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Analytic functionals and Bergman spaces

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I. – Introduction and Definitions.

This paper concerns the problem of representing analytic functionals as analytic functions. Let $\mathcal{O}(D)$ denote the vector space of holomorphic functions defined on a domain D in $\mathbf{C}^N$, $N \geqslant 1$. With the topology of uniform convergence on compact subsets of D, $\mathcal{O}(D)$ is a Fréchet space. An *analytic functional* $T\colon \mathcal{O}(D) \to \mathbf{C}$ is an element of the topological dual space $\mathcal{O}'(D)$. Continuity of T means that there exist a compact set $K \subset D$ and a constant C such that for all $g \in \mathcal{O}(D)$,

$$|Tg| \leqslant C\|g\|_K ,$$

where $\|g\|_K$ denotes the supremum of $|g|$ on K. Under these conditions, K is a *carrier* (¹) of T; more generally, any compactum $K_1 \subset D$ is a *weak carrier* of T if for every open set U with $K_1 \subset U \subset D$, there is a constant C_U such that for all $g \in \mathcal{O}(D)$,

$$|T_g| \leqslant C_U\|g\|_U .$$

These definitions are due to Martineau [27]. In this paper we shall be concerned only with compact weak carriers as defined above.

Let H be a Hilbert subspace of $\mathcal{O}(D)$, with continuous inclusion $i\colon H \to \mathcal{O}(D)$. The restriction of an analytic functional T on D to H corresponds by the Riesz representation theorem to an element f_T of H: for every $g \in H$,

$$T_g = \langle g, f_T \rangle_{H'} .$$

We study the correspondence $T \to f_T$ in this paper.

(¹) For a compactum K to be a carrier of T, it is sufficient but not necessary that T be continuous in the seminorm $\| \ \|_K$. (See, e.g., [27].)

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Analytic functionals have been represented as analytic functions in various ways for special domains D; see, e.g., [19], [16], [36], [2]. In each of the cases cited, the holomorphic function $\tilde{T}$ associated to the analytic functional T—T is called the *indicatrix* of the functional T—is given by an expression of the form

$$\tilde{T}(w) = T_z k(z, w)$$

where $k(z, w)$ is a function holomorphic in z and w. The correspondence we study can be viewed similarly: it follows from the analysis of the next section that

$$\overline{f_T(w)} = T_z k(z, w) ,$$

where $k(z, w)$ is the *reproducing kernel* for the Hilbert space H. Notice that in our case, the indicatrix f_T of an analytic functional T is a holomorphic function on the domain D itself.

Though every $f \in H$ represents a linear functional T_f on H which is continuous in the topology of H, continuity of T_f in the stronger $\mathcal{O}(D)$-topology on H implies in many cases that f can be extended smoothly to or holomorphically across the boundary of D. In the latter case, we obtain a correspondence between analytic functionals and functions holomorphic in a neighborhood of $\bar{D}$. Whether this correspondence exists for a given domain D and Hilbert space H depends upon a certain extendibility property of the reproducing kernel for H. This is the topic of Chapter II.

In Chapter III we study analytic functionals by considering the case $H = H_2(D)$, the Bergman space of holomorphic functions on D which are square-integrable with respect to Lebesgue measure on $\mathbf{C}^N$. In this case, the reproducing kernel is the Bergman kernel function for the domain D, which has been extensively studied by Kerzman [19], Bell [5], and others.

We shall use results about regularity of the Bergman kernel in proving assertions about analytic functionals. Our principal result is as follows: *if $D \subset\subset \mathbf{C}^N$ is strictly pseudoconvex and has real-analytic boundary, then $T \to f$ is a topological isomorphism of $\mathcal{O}'(D)$ and $\mathcal{O}(\bar{D})$.* Here $\mathcal{O}(\bar{D})$ is the space of holomorphic functions defined in a neighborhood of $\bar{D}$, with an inductive limit topology to be defined below; $\mathcal{O}(D)$ has the strong dual topology: basic open neighborhoods of the zero functional are of the form

$$U(A, \varepsilon) = \left\{T \in \mathcal{O}'(D) \colon \sup_{f \in A} |Tf| < \varepsilon\right\} ,$$

where $\varepsilon > 0$ and A is a bounded subset of $\mathcal{O}(D)$.

The appropriate extendibility property of the Bergman kernel is a consequence of real-analytic hypoellipticity of the $\bar{\partial}$-Neumann operator on such domains, which has been established by [23] and [7].

This theorem is false without the hypothesis of real-analyticity. For example, we show that *if $N = 1$ and D has $\mathcal{C}^2$ boundary, the linear functional $I: H_2(D) \to \mathbf{C}$ represented by the constant function* 1 *is continuous in the $\mathcal{O}(D)$-topology if and only if D has real-analytic boundary.*

In the setting $H = H_2(D)$, we show that analytic functions which represent analytic functionals are precisely those of the form $f = Ph$, where $P: L^2(D) \to H_2(D)$ is the orthogonal projection, and $h \in \mathcal{C}_0^\infty(D)$, the space of $\mathcal{C}^\infty$ functions with support compactly contained in D. From this point of view, results about analytic functionals can be couched in terms of the Bergman projection. Thus in the context of the main result above, *a function $f \in H_2(D)$ extends holomorphically to a neighborhood of $\bar{D}$ if and only if $f = Ph$, for some $h \in \mathcal{C}_0^\infty(D)$.* The second result above becomes: *the constant function* 1 *on D is the orthogonal projection of a compactly supported function if and only if D has real-analytic boundary.* These results and other applications of the previous work are presented in Chapter IV.

II. – Analytic functionals, embedded Hilbert spaces and their reproducing kernels.

Let D be a relatively compact domain in $\mathbf{C}^N$ and $H \subset \mathcal{O}(D)$ a Hilbert space with inner product $\langle\, ,\rangle_H$, such that the inclusion $i: H \to \mathcal{O}(D)$ is continuous. We also require that the inclusion $\mathcal{O}(\bar{D}) \hookrightarrow H$ hold and be continuous if $\mathcal{O}(\bar{D})$ is endowed with the topology of uniform convergence on $\bar{D}$. The latter condition is natural inasmuch as it is satisfied whenever the inner product $\langle\, ,\rangle_H$ is given by integration against a finite positive measure supported in $\bar{D}$.

The restriction of an analytic functional $T \in \mathcal{O}'(D)$ to H is continuous and has the form T_f, where for all $g \in H$,

$$T_f g = \langle g, f\rangle_H$$

for some $f \in H$. That T_f is continuous in the topology of $\mathcal{O}(D)$ means that there are a constant C and a compact set $K \subset D$ such that for every $g \in H$,

$$|Tg| = |T_f g| = |\langle g, f\rangle_H| \leqslant C\|g\|_K .$$

Thus K is a *weak carrier* of the restriction of T to H. Suppose $f \in H$ and T_f

is continuous in the $\mathcal{O}(D)$-topology on H. The Hahn-Banach theorem asserts that T_f can be extended to be a continuous functional on $\mathcal{O}(D)$; let T denote the extended functional. If H is dense in $\mathcal{O}(D)$, the extension is unique. Equivalently, if H is dense in $\mathcal{O}(D)$, the map $\Phi\colon \mathcal{O}'(D) \to H$ given by

$$T \to T_f \in H' \to f \in H$$

is injective. Under these conditions, a function f in the image of Φ will be said to *represent an analytic functional on* D, or less precisely, to *be an analytic functional on* D. Our main object is to study the image of the map Φ.

The condition that $i\colon H \hookrightarrow \mathcal{O}(D)$ be continuous is satisfied for Hilbert spaces of the form $H = L^2(\mu) \cap \mathcal{O}(D)$, where μ is a finite positive measure supported in $\bar{D}$. More generally, by the closed graph theorem, *if* $H \subset \mathcal{O}(D)$ *is a Hilbert space, the inclusion* i *is continuous if and only if for each* $p \in D$, *the evaluation functional* $e_p\colon H \to \mathbf{C}$ *defined by*

$$e_p(f) = f(p)$$

is continuous.

From the general theory of separable Hilbert spaces of functions (see, e.g., [35]), it is known that when all evaluation functionals are continuous, there exists a unique kernel function $k(z, w)\colon D \times D \to \mathbf{C}$ which has the reproducing property that for every $f \in H$ and $z \in D$,

$$f(z) = \langle f, k(\cdot, z)\rangle_H .$$

Both $k(\cdot, w)$ and $\overline{k(z, \cdot)}$ are elements of H, and $k(z, w) = \overline{k(w, z)}$. If $\{\varphi_n\}_{n=1}^{\infty}$ is any orthonormal basis for H, then $k(z, w)$ can be written

$$k(z, w) = \sum_{n=1}^{\infty} \varphi_n(z)\,\overline{\varphi_n(w)} .$$

For suitable domains D and embedded Hilbert spaces H, the map $\Phi\colon \mathcal{O}'(D) \to H$ has image contained in $\mathcal{O}(\bar{D})$. This condition turns out to be equivalent to a certain extendibility property of the kernel function $k(z, w)$ for H:

THEOREM II.1. *Let* D *be a relatively compact domain in* $\mathbf{C}^N$ *with* $\mathcal{C}^2$ *boundary, and* H *a dense Hilbert subspace of* $\mathcal{O}(D)$ *for which the inclusions* $i\colon H \hookrightarrow \mathcal{O}(D)$ *and* $j\colon \mathcal{O}(\bar{D}) \hookrightarrow H$ *hold and are continuous. The following conditions are equivalent:*

(1) *For each fixed* $w \in D$, $k(z, w) \in \mathcal{O}(\bar{D})$.

(2) *For any compact set $K \subset D$, there is a domain D_K containing $\bar{D}$ such that*

(i) $k(z, w) \in \mathcal{O}(D_K)$ *for fixed* $w \in K$.

(ii) $k(z, w)$ *is continuous on* $D_K \times K$.

(3) *If* $f \in H$ *and* $T_f: H \to \mathbf{C}$ *is continuous in the* $\mathcal{O}(D)$*-topology, then* $f \in \mathcal{O}(\bar{D})$.

NOTE. We denote by bD the boundary of a domain D.

REMARK. We will show below that when $D \subset\subset \mathbf{C}^N$ is strictly pseudoconvex with real-analytic boundary and H is the Bergman space $H_2(D)$, the Bergman kernel $k(z, w)$ satisfies (1). Without the hypothesis of real-analyticity, (1) usually fails.

The proof of Theorem II.1 involves several steps. In the following lemmas, $P_R(z_0)$ will denote the polydisc in $\mathbf{C}^N$ of polyradius $(R, \dots, R)$ about z_0.

LEMMA II.2. *If* $f(z, w)$ *is holomorphic in the domain*

$$(P_{R_0}(0) \times P_{r_0}(0)) \cup (P_{R_1}(0) \times P_{r_1}(0)) \subset \mathbf{C}^N \times \mathbf{C}^N,$$

where $R_1 < R_0$ *and* $r_1 > r_0$, *then for every* $r \in (r_0, r_1)$, *there exists* $R > R_1$ *so that* f *is holomorphic in* $P_R(0) \times P_r(0)$. *More precisely, if* $r = r_0^t r_1^{(1-t)}$ *for some* $t \in (0, 1)$, *then* $R_0^t\ R_1^{(1-t)}$ *is a satisfactory choice for* R.

PROOF. The domain of convergence of the power series about 0 which represents $f(z, w)$ is logarithmically convex. (See, e.g., [1], p. 21.)

The following elementary lemma is certainly well known, but as we cannot find an explicit statement in the literature, we include a proof.

LEMMA II.3. *Let* $f(z, w)$ *be holomorphic in* $P_a(0) \times P_b(0) \subset \mathbf{C}^N \times \mathbf{C}^N$, *where* a *and* b *are positive numbers. For each* $w \in P_b(0)$, *assume that* $f(\cdot, w)$ *extends holomorphically to the polydisc* $P_{R(w)}(0)$, *where* $R(w) > a$. *Then for every compact set* $K \subset P_b(0)$, *there exists* $R(K) > a$ *such that* $f(z, w)$ *is holomorphic in a neighborhood of* $P_{R(K)}(0) \times K$.

PROOF. We assume for simplicity that $a = b = 1$. Let $U \subset P_1(0)$ be a neighborhood of 0 chosen so small that for any $w_0 \in U$, there is a polydisc $P_{r_1}(w_0) \subseteq P_1(0)$ such that $K \subset\subset P_{r_1}(w_0)$.

The function $f(z, w)$ has the power series expansion

$$f(z, w) = \sum_{j,k \in \mathbf{N}^N} a_{jk} z^j w^k$$

valid in all of $P_1(0)\times P_1(0)$. Rearranging, we can write

$$f(z, w) = \sum_{j\in \mathbf{N}^N} a_j(w) z^j ,$$

where

$$a_j(w) = \sum_{k\in \mathbf{N}^N} a_{jk} w^k .$$

The functions $a_j(w)$ are holomorphic and defined for all $w\in P_1(0)$.

For non-negative integers l and m, we set

$$S_{lm} = \left\{ w\in U\colon |a_j(w)| \leqslant l \left(\frac{1}{1+1/m}\right)^{|j|}, \qquad \text{for all } j\in \mathbf{N}^N \right\} .$$

The sets S_{lm} are closed subsets of U, and by the hypothesis of the lemma, they cover U. By the Baire category theorem some S_{lm} contains a polydisc $P_{r_0}(w_0)$. The definition of S_{lm} shows that the series

$$\sum_{j\in \mathbf{N}^N} a_j(w) z^j$$

converges uniformly in z and w on compact subsets of $P_{1+1/m}(0)\times P_{r_0}(w_0)$; hence f is holomorphic there.

We have shown that f is holomorphic in

$$(P_1(0)\times P_{r_1}(w_0)) \cup (P_{1+1/m}(0)\times P_{r_0}(w_0)) .$$

(Recall that r_1 was chosen so that $K \subset\subset P_{r_1}(w_0)$.) By Lemma II.2, there exists $R(K) > 1$ so that f is holomorphic on $P_{R(K)}(0)\times P_{r_1}(w_0)$. This completes the proof.

LEMMA II.4. *Let $D\subset\subset \mathbf{C}^N$ be a domain with bD of class C^2. If $p\in bD$, then there is a unitary coordinate system $(z_1, \dots, z_n)$ for $\mathbf{C}^N$ and in these coordinates a polydisc $P_\delta(z_0)\subset D$ so that $p\in \overline{P_\delta(z_0)}\subset \overline{D}$.*

PROOF. Since bD is of class C^2, there is a ball contained in D with boundary internally tangent to bD at p, whence the result.

LEMMA II.5. *Let D, K, and $k(z, w)$ be as in the theorem, and assume that condition* (1) *holds. For each $p\in bD$ there exists a neighborhood $U(p)$ of p such that for each $w\in K$, $k(z, w)$ extends holomorphically to $D\cup U(p)$.*

PROOF. Fix a point $w_0 \in K$ and let $P_\varepsilon(w_0)$ be a polydisc neighborhood of w_0. Using Lemma II.4, let $P_\delta(z_0) \subset D$ be a polydisc with $p \in \overline{P_\delta(z_0)}$. Then $k(\cdot, w)$ is holomorphic in $P_\delta(z_0)$ and $k(z, \cdot)$ is conjugate holomorphic in $P_\varepsilon(w_0)$. If we take $P_\varepsilon(w_0)$ to have the analytic structure conjugate to that which it inherits from $\mathbf{C}^N$, we can regard $k(z, w)$ as a function on $P_\delta(z_0) \times P_\varepsilon(w_0)$ which is separately holomorphic in z and w. Since $k(z, w)$ is in $C^\infty(D \times D)$, we can apply Osgood's Lemma to conclude that $k(z, w)$ is jointly holomorphic in z and w on $P_\delta(z_0) \times P_\varepsilon(w_0)$. Condition (1) together with Lemma II.3 implies that there exists $\delta' > \delta$ so that $k(z, w)$ extends to be jointly holomorphic in $P_{\delta'}(z_0) \times P_{\varepsilon/2}(w_0)$. In particular, for $w_0 \in K$ there is an open neighborhood U_{w_0} of p to which $k(\cdot, w)$ extends for all $w \in P_{\varepsilon/2}(w_0)$. The compactness of K implies that there is an open neighborhood $U(p)$ of p to which $k(\cdot, w)$ extends for all $w \in K$. This is the desired assertion.

REMARK. Lemma II.3 and the proof of Lemma II.5 show that, in fact, there is an open set $W_p \subset D$, $W_p \supset K$, so that $k(z, w)$ is jointly holomorphic on $(D \cup U(p)) \times W_p$, where, as before, the conjugate-holomorphic structure is taken in the second factor.

PROOF OF THEOREM II.1. We first prove the equivalence of (1) and (2). Assume condition (1) holds. For each $p \in bD$, let $U(p)$ and W_p be chosen as in the previous remark. The compactness of $\overline{D}$ implies that by shrinking, if necessary, the $U(p)$ can be taken to be polydiscs, such that $U(p) \cap U(q) \cap D$ is non-empty whenever $U(p) \cap U(q)$ is non-empty. Further, we can choose finitely many p_α, $\alpha = 1, \dots, \lambda$, such that

$$\overline{D} \subset \bigcup_{\alpha=1}^{\lambda} (D \cup U(p_\alpha)) .$$

The principle of analytic continuation guarantees that the extensions of $k(z, w)$ to the various $D \cup U(p_\alpha)$ are compatible. Setting

$$W = \bigcap_{\alpha=1}^{\lambda} W_{p_\alpha},$$

we have $K \subset\subset W$. With the conjugate holomorphic structure on W, $k(z, w)$ is jointly holomorphic in $D_K \times W$, where

$$D_K = \bigcup_{\alpha=1}^{\lambda} (D \cup U(p_\alpha)) .$$

With the usual analytic structure on W, $k(z, w)$ is jointly real-analytic, holomorphic in z, and conjugate holomorphic in w. Hence (2) holds. That (2) implies (1) is trivial.

Next we show that (2) implies (3). Suppose $f \in H$ and $T_f: H \to \mathbf{C}$ is $\mathcal{O}(D)$-continuous. Then by the Hahn-Banach theorem, T_f extends to a continuous linear functional on $\mathcal{C}(D)$, the space of continuous complex-valued functions on D. The Riesz representation theorem asserts that there is a compactum $K \subset D$ and a regular Borel measure μ supported on K so that for all $g \in H$,

$$T_f g = \int_K g \, d\mu \, .$$

Let $M(K)$ be the space of complex Borel measures supported on K, $|\mu|$ the total variation of the measure μ. The norm-closed ball

$$B = \{\nu \in M(K): |\nu| \leqslant |\mu|\}$$

of radius $|\mu|$ in $M(K)$ is compact in the weak-* topology on $M(K)$. Since B is also convex, the Krein-Milman theorem asserts that B is the closed convex hull of the set E of its extreme points. The set E consists of the measures on K of the form $\lambda\delta(p)$, where λ is a complex number of modulus $|\mu|$, and $\delta(p)$ is the unit point-mass at $p \in K$.

This means that there exist measures μ_n, $n \in \mathbf{N}$, on K of the form

$$\mu_n = \sum_{j=1}^{L_n} \lambda_{j,n} \delta(w_{j,n}) \, ,$$

where the L_n are positive integers, $\lambda_{j,n}$ are complex numbers, and $w_{j,n}$ are points in K, such that $\mu = \lim_{n\to\infty} \mu_n$, the limit being taken in the weak-* topology on $M(K)$. For each n,

$$\sum_{j=1}^{L_n} |\lambda_{j,n}| \leqslant |\mu| \, .$$

We define functions f_n, $n \in \mathbf{N}$, by setting

$$f_n = \sum_{j=1}^{L_n} \bar{\lambda}_{j,n} k(\cdot, w_{j,n}) \, .$$

Each f_n is holomorphic on D_K because $k(\cdot, w_{j,n})$ is. As $k(z, w)$ is continuous on $D_K \times K$, it is bounded on compact sets $K' \times K$. Hence if $K' \subset\subset D_K$ and $z \in K'$, then

$$|f_n(z)| = \left| \sum_{j=1}^{L_n} \lambda_{j,n} k(z, w_{j,n}) \right| \leqslant \sum_{j=1}^{L_n} |\lambda_{j,n}| \sup_{(z,w)\in K'\times K} |k(z,w)| \leqslant |\mu| \sup_{(z,w)\in K'\times K} |k(z,w)| \,.$$

Thus $\{f_n\}_{n=1}^{\infty}$ is a normal family on D_K, so there is a subsequence $\{f_{n_*}\}$ which converges uniformly on compact subsets of D_K to $f_* \in \mathcal{O}(D_K)$. In particular, the f_{n_*} converge uniformly to f_* on $\bar{D}$. For any $g \in H$,

$$\begin{aligned} T_f g = \langle g, f\rangle_H = \int_K g(w)\, d\mu(w) = \int_K \langle g, k(\cdot, w)\rangle_H\, d\mu(w) = \\ = \lim_{n_*\to\infty} \int_K \langle g, k(\cdot, w)\rangle_H\, d\mu_{n_*}(w) = \lim_{n_*\to\infty} \sum_{j=1}^{L_{n_*}} \lambda_{j,n} \delta(w_{j,n_*}) \langle g, k(\cdot, w)\rangle_H = \\ = \lim_{n_*\to\infty} \left\langle g, \sum_{j=1}^{L_{n_*}} \bar{\lambda}_{j,n_*} k(\cdot, w_{j,n_*}) \right\rangle_H = \lim_{n_*\to\infty} \langle g, f_{n_*}\rangle_H = \langle g, f_*\rangle_H \,. \end{aligned}$$

The last equality holds because of the requirement that $\mathcal{O}(\bar{D})$ be continuously contained in H. Since $f_{*|D}$ and f represent the same element of H', they coincide on D. Hence f admits an extension to D_K. This completes the proof that (2) implies (3).

To complete the proof, we show that (3) implies (1). The evaluation functional $e_w\colon H \to \mathbf{C}$ given by

$$e_w(f) = f(w)$$

is continuous in the $\mathcal{O}(D)$-topology. This functional is represented by a unique element of H, which by (3) extends holomorphically across bD. But for every $f \in H$,

$$f(w) = \langle f, k(\cdot, w)\rangle;$$

hence $k(\cdot, w)$ represents e_w, and so extends holomorphically across bD. This completes the proof of Theorem II.1.

REMARK. The hypothesis in Theorem II.1 that bD be of class $\mathcal{C}^2$ was used only in the proof that (1) implies (2), where its use was limited to finding, for each $p \in bD$, a polydisc $P \subset D$ with $p \in \bar{P}$. Thus (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) hold for arbitrary relatively compact domains in $\mathbf{C}^N$, and (1) $\Rightarrow$ (2) holds whenever polydiscs P can be found as above. In particular, if P is itself a polydisc, then Theorem II.1 holds as stated.

III. – Analytic functionals in Bergman spaces.

For D a relatively compact domain in $\mathbf{C}^N$, the Bergman space $H_2(D)$ is the space of functions $f \in \mathcal{O}(D)$ such that

$$\|f\|^2_{H_2(D)} = \int_D |f|^2$$

is finite. The space $H_2(D)$ satisfies the requirements of the last section.

The orthogonal projection $P\colon L^2(D) \to H_2(D)$ is known as the *Bergman projection* and is given by

$$Pf(w) = \int_D fk(w, \cdot)\, dm_{2N}$$

where $k(z, w)$ is the reproducing kernel for $H_2(D)$, i.e., the *Bergman kernel* for the domain D.

(*Here and in the sequel, dm_k denotes Lebesgue measure in dimension k. Any integral expression omitting an explicit measure is understood to be with respect to Lebesgue measure in the appropriate dimension.*)

REMARK. If D_1 and D_2 are domains in $\mathbf{C}^N$, and if $\Phi\colon D_1 \to D_2$ is a diffeomorphism with inverse Φ^{-1}, then the mapping $\Phi_*\colon L^2(D_1) \to L^2(D_2)$ given by

$$f \to (f \circ \Phi^{-1}) \cdot \mathrm{ac}\, \Phi^{-1}$$

is an isometry. (Jac Φ^{-1} denotes the Jacobian of the mapping Φ^{-1}.) If Φ is a biholomorphism, then Φ_* effects an isometry of $H_2(D_1)$ and $H_2(D_2)$.

The change-of-variables formula together with the expression for the Bergman kernel in terms of an orthonormal basis for $H_2(D)$ implies that if D_1 and D_2 are bounded domains in $\mathbf{C}^N$, $\Phi\colon D_1 \to D_2$ a biholomorphism, and $K_{D_j}\colon D_j \times D_j \to \mathbf{C}$ the Bergman kernel for D_j, $j = 1, 2$, then

$$k_{D_2}(\Phi(z), \Phi(w)) = k_{D_1}(z, w)\, \mathrm{Jac}\, \Phi^{-1}(\Phi(z)) \cdot \overline{\mathrm{Jac}\, \Phi^{-1}(\varphi(w))}\,.$$

For proofs of these elementary facts, see, e.g., [11, Ch. I].

The mapping Φ_* commutes with projection onto the respective subspaces of holomorphic functions in $L^2(D_j)$. More precisely, for $j = 1, 2$, let $P_j\colon L^2(D_j) \to H_2(D_j)$ be the Bergman projection. By direct calculation, we have:

LEMMA III.1. *If $h \in L^2(D_1)$, then*

$$(P_1 h \circ \Phi^{-1}) \cdot \operatorname{Jac} \Phi^{-1} = P_2(\operatorname{Jac} \Phi^{-1} \cdot (h \circ \Phi^{-1})) .$$

It is easily seen that $\Phi_*(\mathcal{C}_0^\infty(D_1)) = \mathcal{C}_0^\infty(D_2)$. The connection between the Bergman projection and analytic functionals follows from this and from a result proved in its present form by Lelong ([25], p. 39):

THEOREM III.2. *Let D be a bounded domain in $\mathbf{C}^N$, $T\colon \mathcal{O}(D) \to \mathbf{C}$ an analytic functional on D, and $K \subset\subset D$ a weak carrier of T. For any neighborhood U of K there exists a function $h \in \mathcal{C}_0^\infty(U)$ so that for all $f \in \mathcal{O}(D)$,*

$$(*) \qquad Tf = \int_D fh .$$

REMARK. It is clear that if $h \in \mathcal{C}_0^\infty(D)$, then $(*)$ defines an analytic functional on D which is weakly carried by $\operatorname{supp} h$.

The connection with Bergman spaces is as follows:

THEOREM III.3. *Let D be a relatively compact domain in $\mathbf{C}^N$ and let $P\colon L^2(D) \to H_2(D)$ be the Bergman projection. If $f \in H_2(D)$, then the linear functional*

$$T_f\colon g \to \int_D \bar{f} g$$

is continuous in the $\mathcal{O}(D)$-topology on $H_2(D)$ if and only if $f = Ph$ for some $h \in \mathcal{C}_0^\infty(D)$.

PROOF. If T_f is $\mathcal{O}(D)$-continuous, then by the previous theorem there is an $h \in \mathcal{C}_0^\infty(D)$ such that for all $g \in H_2(D)$,

$$\int_D \bar{f} g = \int_D \bar{h} g = \int_D \overline{Ph}\, g .$$

Hence, $f = Ph$.

Conversely, if $f = Ph$ then for all $g \in H_2(D)$,

$$\left| \int_D \bar{f} g \right| = \left| \int_D \bar{h} g \right| \leqslant m_{2N}(D) \|h\|_D \|g\|_{\operatorname{supp} h} .$$

Since D has finite measure, we conclude that f represents an analytic funcional with compact weak carrier $\operatorname{supp} h$.

COROLLARY III.4. *Let D_1, D_2, and the biholomorphism Φ be as in the remarks above. The mapping Φ_* is a bijection of* $\{f \in H_2(D_1)\colon T_f$ is an analytic functional$\}$ *and* $\{f \in H_2(D_2)\colon T_f$ is an analytic functional$\}$.

PROOF. Theorem III.3 and Lemma III.1.

In the sequel, we study the relationship between analytic functionals and functions which extend holomorphically across bD. We shall make use of the following fact:

LEMMA III.5. *Suppose bounded domains D_1 and D_2 in $\mathbf{C}^N$ are biholomorphic via a mapping Φ which extends to a biholomorphism of domains D'_1 and D'_2, such that $D_j \subset\subset D'_j$ for $j = 1, 2$. Let $S_j = \{f \in H_2(D_j)\colon T_f\colon H_2(D_j) \to \mathbf{C}$ is $\mathcal{O}(D)$-continuous$\}$.*

If $S_1 \subseteq \mathcal{O}(\bar{D}_1)$, then $S_2 \subseteq \mathcal{O}(\bar{D}_2)$.
If $S_1 \supseteq \mathcal{O}(\bar{D}_1)$, then $S_2 \subseteq \mathcal{O}(\bar{D}_2)$.

PROOF. By Corollary III.4, $S_2 = \Phi_*(S_1)$. If $f \in \mathcal{O}(\bar{D}_1)$, $\Phi_* f = (f \circ \Phi^{-1})\cdot$ $\cdot \operatorname{Jac} \Phi^{-1} \in \mathcal{O}(\bar{D}_2)$. The same argument applied to $(\Phi^{-1})_*$ shows that

$$\Phi_*(\mathcal{O}(\bar{D}_1)) = \mathcal{O}(\bar{D}_2)\,.$$

The assertion follows immediately.

The next lemma is a special case of the main theorem, and is used essentially in the proof of that result.

LEMMA III.6. *Let $U = \{z \in \mathbf{C}\colon |z| < 1\}$. A function $f \in H_2(U)$ represents an analytic functional on U if and only if $f \in \mathcal{O}(\bar{U})$.*

PROOF. The Bergman kernel for U has the form

$$k(z, w) = \frac{1}{\pi}(1 - z\bar{w})^{-2}.$$

Thus for real R,

$$k(Rz, w) = k(z, Rw) \tag{$*$}$$

whenever both sides are defined.

Suppose that T_f is $\mathcal{O}(D)$-continuous. For fixed w with $|w| < A < 1$, the function $k(z/A, Aw)$ is holomorphic in z, and by $(*)$, is an extension across bU of $k(z, w)$. By Theorem II.1, $f \in \mathcal{O}(\bar{U})$.

Conversely, suppose $f \in \mathcal{O}(a\bar{U})$ for some $a > 1$. Define $\Psi\colon \mathbf{C} \to \mathbf{C}$ by

$$\psi(z) = \begin{cases} f(a^2 z)\cdot a^2 & \text{if } z \in \dfrac{1}{a} U \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, if $P: L^2(U) \to H_2(U)$ is the orthogonal projection,

$$P\psi(z) = a^2 \int_{(1/a)U} f(a^2 w)\, k(z, w)\, dm_2(w) = \int_U f(at)\, k\left(\frac{z}{a}, t\right) dm_2(t) = f(z)\,,$$

by (*) and the reproducing property of $k(z, w)$, where the change of variable $t = aw$ was performed. Thus f is the projection of a compactly supported L^2-function on U; by the proof of Theorem III.3, f represents an analytic functional on U.

REMARK. If D is a complete Reinhardt domain in $\mathbf{C}^N$, the monomials $\{z^\alpha: \alpha \in \mathbf{N}^N\}$ form a complete orthogonal set in $H_2(D)$ [11, p. 71]. It follows that (*) holds for $k_D(z, w)$ and hence that Lemma III.6 is valid for such domains.

We now consider plane domains other than the unit disc. All the domains we study will be bounded, with boundaries consisting of finitely many closed Jordan curves. For such domains D, Runge's theorem asserts that the rational functions with poles bounded away from $\bar{D}$ are dense in $\mathcal{O}(D)$. Thus $\mathcal{O}(\bar{D})$ and hence $H_2(D)$ are dense in $\mathcal{O}(D)$, so the mapping

$$T \in \mathcal{O}(D) \to T_f \in H_2'(D) \to f \in H_2(D)$$

is injective.

Suppose D is a simply connected domain whose boundary is a real-analytic simple closed curve. By the Riemann mapping theorem and the Schwarz reflection principle, D is biholomorphically equivalent to the unit disc *via* an extendible biholomorphism. Lemmas III.5 and III.6 imply that there is a one-to-one correspondence between $\mathcal{O}'(D)$ and $\mathcal{O}(\bar{D})$. We shall prove the analogous result in the more general case that D is of finite connectivity.

LEMMA III.7. *Let $A = A(r, R) \subset \mathbf{C}^1$ be the annulus $\{z: r < |z| < R\}$, where $0 < r < R < \infty$. Suppose f is holomorphic on an annulus A' with $A \subset\subset A'$. Then there exist a constant C and a compact $K \subset\subset A$ such that for all $g \in H_2(A)$,*

$$\left|\int_A \bar{f} g\right| \leqslant C \|g\|_K\,,$$

i.e., f is an analytic functional on A.

PROOF. Let $D(R) = \{|z| < R\}$, $C(R) = \{|z| = R\}$. By Cauchy's integral formula, we can write $f = f_1 + f_2$, where for some $\varepsilon > 0$,

$$f_1(z) = \frac{1}{2\pi i} \int_{C(R+\varepsilon)} \frac{f(\xi)}{\xi - z}\, d\xi\,,$$

$$f_2(z) = \frac{-1}{2\pi i} \int_{C(r-\varepsilon)} \frac{f(\xi)}{\xi - z}\, d\xi\,.$$

Then f_1 is holomorphic in $D(R+\varepsilon)$, and f_2 is holomorphic in $\mathbf{C}_\infty \setminus \overline{D(r-\varepsilon)}$, with $f_2(\infty) = 0$. Similarly, $g = g_1 + g_2$, where

$$g_1 \in H_2(D(R)) \qquad \text{and} \qquad g_2 \in \mathcal{O}(\mathbf{C}_\infty \setminus \bar{D}_r) \cap H_2(A)\,.$$

We have

$$\begin{aligned}\int_A \bar{f} g &= \int_A \bar{f}_1 g_1 + \int_A \bar{f}_1 g_2 + \int_A \bar{f}_2 g_1 + \int_A \bar{f}_2 g_2 \\ &= I_1 \quad + \quad I_2 + \quad I_3 + \quad I_4\,.\end{aligned}$$

First,

$$|I_1| \leqslant \left| \int_{D(R)} \bar{f}_1 g_1 \right| + \left| \int_{D(r)} \bar{f}_1 g_1 \right| \leqslant C_1 \|g_1\|_{D(\gamma R)}$$

for some $\gamma < 1$, by Lemma III.6. For $z \in \overline{D(\gamma R)}$,

$$|g_1(z)| = \frac{1}{2\pi} \left| \int_{C(((\gamma+1)/2)R)} \frac{g(\xi)}{\xi - z}\, d\xi \right| \leqslant C_2 \|g\|_{C(((\gamma+1)/2)R)}\,,$$

where C_2 is independent of g.

To compute I_2 and I_3, we observe that integration in polar coordinates shows that if δ_n^m is the Kronecker delta, $m, n \in \mathbf{Z}$, then

$$\int_A \bar{z}^n z^m = \delta_n^m \frac{\pi}{n+1} (R^{2n+2} - r^{2n+2})\,, \qquad \text{if } n \neq -1\,.$$

Writing

$$f_1 = \sum_{n=0}^{\infty} a_n z^n, \qquad g_2 = \sum_{m=1}^{\infty} \frac{b_m}{z^m}\,, \qquad I_2 = \int_A \bar{f}_1 g_2 = \int_A \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \bar{a}_n b_m \frac{\bar{z}^n}{z^m} = 0\,.$$

Similarly, $I_3 = 0$.

To compute I_4, we observe that there exists $\lambda > 1$ such that $f_2(z\lambda^{-1})$ is holomorphic near $\mathbf{C}_\infty \setminus D_r$. Writing $f_2(z) = \sum_{n=1}^{\infty} d_n/z^n$ and using the orthogonality of the z^j, we have

$$I_4=\int_A \bar{f}_2 g_2=\sum_{n=1}^{\infty}\int_A \frac{\bar{d}_n b_n}{|z|^{2n}}=\sum_{n=1}^{\infty}\int_A \frac{\bar{d}_n b_n}{(\bar{z}/\lambda)^n(z\lambda)^n}=$$

$$=\int_A\left(\overline{\sum_{m=1}^{\infty}\frac{d_m}{(z/\lambda)^m}}\right)\left(\sum_{n=1}^{\infty}\frac{b_n}{(\lambda z)^n}\right)=\int_A \bar{f}_2\left(\frac{z}{\lambda}\right)g_2(\lambda z)\,.$$

Hence,

$$|I_4|\leqslant C_3\|f_2\|_{A(r/\lambda,R/\lambda)}\|g_2\|_{A(r\lambda,R\lambda)}=C_4\|g_2\|_{A(r\lambda,R\lambda)}\,,$$

where C_4 is independent of g. But for $|z|>r\lambda$,

$$|g_2(z)|=\frac{1}{2\pi}\left|\int_{C(((\lambda+1)/2)r)}\frac{g(\xi)}{\xi-z}\,d\xi\right|\leqslant C_5\|g\|_{C(((\lambda+1)/2)r)}\,.$$

Thus

$$|I|\leqslant|I_1|+|I_4|\leqslant C_2\|g\|_{C(((\gamma+1)/2)R)}+C_5\|g\|_{C(((\lambda+1)/2)r)}\leqslant C\|g\|_{A(((\lambda+1)/2)r,((\gamma+1)/2)R)}$$

by the maximum modulus principle. This completes the proof.

We use the lemma to prove our next result:

THEOREM III.8. *Let D be a bounded plane domain whose boundary consists of finitely many disjoint simple closed real-analytic curves. A function $f\in H_2(D)$ represents an analytic functional on D if and only if $f\in\mathcal{O}(\bar{D})$.*

PROOF. It is known (see, e.g., [13], p. 237) that every plane domain of finite connectivity can be mapped univalently (by Φ) onto a domain whose boundary consists of finitely many points and finitely many disjoint circles. Since in our case no component of bD is a point, all boundary components of the image $\Phi(D)$ are circles. Since the boundary components of D and $\Phi(D)$ are real-analytic arcs, the Schwarz reflection principle implies that both Φ and Φ^{-1} extend holomorphically across bD and $b(\Phi(D))$. The resulting extension of Φ is a biholomorphism from a neighborhood of $\bar{D}$ to a neighborhood of $\overline{\Phi(D)}$. By Lemma III.5, it suffices to prove our theorem in the case that D has circular boundary arcs.

Let bD consist of circles γ_j, $j=1,\dots,k$, which bound domains D_j. Let γ_1 be the outer boundary of D.

Suppose $f\in\mathcal{O}(\bar{D})$. Let A_j, $j=1,\dots,k$, be disjoint annuli, $A_j\subset D$, with γ_j one component of bA_j. By the lemma, there exist constants C_j and compacta $K_j \subset\subset A_j$ such that for all $g \in H_2(D)$,

$$\left|\int_{A_j} \bar{f} g\right| \leqslant C_j \|g\|_{K_j} .$$

Hence,

$$\left|\int_D \bar{f} g\right| \leqslant \sum_{j=1}^{k}\left|\int_{A_j} \bar{f} g\right| + \left|\int_{D \setminus \bigcup_{j=1}^{k} A_j} \bar{f} g\right| \leqslant \sum_{j=1}^{k} C_j \|g\|_{K_j} + C' \|g\|_{\bar{D} \setminus \bigcup_{j=1}^{k} A_j} .$$

Let $K = \left(\bar{D} \setminus \bigcup_{j=1}^{k} A_j\right) \cup \bigcup_{j=1}^{k} K_j$. Then $K \subset\subset D$ and

$$\left|\int_D \bar{f} g\right| \leqslant C \|g\|_K ,$$

where C is independent of g. Thus f is an analytic functional.

Conversely, suppose f is an analytic functional on D. Using the Cauchy integral formula, we can write $f = f_1 + f_2$, where

$$f_1 \in H_2(D_1) , \quad f_2 \in \mathcal{O}\left(\mathbf{C}_\infty \setminus \bigcup_{j=2}^{k} D_j\right) \cap H_2(D) ,$$

and $f_2(\infty) = 0$. We will show that f_1 extends to a domain larger than D_1; the proof that f_2 extends is similar.

We assume without loss of generality that the outer boundary is the unit circle about the origin. Let $r < 1$ be chosen so that the annulus $A = A_{r,1}(0) \subset D$. In A, $f_2(z)$ can be written

$$f_2(z) = \sum_{m=1}^{\infty} \frac{b_m}{z^m} .$$

Hence for $n \geqslant 0$,

$$\int_A \bar{f}_2 z^n = \sum_{m=1}^{\infty} \int_A \bar{b}_m \frac{z^n}{\bar{z}^m} = 0 ,$$

by the orthogonality on A of $\{z^j : j \in \mathbf{Z}\}$. This implies that

$$\int_D \bar{f}_2 z^n = \int_{D \setminus A} \bar{f}_2 z^n .$$

Therefore, writing $f_1(z) = \sum_{m=0}^{\infty} a_m z^m$,

$$\int_D \bar{f} z^n = \int_D \bar{f}_1 z^n + \int_{D \setminus A} \bar{f}_2 z^n = \int_{D_1} \bar{f}_1 z^n - \sum_{j=2}^{k} \int_{D_j} \bar{f}_1 z^n + \int_{D \setminus A} \bar{f}_2 z^n .$$

This implies that for some constants C_j and some $R_1 < 1$,

$$\frac{\pi}{n+1}|a_n| = \left|\int_{D_1} \bar{f}_1 z^n\right| \leqslant \left|\int_D \bar{f} z^n\right| + \sum_{j=2}^{k}\left|\int_{D_j} \bar{f}_1 z^n\right| + \left|\int_{D\setminus A} \bar{f}_2 z^n\right| \leqslant$$

$$\leqslant C_1 \|z\|^n_{\{|z|<R_1\}} + \sum_{j=2}^{k} C_j \|f_1\|_{\bar{D}_j}\|z^n\|_{\bar{D}_j} + \|f_2\|_{H_2(D\setminus A)}\|z^n\|_{H_2(D\setminus A)},$$

where the last inequality uses the fact that f is an analytic functional on D and the Cauchy-Schwarz inequality for the space $H_2(D\setminus A)$. Since $D_j \subset \{|z| < R_2\}$, $j = 1, \ldots, k$, for some $R_2 < 1$, and

$$\|z^n\|_{H_2(D\setminus A)} \leqslant m_2(D\setminus A) \cdot R_3^n$$

for some $R_3 < 1$, setting $R = \max\{R_1, R_2, R_3\}$ gives

$$\frac{\pi}{n+1}|a_n| \leqslant C' \|z^n\|_{\{|z|<R\}} = C'R^n,$$

where C' is independent of n. Thus the power series for $f_1(z)$ converges for $|z| < R^{-1}$. This completes the proof of Theorem III.8.

We are now able to state and prove our main result (Theorem III.14) that the one-to-one correspondence between analytic functionals and extendible functions already demonstrated for real-analytic bounded plane domains continues to hold for strictly pseudoconvex domains in $\mathbf{C}^N$ with boundaries that are $(2N-1)$-dimensional real-analytic submanifolds of $\mathbf{C}^N$. It will also be seen that $\mathcal{O}'(D)$ and $\mathcal{O}(\bar{D})$ are isomorphic as topological vector spaces.

We shall repeatedly use the following fact: *if $D \subset\subset \mathbf{C}^N$ is strictly pseudoconvex with C^2 boundary, then there exists a domain Ω such that $D \subset\subset \Omega$ and $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(D)$.* (See [20, Th. 1.4.1] and [16, Th. 1.3].)

THEOREM III.9. *Let D be a convex domain in $\mathbf{C}^N$ with real-analytic boundary. Let f be a holomorphic function defined on a domain D_1 such that $D \subset\subset D_1$. Then f represents an analytic functional on D, i.e., there are a compact set $K \subset\subset D$ and $C > 0$ such that for all g in $H_2(D)$,*

$$\left|\int_D \bar{f} g\right| \leqslant C\|g\|_K.$$

THEOREM III.10. *Let $D \subset\subset \mathbf{C}^N$ be strictly pseudoconvex with bD a real-analytic submanifold of $\mathbf{C}^N$ of dimension $(2N-1)$. Let f be a holomorphic* *function defined on a domain* D_1 *with* $D \subset\subset D_1$. *Then* f *represents an analytic functional on* D.

REMARK. Both theorems are, in general, false without the hypothesis of real-analyticity, as we will show below. Observe also that Theorem III.9 does not require that D be strictly convex, so *strict* pseudoconvexity is not a necessary condition, at least when D is convex. On the other hand, the proof of Theorem III.10 uses the property of strictly pseudoconvex domains that boundary points are peak points. For *strictly* convex domains, the result of Theorem III.9 is a corollary of Theorem III.10.

PROOF OF THEOREM III.9. It suffices to show that there exist compact K and $C > 0$ such that

$$\left|\int_D \bar{f} g\right| \leqslant C\|g\|_K \tag{1}$$

for all g in a dense subset of $H_2(D)$. Since D is convex, the polynomials are dense in $H_2(D)$, so we may assume that g is a polynomial. Let $\tilde{D}$ be any fixed domain with $D \subset\subset \tilde{D}$. Since both sides of (1) are positively homogeneous in g, it suffices to prove (1) for polynomials g with $\|g\|_{\tilde{D}} = 1$; such g will be called *admissible.*

We assume without loss of generality that $0 \in D$. For every g in $H_2(D)$, the integral in (1) can be represented as a double integral:

$$\int_D \bar{f} g = \int_{\alpha \in \mathbf{CP}^{N-1}} \left\{ \int_{\pi^{-1}(\alpha) \cap D} \gamma \right\} \omega^{N-1}$$

where γ is a suitable (1,1)-form on $\mathbf{C}^N$ and ω is the fundamental form of the Fubini-Study metric on $\mathbf{CP}^{N-1}$. The map $\pi\colon \mathbf{C}^N \setminus \{0\} \to \mathbf{CP}^{N-1}$ is the projection given in homogeneous coordinates on $\mathbf{CP}^{N-1}$ by

$$\pi(z_1, \ldots, z_N) = [z_1 : \ldots : z_N]\,.$$

Estimates for the fiber integrals lead to a proof of (1).

According to [14, p. 30],

$$\pi^* \omega = \frac{i}{2\pi} \partial\bar{\partial}(\log |z|^2) = \frac{i}{2\pi}\left(\frac{dz_1 \wedge d\bar{z}_1 + \ldots + dz_N \wedge d\bar{z}_N}{|z|^2} - \frac{\sum_{j=1}^N \bar{z}_j\, dz_j \wedge \sum_{j=1}^N z_j\, d\bar{z}_j}{|z|^4}\right).$$

A computation shows that

$$(\pi^* \omega)^{N-1} \wedge \partial\bar{\partial}(|z|^2) = \left(\frac{i}{2\pi}\right)^{N-1} (N-1)!\, |z|^{2-2N} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N\,. \tag{2}$$

By a version of Fubini's theorem for differential forms ([30, p. 210]), if H is any differentiable function defined near $\bar{D}$, we conclude from (2) that

$$(3)\qquad (N-1)!\left(\frac{i}{2\pi}\right)^{N-1}\int_{D\setminus\{0\}} H\, dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_N\wedge d\bar{z}_N = \\ = (N-1)!\,\frac{2}{i\pi^{N-1}}\int_D H\, dm_{2N} = \int_{\mathbf{CP}^{N-1}}\left\{\int_{\pi^{-1}(\alpha)\cap D} H|z|^{2N-2}\,\partial\bar{\partial}(|z|^2)\right\}\omega^{N-1}.$$

For any $\alpha\in\mathbf{CP}^{N-1}$, fix $u\in\pi^{-1}(\alpha)$ with $|u|=1$. Then $\pi^{-1}(\alpha)=\{\tau u\colon \tau\in\mathbf{C}\setminus\{0\}\}$. Taking τ to be the coordinate on $\pi^{-1}(\alpha)$, we have

$$\int_{\pi^{-1}(\alpha)\cap D} H|z|^{2N-2}\,\partial\bar{\partial}(|z|^2) = \int_{\pi^{-1}(\alpha)\cap D} H(\tau)|\tau|^{2N-2}\,d\tau\wedge d\bar{\tau} = \frac{2}{i}\int_{\alpha\cap D} H(\tau)|\tau|^{2N-2}\,dm_2(\tau)\,,$$

where $dm_2(\tau)$ is Lebesgue measure on the complex line α.

We apply this formula in the case $H=\bar{f}g$, where f is as in the hypothesis and g is admissible. Let $\alpha_0\in\mathbf{CP}^{N-1}$ be fixed. Since D is convex with real-analytic boundary, $D_{\alpha_0}=D\cap\alpha_0$ is also convex with real-analytic boundary. The restrictions of f and g to $D_1\cap\alpha_0$ are holomorphic. By the result in complex dimension one,

$$\left|\int_{D_{\alpha_0}}\bar{f}g|\tau|^{2N-2}\,dm_2(\tau)\right| = \left|\int_{D_{\alpha_0}}\overline{f\cdot\tau^{N-1}}\,g\cdot\tau^{N-1}\,dm_2(\tau)\right| \leqslant \\ \leqslant C_{\alpha_0}\|g\tau^{N-1}\|_{D(\alpha_0,\varepsilon_{\alpha_0})} < N_{\alpha_0}\|g\|_{D(\alpha_0,1/N_{\alpha_0})}$$

holds for all admissible g and some $N_{\alpha_0}\in\mathbf{N}$, where $D(\alpha_0,\varepsilon)=\{z\in D_{\alpha_0}\colon \operatorname{dist}(z, bD_{\alpha_0})\geqslant\varepsilon\}$. The second inequality holds because $\bar{D}$ is compact.

We consider the inequality

$$(4)\qquad \left|\int_{D_\alpha}\bar{f}gr^{2N-2}\,dm_2\right| < N_{\alpha_0}\|g\|_{D(\alpha,1/N_{\alpha_0})}.$$

This is valid for $\alpha=\alpha_0$ and arbitrary admissible g. For a fixed g, both sides of (4) are continuous in α, so there is a neighborhood U_{α_0} of α_0 in $\mathbf{CP}^{N-1}$ such that (4) holds for all $\alpha\in U_{\alpha_0}$.

The admissible g are a normal family on $\tilde{D}$ and are hence uniformly equicontinuous on $\bar{D}$. This implies that the neighborhoods U_{α_0} can be chosen so that (4) holds for all admissible g and all $\alpha\in U_{\alpha_0}$. Since $\mathbf{CP}^{N-1}$ is compact, finitely many such neighborhoods $U_{\alpha_1},\ldots,U_{\alpha_k}$ form an open cover of $\mathbf{CP}^{N-1}$. Taking $M = \max\{N_{\alpha_1}, \ldots, N_{\alpha_k}\}$, we have

$$\left|\int_{D_\alpha} \bar{f} g r^{2N-2}\, dm_2\right| \leqslant M \|g\|_{D(\alpha, 1/M)}$$

for all α and all admissible g.

Since D is convex, all the D_α meet bD transversally, so $K = \bigcup_{\alpha \in \mathbf{CP}^{N-1}} D(\alpha, 1/M)$ is relatively compact in D. From (3) and (4),

$$\begin{aligned}\left|\int_D \bar{f} g\right| = \frac{\pi^{N-1}}{(N-1)!}\left|\int_{\mathbf{CP}^{N-1}} \left\{\int_{D_\alpha} \bar{f} g\, r^{2N-2}\, dm_2\right\} \omega^{N-1}\right| &\leqslant \\ \leqslant \frac{\pi^{N-1}}{(N-1)!}\left|\int_{\mathbf{CP}^{N-1}}\right| \int_{D_\alpha} \bar{f} g\, r^{2N-2}\, dm_2\, |\omega^{N-1}| &\leqslant \\ \leqslant \frac{\pi^{N-1}}{(N-1)!}\left|\int_{\mathbf{CP}^{N-1}} M\|g\|_K\, \omega^{N-1}\right| \leqslant \frac{\pi^{N-1}}{(N-1)!} M\|g\|_K \left|\int_{\mathbf{CP}^{N-1}} \omega^{N-1}\right|.\end{aligned}$$

By the Wirtinger theorem [14, p. 31], $1/(N-1)!\int_{\mathbf{CP}^{N-1}} \omega^{N-1}$ is the (finite) volume of $\mathbf{CP}^{N-1}$. This completes the proof of Theorem III.9.

PROOF OF THEOREM III.10. Let Ω be a domain of holomorphy in $\mathbf{C}^N$ such that $D \subset\subset \Omega$ and $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(D)$. By density, it suffices to show that the analytic functional $\Lambda\colon \mathcal{O}(\Omega) \to \mathbf{C}$ given by

$$g \to \int_D g\bar{f}$$

is weakly carried by some compactum $K \subset\subset D$.

LEMMA III.11. *Let $p \in bD$. There is an open neighborhood U_p of p in Ω such Λ is weakly carried by $K_p = \bar{D}\setminus U_p$.*

PROOF. There is a complex line λ through p which meets bD transversally. To see this, we assume without loss of generality that $p = 0$, and define $\pi\colon bD\setminus\{0\} \to \mathbf{CP}^{N-1}$ by

$$z \to [z],$$

where $[z]$ denotes the complex line through 0 and z. The mapping π is smooth (indeed, real-analytic) so by Sard's theorem, almost all values of π in $\mathbf{CP}^{N-1}$ are regular values. The range of π on $bD\setminus\{0\}$ contains all lines

$\lambda \in \mathbf{CP}^{N-1}$ except perhaps those contained in $T_0(bD)$, and so contains almost all of $\mathbf{CP}^{N-1}$. Choose a line $\lambda_0 \in \mathbf{CP}^{N-1}$ transverse at 0 to bD. If U is a small neighborhood of λ_0, then each $\lambda \in U$ is transverse to bD at 0. As U is open, it contains a regular value, say λ_1, of π. This λ_1 meets bD transversally.

We assume further (by a change of coordinates in $\mathbf{C}^N$, if necessary) that $\lambda_0 = \{z: z_2 = \ldots = z_N = 0\}$ meets bD transversally. For $a = (a_2, \ldots, a_N) \in \mathbf{C}^{N-1}$ sufficiently near 0 in $\mathbf{C}^{N-1}$, say for $a \in V \subset \mathbf{C}^{N-1}$, V open, the line $\lambda_a = \{z: z_2 = a_2, \ldots, z_N = a_N\}$ is transverse to bD. The set $W = \{z \in \mathbf{C}^N: z \in \lambda_a, \text{ some } a \in V\}$ contains an open neighborhood of 0 in $\mathbf{C}^N$.

To prove that Λ is weakly carried away from 0, we show that there is a compact set $A \subset \bar{D}$, $0 \notin A$, and a number $C > 0$ such that for any $g \in \mathcal{O}(\Omega)$,

$$|\Lambda g| = \left|\int_D \bar{f} g\right| \leqslant C\|g\|_A .$$

Let $\tilde{D}$ be a domain chosen so that $D \subset\subset \tilde{D} \subset\subset \Omega$. It suffices to prove that the inequality holds for functions g with $\|g\|_{\tilde{D}} \leqslant 1$.

Let such a g be given. Then

$$|\Lambda g| = \left|\int_D \bar{f} g\right| \leqslant \left|\int_{D\setminus W} \bar{f} g\right| + \left|\int_{W\cap D} \bar{f} g\right| \leqslant C_1\|g\|_{\bar{D}\setminus W} + \left|\int_{W\cap D} \bar{f} g\right| \tag{1}$$

where C_1 is independent of g. To estimate the second integral, we use Fubini's theorem:

$$\int_{W\cap D} \bar{f} g \, dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N = \int_V \left\{ \int_{\lambda_{(z_2,\ldots,z_N)} \cap D} \bar{f} g \, dz_1 \wedge d\bar{z}_1 \right\} dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N .$$

By transversality, for each $z' = (z_2, \ldots, z_N) \in V$, $\lambda_{z'} \cap D$ is a plane domain with real-analytic boundary. In particular, for $z' = 0$ the duality result in plane domains implies that for some constant M independent of g,

$$\left|\int_{\lambda_0 \cap D} \bar{f} g \, dz_1 \wedge d\bar{z}_1\right| < M\|g\|_{(\lambda_0 \cap D)_{M^{-1}}} ,$$

where $(\lambda_{z'} \cap D)_\varepsilon = \{z \in \lambda_{z'} \cap D: \operatorname{dist}(z, b(\lambda_{z'} \cap D)) \geqslant \varepsilon\}$. We consider the inequality

$$\left|\int_{\lambda_{z'} \cap D} \bar{f} g \, dz_1 \wedge d\bar{z}_1\right| < M\|g\|_{(\lambda_{z'} \cap D)_{M^{-1}}} , \tag{2}$$

which is valid for $z' = 0$ and arbitrary g. For a *fixed* g, both sides of (2) are continuous in z', by the continuity of g and smoothness of bD. Thus for fixed g there is a neighborhood $\tilde{V}$ of 0 in $\mathbf{C}^{N-1}$ so that (2) holds for all $z' \in \tilde{V}$. We assume $\tilde{V} \subset\subset V$. The family $\{g \in \mathcal{O}(\Omega): \|g\|_{\tilde{D}} \leqslant 1\}$ is normal on $\tilde{D}$, hence uniformly equicontinuous on $\bar{D}$. Therefore $\tilde{V}$ can be chosen so that (2) holds for all such g and all $z' \in \tilde{V}$.

Since all the $\lambda_{z'}$, $z' \in \tilde{V}$, meet bD transversally, the set

$$\bigcup_{\tau \in \tilde{V}} (\lambda_{z'} \cap D)_{M^{-1}}$$

is relatively compact in $W \cap D$. Define

$$A = \bigcup_{z' \in \tilde{V}} (\lambda_{z'} \cap D)_{M^{-1}} \cup \bigcup_{z' \in V \setminus \tilde{V}} (\lambda_{z'} \cap \bar{D}) .$$

Then there is a constant C_2 so that for all $z' \in V$,

$$\Big| \int_{\lambda_{z'} \cap D} \bar{f} g \Big| \leqslant C_2 \|g\|_A .$$

Thus

$$\begin{aligned} \Big| \int_{W \cap D} \bar{f} g \Big| &= \Big| \int_{W \cap D} \bar{f} g \, d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N \Big| \\ &= \Big| \int_V \Big\{ \int_{\lambda_{z'} \cap D} \bar{f} g \, dz_1 \wedge d\bar{z}_1 \Big\} dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N \Big| \\ &\leqslant \Big| \int_V C_2 \|g\|_A \, dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_N \wedge d\bar{z}_N \Big| \\ &\leqslant \text{volume } (V) \cdot C_2 \|g\|_A . \end{aligned}$$

From the inequality (1), we conclude that

$$|\Lambda g| < C \|g\|_{A \cup (D \setminus W)} .$$

Any neighborhood U_0 of 0 such that $U_0 \subset W \setminus A$ satisfies the assertion of the lemma, and the proof is complete.

By the compactness of bD, there is a finite collection $K_{p_1}, \ldots, K_{p_n}$ of compact weak carriers of Λ such that

$$\Big(\bigcap_{j=1}^{n} K_{p_j} \Big) \cap bD = \emptyset .$$

LEMMA III.12. *Let K_1 and K_2 be closed, $K_i \subset \bar{D}$ for $i = 1, 2$, and let each K_i be a weak carrier of Λ. There is a closed set K which carries Λ weakly such that $K \cap bD = K_1 \cap K_2 \cap bD$.*

Let us assume for the moment that III.12 is proved. Applying it to K_{p_1} and K_{p_2}, we obtain K_{12} carrying Λ weakly, with $K_{12} \cap bD = K_{p_1} \cap \cap K_{p_2} \cap bD$. Now we apply IV.4 to K_{p_3} and K_{12}, obtaining a weak carrier K_{123} with $K_{123} \cap bD = K_{p_1} \cap K_{p_2} \cap K_{p_3} \cap bD$. Continuing this process produces a weak carrier K for Λ which is bounded away from bD. This yields the theorem. It remains only to prove III.12. We use a result of Kiselman [21].

LEMMA III.13. *Let F_1 and F_2 be compact subsets of Ω, and let L be the $\mathcal{O}(\Omega)$-hull of $F_1 \cup F_2$. Let K be a compact set separating F_1 and F_2 in the sense that $L \setminus K = M_1 \cup M_2$, where for $j = 1, 2$, M_j is closed in $L \setminus K$, $F_j \setminus K \subset M_j$, and $M_1 \cap M_2 = \emptyset$. Every analytic functional weakly carried by each F_j is weakly carried by K.*

PROOF OF LEMMA III.12. Let $A = K_1 \cup K_2 = \bar{D} \setminus U$, where for some open $U_i \subset \mathbf{C}^N$, $U = U_1 \cap U_2$, and $K_i = \bar{D} \setminus U_i$. Let L be the $\mathcal{O}(\Omega)$-hull of A. Since each point of bD is a peak point for $A(D)$, (indeed, for $\mathcal{O}(\Omega)|_{\bar{D}}$), $L \cap bD = A \cap bD$. Let $V \subsetneq U$ be an open set such that $L = \bar{D} \setminus V$. Define

$$V_1 = V \cup (U_1 \setminus \bar{U}_2) \cup \{z \in \bar{U} \colon \operatorname{dist}(z, bU_2) < \operatorname{dist}(z, bU_1)\},$$

$$V_2 = V \cup (U_2 \setminus \bar{U}_1) \cup \{z \in \bar{U} \colon \operatorname{dist}(z, bU_1) < \operatorname{dist}(z, bU_2)\}.$$

For $j = 1, 2$, the V_j have the following properties:

(i) V_j is open;

(ii) $V_1 \cap V_2 = V$;

(iii) $V_j \subset U_j$;

(iv) $(V_1 \cup V_2) \cap bD = (U_1 \cup U_2) \cap bD$.

To see (iv), assume that $p \in U_1 \cap bD$. Then $p \in V$ or $p \in bU_2$ or $p \in U_1 \setminus \bar{U}_2$: In each case, $p \in V_1 \cap bD$. Similarly, if $p \in U_2 \cap bD$, then $p \in V_2 \cap bD$. Properties (i)-(iii) are immediate from the definitions.

Let $F_j = \bar{D} \setminus V_j$, $j = 1, 2$. Then by (iii), Λ is weakly carried on each F_j. By (ii), $F_1 \cup F_2 = \bar{D} \setminus V = L$, a holomorphically convex set in Ω. Let $K = F_1 \cap F_2$; then $L \setminus K = (F_1 \setminus K) \cup (F_2 \setminus K)$. Setting $M_j = F_j \setminus K$, we observe that K separates F_1 and F_2 in the sense of Lemma III.13. Thus Λ is weakly carried by K, and by (iv), $K \cap bD = F_1 \cap F_2 \cap bD = K_1 \cap K_2 \cap \cap bD$. The lemma is proved.

THEOREM III.14. *Let $D \subset\subset \mathbf{C}^N$ be a strictly pseudoconvex domain with bD a real-analytic manifold of dimension $2N-1$. A function $f \in H_2(D)$ represents an analytic functional on D if and only if $f \in \mathcal{O}(\bar{D})$. The mapping $\Phi\colon \mathcal{O}'(D) \to \mathcal{O}(\bar{D})$ given by*

$$\Phi(T_f) = f$$

is an isomorphism of topological vector spaces.

REMARK. The map Φ^{-1} is well-defined because $H_2(D)$ is dense in $\mathcal{O}(D)$.

Various topologies can be given for $\mathcal{O}(\bar{D})$ and $\mathcal{O}'(D)$. For the purpose of Theorem III.14, $\mathcal{O}(\bar{D})$ will be taken to be the inductive limit (in the category of locally convex topological vector spaces) of the Fréchet spaces $\mathcal{O}(\Omega)$, as Ω ranges through the set of open neighborhoods of $\bar{D}$. On $\mathcal{O}'(D)$ we take the topology of strong dual to $\mathcal{O}(D)$.

PROOF OF THEOREM III.14. That $f \in \mathcal{O}(\bar{D})$ represents an analytic functional on D is the assertion of Theorem II.10. In view of Theorem II.1, the converse is a corollary of the next theorem on extendibility of the Bergman kernel. Regularity of the kernel for strictly pseudoconvex domains with C^∞ boundary has been studied by Kerzman [19], Bell [5], and others. The general idea of the proof of the following theorem was suggested to me by Dr. Steven Bell.

THEOREM III.15. *Let $D \subset\subset \mathbf{C}^N$, $N \geqslant 1$, be a strictly pseudoconvex domain, bD a real-analytic manifold of dimension $2N-1$. Let $k(z, w)$ be the Bergman kernel for D. If K is any compact subset of D, then there exists a domain D_K with $D_K \supset\supset D$, such that*

(i) *for $w \in K$ fixed, $k(z, w)$ extends to be holomorphic in D_K;*

(ii) *for $z \in D_K$ fixed, $\overline{k(z, w)} \in \mathcal{O}(K)$;*

(iii) *$k(z, w)$ extends to be jointly real-analytic in $D_K \times K$.*

The proof of Theorem II.1 shows that if for fixed $w \in D$, $k(z, w)$ extends holomorphically to a domain D_w with $D_w \supset\supset D$, then (i), (ii), and (iii) hold. The proof that $k(z, w)$ extends in z across bD involves several lemmas.

LEMMA III.16. *For w fixed in D, there exists $\varphi_w \in C_0^\infty(D)$ such that $k(z, w) = (P\varphi_w)(z)$, where P is the orthogonal projection of $L^2(D)$ onto the subspace of holomorphic functions.*

PROOF. For fixed $w \in D$, $k(z, w)$ represents the evaluation functional at w. By Theorem III.3, φ_w exists as claimed.

By the work of Kerzman [19], we know that $k(\cdot, w) \in A^\infty(D)$, the space of holomorphic functions on D which, together with all derivatives, extend continuously to $\bar{D}$. (Indeed, $k \in C^\infty(\bar{D}\times\bar{D}\setminus\{(z, z): z \in bD\})$. More is true in our case:

LEMMA III.17. *For fixed $w \in D$, $k(\cdot, w)$ has real-analytic boundary values.*

PROOF. According to Kohn [22] (cf. Kerzman [20]), the projection $P\colon L^2(D) \to H_2(D)$ is given by

$$P = I - \vartheta N \bar{\partial}\,,$$

where N is the Neumann operator, and ϑ is the formal adjoint of $\bar{\partial}$. The operator ϑ takes $(0, 1)$-forms to functions: on smooth forms, it is given explicitly by

$$(*) \qquad \vartheta\left(\sum_{j=1}^{N} f_j d\bar{z}_j\right) = -\sum_{j=1}^{N} \frac{\partial f_j}{\partial z_j}\,.$$

Thus, by Lemma III.16,

$$\begin{aligned} k(\cdot, w) &= P\varphi_w \\ &= \varphi_w - \vartheta N \bar{\partial}\varphi_w\,. \end{aligned}$$

The function φ_w vanishes near bD, so the same is true of the form $\bar{\partial}\varphi_w$. It follows from the expression $(*)$ that $N\bar{\partial}\varphi_w$ (and hence $k(\cdot, w)$) will be real-analytic near and on bD provided that $N\bar{\partial}\varphi_w$ is.

The Neumann operator N takes $(0, 1)$-forms to $(0, 1)$-forms. For smooth compactly supported $(0, 1)$-forms η, there is the decomposition

$$\eta = (\vartheta\bar{\partial} + \bar{\partial}\vartheta)N\eta \oplus H\eta\,,$$

where H is the orthogonal projection onto the space of harmonic $(0, 1)$-forms, i.e. forms α satisfying $\Box\alpha = 0$, where $\Box = \vartheta\bar{\partial} + \bar{\partial}\vartheta$. In our case $\eta = \bar{\partial}\varphi_w$, $H\eta = 0$, since the harmonic space is orthogonal to the range of $\bar{\partial}$ on compactly supported forms. (A reference for these facts is [10, p. 51].)

Thus, applying the decomposition above to $\eta = \bar{\partial}\varphi_w$, we have

$$\bar{\partial}\varphi_w = \Box N \bar{\partial}\varphi_w\,.$$

Hence $\beta = N\bar{\partial}\varphi_w$ is a solution to the differential equation

$$\bar{\partial}\varphi_w = \Box\beta\,.$$

The form $\bar{\partial}\varphi_w$ is supported away from bD, so as $\Box$ is *globally real-analytic hypoelliptic*, it follows that $N\bar{\partial}\varphi_w$ is real-analytic at and near bD, which is the desired assertion.

NOTE. Global real-analytic hypoellipticity of $\Box$ means that if f is real-analytic near and on bD, and $\Box g = f$, then g is real-analytic near and on bD. This property of $\Box$ for a class of domains including real-analytic bounded strictly pseudoconvex domains was proved by Komatsu [23] and independently by Derridj and Tartakoff [7]. Local real-analytic hypoellipticity of $\Box$ has more recently been established by Tartakoff [31] and independently by Treves [33].

In order to obtain a holomorphic extension of $k(\cdot, w)$ to a neighborhood D_w of $\bar{D}$, we will use a form of Schwarz reflection principle which follows from a lemma of Tomassini [32]. To this end, we consider a real-analytic submanifold X^p of $\mathbf{C}^N$, with real dimension p. Near a point $x \in X^p$, X^p is defined by parametric equations of the form

$$
\begin{aligned}
z_1 &= f_1(t_1, \ldots, t_p) \\
&\cdots \\
z_N &= f_N(_1, \ldots, t_p)\,,
\end{aligned}
$$

where the t_i are real coordinates in a neighborhood of 0 in $\mathbf{R}^p$ and the f_i are complex-valued real-analytic functions. The rank of x, denoted $r(x)$, is defined to be the complex rank of the $N \times p$ complex matrix

$$
\frac{\partial(f_1, \ldots, f_N)}{\partial(t_1, \ldots, t_p)}(x)\,.
$$

The rank $r(x)$ is independent of the choice of parameter t. We use this notation in the next lemma.

LEMMA III.18. (Tomassini) *Suppose $r(x) = N$ for each x in X^p. Let f be a real-analytic function defined on X^p. Then f is the restriction to X^p of a holomorphic function F defined in a neighborhood U of X^p if and only if*

$$
df \wedge (dz_1 \wedge \ldots \wedge dz_N|_{X^p}) \equiv 0\,.
$$

REMARK. Tomassini proves a more general result for X^n a real-analytic submanifold of an N-dimensional complex manifold.

LEMMA III.19. *Let D be a domain in $\mathbf{C}^N$, bD a $(2N-1)$-dimensional* *real-analytic manifold. If $f \in A^\infty(D)$ has real-analytic boundary values, then f extends holomorphically to a neighborhood of $\bar{D}$.*

PROOF. We apply Lemma III.18 with $X^{2N-1} = bD$. Since f is holomorphic in D, we have $df \wedge dz_1 \wedge \ldots \wedge dz_N \equiv 0$ in D; by continuity, the equality persists on $\bar{D}$. In order to apply III.18 we must show that $r(p) = N$ for all $p \in bD$. To see this, we choose holomorphic coordinates $z_1, \ldots, z_N$ near p, with $z_j = t_{2j-1} + it_{2j}$, $j = 1, \ldots, N$, so that near p,

$$bD = \{t_{2N} = G(t_1, \ldots, t_{2N-1})\},$$

where G is some real-analytic function defined near 0 in $\mathbf{R}^{2N-1}$, and $dG \neq 0$ at 0. Then $r(p)$ is the rank of the following $N \times (2N-1)$ matrix:

$$\begin{bmatrix} 1 & i & 0 & 0 & \ldots & & \ldots & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & \ldots\ \ldots & 0 & 0 \\ \vdots & \vdots & & & & & & \vdots & \vdots \\ i\dfrac{\partial G}{\partial t_1}(p) & i\dfrac{\partial G}{\partial t_2}(p) & \ldots & & & & \ldots & i\dfrac{\partial G}{\partial t_{2N-2}}(p) & 1 + i\dfrac{\partial G}{\partial t_{2N-1}}(p) \end{bmatrix}$$

The $N \times N$ minor consisting of the N odd-numbered columns of this matrix is non-singular wherever G is defined, so $r(p) = N$.

Lemma III.18 implies that there is a neighborhood U of bD in $\mathbf{C}^N$ and a holomorphic function F which extends the boundary values of f. That F extends f near bD is a consequence of the following fact applied to $g = F - f$:

If $g \in \mathcal{O}(\Omega)$, Ω a domain in $\mathbf{C}^N$, and g has boundary values identically zero in an open set $E \subset b\Omega$, then g is identically zero. (For a proof, see [17].) This completes the proof of Lemma III.19.

PROOF OF THEOREM III.15. Lemma III.17 asserts that $k(z, w)$ satisfies the hypothesis of Lemma III.19, so $k(z, w)$ extends in z to a domain $D_w \supset\supset D$. This completes the proof.

PROOF OF THEOREM III.14 CONCLUDED. We have seen above that $\Phi\colon \mathcal{O}'(D) \to \mathcal{O}(\bar{D})$ is a bijection; it is clear from the definition that Φ is an algebraic isomorphism. It remains to show that Φ and Φ^{-1} are continuous.

Let $\{\Omega_n\}_{n=1}^\infty$ be a sequence of neighborhoods of $\bar{D}$ such that for all n, $\Omega_{n+1} \subset\subset \Omega_n$, and $\bigcap_{n=1}^\infty \Omega_n = \bar{D}$. Then $\mathcal{O}(\bar{D})$ is the inductive limit in the category of locally convex spaces of the increasing sequence $\mathcal{O}(\Omega_1) \subset \mathcal{O}(\Omega_2) \subset \ldots$ of Fréchet spaces, i.e., $\mathcal{O}(\bar{D})$ is a generalized LF-space. As the dual space to the reflexive Fréchet space $\mathcal{O}(D)$, $\mathcal{O}'(D)$ with its strong dual topology is also a generalized LF-space [8, p. 513]. According to [24], the open mapping theorem is valid for maps between generalized LF-spaces, so it suffices for our theorem to show that Φ is continuous.

A linear map $A: X \to Y$ of generalized LF-spaces is continuous if it preserves convergent *sequences*; this follows from the fact that if X is the inductive limit of Fréchet spaces X_n, $n \in \mathbb{N}$, A is continuous whenever its restriction to each X_n is [8, p. 434].

Let $\{K_j\}_{j=1}^{\infty}$ be a sequence of compact subsets of D such that for all j, $K_j \subset\subset K_{j+1}^0$, and $\bigcup_{j=1}^{\infty} K_j = D$. The topology in $\mathcal{O}(D)$ is given by the seminorms $\| \ \|_{K_j}$, $j \in \mathbb{N}$. Since $\mathcal{O}(D)$ is a perfect space, it follows (see, e.g., [12, p. 57]) that a sequence $\{T_n\}_{n=1}^{\infty}$ converges to 0 in the strong topology on $\mathcal{O}'(D)$ if and only if

(i) for some fixed j, all T_n are continuous in the seminorm $\| \ \|_{K_j}$;

(ii) $\|T_n\|_{K_j} \to 0$, where $\|T_n\|_{K_j} = \sup\limits_{\substack{g \in \mathcal{O}(D) \\ \|g\|_{K_j}=1}} |T_n g|$.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{O}(\bar{D})$, and suppose that $T_{f_n} \to 0$ in $\mathcal{O}'(D)$. To prove that Φ is continuous it suffices to show that $f_n \to 0$ in $\mathcal{O}(\bar{D})$.

Let $K \subset\subset D$ be chosen so that (i) and (ii) hold for $K = K_j$, and $T_n = T_{f_n}$. Each T_{f_n} is represented by a regular Borel measure μ_n supported on K; by (ii), the μ_n can be chosen so that the total variation $|\mu_n|$ tends to zero. According to Theorem III.3, $f_n = P\varphi_n$, where $\varphi_n \in \mathbb{C}^{\infty}$ and is supported near K. More precisely, if $\delta = \operatorname{dist}(K, bD)$ and $\varphi: \mathbb{C}^N \to \mathbb{R}$ is a smooth, radially symmetric function supported in $\{z: |z| < \delta/2\}$, satisfying $\int_{\mathbb{C}^N} \varphi = 1$, then the functions φ_n can be written

$$\varphi_n = \mu_n * \varphi ,$$

where μ_n are regarded as compactly supported distributions. Thus $K' \subset\subset D$ can be chosen so that for all n, $\operatorname{supp} \varphi_n \subset K'$, and $\lim\limits_{n\to\infty} \|\varphi_n\|_{K'} = 0$.

Let a neighborhood $D_{K'}$ of $\bar{D}$ be chosen according to Theorem III.15. By shrinking $D_{K'}$ slightly if necessary, we can assume that the Bergman kernel for D satisfies $k(z, w) \in \mathbb{C}^{\infty}(\bar{D}_{K'} \times K')$. Hence for each n,

$$f_n(z) = P\varphi_n = \int_D \varphi_n(w) k(z, w)$$

is holomorphic in a neighborhood of $D_{K'}$, and

$$\|f_n(z)\|_{D_{K'}} \leqslant C \|\varphi_n\|_{K'} \|k\|_{\bar{D}_{K'} \times K'} ,$$

where C is a constant independent of n. This shows that $f_n \to 0$ in the topology of $\mathcal{O}(\bar{D})$, and completes the proof of Theorem III.14.

What can be said about functions which represent analytic functionals on domains with smooth but not real-analytic boundaries? In view of Theorem III.3, it is natural to consider the properties of the orthogonal projection $P\colon L^2(D) \to H_2(D)$ on such domains.

For any positive integer t, let $W^t(D)$ denote the Sobolev space of complex-valued functions with square-integrable distribution derivatives up to order t. For $f \in W^t(D)$,

$$\|f\|_t^2 = \sum_{|\alpha| \leqslant t} \int_D |D^\alpha f|^2.$$

The subspace of holomorphic functions in $W^t(D)$ is denoted $H^t(D)$. In this notation, $H_2(D) = H^0(D)$, and $L^2(D) = W^0(D)$; *a priori*, the orthogonal projection P maps $W^t(D)$ into $H^0(D)$. We now consider a class of domains studied by Bell [3, 4]:

DEFINITION. *A smooth bounded domain D in $\mathbb{C}^N$ is said to satisfy condition R if for each positive integer s, there is an integer $M = M(s)$ such that P is a bounded operator from $W^{s+M}(D)$ to $W^s(D)$.*

Domains of type R include strictly pseudoconvex smooth domains, smooth pseudoconvex domains with real-analytic boundary, and domains of finite type in $\mathbb{C}^2$.

THEOREM III.20. *If D is a domain satisfying condition R, then any $f \in H_2(D)$ which represents an analytic functional has C^∞ extension to $\bar{D}$.*

PROOF. Let s be a fixed positive integer. Given f as above, we write $f = Ph$, where $h \in C_0^\infty(D)$. In particular, $h \in W^{s+M(s)}$, so $Ph \in W^s(D)$. Since s is arbitrary, Sobolev's lemma implies that $f \in A^\infty(D)$.

COROLLARY. *If D is type R, the Bergman kernel $k(z, w)$ extends smoothly to bD as a function of z for each fixed w.*

PROOF. The analytic functional $f \to f(w)$ is represented on $H_2(D)$ by $k(z, w)$.

On strictly pseudoconvex smoothly bounded domains D, the functions which represent analytic functionals are dense in $\mathcal{O}(D)$. A result of Bell [5] implies a stronger density theorem.

THEOREM III.21. *Let $D \subset\subset \mathbf{C}^N$ be strictly pseudoconvex with smooth boundary, $K \subset\subset D$ any compactum with interior. The set*

$$S_K = \{f \in H_2(D):\ T_f \text{ is weakly carried by } K\}$$

is dense in $\mathcal{O}(D)$.

PROOF. Since K has interior, it is a set of determinacy for holomorphic functions on D. According to [5], finite linear combinations of the functions $k(z, w_j)$, $w_j \in K$, are dense in $H_2(D)$. If $f(z) = \sum_{j=1}^{k} a_j k(z, w_j)$, then there is a $C > 0$ such that for all $g \in H_2(D)$,

$$|T_f g| = \left| \sum_{j=1}^{K} \bar{a}_j \int_D g(z)\, k(w_j, z)\, dm_{2N}(z) \right| = \left|\sum \bar{a}_j g(w_j)\right| \leqslant C \|g\|_K .$$

Hence $f \in S_K$. Since $H_2(D)$ is dense in $\mathcal{O}(D)$, the proof is complete.

IV. – Applications and Examples.

It follows from Theorems III.3 and III.10 that when U is the unit disc in $\mathbf{C}^1$, functions $f \in \mathcal{O}(\overline{U})$ are precisely those of the form $f = Pu$, where $u \in \mathcal{C}_0^\infty(U)$. Every such f represents an analytic functional on U which is weakly carried on $\operatorname{supp} u$. In this setting, an explicit solution $u \in \mathcal{C}_0^\infty(U)$ to the equation $Pu = f$ can be given as follows. Suppose $f \in \mathcal{O}(a\overline{U})$ for some $a > 1$, and let

$$\psi(z) = \begin{cases} f(a^2 z)\cdot a^2 & \text{if } |z| < \dfrac{1}{a} \\ 0 & \text{if } |z| \geqslant \dfrac{1}{a} . \end{cases}$$

By the proof of Lemma III.6, $P\psi = f$. The function ψ is not $\mathcal{C}^\infty$, but this is easily remedied.

LEMMA IV.1. *Let a, f, U, and ψ be as above. Let $\delta > 0$ be chosen so that $1/a + \delta < 1$. Let $\varphi\colon \mathbf{C} \to \mathbf{C}$ be any $\mathcal{C}^\infty$ function, radially symmetric about* 0, *with* $\operatorname{supp}\varphi \subset \delta U$ *and* $\int_{\mathbf{C}} \varphi = 1$. *Set* $u = \psi * \varphi$. *Then* $\operatorname{supp} u \subset (1/a + \delta)U$ *and* $Pu = f$.

PROOF. The first statement is clear. For the second, we compute:

$$\begin{aligned} Pu(z) &= \int_U k(z, w) \int_{\mathbf{C}} \psi(t)\varphi(w-t)\, dm_2(t)\, dm_2(w) \\ &= \int_{\mathbf{C}}\int_U k(z, w)\, \varphi(w-t)\, dm_2(w)\, \psi(t)\, dm_2(t) \\ &= \int_{\mathbf{C}} k(z, t)\, \psi(t)\, dm_2(t) \\ &= P\psi(z) \\ &= f(z)\,. \end{aligned}$$

The third equality follows from the facts that $k(z, w)$ is harmonic in w for fixed z and $\varphi(w-t)$ is radially symmetric in w about t. This completes the proof.

REMARK. Lemma IV.1 shows that a function $f \in \mathcal{O}(\overline{aU})$ represents an analytic functional on U which is weakly carried on $(1/a)\overline{U}$. In particular, this implies that an entire function corresponds to an analytic functional weakly carried by the singleton $\{0\}$. Equivalently, any entire function f can be written

$$f(z) = Pu(z)\,,$$

(the equation holds in the unit disc), where u can be chosen to be supported arbitrarily near the origin. Can f be written $f = Pv$, where v has small support away from the origin? Equivalently, can the analytic functional T_f be weakly carried by a point $z_0 \neq 0$?

Questions of uniqueness for carriers of analytic functionals in several variables have been treated by Martineau [27], Kiselman [21], and others. In one dimension, a special case answers the question raised above.

LEMMA IV.2. *Let D be a domain in $\mathbf{C}$ and $T \in \mathcal{O}'(D)$. If T is weakly carried by compacta K_1 and K_2, where $D \setminus (K_1 \cup K_2)$ has no components compact in D, then T is the zero functional.*

PROOF. Let U_1 and U_2 be neighborhoods of K_1 and K_2 such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ and $D \setminus (\overline{U}_1 \cup \overline{U}_2)$ has no components compact in D. By Runge's theorem, there exist functions $h_n \in \mathcal{O}(D)$, $n \in \mathbf{N}$, such that

$$h_n \to 1 \text{ uniformly on } \overline{U}_1\,,$$

$$h_n \to 0 \text{ uniformly on } \overline{U}_2\,.$$

Let $f \in \mathcal{O}(D)$ be arbitrary. Then $h_n f \to f$ uniformly on $\overline{U}_1$; since T is weakly carried on K_1, $T_{h_n} f \to Tf$. But $h_n f \to 0$ uniformly on $\overline{U}_2$; since T is weakly carried on K_2, $T_{h_n} f \to 0$. Thus $Tf = 0$, which completes the proof.

Lemma IV.1 implies in our case $D = U$ that *no non-zero entire function f can satisfy $f|_U = Pv$, if v has support in a disc disjoint from the origin.*

We have shown that a one-to-one correspondence exists between extendible functions and analytic functionals on a plane domain with real-analytic boundary. Perhaps surprisingly, real-analyticity is also a necessary condition, at least for domains with C^2 boundary:

THEOREM IV.3. *Let D be a bounded plane domain with C^2 boundary. The following conditions are equivalent:*

(i) $\mathcal{O}'(D) \subset \mathcal{O}(\overline{D})$; *more precisely, every $f \in H_2(D)$ which represents an analytic functional on D extends across bD.*

(ii) *The constant function $1: D \to \mathbf{C}$ represents an analytic functional on D.*

(iii) *bD is real-analytic.*

PROOF. It suffices to show that (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii). Let D_1 be a domain whose boundary consists of disjoint circles, and $\Phi: D_1 \to D$ a biholomorphism. Since bD is C^2, a result of Warschawski [34] implies that Φ extends to a homeomorphism of $\overline{D}_1$ and $\overline{D}$, and that Φ' and $(\Phi^{-1})'$ extend continuously to bD_1 and bD. By the chain rule, neither derivative vanishes on its respective boundary.

The constant function $1: D_1 \to \mathbf{C}$ is an analytic functional on D_1, so by Corollary III.4, $\Phi_*(1) = (\Phi^{-1})'$ is an analytic functional on D. If (i) holds, then $(\Phi^{-1})'$ extends holomorphically across bD.

Let p be a point of bD and U_p a simply connected neighborhood of p to which $(\Phi^{-1})'$ extends. Then Φ^{-1} also extends to U_p. Since $(\Phi^{-1})'(p) \neq 0$, Φ^{-1} is univalent near p, mapping bD to bD_1, so bD is real-analytic near p. Thus (i) $\Rightarrow$ (iii).

If (ii) holds, then $\Phi_*^{-1}(1) = \Phi'$ extends holomorphically across bD_1. For $p \in bD$, we apply the argument of the last paragraph to $\Phi^{-1}(p) \in bD_1$, to conclude that near p, bD is the image under a conformal map of bD_1 near $\Phi^{-1}(p)$. Thus (iii) holds, and the proof is complete.

REMARK. It is not necessary that bD be of class C^2 for (ii) and (iii) to be equivalent. We use this fact in our next result:

THEOREM IV.4. *Let D be a bounded plane domain each of whose boundary components C_j is a simple closed curve. Suppose further that each C_j is the union of $\mathcal{C}^1$ arcs which intersect transversally. Then 1 is an analytic functional on D if and only if bD is real-analytic, i.e., if and only if each C_j is a simple closed real-analytic curve.*

PROOF. Let D_1 be a domain with circular boundary components and $\Phi\colon D_1 \to D$ a biholomorphism. If 1 is an analytic functional on D, then $(\Phi^{-1})_*(1) = \Phi'$ (and hence also Φ) extends holomorphically across bD_1. Suppose that $\Phi'(p) = 0$ for some $p \in bD_1$, and assume without loss of generality that $\Phi(p) = 0$. Let U be any neighborhood of p on which Φ is defined. Since bD_1 is smooth at p, $w = \arg \Phi(z)$ assumes on $U \cap D_1$ all values in $[0, 2\pi]$ with at most one exception. By the assumption on bD, we can choose a disc neighborhood V of $\Phi(p) = 0$ such that for some θ_j, $j = 1, 2$, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, $D \cap V$ omits the sector $\{\arg w \in (\theta_1, \theta_2)\}$. This is a contradiction; hence Φ' does not vanish on bD_1. The proof of Theorem IV.3 now applies to show that bD is real-analytic, which is the desired assertion.

REMARK. In view of Theorem III.8, Theorem IV.4 implies that on domains of this kind, 1 represents an analytic functional only if *every* extendible holomorphic function does.

An example of a non-smooth domain on which 1 represents an analytic functional may be given as follows: Let D be the image under the map $\Phi(z) = z^2$ of the disc $D_1 = \{z\colon |z - i| < 1\}$. Note that the cardioid bD is piecewise real-analytic but that the transversality hypothesis of Theorem IV.4 fails to hold at $0 \in bD$. Since $\Phi'(z) = 2z$, Φ' represents an analytic functional on D_1. Hence $1 = \Phi_*(\Phi')$ represents an analytic functional on D.

Consider the map $I\colon H_2(D) \to \mathbb{C}$ given by

$$g \to \int_D g = \int_D 1g\,.$$

Whenever I is continuous in the $\mathcal{O}(D)$-topology, i.e., whenever there exist $C > 0$ and a compact $K \subset\subset D$ such that for all $g \in H_2(D)$,

$$\left| \int_D g \right| \leqslant C \|g\|_K,$$

I extends to a (unique) element of $\mathcal{O}'(D)$. Reworded in these terms, Theorem IV.4 gives a criterion for I to be an analytic functional:

COROLLARY IV.5. *Let D be as in Theorem* IV.4. *Integration over D defines an analytic functional on D if and only if bD is real-analytic.*

We illustrate these ideas with an example. Let U be the unit disc, $V = U \cap \{\operatorname{Re} z > 0\}$. According to Theorem IV.4, integration over the domain defines an analytic functional on U but not on V. If $f = \sum_{n=0}^{\infty} a_n z^n \in H_2(U)$, then

$$\int_U f = \int_U \sum_{n=0}^{\infty} a_n z^n = \pi \cdot a_0 = \pi \cdot f(0)\,.$$

Thus integration over U is, up to a constant, evaluation at the origin, which certainly defines an analytic functional on U.

On V, we consider the integrals $\int_V z^n$ for odd n:

$$\left|\int_V z^n\right| = \left|\int_0^1 \int_0^\pi r^n e^{in\theta} r\, dr\, d\theta\right| = \left|\frac{-1}{n+2} \cdot \frac{2}{in}\right| = \frac{2}{n(n+2)}\,.$$

If 1 were an analytic functional on V, then for some $C > 0$ and $K \subset\subset V$ we would have

$$\frac{2}{n(n+2)} = \left|\int_V z^n\right| \leqslant C\|z^n\|_K = C\lambda^n$$

for all odd n, where $\lambda < 1$ is a constant. This is absurd, so integration over V does not define an analytic functional on V.

We have seen that integration against 1 does not define an analytic functional on domains with « corners ». Does integration against an extendible function vanishing to high order at the corners define an analytic functional? The half-disc example shows that, in general, the answer is negative:

THEOREM IV.6. *If V is the half-disc, then no non-zero polynomial represents an analytic functional on V.*

PROOF. Suppose $p(z) = a_0 + a_1 z + \ldots + a_m z^m$ represents an analytic functional on V. Then there exist $C > 0$ and $r < 1$ such that for any $n \in \mathbb{N}$,

$$\left|\int_V \bar{p} z^n\right| \leqslant C r^n\,.$$

We will show that all odd coefficients of p vanish; the proof for the even coefficients is similar. A computation in polar coordinates shows that

$$\int_V \bar{z}^j z^n = \begin{cases} \dfrac{2i}{(n+j+2)(n-j)} & \text{if } (n-j) \text{ is odd.} \\ 0 & \text{if } (n-j) \text{ is even, } \neq 0. \end{cases}$$

Thus for all $n > m$, n even, we have

$$\left|\int_V \bar{p} z^n\right| = 2\left|\frac{a_1}{(n+3)(n-1)} + \frac{a_3}{(n+5)(n-3)} + \dots\right| = \\ = 2\left|\sum_{\substack{l \text{ odd} \\ l \leqslant m}} \frac{a_l}{(n+l+2)(n-l)}\right| \leqslant Cr^n.$$

Since the left-hand of the last inequality is a rational function of n and the right hand side is exponential, we must have that for every $n \in \mathbf{N}$,

$$\sum_{\substack{l \text{ odd} \\ l \leqslant m}} \frac{a_l}{(n+l+2)(n-l)} = 0.$$

Thus, $a_l = 0$ for each l, and the proof is complete.

A result similar to IV.3 holds for certain smoothly bounded domains in $\mathbf{C}^N$. In the absence of a conformal mapping theorem in several variables, the hypothesis is strengthened.

THEOREM IV.7. *Let $D \subset\subset \mathbf{C}^N$ be strictly pseudoconvex with smooth boundary. Assume there to be a biholomorphism $\Phi: D_1 \to D$, where $D_1 \subset\subset \mathbf{C}^N$ is strictly pseudoconvex with bD_1 a $(2N-1)$-dimensional real-analytic submanifold of $\mathbf{C}^N$. If the monomials of degree 0 and 1 represent analytic functionals on D, then bD is a $(2N-1)$-dimensional real-analytic submanifold of $\mathbf{C}^N$.*

COROLLARY. *Let D be as in the theorem. If the monomials of degree 0 and 1 represent analytic functionals, then all extendible functions $f \in \mathcal{O}(\bar{D})$ do.*

PROOF OF THEOREM IV.7. Let $\Phi_1, \dots, \Phi_N$ be the components of Φ. Since 1 and z_j, $j = 1, \dots, N$, are analytic functionals on D, the functions $\operatorname{Jac} \Phi = \Phi_*^{-1}(1)$ and $\operatorname{Jac} \Phi \cdot \Phi_j = \Phi_*^{-1}(z_j)$ represent analytic functionals on D_1. By Theorem III.14, they extend holomorphically across bD_1.

It is known ([9], cf. [28]) that biholomorphisms between strictly pseudoconvex smoothly bounded domains extend smoothly to the boundary. Thus $\operatorname{Jac} \Phi$ and $\operatorname{Jac} \Phi^{-1}$ extend smoothly to bD_1 and bD; by the chain rule neither vanishes on the boundary. It follows that for each j, Φ_j extends holomorphically across bD_1. In the neighborhood of any point $p \in bD_1$, the extended function Φ is a biholomorphism, mapping bD_1 to bD. This completes the proof.

We consider next some consequences of our duality results on domains in several variables.

THEOREM IV.8. *Let $D \subset\subset \mathbf{C}^N$ be as in Theorem* III.14 *and assume $f \in \mathcal{O}(D)$. The following conditions are equivalent:*

(i) $f \in \mathcal{O}(\bar{D})$;

(ii) *$f \in A^\infty(D)$ and f has real-analytic boundary values;*

(iii) *$f = P\varphi$ for some $\varphi \in \mathcal{C}_0^\infty(D)$.*

PROOF. Lemma III.19 and Theorem III.3.

REMARK. Theorem IV.8 provides an analytic counterpart to a result of Bell [3], who proved that *for D strictly pseudoconvex with smooth boundary, each $u \in A^\infty(D)$ can be written $u = P\varphi$, where φ vanishes to infinite order on bD.*

From facts about weak carriers of analytic functionals, we can infer results about the supports of functions which project to analytic functionals. Following are two examples; note that analyticity of the boundary is not required.

THEOREM IV.9. *Let $D \subset\subset \mathbf{C}^N$ be strictly pseudoconvex. Let K_1 and K_2 be disjoint compacta in D, such that $K_1 \cup K_2$ is holomorphically convex in D. If $f = P\varphi_1 = P\varphi_2$, where* $\operatorname{supp} \varphi_j \subset K_j$, *$j = 1, 2$, then $f = 0$.*

PROOF. Since D is strictly pseudoconvex, $H_2(D)$ is dense in $\mathcal{O}(D)$, so f represents a unique analytic functional on D. Since $K_1 \cup K_2$ is holomorphically convex in D, Theorem 4.3.2 of [18] implies that there exist functions $h_n \in \mathcal{O}(D)$, $n \in \mathbf{N}$, such that

$$h_n \to 1 \quad \text{uniformly on } K_1 ,$$

$$h_n \to 0 \quad \text{uniformly on } K_2 .$$

The proof of Lemma IV.2 applies to show that T_f is the zero functional; hence $f = 0$.

THEOREM IV.10. *Let D be as in Theorem* IV.9. *For $j = 1, 2$, assume $K_j \subset\subset D$, $K_1 \cup K_2$ is $\mathcal{O}(D)$-convex, and $\varphi_j \in \mathcal{C}^\infty(D)$ with* $\operatorname{supp} \varphi_j \subset K_j$. *If $f =$*

$= P\varphi_1 = P\varphi_2$, then for any open set U such that $U \supset K_1 \cap K_2$, there exists $\varphi \in C_0^\infty(U)$ such that $f = P\varphi$.

PROOF. The analytic functional T_f is weakly carried on each K_j. Lemma III.15 implies that T_f is weakly carried on $K_1 \cap K_2$. The conclusion follows from the remarks at the beginning of this section.

Theorem III.21 also has a counterpart in this setting:

THEOREM IV.11. *Let $D \subset\subset \mathbf{C}^N$ be strictly pseudoconvex with smooth boundary. If $U \subset\subset D$ is any open set, then*

$$\{P\varphi\colon \varphi \in C_0^\infty(U)\}$$

is dense in $\mathcal{O}(D)$.

PROOF. Let $V \subset\subset U$ be open. Every analytic functional T_f weakly carried by $\overline{V}$ satisfies $P\varphi = f$, for some $\varphi \in C_0^\infty(U)$. By Theorem III.21, such f are dense in $\mathcal{O}(D)$. This completes the proof.

Theorems III.14 and III.20 assert that functions in $H_2(D)$ extend holomorphically across bD or smoothly to bD when they represent analytic functionals. The proof of Theorem III.9 shows that if $D \subset\subset \mathbf{C}^N$ is convex with smooth boundary and $0 \in D$, a function $f \in H_2(D)$ represents an analytic functional if its restriction to each complex line λ through 0 satisfies the following conditions:

(i) $f|_\lambda \in H_2(D \cap \lambda)$;

(ii) there exists $C_\lambda > 0$ and a compact $K_\lambda \subset\subset D \cap \lambda$ such that for all polynomials p in the single variable z (the coordinate on λ),

$$\left| \int_{D\cap\lambda} \bar{f} p |z|^{2N-2}\, dm_2(z) \right| \leqslant C_\lambda \|p\|_{K_\lambda}.$$

(We have used the fact that $D \cap \lambda$ is polynomially convex.) We have shown:

THEOREM IV.12. *Let $D \subset\subset \mathbf{C}^N$ be convex, $0 \in D$, bD smooth, $f \in H_2(D)$. If for all complex lines λ through 0,* (i) *and* (ii) *hold, then $f \in A^\infty(D)$. If, in addition, bD is a $(2N-1)$-dimensional real-analytic submanifold of $\mathbf{C}^N$, then $f \in \mathcal{O}(\overline{D})$.*

The duality theory can also be used to deduce results about boundary behavior of biholomorphisms between certain domains.

THEOREM IV.13. *Let D_1 and D_2 be strictly pseudoconvex domains, each with boundary a $(2N-1)$-dimensional real-analytic submanifold of $\mathbf{C}^N$. Let $\Phi\colon D_1 \to D_2$ be a biholomorphism. Both* Jac Φ *and Φ extend holomorphically across bD_1.*

REMARK. Theorem IV.13 was proved originally by Pinčuk [29].

PROOF. The constant function 1 is an analytic functional on D_2, so $\Phi_*^{-1}(1) = \text{Jac}\,\Phi$ extends holomorphically across bD_1. Similarly, Jac Φ^{-1} extends holomorphically across bD_2. Hence Jac $\Phi \neq 0$ on $\overline{D_1}$, and indeed does not vanish on a neighborhood of $\overline{D_1}$.

The functions z_j, $j = 1, \ldots, N$, are analytic functionals on D_2, so $\Phi_*^{-1}(z_j) = \Phi_j \cdot \text{Jac}\,\Phi$ is an analytic functional on D_1 for each j, and hence extends holomorphically across bD_1. Since Jac $\Phi \neq 0$ near $\overline{D_1}$ and extends holomorphically across bD_1, Φ_j itself extends holomorphically across bD_1. This completes the proof.

THEOREM IV.14. *Let D_1 and D_2 be strictly pseudoconvex domains, D_2 as above, and D_1 with C^∞ boundary. Let $\Phi\colon D_1 \to D_2$ be a biholomorphism. Then* Jac Φ *extends smoothly to bD_1.*

PROOF. The constant 1 is an analytic functional on D_2, so $\text{Jac}\,\Phi = \Phi_*^{-1}(1)$ is an analytic functional on D_1. By Theorem III.20, Jac Φ extends smoothly to bD_1.

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