

Bounding (5.6) (b) by integrations by parts similar to the proof of Lemma 4.1, and throwing the leftovers (5.6) (c) onto the right hand side, the limit as $h \to 0$ gives us the estimate

$$
\nu^2 |\eta u_{t^3 x_k}|^2_{H^2} + \frac{\alpha}{3} |\eta u_{x_k}|^2_{H^2} - R\Big[\|a_{ij}\|_1 + \Big\|\frac{\partial}{\partial x_k} a_i\Big\|_1\Big] |\eta u_{x_k}|^2_{H^2} \\
\leqslant \text{const}\, [|G|^2 + |G_{x_k}|^2 + |G_t|^2 + |G_{x_k t}|^2 + |G_{tt}|^2]
$$

where R is a constant independent of $\eta(x)$ and $\nabla\eta(x)$. For $\|a_{ij}\|_2$ sufficiently small this finishes the proof of the lemma. □

Normal derivatives at the boundary.

Let us suppose that in a neighborhood of a straightened section of the boundary we have coordinates $(t, x_1, \ldots, x_{n-1}, y)$, with y the normal direction.

Take as before the Laplacian in these new coordinates

$$\sum_{l,m} b_{lm}(x) D_{x_l} D_{x_m} + b_l(x) D_{x_l} .$$

Rewriting $L_\nu u = G$,

$$\text{(5.7)} \qquad \nu b_{nn}(x) u_{yyt^3} - b_{nn}(x) u_{yy} + Pa_{02}^{nn}(x, t) u_{yy}$$
$$= -\nu \Big[u_{t^5} + \sum_{(l,m)\neq(n,n)} b_{lm} u_{x_l x_m t^3} + b_l(x) u_{x_l t^3} \Big] - u_{tt} + \sum_{(l,m)\neq(n,n)} b_{lm}(x) u_{x_l x_m} + b_l(x) u_{x_l}$$
$$- \alpha u_t + mu - Pa_{ij} u_{t^i x^j} + G .$$

Suppose that by interior regularity we already know that the y derivative of any of the above terms exists. Take the y derivative, and integrate against $-\eta^2 u_{yyy}$, to find

$$(\eta^2 b_{nn} u_{y^3}, u_{y^3}) - (Pa_{02}^{nn} u_{y^3}, \eta^2 u_{y^3})$$
$$= \nu \Big(\frac{\partial}{\partial y} b_{nn} u_{y^2 t^3}, \eta^2 u_{y^3} \Big) - \Big(\frac{\partial}{\partial y} \varphi_1, \eta^2 u_{y^3} \Big) \int_0^\tau \int_\Omega \varphi_1 a_{02}^{nn} u_{yy} \, dx \, dt$$
$$+ \Big(\frac{\partial}{\partial y} [\text{right hand side of (5.7)}], \eta^2 u_{y^3} \Big) + (G_y, \eta^2 u_{y^3}) .$$

Remark that the « [*right hand side of* (5.7)] » contains at most one derivative of u with respect to y. The usual estimates prove the following lemma.

LEMMA 5.3. *If the hypotheses of lemmas* 5.1 *and* 5.2 *hold for all* x_l, $l \neq n$, *and if* $|G_y| < \infty$, *then*

$$\text{(5.8)} \quad |\eta u_{y^3}|^2 \leqslant \text{const} \Big[|G|^2 + |G_t|^2 + \sum_{l\neq n} |G_{x_l}|^2 + |G_{tt}|^2 + \sum_{l\neq n} |G_{tx_l}|^2 + |G_y|^2 \Big] . \qquad \square$$

Integrating (5.7) against $\eta^2 u_{y^3 t^3}$ instead, we can use the results of Lemma 5.3 to prove that;

LEMMA 5.3 *bis*. *Same hypotheses as Lemma* 5.3. *Then*

$$\text{(5.8) } bis \quad \nu^2 |\eta u_{y^3 t^3}|^2 \leqslant \text{const} \Big[|G|^2 + |G_t|^2 + \sum_t |G_{x_l}|^2 + \sum_{l\neq m} |G_{tx_l}|^2 + |G_{tt}|^2 \Big] . \qquad \square$$

Using the above three lemmata, the main regularity results needed for the nonlinear theorem can be proven.

Induction to obtain higher regularity results.

From the existence theorem we know that if $\|a_{ij}\|_1$ are sufficiently small, then the estimate holds;

$$\nu^2|u_{t^3}|^2_{H^2} + |u|^2_{H^2} \leqslant \text{const}\left[|g|^2_{L^2} + \left|\frac{\partial}{\partial t} g\right|^2_{L^2}\right].$$

Furthermore $u \in H^1_0$, so that one may immediately apply lemmas 5.1, 5.2 and 5.3 to conclude

$$\nu^2|u_{t^3}|^2_{H^3} + |u|^2_{H^3} \leqslant \text{const}\left[|g|^2_{H^1} + \left|\frac{\partial}{\partial t} g\right|^2_{H^1}\right].$$

We now proceed by induction. Suppose that

$$(5.9)\ (r-1) \quad \nu^2|u_{t^3}|^2_{H^{r+1}} + |u|^2_{H^{r+1}} \leqslant \text{const}\left[|g|^2_{H^{r-1}} + \left|\frac{\partial}{\partial t} g\right|^2_{H^{r-1}} + \|a_{ij}\|^2_{r-1} + \left\|\frac{\partial}{\partial t} a_{ij}\right\|^2_{r-1}\right].$$

Time derivatives do not affect the boundary conditions, so take D_t^{r-1} of the equation and use Lemma 5.1. Now we can satisfy the hypotheses of Lemma 5.2 when D_t^{r-1} is applied to the equation. Proceed by induction on $|p|$ to take $D_t^{r-|p|-1}D_x^p$ of the equation and applying Lemma 5.2, where D_x^p near the boundary involves tangential derivatives only. Finally apply $D_t^{r-|p|-|q|-1}D_x^pD_y^q$ to the equation and use Lemma 5.3 repeatedly, where induction now is an $|q|$. Collecting all terms, statement (5.9) (r) follows.

The appearance of the sup norms on the right hand side is inconvenient for $r > [(n+1)/2]$. Using Lemma A.5 from the appendix it is possible to do somewhat better. Results are stated in the following theorem.

THEOREM 5. *If $\|a_{ij}\|_2$ are sufficiently small, solutions of the equation $L_\nu u = g$ satisfy the estimates;*

$$(5.10) \quad (1) \quad \nu^2\left|\frac{\partial^3}{\partial t^3} u\right|^2_{H^{r+2}} + |u|^2_{H^{r+2}} \leqslant \text{const}\left[|g|^2_{H^r} + \left|\frac{\partial}{\partial t} g\right|^2_{H^r} + \|a_{ij}\|^2_r + \left\|\frac{\partial}{\partial t} a_{ij}\right\|^2_r\right].$$

If $r > [(n+1)/2] \equiv m$, and $\|D^2u\|_1 < 1$, while the quantity

$$\|a_{ij}\|^2_{m-1} + \left\|\frac{\partial}{\partial t} a_{ij}\right\|^2_{m-1} + |a_{ij}|^2_{m+1}$$

is sufficiently small, then

$$(5.10) \quad (2) \quad \nu^2\left|\frac{\partial^3}{\partial t^3} u\right|^2_{H^{r+2}} + |u|^2_{H^{r+2}} \leqslant \text{const}\left[|g|^2_{H^r} + \left|\frac{\partial}{\partial t} g\right|^2_{H^r} + |a_{ij}|^2_{H^r} + \left|\frac{\partial}{\partial t} a_{ij}\right|^2_{H^r}\right]. \qquad \square$$

6. – Stability.

Denote by $v(x, t)$ a classical solution of the initial value problem

$$(6.1) \qquad 0 = \mathcal{F}(v, m) = v_{tt} - \Delta v + \alpha v_t - mv + F(x, t; v, Dv, D^2 v)$$

$$v(x, T_1) = f_1(x)$$

$$v_t(x, T_1) = f_2(x)$$

$$v(\partial\Omega, t) = 0 .$$

The periodic solution $u(x, t)$ will be called *stable* if there is a δ such that if

$$\sup_{\substack{|\alpha|\leq 2\\ x\in\Omega}} |D_x^\alpha(u(x, T_1) - f_1(x))| < \delta$$

$$\sup_{\substack{|\alpha|\leq 3\\ \in x\Omega}} |D^\alpha(u_t(x, T_1) - f_2(x))| < \delta$$

then it is true that for all $t > T_1$

$$\sup_{\substack{|\alpha|\leq 3\\ x\in\Omega}} |D^\alpha(u(x, t) - v(x, t))| < \text{const}\ \delta .$$

In the case $F = F(x, t; u, Du)$, $m = 0$, the problem of stability of a small periodic solution has been answered in the paper Rabinowitz [12]. In case $F = F(x, t; u, Du, D^2u)$ is fully nonlinear, the general question of stability relies on an existence theory for the initial value problem, a subject not taken up in this paper. However it is straightforward to obtain the following consequences.

LEMMA 6.1. *Let $u(x, t)$ be the periodic solution of Theorem* 1. *There exists a small constant δ such that if $v(x, t)$ satisfies*

$$v_{tt} - \Delta v + \alpha v_t - mv + F(x, t; v, Dv, D^2 v) = 0$$

$$\sup_{\substack{|\alpha|\leq 3\\ (x,t)\in\Omega\times[T_1,T_2]}} |D^\alpha v(x, t)| < \delta$$

$$m < \lambda_1 .$$

Then the difference $w = (u - v)$ *satisfies*

$$\int_\Omega |w_t(t)|^2 + |\nabla w(t)|^2 + |w(t)|^2 dt$$
$$\leqslant \text{const}\cdot\exp[-\gamma(t - T_1)] \int_\Omega |w_t(T_1)|^2 + |\nabla w(T_1)|^2 + |w(T_1)|^2 dx$$

for any $t \in [T_1, T_2]$. *The rate is*

$$\gamma = \frac{\alpha(\lambda_1 - m)}{2(\lambda_1 + 2\alpha)^2}\cdot\frac{(1-\delta)}{(1+\delta)}. \qquad \square$$

An immediate result of this lemma is the following.

THEOREM 6. *If* $v(x, t)$ *is a solution of* (6.1), $m < \lambda_1$, *and*

$$\sup_{\substack{|\alpha|\leqslant 3\\ (x,t)\in\Omega\times\mathbf{R}^+}} |D^\alpha v| \leqslant \delta .$$

Then v *is attracted in* $H^1(\Omega)$ *to* u *with exponential rate* γ *for all time* $t > 0$. $\square$

REMARK. If we demanded that higher derivatives of v be bounded, the result would be true for higher energy norms $H^k(\Omega)$.

PROOF OF LEMMA 6.1. For simplicity assume $F = F(x, t;\, D^2 u)$. $(u - v) = w$ satisfies the following equation

$$w_{tt} - \Delta w + \alpha w_t - m w + a_{ij}(x, t)\, w_{t^i x^j} = 0$$

where $a_{ij}(x, t) = F_{u_{t^i x^j}}$ $(x, t;$ intermediate point). Since $\sup\limits_{\substack{|\alpha|\leqslant 3\\ (x,t)\in\Omega\times[T_1,T_2]}} |D^\alpha v| < \delta$ we can bound the size of the variable coefficients, $\sup\limits_{|\alpha|\leqslant 1} |D^\alpha a_{ij}| < \delta$. Integrating the equation against $\Delta w = w_t - \mu w$ we obtain

$$\text{(6.2)} \qquad \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |w_t|^2 + |\nabla w|^2 + (\alpha\mu - m)|w|^2 + 2\mu w w_t\, dx$$
$$+ \int_\Omega (\alpha - \mu)|w_t|^2 + \mu|\nabla w|^2 - \mu m |w|^2 dx + \int_\Omega a_{ij}(x, t)\, w_{t^i x^j}[w_t + \mu w]\, dx = 0 .$$

Integrating by parts the last term, we compute for example that

$$\int_\Omega a_{02}^{lm}(x,t)\,w_{x_l x_m}[w_t+\mu w]\,dx = -\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega a_{02}^{lm}(x,t)\,w_{x_l}w_{x_m}\,dx$$
$$-\int_\Omega\left(\frac{\partial}{\partial t}a_{02}^{lm}+\mu a_{02}^{lm}\right)w_{x_l}w_{x_m}+\frac{\partial}{\partial x_l}a_{02}^{lm}w_{x_m}[w_t+\mu w]\,dx\,.$$

Including this information in the above quadratic forms (6.2), noticing that the expressions with variable coefficients only involve first derivatives of w, we may write

$$\frac{1}{2}\frac{\partial}{\partial t}Q_1(w,w)+Q_2(w,w)=0\,.$$

If $\sup\limits_{|\alpha|\leqslant 1}|D^\alpha a_{ij}(x,t)|<\delta$, $m<\lambda_1$ and e.g. $\mu=\alpha/4$ both $Q_k(w,w)$ are positive definite, and

$$\frac{\gamma}{2}Q_1(w,w)\leqslant Q_2(w,w)$$

where

$$\gamma=\frac{\alpha}{2}\left(\frac{\lambda_1-m}{\lambda_1+2\alpha^2}\right)\left(\frac{1-\delta}{1+\delta}\right).$$

Integrating the differential inequality

$$\frac{1}{2}\frac{\partial}{\partial t}Q_1(w,w)+\frac{\gamma}{2}Q_1(w,w)\leqslant 0$$

and using the positivity of Q_1, the lemma follows. □

Notice that as $\alpha\to 0$, $\gamma\sim\alpha/4$, ($\delta\to 0$ as well) and as $\alpha\to\infty$, $\gamma\sim(\lambda_1-m)/4\alpha$.

7. – Perturbation results.

When $\mathcal{F}(u,m)$ is linearized at $u=0$, $m=\lambda_1$ there is a one dimensional kernel. In a Newton scheme the method involves linearization at u small but nonzero. Although it is possible to avoid inverting directly the equations

$$d\mathcal{F}(u,m)\cdot v=g$$

it is natural to ask whether for small perturbations u the linear operator $d\mathcal{F}(u, m)$ continues to have a one dimensional eigenspace with small eigenvalue. In fact we have the following result.

THEOREM 7. *Consider the linear equation*

$$L(a)v = v_{tt} - \Delta v + \alpha v_t + a_{ij}(x, t) v_{t^i x^j} = mv . \tag{7.1}$$

For $\|a_{ij}(x, t)\|_2 < \delta$ *there exists an eigenvector* $v(a_{ij})$ *and corresponding eigenvalue* $m(a_{ij})$, *such that*

$$m(0) = \lambda_1 \quad v(0) = \varphi_1 .$$

Furthermore $v(a_{ij})$ *and* $m(a_{ij})$ *are locally Lipschitz functions of* a_{ij}, *and*

$$v(a_{ij}) = \varphi_1 + w(a_{ij})$$

where

$$w \perp \varphi_1 , \quad |w|_{H^2} < \infty . \quad \square$$

When the nonlinear operator is linearized about a given function u, the coefficients are $a_{ij}(x, t) = F_{u_{t^i x^j}}(x, t; u \ldots)$. The theorem states that if $\|u\|_4$ are small, the perturbed operator has a kernel that is Lipschitz in $u \in C^4$.

PROOF. An eigenfunction of (7.1) of the form $\varphi_1 + w(x, t)$ must solve the equation

$$w_{tt} - \Delta w + \alpha w_t - mw + a_{ij}(x, t) w_{t^i x^j} = (m - \lambda_1)\varphi_1 - a_{02}^{lm}(x, t)\varphi_{1 x_l x_m} . \tag{7.2}$$

Projecting this with P

$$w_{tt} - \Delta w + \alpha w_t - mw + Pa_{ij}(x, t) w_{t^i x^j} = - Pa_{02}\varphi_{1 x^j} .$$

Using the linear estimates, this can be solved for $w = w(m, a) \perp \varphi_1$ if the coefficients satisfy $\|a_{ij}\|_2 < \delta$.

The solution admits the estimate

$$|w|^2_{H^2} \leqslant \text{const}\left[|Pa_{02}\varphi_{1x^j}|^2_{L^2} + \left| \frac{\partial}{\partial t}(Pa_{02}\varphi_{1x^j}) \right|^2_{L^2} \right] .$$

Now project (7.2) onto the corange

$$0 = (\lambda_1 - m) - [I - P] a_{ij}(x, t) w_{t^i x^j} - [I - P] a_{02}(x, t)\varphi_{1x^j} . \tag{7.3}$$

Assume for the moment that the right hand side of (7.3) is continuously differentiable with respect to m, and locally Lipschitz with respect to a_{ij}. A trivial solution is given by $(m, a_{ij}) = (\lambda_1, 0)$. The derivative with respect to m at $m = \lambda_1$, $a_{ij} = 0$ is just -1. Hence the implicit function theorem implies a solution $(m(a_{ij}), a_{ij})$ of (7.3) which is locally Lipschitz in a_{ij}. The above regularity results will be demonstrated in the following lemmata. □

LEMMA 7.1. *$w(m, a_{ij})$; $\mathbb{R}\times C^2 \to H^1$ is locally Lipschitz.* □

PROOF. Let $w^k(x, t)$ solve the equations

$$w_{tt} - \Delta w + \alpha w_t - m_k w + Pa_{ij}(x, t) w_{t^i x^j} = -Pa_{02}(x, t)\varphi_{1_{x^j}}$$

$v = (w^1 - w^2)$ then satisfies

$$v_{tt} - \Delta v + \alpha v_t - m_1 v + Pa_{ij}(x, t) v_{t^i x^j} = (m_1 - m_2) w^2 .$$

Applying the linear estimates

$$|v|^2_{H^2} \leqslant \text{const } |m_1 - m_2^2|^2 \left(|w^2|^2_{L^2} + |w^2_t|^2_{L^2}\right) ,$$

where we know that $|w^2|_{H^2}$ has been bounded by $\|a_{ij}(x, t)\|_1$.

Now let $w^k(x, t)$ solve the equations with a_{ij} varying.

$$w_{tt} - \Delta w + \alpha w_t - mw + Pa^k_{ii}(x, t) w_{t^i x^j} = -Pa^k_{02}(x, t)\varphi_{1_{x^j}} .$$

Denoting $v = w^1(x, t) - w^2(x, t)$, it satisfies

$$\begin{aligned} v_{tt} - \Delta v + \alpha v_t - mv + Pa^1_{ij}(x, t) v_{t^i x^j} \\ = -P\left(a^1_{ij}(x, t) - a^2_{ij}(x, t)\right) w^2_{t^i x^j} - P\left(a^1_{02}(x, t) - a^2_{02}(x, t)\right)\varphi_{1_{x^j}} . \end{aligned}$$

The linear estimates tell us that

$$\begin{aligned} |v|^2_{H^1} &\leqslant \text{const } \left[|(a^1_{ij} - a^2_{ij}) w^2_{t^i x^j}|^2_{L^2} + |(a^1_{02} - a^2_{02})\varphi_{1_{x^j}}|^2_{L^2}\right] \\ &\leqslant \text{const } \|a^1_{ij} - a^2_{ij}\|^2 \end{aligned}$$

and the lemma is finished. □

LEMMA 7.2. *For $a_{ij}(x, t)$ fixed, $w(m)$; $\mathbb{R} \to H^2$ has a derivative which is locally Lipschitz.*

PROOF. The derivative with respect to m exists, a.e. Setting $v = \partial w/\partial m$, we find

$$v_{tt} - \Delta v + \alpha v_t - mv + Pa_{ij}(x, t) v_{t^i x^j} = w \, .$$

For $|w|_{H^2}$ bounded we can show v to be Lipschitz as before. □

LEMMA 7.3. *The expression*

$$\int_0^\tau \int_\Omega \varphi_1 a_{ij}(x, t) w_{t^i x^j} \, dx \, dt \tag{7.4}$$

is locally Lipschitz in a_{ij} and has a locally Lipschitz derivative with respect to m.

PROOF. Integrate (7.4) by parts once, to get

$$-\int_0^\tau \int_\Omega \frac{\partial}{\partial x_p} [\varphi_1 a_{ij}(x, t)] \frac{\partial}{\partial x_q} w \, dx \, dt \, .$$

Now the result for $w \in H^1$ will suffice. □

REMARK. The kernel remains one dimensional under the perturbation, since $w(a, m)$ is unique and $|w|_{H^2}^2$ is small for small $\|a_{ij}\|_2$.

REMARK. If more smoothness is required for $a_{ij}(x, t)$, it can be shown that $w(a_{ij})$ and $m(a_{ij})$ are smoother with respect to the perturbation.

Appendix.

In this chapter will be proved well known estimates [9] for composition of functions, and interpolation inequalities in both Sobolev and the supremum norm. We will also define the smoothing operators.

LEMMA A1.

$$|D^\varrho(u^\alpha(x))|_{L^2}^2 \leqslant C|u|_{L^\infty}^{2\alpha-2} |D^\varrho u|_{L^2}^2 \, . \quad \square$$

PROOF.

$$D^\varrho u^\alpha(x) = \sum \prod_{\lambda=0}^{\varrho} (D^\lambda u)^{\alpha_\lambda}$$

where

$$\sum_{\lambda=0}^{\varrho} \alpha_\lambda = \alpha, \qquad \sum_{\alpha=0}^{\varrho} \lambda \alpha_{\alpha_\lambda} = \varrho$$

$$\int \prod_{\lambda=0}^{\varrho} (D^\lambda u)^{2\alpha_\lambda} dx \leqslant \prod_{\lambda=0}^{\varrho} \left[\left[\int |D^\lambda u|^{2\alpha_\lambda p_\lambda} dx\right]^{1/p_\lambda} \leqslant \prod_{\lambda=0}^{\varrho} |D^\lambda u|_{L^{2\varrho/\lambda}}^{2\alpha_\lambda}\right.$$

where

$$p_\lambda = \frac{\varrho}{\lambda \alpha_\lambda}, \qquad \text{so that } \sum \frac{1}{p_\lambda} = 1 .$$

Using the Nirenberg-Gagliardo inequalities

$$\left| \int \prod_{\lambda=0}^{\varrho} (D^\lambda n)^{2\alpha_\lambda} dx \right| \leqslant \prod_{\lambda=0}^{\varrho} C |u|_{L^\infty}^{2\alpha_\lambda(1-\lambda/\varrho)} |D^\varrho u|_{L^2}^{2\alpha_\lambda(\lambda/\varrho)} .$$

Hence

$$|D^\varrho(u^\alpha(x))|_{L^2}^2 \leqslant C |u|_{L^\infty}^{2\alpha-2} |D^\varrho u|_{L^2}^2 . \qquad \square$$

LEMMA A2. *If* $|u|_{L^\infty}$ *is bounded, then*

$$|D^r f(x, u)|_{L^2}^2 \leqslant \sum_{\alpha=0}^{r+1} \sum_{\varrho=1}^{r} C |u|_{L^\infty}^{2\alpha-2} |D^\varrho u|_{L^2}^2 + |D^r f(x, 0)|_{L^2}^2$$

where $|\alpha| \leqslant r$, $|\sigma| \leqslant r$, *and* C *depends on*

$$\sup_x \left| \frac{\partial^\sigma}{\partial x^\sigma} \frac{\partial^\alpha}{\partial u^\alpha} f(x, 0) \right|_{L^\infty}^2$$

and

$$\sup_{|s| \leqslant |u|_{L^\infty}} \left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^{r+1}}{\partial u^{r+1}} f(x, s) \right|_{L^\infty}^2 . \qquad \square$$

PROOF. Write $f(x, u)$ in its Taylor series in u up to r terms.

$$f(x, u) = \sum_{\alpha=0}^{r} \frac{\partial^\alpha f(x, 0)}{\partial u^\alpha} \frac{u^\alpha(x)}{\alpha!} + \int_0^u \frac{(u-s)^r}{r!} \frac{\partial^{r+1} f(x, s)}{\partial u^{r+1}} ds$$

$$D^r f(x, u) = \sum_{\alpha=0}^{r} \sum_{\sigma+\varrho=r} \left[\frac{\partial^\sigma}{\partial x^\sigma} \frac{\partial^\alpha}{\partial u^\alpha} f(x, 0)\right] \cdot \frac{\partial^\varrho}{\partial x^\varrho} \left[\frac{u^\alpha(x)}{\alpha!}\right]$$
$$+ \sum_{\sigma+\varrho=r} \int \frac{\partial^\varrho}{\partial x^\varrho} \left[\frac{(u-s)^r}{r!}\right] \cdot \left[\frac{\partial^\sigma}{\partial x^\sigma} \frac{\partial^{r+1} f(x, s)}{\partial u^{r+1}}\right] ds$$

$$|D^r f(x, u)|^2_{L^2} \leqslant \sum_{\alpha=0}^{r+1} \sum_{\varrho=1}^{r} C|u|^{2\alpha-2}_{L^\infty} |D^\varrho u|^2_{L^2} + |D^r f(x, 0)|^2_{L^2}$$

by Lemma 1. □

COROLLARY. *If* $(\partial/\partial u) f(x, 0) = 0$, $f(x, 0) = 0$, *then*

$$|D^r f(x, u)|^2_{L^2} \leqslant C \sum_{\alpha=2}^{r+1} \sum_{\varrho=1}^{r} |u|^{2\alpha-2}_{L^\infty} |D^\varrho u|^2_{L^2}. \quad \square$$

LEMMA A3. (i) $\|D^\lambda u\|_0 \leqslant C\|u\|_0^{1-\lambda/r}[\|u\|_0 + \|D^r u\|_0]^{\lambda/r}$

(ii) $\|D^\varrho[u^\alpha(x)]\|_0 \leqslant C \sum_{\alpha=0}^{\varrho} \|u\|_0^{\alpha-1}[\|u\|_0 + \|D^\varrho u\|_0]$.

PROOF. (i) Is a classical interpolation inequality.

(ii) Follows from (i) just as in Lemma 1. □

LEMMA A4.

$$\|D^r f(x, u)\|_0 \leqslant \sum_{\substack{\alpha=1, r+1 \\ \varrho=1, r}} C\|u\|_0^{\alpha-1}[\|u\|_0 + \|D^\varrho u\|_0] + \|D^r f(x, 0)\|_0$$

if $\|u\|_0$ *is bounded.* □

PROOF. Expand f in a Taylor series in u up to the r-th term, and then interpolate as in Lemma 2. □

COROLLARY. *If* $f(x, 0) = (\partial/\partial f) f(x, 0) = 0$, *then*

$$\|D^r f(x, u)\|_0 \leqslant C \sum_{\alpha=2}^{r+1} \sum_{\varrho=1}^{r} \|u\|_0^{\alpha-1}[\|u\|_0 + \|D^\varrho u\|_0]. \quad \square$$

LEMMA A.5. *For a, u sufficiently differentiable that all the following norms are bounded, then for $r \geqslant m$,*

(i) $\left|\sum_{k=0}^{r-m} D^{r-k-1} a \cdot D^k u\right|_{L^2} \leqslant C_{r-m}[|D^{r-1} a|_{L^2}\|u\|_0 + |D^{m-1} a|_{L^2}\|D^{r-m} u\|_0\}$

(ii) $\left|\sum_{k=r-m}^{r-1} D^{r-k-1} a \cdot D^k u\right|_{L^2} \leqslant C_{m-1}[\|a\|_0 |D^{r-1} u|_{L^2} + \|D^{m-1} a\|_0 |D^{r-m} u|_{L^2}]$. □

PROOF. (1) $|D^{r-k-1} a \cdot D^k u|_{L^2} \leqslant |D^{r-k-1} a|_{L^2} \|D^k u\|_0$. Use the classical interpolation inequality for $0 \leqslant k \leqslant r-m$.

$$|D^{r-k-1}a|_{L^2} \leqslant C_{r-m}|D^{m-1}A|_{L^2}^{1-(r-m-k)/(r-m)}|D^{r-1}a|_{L^2}^{(r-k-m)/(r-m)}$$

$$\|D^k u\|_0 \leqslant C_{r-m}\|u\|_0^{1-k/(r-m)}\|D^{r-m}u\|_0^{k/(r-m)}.$$

Hence

$$\begin{aligned}
&\left|\sum_{k=0}^{r-m} D^{r-k-1}a\cdot D^k u\right|_{L^2} \\
&\qquad \leqslant C_{r-m}\sum_{k=0}^{r-m}\left(|D^{m-1}a|_{L^2}\|D^{r-m}u\|_0\right)^{k/(r-m)}\left(|D^{r-1}a|_{L^2}\|u\|_0\right)^{(r-k-m)/(r-m)} \\
&\qquad \leqslant C_{r-m}\left[|D^{m-1}a|_{L^2}\|D^{r-m}u\|_0 + |D^{r-1}a|_{L^2}\|u\|_0\right].
\end{aligned}$$

Similarly, for $r-m \leqslant k \leqslant r-1$.

$$\begin{aligned}
(2)\quad &\left|\sum_{k=r-m}^{r-1} D^{r-k-1}a\cdot D^k u\right|_{L^2} \\
&\leqslant \sum_{k=r-m}^{r-1}\|D^{r-k-1}a\|_0|D^k u|_{L^2} \\
&\leqslant C_{m-1}\sum_{k=r-m}^{r-1}\left(\|a\|_0|D^{r-1}u|_{L^2}\right)^{(m-r+k)/(m-1)}\left(\|D^{r-1}a\|_0|D^{r-m}u|_{L^2}\right)^{(r-k-1)/(m-1)} \\
&\leqslant C_{m-1}\left[\|a\|_0|D^{r-1}u|_{L^2} + \|D^{m-1}a\|_0|D^{r-m}u|_{L^2}\right]. \quad \square
\end{aligned}$$

Smoothing Estimates.

Here is an explicit construction of Galerkin truncation smoothing operators P_S. We want to satisfy the following inequalities for $S > 1$, $\varphi \in H^r$.

(i) $|P_S\varphi|_{H^r} \leqslant S^r|P_S\varphi|_{H^0} \leqslant S^r|\varphi|_{H^0}$.

(ii) $|[I-P_S]\varphi|_{H^0} \leqslant S^{-r}|[I-P_S]\varphi|_{H^r} \leqslant S^{-r}|\varphi|_{H^r}$.

(iii) For $r > [(n+1)/2]$, $\|[I-P_S]\varphi\|_0 \leqslant C|[I-P_S]\varphi|_{H^{[(n+1)/2]}} \leqslant CS^{-r+[(n+1)/2]}|\varphi|_{H^r}$.

Since the base space $\Omega\times[0,\tau)$ is bounded, it suffices to define the H^r norms by:

$$|\varphi|_{H^r}^2 = |\Delta^{r/2}\varphi|_{L^2}^2 = \sum_{k=1}^{\infty}\mu_k^r(\varphi,\psi_k)^2$$

where ψ_k, μ_k are eigenfunctions and eigenvalues of $\Delta = \partial_t^2 + \partial_{x_1}^2 + \dots + \partial_{x_n}^2$. Define the projections $P_S\varphi = \sum_{k\in\Sigma}\psi_k\cdot(\varphi,\psi_k)$, where $\sum = \{\mu_k\in\sigma(\Delta);\ |\mu_k| \leqslant S\}$. Now (i) and (ii) are immediate, while (iii) follows using the Sobolev Lemma.

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