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On inseparable descent

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Introduction.

Let k be a perfect field of characteristic $p \neq 0$. Put $R = k[\![x]\!]$ = the ring of formal power series in x with coefficients in k, $R' = k[\![x^{p^{-\infty}}]\!] = \bigcup_{n=0}^{\infty} k[\![x^{p^{-n}}]\!]$, and Q, Q' = the quotient fields of R, R', respectively. We also use the notation $W(\cdot)$ to denote the ring of infinite Witt vectors (relative to the prime number p) with components in $\cdot$, and put $K = W(k)$. Let A denote the ring $K[\![X]\!]$ of formal power series in X with coefficients in K, and let B denote the p-adic completion of the ring $A[1/X]$. We will define in section 3 embeddings of rings $A \to W(R')$ and $B \to W(Q')$.

The purpose of this paper is to give a manageable expression for descent data on modules relatively to the extensions $R \to R'$, $Q \to Q'$ and on p-adically separated and complete modules relatively to the extensions $A \to W(R')$, $B \to W(Q')$.

The simple form of the results obtained, say in the case $R \to R'$, depends on the following fact. Let $S = \operatorname{Spec} R$, $X = \operatorname{Spec} R'$, G = the affine S-group Cartier dual to $(\mathbf{Q}_p/\mathbf{Z}_p)_S$ (the standard étale p-divisible group of height 1, viewed over S). Then it is possible to define a morphism of schemes: $G \times_S X \to X$, making $X \to S$ into a principal homogeneous space under G. We do not pursue in the present paper this geometric viewpoint: our aim here is not towards greatest generality but towards a complete understanding of the extensions of rings mentioned above.

We will apply the, results obtained here in subsequent papers to give a generalization of Dieudonné theory for p-divisible groups defined over R or Q.

This paper is essentially self-contained: we send to the references only for the proof of two theorems. Some computations are however left to the reader.

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In writing this paper we have been stronlgy influenced by the work of Barsotti: some of the constructions we use are due to him and others are direct generalizations of the former worked out in the same spirit.

1. – In this paper the word « ring » means « commutative ring with 1 »; a morphism of rings always sends 1 to 1 and « module » means « unitary module ». If k is a ring, a k-algebra will always be associative with a right and left identity element 1, and a morphism of k-algebras (a *representation*) will always send 1 to 1. If A, B are k-algebras, an *antirepresentation* $f\colon A \to B$ is a representation of the opposite k-algebra A^* of A in B.

If A is a linearly topologized (l.t.) ring and M, N are linearly topologized (l.t.) A-modules, the *usual topology* of $\mathrm{Hom}_A(M, N)$ will be the topology of simple convergence on the elements of M. The usual topology of $\mathrm{Hom}_A(M, N)$ is A-linear and a fundamental system of open submodules of $\mathrm{Hom}_A(M, N)$ in that topology is given by the set of the

$$\mathcal{U}(S, V) = \{f \in \mathrm{Hom}_A(M, N)\colon f(S) \subseteq V\}$$

as S varies among the finite subsets of M and V among the open submodules of N. Unless otherwise specified A, M, N will be equipped with the discrete topology and $\mathrm{Hom}_A(M, N)$ with the usual topology.

If $f\colon A \to B$ is a morphism of rings and $g\colon M \to N$ is a morphism of A-modules, we denote by $g_{(f)}\colon M \otimes B \to N \otimes B$ the morphism of B-modules obtained by the scalar extension f.

(1.1) DEFINITION. Let k be a l.t. ring, separated and complete. A *linearly topologized* (l.t.) *k-hyperalgebra* is a structure $(A, i, \mu, \mathbf{P}, \varepsilon, \varrho)$, that we usually denote simply by A, where:

(1.1.i) A is a separated and complete l.t. k-module;

(1.1.ii) $\mu\colon A \hat{\otimes}_k A \to A$ (« product ») and

$i\colon k \to A$ (« structural morphism »)

are continuous morphisms of k-modules satisfying:

(1.1.ii.1) $\mu(\mathrm{id}_A \hat{\otimes} \mu) = \mu(\mu \hat{\otimes} \mathrm{id}_A)$ (associativity),

(1.1.ii.2) $\mu(i \hat{\otimes} \mathrm{id}_A) = \mu(\mathrm{id}_A \hat{\otimes} i) = \mathrm{id}_A$ (existence of 1_A);

therefore (A, i, μ) is a l.t. k-algebra;

(1 1.iii)
$$\mathbf{P}\colon A \to A \hat{\otimes}_k A \qquad (\text{« coproduct »}),$$
$$\varepsilon\colon A \to k \qquad (\text{« augmentation »}),$$
$$\varrho\colon A \to A \qquad (\text{« antipodism »}),$$

are continuous morphisms of k-algebras such that:

(1.1.iii.1) $(\mathrm{id}_A \hat{\otimes} \mathbf{P})\mathbf{P} = (\mathbf{P} \hat{\otimes} \mathrm{id}_A)\mathbf{P}$ (coassociativity),

(1.1.iii.2) $(\mathrm{id}_A \hat{\otimes} \varepsilon)\mathbf{P} = (\varepsilon \hat{\otimes} \mathrm{id}_A)\mathbf{P} = \mathrm{id}_A$,

(1.1.iii.3) $\mu(\varrho \hat{\otimes} \mathrm{id}_A)\mathbf{P} = \mu(\mathrm{id}_A \hat{\otimes} \varrho)\mathbf{P} = i\varepsilon$.

Let $\varkappa\colon A \hat{\otimes}_k A \to A \hat{\otimes}_k A$ be the continuous k-linear map defined by $\varkappa(a \hat{\otimes} b) = b \hat{\otimes} a$. Then the l.t. k-hyperalgebra A is *commutative* if $\mu = \mu\varkappa$, and it is *cocommutative* if $\varkappa\mathbf{P} = \mathbf{P}$.

Morphisms of l.t. k-hyperalgebras are defined in the obvious way. The kernel of $\varepsilon\colon A \to k$ will be denoted by A^+ and will be called the *augmentation ideal* of A.

If k is a ring, a ***k*-hyperalgebra** is a discrete l.t. k-hyperalgebra, where k is equipped with the discrete topology.

(1.2) DEFINITION. Let $k \to A$ be a morphism of rings and M be an A-module. A *descent datum* on M relatively to $k \to A$ is a homomorphism of $A \otimes_k A$-modules:

(1 2.0)
$$\Theta\colon M \otimes_k A \to A \otimes_k M$$

such that:

(1.2.1)
$$\Theta_{(\mu_A)} = \mathrm{id}_M;$$

(1.2.2) if $p_{ij}\colon A \otimes A \to A \otimes A \otimes A$ (e.g. $p_{32}(a \otimes b) = 1 \otimes b \otimes a$)

and $\Theta_{ij} = \Theta_{(p_{ij})}$, the diagram:

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{\Theta_{13}} & A \otimes A \otimes M \\ & {\scriptstyle\Theta_{12}}\searrow \quad \nearrow{\scriptstyle\Theta_{23}} & \\ & A \otimes M \otimes A & \end{array}$$

is commutative.

It follows from (1.2.1), (1.2.2) that:

(1.2.3)
$$\Theta \text{ is an isomorphism}.$$

Let us prove (1.2.3). Let $\mu_{13}\colon A\otimes A\otimes A\to A\otimes A$, $\mu_{13}(a\otimes b\otimes c)=ac\otimes b$ and $\Theta'_{ij}=(\Theta_{ij})_{(\mu_{13})}$ Then: $\Theta'_{13}=\mathrm{id}_{M\otimes A}$, $\Theta'_{12}=\Theta$, $\Theta'_{23}=\varkappa\Theta\varkappa \underset{\text{def.}}{=} \Theta_\varkappa$; therefore $\Theta_\varkappa\Theta=\mathrm{id}_{M\otimes A}$. Analogously $\Theta\Theta_\varkappa=\mathrm{id}_{A\otimes M}$: Q.E.D.

(1.3) GROTHENDIECK'S DESCENT THEOREM. *Let $k\to A$ be a morphism of rings and M_0 be a k-module. Then $M=M_0\otimes_k A$ is automatically equipped with a descent datum $\Theta=\Theta_{M_0}$, relative to $k\to A$, namely:*

(1.3.1)
$$\Theta_{M_0}\colon (M_0\otimes_k A)\otimes_k A\to A\otimes_k(M_0\otimes_k A)$$
$$(m\otimes a)\otimes b\mapsto a\otimes(m\otimes b)$$

for $m\in M_0$, $a,b\in A$.

Assume that $k\to A$ is faithfully flat and let M, Θ be as in (1.2). *Then,*

(1.3.2)
$$M_0=\{m\in M/\ \Theta(m\otimes 1)=1\otimes m\}$$

is a k-submodule of M, $M=M_0\otimes_k A$, and $\Theta=\Theta_{M_0}$. Moreover, if (M,Θ), (M',Θ') are two data as in (1.2), *if M_0 is given by* (1.3.2) *and M'_0 is given analogously, then an A-linear morphism $f\colon M\to M'$ is obtained by the scalar extension $k\to A$ from a k-linear morphism $f_0\colon M_0\to M'_0$ if and only if the following diagram commutes:*

(1.3.3)
$$\begin{array}{ccc} M\otimes_k A & \xrightarrow{f\otimes \mathrm{id}_A} & M'\otimes_k A\\ \Big\downarrow{\scriptstyle\Theta} & & \Big\downarrow{\scriptstyle\Theta'}\\ A\otimes_k M & \xrightarrow{\mathrm{id}_A\otimes f} & A\otimes_k M'\,. \end{array}$$

(See [2] for the proof).

Let k be a ring, A a commutative faithfully flat k-algebra and D a commutative (but not necessarily cocommutative) k-hyperalgebra. Let

(1.4)
$$u\colon A\to D\otimes_k A$$

be a k-algebra morphism such that:

(1.4.1)
$$(\mathbf{P}_D\otimes\mathrm{id}_A)u=(\mathrm{id}_D\otimes u)u;$$

(1.4.2) if $p_2\colon A\to D\otimes A$ is defined by $p_2(a)=1\otimes a$, then

$$\chi=u\otimes p_2\colon A\otimes A\to D\otimes A$$

is a k-algebra isomorphism.

From (1.4.1) and (1.4.2) it follows that

$$(1.4.3) \qquad (\varepsilon_D \otimes \mathrm{id}_A)u = \mathrm{id}_A .$$

Let us prove (1.4.3). Let $f = (\varepsilon_D \otimes \mathrm{id}_A)u$; we have: $uf = (\varepsilon_D \otimes u)u = (\varepsilon_D \otimes \mathrm{id}_D \otimes \mathrm{id}_A)(\mathrm{id}_D \otimes u)u = (\varepsilon_D \otimes \mathrm{id}_D \otimes \mathrm{id}_A)(\mathbb{P}_D \otimes \mathrm{id}_A)u = u$; but u is injective. Q.E.D.

Let τ denote the χ-linear isomorphism:

$$(1.4.4) \qquad \tau\colon A \otimes_k M \to D \otimes_k M$$
$$a \otimes m \mapsto u(a)(1 \otimes m) .$$

Suppose we are given M, Θ as in (1.2). Let us put:

$$(1.5) \qquad \varphi\colon M \to D \otimes_k M$$
$$m \to \tau\big(\Theta(m \otimes 1)\big) .$$

Then:

(1.5.1) $\varphi(am) = \tau\big((a \otimes 1)\Theta(m \otimes 1)\big) = u(a)\varphi(m)$;

(1.5.2) Let $m \in M$ and $\Theta(m \otimes 1) = \sum a_i \otimes m_i$: Then: $(\varepsilon_D \otimes \mathrm{id}_M)\varphi(m) = \sum_i (\varepsilon_D \otimes \mathrm{id}_M)\tau(a_i \otimes m_i) = \sum_i (\varepsilon_D \otimes \mathrm{id}_M)\big(u(a_i)(1 \otimes m_i)\big) = \sum_i \big((\varepsilon_D \otimes \mathrm{id}_A)u(a_i)\big)m_i = \sum_i a_i m_i = m$,

(the last equality follows from (1.2.1));

(1.5.3) Let $m \in M$, $\Theta(m \otimes 1) = \sum_i a_i \otimes m_i$, $\Theta(m_i \otimes 1) = \sum_j a_{ij} \otimes m_{ij}$.

According to (1.2.2) we have:

$$(1.5.3.1) \qquad \sum_i a_i \otimes 1 \otimes m_i = \sum_{i,j} a_i \otimes a_{ij} \otimes m_{ij} .$$

Therefore:

$$(\mathbb{P}_D \otimes \mathrm{id}_M)\varphi(m) = (\mathbb{P}_D \otimes \mathrm{id}_M)\sum_i u(a_i)(1 \otimes m_i) =$$
$$\sum_i \big((\mathbb{P}_D \otimes \mathrm{id}_A)u(a_i)\big)(1 \otimes 1 \otimes m_i) = \sum_i \big((\mathrm{id}_D \otimes u)u(a_i)\big)(1 \otimes 1 \otimes m_i) =$$
$$(\mathrm{id}_D \otimes \tau)\sum_i u(a_i) \otimes m_i = (\mathrm{id}_D \otimes \tau)\sum_i (\chi \otimes \mathrm{id}_M)(a_i \otimes 1 \otimes m_i) =$$

$$(\mathrm{id}_D\otimes\tau)(\chi\otimes\mathrm{id}_M)\sum_{i,j}a_i\otimes a_{ij}\otimes m_{ij}=\sum_{i,j}\big((\mathrm{id}_D\otimes u)\big(u(a_i)(1\otimes a_{ij})\big)\big)(1\otimes 1\otimes m_{ij})=$$
$$\sum_{i,j}\big((\mathrm{id}_D\otimes u)u(a_i)\big)\big(1\otimes\big(u(a_{ij})(1\otimes m_{ij})\big)\big)=$$
$$\sum_i\big((\mathrm{id}_D\otimes u)u(a_i)\big)\big(1\otimes\varphi(m_i)\big)=\sum_i(\mathrm{id}_D\otimes\varphi)\big(u(a_i)(1\otimes m_i)\big)=(\mathrm{id}_D\otimes\varphi)\varphi(m).$$

We conclude from the computations (1.5) that from a descent datum Θ on M relatively to $k\to A$, if the morphism $u\colon A\to D\otimes_k A$, as in (1.4), is given, we obtain a k-linear morphism:

(1.6) $$\varphi\colon M\to D\otimes_k M$$

satisfying:

(1.6.1) $\varphi(am)=u(a)\varphi(m)$, for a in A and m in M;

(1.6.2) $(\varepsilon_D\otimes\mathrm{id}_M)\varphi=\mathrm{id}_M$;

(1.6.3) $(\mathbf{P}_D\otimes\mathrm{id}_M)\varphi=(\mathrm{id}_D\otimes\varphi)\varphi$.

Conversely, let φ as in (1.6) be given, and define $\Theta\colon M\otimes A\to A\otimes M$, by:

(1.7) $$\Theta(m\otimes a)=\tau^{-1}\big((1\otimes a)\varphi(m)\big).$$

Then Θ is $A\otimes A$-linear, and:

(1.7.1) if $m\in M$ and $\Theta(m\otimes 1)=\tau^{-1}\varphi(m)=\sum_i a_i\otimes m_i$, then $\varphi(m)=\sum_i u(a_i)(1\otimes m_i)$ so that $\sum_i a_i m_i=(\varepsilon_D\otimes\mathrm{id}_M)\sum_i u(a_i)(1\otimes m_i)=(\varepsilon_D\otimes\mathrm{id}_M)\varphi(m)=m$;

(1.7.2) if $\Theta(m\otimes 1)=\sum_i a_i\otimes m_i$ and $\Theta(m_i\otimes 1)=\sum_j a_{ij}\otimes m_{ij}$, by inverting the reasoning used in (1.5.3) one proves that $\sum_i a_i\otimes 1\otimes m_i=\sum_{i,j}a_i\otimes a_{ij}\otimes m_i$, and therefore (1.2.2) for this Θ.

It follows from Grothendieck's theory of descent $\big((1.3)\big)$, that if M, φ are as in (1.6) and one puts:

(1.8) $$M_0=\{m\in M/\ \varphi(m)=1\otimes m\}$$

then $M = A \otimes_k M_0$ and $\varphi(a \otimes m) = u(a)(1 \otimes m) = u(a) \otimes m$. If now we denote φ by φ_M and let (N, φ_N) be a datum analogous to (M, φ_M), a A-linear homomorphism $f: M \to N$ is the extension by A-linearity of a k-linear homomorphism $f_0: M_0 \to N_0$ iff the following diagram commutes:

(1.8.1)
$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_N} \\ D \otimes_k M & \xrightarrow{\mathrm{id}_D \otimes f} & D \otimes_k N . \end{array}$$

We will assume in the rest of this section that D, as a k-module, is the direct limit of a direct system of finite locally free k-modules, say $D = \varinjlim_\alpha D_\alpha$. Then if k is given the discrete topology and the k-modules $C_\alpha = \mathrm{Hom}_k(D_\alpha, k)$, $C = \mathrm{Hom}_k(D, k)$ are given the usual topology, $C = \varprojlim_\alpha C_\alpha$, with the inverse limit topology (the topology of C_α is the discrete). Besides $\mathrm{Hom}_k(D_\alpha \otimes D_\alpha, k) = C_\alpha \otimes_k C_\alpha$ and $\mathrm{Hom}_k(D \otimes D, k)$, with the usual topology, equals $\varprojlim_\alpha C_\alpha \otimes C_\alpha = C \hat{\otimes}_k C$. Under our assumptions, C, endowed with the operations dualizing those of D, is a l.t. k-hyperalgebra (commutative but not necessarily commutative) called the *Cartier dual* of D. Let us put:

(1.9)
$$\begin{aligned} S&: C \to \mathrm{End}_k D \\ &c \mapsto S_c, \qquad \text{where} \\ S_c(d) &= (c \otimes \mathrm{id}_D)\mathbf{P}_D d, \qquad \text{for } d \in D, \end{aligned}$$

a continuous injective antirepresentation of k-algebras (the topology of D is the discrete). If $c \in C$ and $d \in D$ we will denote $c(d) \in k$ by $c \circ d$ and $S_c(d) \in D$ by cd; the previous formula then reads:

(1.9.1)
$$cd = (c\circ \otimes \mathrm{id}_D)\mathbf{P}_D d .$$

One can prove that:

(1.9.2)
$$\begin{aligned} \mathbf{P}_D(cd) &= (c \otimes \mathrm{id}_D)\mathbf{P}_D d \\ c(dd') &= \mu_D\big((\mathbf{P}_C c)(d \otimes d')\big) \\ (cc')d &= c'(cd) \\ c \circ d &= \varepsilon_D(cd) \end{aligned}$$

for $c, c' \in C$ and $d, d' \in D$. The second formula in (1.9.2), as many similar formulas to come, should be interpreted as follows. Suppose $\mathbb{P}_C c = \sum_{j,h} c_j \hat{\otimes} c_h$ (a converging sum in $C \hat{\otimes}_k C$); then $(\mathbb{P}_C c)(d \otimes d') = \sum_{j,h} c_j d \otimes c_k d' = \sum_{j,h} S_{c_j}(d) \otimes \otimes S_{c_h}(d')$ (a finite sum in $D \otimes_k D$). Therefore $c(dd') = \sum_{j,h} (c_j d)(c_k d')$.

Furthermore, if D is free with basis $\{d_i, i \in I\}$ over k and if $\{c_i, i \in I\}$ denotes the dual topological k-basis of C, we have:

$$\mathbb{P}_D d = \sum_{i \in I} d_i \otimes c_i d \,. \tag{1.10}$$

Given $u: A \to D \otimes A$ as in (1.4), let us put:

$$\begin{aligned} &T: C \to \operatorname{End}_k A \\ &c \mapsto T_c\,, && \text{where} \\ &T_c(a) = (c \circ \otimes \mathrm{id}_A) u(a)\,, && \text{for } a \in A\,. \end{aligned} \tag{1.11}$$

Again, T is a continuous antirepresentation of k-algebras (the topology of A being the discrete). If $T_c(a)$ is denoted by ca, one has, for $c \in C$ and $a, a' \in A$:

$$\begin{aligned} u(ca) &= (c \otimes \mathrm{id}_A) u(a) \\ c(aa') &= \mu_A\big((\mathbb{P}_C c)(a \otimes a')\big)\,. \end{aligned} \tag{1.12}$$

The second formula in (1.12) means that $c(aa') = \sum_{j,h} (c_j a)(c_h a')$, if $\mathbb{P}_C c = \sum_{j,h} c_j \hat{\otimes} c_h$.

If D is free we have again:

$$u(a) = \sum_{i \in I} d_i \otimes c_i a\,. \tag{1.13}$$

Analogously, given $\varphi: M \to D \otimes M$ as in (1.6), we define:

$$\begin{aligned} &U: C \to \operatorname{End}_k M \\ &c \mapsto U_c\,, && \text{where} \\ &U_c(m) = (c \circ \otimes \mathrm{id}_M) \varphi(m)\,, && \text{for } m \in M\,. \end{aligned} \tag{1.14}$$

Once again, U is a continuous antirepresentation of k-algebras (the topology of M being the discrete) and, after writing cm for $U_c(m)$, it satisfies:

$$\begin{aligned} \varphi(cm) &= (c \otimes \mathrm{id}_M) \varphi(m) \\ c(am) &= \mu_{sc}\big((\mathbb{P}_C c)(a \otimes m)\big) \end{aligned} \tag{1.15}$$

for any $c \in C$, $a \in A$, $m \in M$, where $\mu_{sc}\colon A \otimes_k M \to M$ denotes the scalar products. So $c(am) = \sum_{j,h} (c_j a)(c_h m) = \sum_{j,h} T_{c_j}(a)\, U_{c_h}(m)$, if $\mathbf{P}_C c = \sum_{j,h} c_j \hat{\otimes} c_h$. If D is free, we have again:

$$\varphi(m) = \sum_{i \in I} d_i \otimes c_i m \,. \tag{1.16}$$

Conversely, let $U\colon C \to \operatorname{End}_k M$, for a (discrete) A-module M, be a continuous antirepresentation of k-algebras satisfying the second formula in (1.15); assume that D is free. Then $\varphi\colon M \to D \otimes M$ defined by (1.16) satisfies (1.6.1, 2, 3). Formula (1.7) now becomes:

$$M_0 = \{m \in M \,/\, cm = 0, \text{ for all } c \in C^+\} \,. \tag{1.17}$$

Clearly:

$$M_0 = \{m \in M \,/\, cm = (\varepsilon_C c)m, \text{ for all } c \text{ in } C\} \,, \tag{1.18}$$

and, if $c \in C$, $m = m_0 \otimes a \in M = M_0 \otimes_k A$, then:

$$cm = U_c(m) = m_0 \otimes T_c(a) = m_0 \otimes ca \,. \tag{1.19}$$

As a consequence of the considerations above we will say that the map U of (1.14) is a *descent datum* on M relatively to $k \to A$. Let now M and N be two A-modules with descent data relatively to $k \to A$. An A-linear map $f\colon M \to N$ is the extension by A-linearity of a k-linear map $f_0\colon M_0 \to N_0$ iff:

$$f(cm) = cf(m) \,, \qquad \text{for all } c \text{ in } C \text{ and } m \text{ in } M \,. \tag{1.20}$$

One verifies immediately that the induced descent data on $M \otimes_A N$ and $\operatorname{Hom}_A(M, N)$ can be respectively expressed as follows:

$$c(m \otimes n) = (\mathbf{P}c)(m \otimes n) \,, \tag{1.21}$$

$$(cf)(m) = \sum_{j,h} c_j\big(f((\varrho_C c_h) m)\big) \,, \tag{1.22}$$

if $c \in C$, $\mathbf{P}c = \sum_{j,h} c_j \hat{\otimes} c_h$ (a converging series in $C \hat{\otimes} C$), $m \in M$, $n \in N$, $f \in \operatorname{Hom}_A(M, N)$. The right-hand term in (1.21) is to be interpreted in the following way. Let $t\colon M \otimes_k N \to M \otimes_A N$ be the canonical map. Then $c(m \otimes n) = c(m \otimes_A n) = t\big((\mathbf{P}c)(m \otimes_k n)\big) = \sum_{j,h} (c_j m)(c_h n)$ (a finite sum in $M \otimes_A N$). Notice that this is a good definition.

2. – Let k be a perfect field of characteristic $p \neq 0$ and, for $n \in \mathbf{N}$, let $K_n = W_n(k)$ be the ring of Witt vectors (relative to the prime number p) of length n with components in k; in particular $K_1 = k$. Let $K_n[\![x]\!]$ be the affine algebra of the standard multiplicative formal group G_{K_n} over K_n: $K_n[\![x]\!]$ is endowed with the (x)-adic topology and it is the l.t. K_n-hyperalgebra (K_n is discrete) whose coproduct $\mathbf{P}$ and augmentation ε are given by:

$$\mathbf{P}(x) = 1 \hat{\otimes} x + x \hat{\otimes} 1 + x \hat{\otimes} x\,, \qquad \varepsilon(x) = 0\,. \tag{2.1}$$

For any m in $\mathbf{N}$, the multiplication by p^m of G_{K_n} (in additive notation) is expressed by the continuous morphism of K_n-algebras:

$$\begin{aligned} P_m\colon K_n[\![x]\!] &\to K_n[\![x]\!] \\ x &\mapsto (1+x)^{p^m} - 1\,. \end{aligned} \tag{2.2}$$

In fact P_m is an injective homomorphism of l.t. K_n-hyperalgebras. Let us regard P_m as an embedding of $K_n[\![x]\!]$ in another copy of itself that we denote by $K_n[\![x_m]\!]$; namely we put:

$$\begin{aligned} &P_m\colon K_n[\![x]\!] \hookrightarrow K_n[\![x_m]\!] \\ &P_m(x) = x = (1+x_m)^{p^m} - 1\,. \end{aligned} \tag{2.3}$$

One immediately checks that $K_n[\![x_m]\!]$ is freely generated as a $K_n[\![x]\!]$-module by $\{1, x_m, x_m^2, \dots, x_m^{p^m-1}\}$. Let us denote by $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ the group hyperalgebra of the group $p^{-m}\mathbf{Z}_p/\mathbf{Z}_p$ (that is the Cartier dual of its affine algebra). Explicitly we have: $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ is the free K_n-module generated by the symbols $\{Y_g,\ g \in p^{-m}\mathbf{Z}_p/\mathbf{Z}_p\}$, and:

$$\begin{aligned} Y_g Y_h &= Y_{g+h} \\ \mathbf{P} Y_g &= Y_g \otimes Y_g \\ \varepsilon Y_g &= 1 \end{aligned} \tag{2.4}$$

for any g, h in $p^{-m}\mathbf{Z}_p/\mathbf{Z}_p$. It is clear that the K_n-module homomorphism:

$$\begin{aligned} K_n[\![x_m]\!] \underset{K_n[\![x]\!]\nearrow\varepsilon}{\otimes} K_n &\to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \\ (1+x_m)^i \otimes 1 &\mapsto Y_{ip^{-m}+\mathbf{Z}_p} \end{aligned} \tag{2.5}$$

is an isomorphism of K_n-hyperalgebras (notice that $K_n[\![x_m]\!] \underset{K_n[\![x]\!]/\varepsilon}{\otimes} K_n$ is naturally a K_n-hyperalgebra). We deduce from (2.5) a surjective morphism of l.t. K_n-hyperalgebras ($K_n[p^{-m}\,\mathbf{Z}_p/\mathbf{Z}_p]$ is discrete), with kernel $xK_n[\![x_m]\!]$:

$$\text{(2.6)} \qquad \begin{aligned} \sigma_m\colon K_n[\![x_m]\!] &\to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \\ x_m &\mapsto Y_{p^{-m}+\mathbf{Z}_p} - 1\,. \end{aligned}$$

If we denote by ${}_{p^m}G_{K_n}$ the finite multiplicative K_n-group whose affine algebra is $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$, we have proved above that the sequence:

$$\text{(2.7)} \qquad 0 \longrightarrow {}_{p^m}G_{K_n} \longrightarrow G_{K_n} \xrightarrow{p^m} G_{K_n} \longrightarrow 0$$

is exact (in the category of (faithfully) flat sheaves of abelian groups on finite K_n-algebras).

Let us define now:

$$\text{(2.8)} \qquad \begin{aligned} u_m\colon K_n[\![x_m]\!] &\to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \underset{K_n}{\hat{\otimes}} K_n[\![x_m]\!] \\ f &\mapsto (\sigma_m \hat{\otimes}\, \mathrm{id})\,\mathbf{P}f \end{aligned}$$

(notice that, since $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ is a finite K_n-module and $K_n[\![x_m]\!]$ is complete, we could replace $\hat{\otimes}$ by $\otimes$ in (2.8)). Clearly, u_m is a $K_n[\![x]\!]$-algebra morphism and it is determined, as a $K_n[\![x]\!]$-linear map, by:

$$\text{(2.8.1)} \qquad u_m(1+x_m)^a = Y_{ap^{-m}+\mathbf{Z}_p} \otimes (1+x_m)^a\,, \quad \text{for } a \in \mathbf{N}\,.$$

We would like to prove that u_m satisfies to the properties required for u in (1.4), with the following replacements: (in the left-hand column of (2.9) find the symbols of section 1 while in the right-hand one find the symbols replacing them)

$$\text{(2.9)} \qquad \begin{array}{ll} k\,, & K_n[\![x]\!] \\ A\,, & K_n[\![x_m]\!] \\ D\,, & K_n[\![x]\!] \otimes_{K_n} K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \\ D \otimes_k A\,, & K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \otimes_{K_n} K_n[\![x_m]\!]\,. \end{array}$$

In the first place since $K_n[\![x]\!] \hookrightarrow K_n[\![x_m]\!]$ is free, it is also faithfully flat. (1.4.1) is obvious. Let us check (1.4.2). We observe first that:

$$\text{(2.10)} \qquad \begin{aligned} l\colon K_n[\![x_m]\!] \underset{K_n}{\hat{\otimes}} K_n[\![x_m]\!] &\to K_n[\![x_m]\!] \underset{K_n}{\hat{\otimes}} K_n[\![x_m]\!] \\ f \hat{\otimes} g &\to (\mathbf{P}f)(1 \hat{\otimes} g) \end{aligned}$$

is a continuous K_n-algebra isomorphism. The inverse of l is:

$$r\colon f\mathbin{\hat{\otimes}} g \to \big((\mathrm{id}\mathbin{\hat{\otimes}}\varrho)\mathbf{P}f\big)(1\otimes g)\,. \tag{2.11}$$

Let us now consider the diagram:

$$\begin{array}{ccc} K_n[\![x_m]\!]\ \hat{\bigotimes}_{K_n}\ K_n[\![x_m]\!] & \underset{r}{\overset{l}{\rightleftarrows}} & K_n[\![x_m]\!]\ \hat{\bigotimes}_{K_n}\ K_n[\![x_m]\!] \\ \big\downarrow{\scriptstyle \text{can.}} & & \big\downarrow{\scriptstyle \sigma_m\hat{\otimes}\mathrm{id}} \\ K_n[\![x_m]\!]\ \bigotimes_{K_n[\![x]\!]}\ K_n[\![x_m]\!] & \underset{\bar{r}}{\overset{\bar{l}}{\rightleftarrows}} & K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\ \hat{\bigotimes}_{K_n}\ K_n[\![x_m]\!]\,. \end{array} \tag{2.12}$$

If we prove that in (2.12) the barred arrows exist in such a way that the resulting diagram is commutative, (1.4.2) will follow for u_m. We would have in fact then $\bar{l}(f\otimes g) = u_m(f)(1\otimes g)$ and $\bar{r}\bar{l} = \mathrm{id}_{K_n[\![x]\!]\underset{K_n[[x]]}{\otimes}K_n[\![x_m]\!]}$ and $\bar{l}\bar{r} = \mathrm{id}_{K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\underset{K_n}{\otimes}K_n[\![x_m]\!]}$. To show the existence of $\bar{l}$ it is enough to prove that $(\sigma_m\mathbin{\hat{\otimes}}\mathrm{id})\, l(fg\mathbin{\hat{\otimes}} h) = (\sigma_m\mathbin{\hat{\otimes}}\mathrm{id})\, l(g\mathbin{\hat{\otimes}} fh)$, for any f in $K_n[\![x]\!]$ and g, h in $K_n[\![x_m]\!]$. Now l is right $K_n[\![x_m]\!]$-linear so that we can put $h=1$. We have to prove that: $(\sigma_m\mathbin{\hat{\otimes}}\mathrm{id})(\mathbf{P}f)(\mathbf{P}g) = (1\otimes f)(\sigma_m\mathbin{\hat{\otimes}}\mathrm{id})\mathbf{P}g$, if $f\in K_n[\![x]\!]$ and $g\in K_n[\![x_m]\!]$. This follows from the fact that the kernel of σ_m is $xK_n[\![x_m]\!]$.

For the existence of $\bar{r}$, it is enough to prove that $r(x\mathbin{\hat{\otimes}}1)$ has zero image in $K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]}K_n[\![x_m]\!]$. Now we have $r(x\mathbin{\hat{\otimes}}1) = (\mathrm{id}\mathbin{\hat{\otimes}}\varrho)\mathbf{P}x\in K_n[\![x]\!]\ \hat{\bigotimes}_{K_n}\ K_n[\![x]\!]$; its image in $K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]}K_n[\![x_m]\!]$ coincides with $\mu_{K_n[\![x]\!]}(\mathrm{id}\otimes\varrho)\mathbf{P}x = i\varepsilon x = 0$ ($i = i_{K_n[\![x]\!]}$ is as usual the structural morphism of $K_n[\![x]\!]$).

We conclude that the map u_m defined in (2.8) is a $K_n[\![x]\!]$-algebra homomorphism:

$$u_m\colon K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\ \hat{\bigotimes}_{K_n}\ K_n[\![x_m]\!] \tag{2.13}$$

such that:

$$(\mathrm{id}\otimes u_m)\,u_m = (\mathbf{P}\otimes\mathrm{id})\,u_m \tag{2.13.1}$$

and

(2.13.2) the map: $K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]}K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\bigotimes_{K_n}K_n[\![x_m]\!]$

$$f\otimes g \to u_m(f)(1\otimes g)$$

is an isomorphism of $K_n[\![x]\!]$-algebras.

We are exactly in the situation of section 1 with the substitutions indicated in (2.9).

At this point we need to make a typographical specification: the x_m belonging to $K_n[\![x_m]\!]$ will be denoted by $x_m^{(n)}$; the symbol x_m will denote only $x_m^{(1)}$, and we also write x for x_0, so that $x_m = x^{p^{-m}}$, for m in N. We also put $x_0^{(n)} = x^{(n)}$. Let $A_n = \bigcup_{m=0}^{+\infty} K_n[\![x_m^{(n)}]\!]$. We want to prove that $A_n = W_n(A_1)$, where A_1 obviously coincides with the perfectionate $k[\![x^{p^{-\infty}}]\!]$ of $k[\![x]\!]$. Let us denote by $\varphi_{n,m}$ the continuous ring homomorphism $\varphi_{n,m}$: $K_{n+1}[\![x_m^{(n+1)}]\!] \to K_n[\![x_m^{(n)}]\!]$ extending the natural map (reduction modulo p^n) $K_{n+1} \to K_n$, such that $\varphi_{n,m}(x_m^{(n+1)}) = x_m^{(n)}$. Let then φ_n: $A_{n+1} \to A_n$ be defined by $\varphi_n(a) = \varphi_{n,m}(a)$ if $a \in K_{n+1}[\![x_m^{(n+1)}]\!]$. Let us regard A_n as a discrete l.t. ring. Then $A = \varprojlim_n (A_n, \varphi_n)$ is a strict p-ring in the sense of [1], chap. II, sect. 5, and it coincides then with $W(A_1)$. It follows that $A_n = W_n(A_1)$. If the symbol $[a]$ denotes the multiplicative representative in $W_n(A_1)$ of $a \in A_1$, we have $[1 + x] = 1 + x^{(n)}$ and, in general, $[1 + x^{p^{-m}}] = 1 + x_m^{(n)}$, in the identification above. A word of caution on the embedding of $K_n[\![x_m^{(n)}]\!]$ in $W_n(A_1)$. Let us (provisionally) topologize A_1 with the (x)-adic topology and $W_n(A_1)$ (in $1-1$ correspondence with A_1^n) with the product topology (notice that this topology coincides with the $([x])$-adic). Then the embedding above is characterized as a continuous K_n-algebra morphism ($K_n[\![x_m^{(n)}]\!]$ being endowed with the $(x_m^{(n)})$-adic topology) by the assignment $x_m^{(n)} \mapsto$ $\mapsto [1 + x^{p^{-m}}] - 1$. In the sequel, no topology will be given to $K_n[\![x_m^{(n)}]\!]$ or to $W_n(A_1)$, but by « the embedding $x_m^{(n)} \mapsto [1 + x^{p^{-m}}] - 1$ », we will always mean the one described above.

Let us fix n and put $x_m^{(n)} = X_m$, $X_0 = X$. By taking direct limits in (2.13) we get a morphism of $K_n[\![X]\!]$-algebras:

$$u\colon W_n(A_1) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n(A_1)\,, \tag{2.14}$$

where $K_n[\mathbf{Q}_p/\mathbf{Z}_p]$ is the free K_n-module generated by $\{Y_g,\ g \in \mathbf{Q}_p/\mathbf{Z}_p\}$, endowed with the hyperalgebra operations defined by formulas (2.4). The morphism of $K_n[\![X]\!]$-algebras u is determined by the relations:

$$u([1 + x^{p^{-m}}]) = Y_{p^{-m}+\mathbf{Z}_p} \otimes [1 + x^{p^{-m}}]\,, \tag{2.14.0}$$

for $m \in \mathbf{N}$. It satisfies:

$$(\mathbf{P} \otimes \mathrm{id})u = (\mathrm{id} \otimes u)u; \tag{2.14.1}$$

$$\begin{gathered} W_n(A_1) \bigotimes_{K_n[\![X]\!]} W_n(A_1) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n(A_1) \\ f \otimes g \mapsto u(f)(1 \otimes g)\,, \end{gathered} \tag{2.14.2}$$

is an isomorphism of $K_n[\![X]\!]$-algebras.

Moreover the morphism $K_n[\![X]\!] \hookrightarrow W_n(A_1)$ is faithfully flat. (The facts just stated follow from the properties of direct limits). We are then in a position to apply the theory of section 1 to get information on the descent relatively to $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$, $X \mapsto [1+x]-1$.

Let us carry out in detail the constructions of section 1 for the present data. The next diagram indicates the replacements to be operated (as before the right-hand column replaces the left-hand one):

$$(2.15)\qquad \begin{array}{rl} k\ , & K_n[\![X]\!] \\ A\ , & W_n(k[\![x^{p^{-\infty}}]\!]) \\ k\to A\ , & X\mapsto[1+x]-1 \\ D\ , & K_n[\![X]\!]\bigotimes_{K_n} K_n[\mathbf{Q}_p/\mathbf{Z}_p] \\ D\bigotimes_k A\ , & K_n[\mathbf{Q}_p/\mathbf{Z}_p]\bigotimes_{K_n} W_n(k[\![x^{p^{-\infty}}]\!])\ . \end{array}$$

Let $F_n = \mathrm{Hom}_{K_n}(K_n[\mathbf{Q}_p/\mathbf{Z}_p], K_n)$ be the l.t. K_n-hyperalgebra Cartier dual to $K_n[\mathbf{Q}_p/\mathbf{Z}_p]$. We can obviously identify F_n with the l.t. K_n-hyperalgebra of functions defined on the group $\mathbf{Q}_p/\mathbf{Z}_p$ taking values in K_n, endowed with the topology of simple convergence on $\mathbf{Q}_p/\mathbf{Z}_p$ with respect to the discrete topology of K_n. (We recall that $\mathbb{P}\colon F_n \to F_n \hat{\otimes} F_n$ is defined by identifying $F_n \hat{\bigotimes}_{K_n} F_n$ with the K_n-algebra of functions from $\mathbf{Q}_p/\mathbf{Z}_p \times \mathbf{Q}_p/\mathbf{Z}_p$ to K_n, endowed with the topology of simple convergence, and by putting $(\mathbb{P}f)(a,b) = f(a+b)$, for f in F_n and a, b in $\mathbf{Q}_p/\mathbf{Z}_p$, and $\varepsilon f = 0$.)

Such an identification is obtained by interpreting $f\colon \mathbf{Q}_p/\mathbf{Z}_p \to K_n$ as the K_n-linear map: $\sum_{i\in\mathbf{Q}_p/\mathbf{Z}_p} a_i Y_i \mapsto \sum_{i\in\mathbf{Q}_p/\mathbf{Z}_p} a_i f(i)$, from $K_n[\mathbf{Q}_p/\mathbf{Z}_p]$ to K_n. Notice that F_n can naturally be identified with $W_n(F_1)$ as a K_n-algebra: $f \in F_n$ is identified with $(f_0, \ldots, f_{n-1}) \in W_n(F_1)$ if $f_i = c_i f$, where for $i = 0, \ldots, n-1$, $c_i\colon K_n = W_n(k) \to k$ is the function « i-th component » of a Witt vector. The topology of F_n corresponds then to the product topology of the topology of F_1, in the natural bijection $W_n(F_1) \leftrightarrow F_1^n$. We can also identify $W_n(F_1 \hat{\bigotimes}_{K_n} F_1)$ with $W_n(F_1) \hat{\bigotimes}_{K_n} W_n(F_1)$, since they are both isomorphic to the K_n-algebra of functions $f\colon \mathbf{Q}_p/\mathbf{Z}_p \times \mathbf{Q}_p/\mathbf{Z}_p \to K_n$ endowed with the topology of simple convergence. The coproduct of F_n then corresponds to the map:

$$(2.16)\qquad W_n(\mathbb{P}_{F_1})\colon W_n(F_1) \to W_n\big(F_1 \hat{\bigotimes}_{K} F_1\big) = W_n(F_1) \hat{\bigotimes}_{K_n} W_n(F_1)$$

$$(f_0, \ldots, f_{n-1}) \mapsto (\mathbb{P}_{F_1} f_0, \ldots, \mathbb{P}_{F_1} f_{n-1})\ .$$

To pursue the correspondence with section 1, we see that the l.t. k-hyper-algebra C is now replaced by $K_n[\![X]\!] \hat{\bigotimes}_{K_n} F_n$ (or $K_n[\![X]\!] \hat{\bigotimes}_{K_n} W_n(F_1)$) where $\hat{\bigotimes}_{K_n}$ is taken with respect to the discrete topology of $K_n[\![X]\!]$. The representation S of (1.9) is now the extension by $K_n[\![X]\!]$-linearity of the K_n-linear continuous representation:

$$\begin{aligned} S'\colon F_n &\to \operatorname{End}_{K_n} K_n[\mathbf{Q}_p/\mathbf{Z}_p] \\ f &\mapsto S'_f\,, \qquad \text{where} \\ S'_f(Y_g) &= f(g)\,Y_g\,, \qquad \text{for } g \text{ in } \mathbf{Q}_p/\mathbf{Z}_p\,. \end{aligned} \tag{2.17}$$

Similarly the T of (1.11) is the extension by $K_n[\![X]\!]$-linearity of the K_n-linear continuous representation:

$$\begin{aligned} T'\colon F_n &\to \operatorname{End}_{K_n[\![X]\!]} W_n(k[\![x^{p^{-\infty}}]\!]) \\ f &\mapsto T'_f\,, \qquad\qquad \text{where} \\ T'_f((1+X)^{ap^m}) &= f(ap^m+\mathbf{Z}_p)(1+X)^{ap^m}\,, \quad \text{for } a, m \text{ in } \mathbf{Z}\,. \end{aligned} \tag{2.18}$$

We leave to the reader the verification of the formulas in (2.17) and (2.18). We then conclude from section 1, that a descent datum on a $W_n(k[\![x^{p^{-\infty}}]\!])$-module M relatively to $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$, $X \mapsto [1+x]-1$, is equivalent to a continuous representation of K_n-algebras:

$$\begin{aligned} U'\colon F_n &\to \operatorname{End}_{K_n} M \\ f &\mapsto U'_f \end{aligned} \tag{2.19}$$

such that (after skipping the symbols U', T' and denoting by

$$\mu_{sc}\colon W_n(k[\![x^{p^{-\infty}}]\!]) \bigotimes_{K_n} M \to M$$

the scalar product):

$$d(rm) = \mu_{sc}\Big((\mathbf{P}d)\big(r \bigotimes_{K_n} m\big)\Big), \tag{2.20}$$

for any d in F_n, r in $W_n(k[\![x^{p^{-\infty}}]\!])$, m in M. Notice that each U'_f, for f in F_n, is then in fact $K_n[\![X]\!]$-linear.

Since $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$ is faithfully flat, all descent data with respect to it are effective and therefore if one puts:

$$M_0 = \{m \in M \,/\, dm = 0 \text{ for } d \in F_n^+\}\,, \tag{2.21}$$

one concludes that M_0 is a $K_n[\![X]\!]$-module, that $M = W_n(k[\![x^{p^{-\infty}}]\!]) \bigotimes_{K^n[\![X]\!]} M_0$, and that $d(r \otimes m) = dr \otimes m$ for each d in F_n, r in $W_n(k[\![x^{p^{-\infty}}]\!])$, and m in M_0.

Since u_m in (2.13) is a $K_n[\![X]\!]$-algebra morphism, we can extend it by $K_n[\![X]\!][1/X]$-linearity to:

$$(2.22) \qquad u'_m \colon K_n[\![X_m]\!][1/X] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \bigotimes_{K_n} K_n[\![X_m]\!][1/X]$$

satisfying:

$$(2.22.1) \qquad (\mathrm{id} \otimes u'_m)\, u'_m = (\mathbf{P} \otimes \mathrm{id})\, u'_m;$$

(2.22.2) the map:

$$K_n[\![X_m]\!][1/X] \bigotimes_{K_n[\![X]\!][1/X]} K_n[\![X_m]\!][1/X] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \bigotimes_{K_n} K_n[\![X_m]\!][1/X]$$
$$f \otimes g \mapsto u'_m(f)(1 \otimes g),$$

is an isomorphism of $K_n[\![X]\!][1/X]$-algebras.

Moreover $K_n[\![X]\!][1/X] \hookrightarrow K_n[\![X_m]\!][1/X]$ is free and therefore faithfully flat. Notice that $1/X_m = \big(1 + (1 + X_m) + \ldots + (1 + X_m)^{p^m - 1}\big)/X$ so that $K_n[\![X_m]\!][1/X] = K_n[\![X_m]\!][1/X_m]$. By passing to the direct limit for m going to infinity, we get: $\varinjlim_n K_n[\![X_m]\!][1/X] = A_n[1/X] = W_n\big(k((x^{p^{-\infty}}))\big)$, where $k((x^{p^{-\infty}}))$ denotes the perfect closure of the field $k((x))$. We obtain again a morphism of $K_n[\![X]\!][1/X]$-algebras:

$$(2.23) \qquad u' \colon W_n\big(k((x^{p^{-\infty}}))\big) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n\big(k((x^{p^{-\infty}}))\big);$$

given by:

$$(2.23.0) \qquad u'([1 + x^{p^{-m}}]) = Y_{p^{-m}+\mathbf{Z}_p} \otimes [1 + x^{p^{-m}}] \quad \text{for } m \text{ in } \mathbf{N},$$

and satisfying:

$$(2.23.1) \qquad (\mathbf{P} \otimes \mathrm{id})\, u' = (\mathrm{id} \otimes u')\, u';$$

(2.23.2) the map:

$$W_n\big(k((x^{p^{-\infty}}))\big) \bigotimes_{K_n[\![X]\!][1/X]} W_n\big(k((x^{p^{-\infty}}))\big) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n\big(k((x^{p^{-\infty}}))\big)$$
$$f \otimes g \mapsto u'(f)(1 \otimes g)$$

is an isomorphism of $K_n[\![X]\!][1/X]$-algebras.

Moreover $K_n[\![X]\!][1/X] \hookrightarrow W_n(k((x^{p^{-\infty}})))$ is faithfully flat. We can therefore apply to the descent relatively to that extension the same criteria we proved for $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$.

3. – We keep the notation of section 2. Let us denote by R the ring $k[\![x]\!]$ and by R' its perfectionate $k[\![x^{p^{-\infty}}]\!]$. Let Q, Q' denote the quotient fields of R, R', respectively. We also put $K = W(k) =$ the ring of infinite Witt vectors with components in k. Let $K[\![X]\!]$ be the ring of formal power series in X with coefficients in K; there is a unique morphism of K-algebras:

$$K[\![X]\!] \to W(R') \tag{3.1}$$

sending X to $[1+x]-1$, which is continuous for, say, the (p, X)-adic topology in $K[\![X]\!]$ and the $(p, [x])$-adic one in $W(R')$. We will always regard $K[\![X]\!]$ as embedded in $W(R')$ by means of (3.1). The embedding (3.1) can obviously be uniquely extended, as a ring homomorphism, to give an embedding:

$$K[\![X]\!][1/X] \to W(Q'), \tag{3.2}$$

that we will always use in the sequel. Notice that (3.2) can again be extended by p-adic continuity, to an embedding of the p-adic completion B of $K[\![X]\!][1/X]$ in $W(Q')$:

$$B \to W(Q'). \tag{3.3}$$

The embeddings (3.1) and (3.3) reduce modulo p^n, to the embeddings used in section 2, for which we were able to give simple descent criteria. Let us restate those results in a more manageable form.

Formula (2.18) provides us with a map:

$$\begin{cases} F_n \times W_n(R') \to W_n(R') \\ (d, r) \qquad \mapsto T'_d(r) = dr. \end{cases} \tag{3.4}$$

Analogously, using (2.22), we get a map (that extends (3.4)):

$$\begin{cases} F_n \times W_n(Q') \to W_n(Q') \\ (d, r) \qquad \mapsto dr. \end{cases} \tag{3.5}$$

The map (3.5) can be characterized by the properties:

(3.5.1) $r \mapsto dr$ is $K_n[\![X]\!][1/X]$-linear;

(3.5.2) $d[(1+x^{ap^m})] = d(ap^m + \mathbb{Z}_p)[(1+x)^{ap^m}]$, for a, m in $\mathbb{Z}$.

Moreover (3.5) makes $W_n(Q')$, endowed with the discrete topology, into a l.t. F_n-module and satisfies:

$$d(rr') = \mu\big((\mathbb{P}d)(r \otimes r')\big)\,, \quad \text{for } d \in F_n,\ r,\ r' \in W_n(Q')\,. \tag{3.5.3}$$

The right-hand term in (3.5.3) is to be interpreted in the following way. Suppose $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$, a converging sum in $F_n \underset{K_n}{\hat{\otimes}} F_n$, then: $d(rr') = \sum_{i,j} (d_i r)(d_j r')$, a finite sum in $W_n(Q')$. A descent datum on a $W_n(R')$-(resp. $W_n(Q')$-) module M, relatively to $K_n[\![X]\!] \hookrightarrow W_n(R')$ (resp. $K_n[\![X]\!]\cdot [1/X] \hookrightarrow W_n(Q')$) is equivalent to a K_n-bilinear map:

$$\begin{cases} F_n \times M \to M \\ (d, m) \mapsto dm \end{cases} \tag{3.6}$$

making M, endowed with the discrete topology, into a topological F_n-module and satisfying:

$$d(rm) = \mu_{sc}\big((\mathbb{P}_{F_n} d)(r \underset{K_n}{\otimes} m)\big)\,, \tag{3.7}$$

for $d \in F_n$, $r \in W_n(R')$ (resp. $W_n(Q')$), $m \in M$. Here, as usual, $\mu_{sc}\colon W_n(R') \underset{K_n}{\otimes} M \to M$ (resp. $W_n(Q') \underset{K_n}{\otimes} M \to M$) is the scalar product, and, if $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ (a converging sum in $F_n \underset{K_n}{\hat{\otimes}} F_n$), the right-hand term of (3.7) is to be interpreted as $\sum_{i,j} (d_i r)(d_j m)$ (a finite sum in M), through (3.5) and (3.6). Notice that $m \mapsto dm$ is then automatically $K_n[\![X]\!]$–(resp. $K_n[\![X]\!][1/X]$–) linear.

Let F be the l.t. K-hyperalgebra (K being endowed with the p-adic topology) of functions from $\mathbb{Q}_p/\mathbb{Z}_p$ to K, with the topology of simple convergence. A fundamental system of open K-submodules (ideals, in fact) of F is given by the

$$U_{m,n} = \{f \in F /\ f(p^{-m}\mathbb{Z}_p/\mathbb{Z}_p) \subseteq p^n K\}\,,$$

as m, n vary in $\mathbb{N}$. Clearly, $F = \varprojlim_n F_n$, as a topological ring. The identification $F_n \xrightarrow{\sim} W_n(F_1)$ of section 2, now carries over to an identification $F \xrightarrow{\sim} W(F_1)$, the last being equipped with the product topology of the topology of F_1.

By taking inverse limits for $n \to +\infty$ in (3.5), we obtain a map:

$$(3.8) \qquad \begin{cases} F \times W(Q') \to W(Q') \\ (d, r) \qquad \mapsto dr \end{cases}$$

that can be characterized by the following properties (3.8.1) and (3.8.2):

(3.8.1) $\quad r \mapsto dr \quad$ is B-linear;

(3.8.2) $\quad d[(1 + x^{ap^m})] = d(ap^m + \mathbb{Z}_p)[(1 + x)^{ap^m}]$, $\quad$ for a, m in $\mathbb{Z}$.

Moreover, the map (3.8) makes $W(Q')$, endowed with the p-adic topology, into a topological F-module and satisfies:

(3.8.3) $\qquad d(rr') = \mu\big((\mathbb{P}d)(r \otimes r')\big)$, $\quad$ for $d \in F$, $r, r' \in W(Q')$.

The right-hand term of (3.8.3) should be interpreted as follows. Let $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ (a converging sum in $F \hat{\otimes}_K F$); then $d(rr') = \sum_{i,j} (d_i r)(d_j r')$ (a p-adically convergent sum in $W(Q')$).

Let M be a $W(R')$-(resp. $W(Q')$-) module, p-adically separated and complete. Let

$$(3.9) \qquad \begin{cases} F \times M \to M \\ (d, m) \mapsto dm \end{cases}$$

be a K-bilinear map, making M, endowed with the p-adic topology, into a topological F-module, and satisfying:

$$(3.10) \qquad d(am) = \mu_{sc}\Big((\mathbb{P}_F d)\big(a \hat{\otimes}_K m\big)\Big)$$

for $d \in F$, $a \in W(R')$ (resp. $W(Q')$), $m \in M$. Here $W(R') \hat{\otimes}_K M$ (resp. $W(Q') \hat{\otimes}_K M$) denotes the p-adic completion of $W(R') \otimes_K M$ (resp. $W(Q') \otimes_K M$), $\mu_{sc}\colon W(R') \hat{\otimes}_K M \to M$ (resp. $W(Q') \hat{\otimes}_K M \to M$) denotes the scalar product, and, if $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ (a converging sum in $F \hat{\otimes}_K F$) the right-hand member of (3.10) is to be interpreted as $\sum_{i,j} (d_i a)(d_j m)$ (a p-adically convergent sum in M) through (3.8) and (3.9).

It is clear that, by reduction modulo p^n, the datum (3.9) satisfying (3.10), provides a series of compatible data on $M/p^n M$ of the type (3.7).

We then easily conclude from the previous section that if we put:

$$M_0 = \{m \in M /\ dm = 0, \text{ if } d(0) = 0\}, \tag{3.11}$$

M_0 is a $K[\![X]\!]$– (resp. B–) submodule of M, M_0 is p-adically separated and complete, $M = M_0 \hat{\otimes}_{K[\![X]\!]} W(R')$ (resp. $M_0 \hat{\otimes}_B W(Q')$) where $\hat{\otimes}$ means p-adic completion of $\otimes$, and $d(m \hat{\otimes} a) = m \hat{\otimes} da$ for $d \in F$, $m \in M_0$, $a \in W(R')$ (resp. $W(Q')$). Analogous results hold for the descent of morphisms of modules.

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