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Type I criteria and the Plancherel formula for Lie groups with co-compact radical

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Type I Criteria and the Plancherel Formula for Lie Groups with Co-Compact Radical (*).

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1. – Introduction.

This paper is another devoted to the Orbit Method—that is, the construction, parameterization and characterization of the ingredients of harmonic analysis on Lie groups by means of co-adjoint orbits. Here we consider Lie groups G having co-compact radical. We shall accomplish two objectives for such groups: (1) obtain criteria for them to be type I; and (2) give a description of their Plancherel formula.

To derive the Plancherel formula we shall need to impose a certain hypothesis first enunciated by Charbonnel and Khalgui (see Condition (A) in Section 2). The condition holds automatically if the group is connected, and in the disconnected case it should be thought of as the analog of the Harish-Chandra class condition for disconnected reductive Lie groups. The condition guarantees the existence of invariant positive polarizations and so enables us to explicitly realize the irreducible unitary representations by holomorphic induction. However we shall not need to assume Condition (A) in order to obtain the type I criteria.

The paper has two sections (besides this introduction). Section two contains the derivation of the type I criteria and Section three the computation of the Plancherel formula. In Section two we construct a generalization of the Kirillov mapping for nilpotent groups. That mapping (or parameterization) involves the co-adjoint orbits $G\cdot\varphi$ of allowable linear functionals (in the dual of the Lie algebra), as well as certain irreducible projective representations of the discrete group G_φ/G_φ^0. The type I criteria we derive, that is the necessary and sufficient conditions for a co-compact radical Lie group to be type I, are that the orbits $G\cdot\varphi$ should be locally

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closed and the projective duals of G_φ/G_φ^0 should be type I. We then turn the latter condition into a structural condition on the group G_φ/G_φ^0. The main results are Theorems 2.7 and 2.8. In Section three we derive the Plancherel formula for G. We make critical use of the preliminary case of Lie groups having co-compact nilradical. Those groups have been treated in [15]. In addition we make another assumption on G, namely that the nilradical is regularly embedded almost everywhere (see Condition (B) in Section 3). This enables us to avoid the algebraic hull and to dramatically simplify Charbonnel's method of proof for solvable groups. It is an open question as to whether there exist any groups that do not satisfy Condition (B). The main theorem is Theorem 3.5.

Finally I call attention to Duflo's paper [6]. He gives there an inductive procedure for associating irreducible representations to admissible orbits for a general Lie group. (See [15, § 8, Remark 2] for a discussion of the relative merits of admissible versus allowable orbits. Recall that for amenable groups—if one is considering only the irreducible representations and Plancherel measure, not the characters—the latter are more convenient.) Duflo indicates necessary conditions for the group to be type I, but not sufficient ones. He omits the Harish-Chandra type condition (A)—but without it, there is no explicit realization of the representations, only the inductive procedure for describing them.

Notation and Terminology. G will be a locally compact separable group, usually a Lie group. G^0 denotes the neutral component. We write dg for right Haar measure. The symbol $\hat{G}$ indicates the set of equivalence classes of irreducible unitary representations. If G is type I and unimodular, we denote by μ_G the Plancherel measure on $\hat{G}$ corresponding to dg. We shall write Cent G for the center of G. G is called a *central group* if $G/\text{Cent}\ G$ is compact. This is not to be confused with the terminology: Γ is a *central subgroup* of G, which means only that $\Gamma \subseteqq \text{Cent}\ G$. If χ is a unitary character of a central subgroup Γ of G, we write $\hat{G}_\chi = \hat{G}_{\Gamma,\chi}$ to denote the classes of irreducibles $\pi \in \hat{G}$ that satisfy $\pi|_\Gamma = (\dim \pi)\chi$. If $\chi \equiv 1$, we let $\hat{G}_\Gamma = \hat{G}_{\Gamma,1}$. We say G_Γ (or $G_{\Gamma,\chi}$) is type I if the space $\hat{G}_\Gamma$ (or $\hat{G}_{\Gamma,\chi}$) is countably separated.

2. – Type I criteria.

In this section we shall describe necessary and sufficient conditions for a Lie group having co-compact radical to be type I. More precisely, G will denote a Lie group, R its solvradical and N the nilradical. When we refer to G as having co-compact radical we mean of course that G/R is compact.

We do not require connectivity of G, but the co-compactness of R insures that G/G^0 is finite. We shall often assume the following property—which is automatic if G is connected:

$$(A) \qquad [G, R] \subseteqq N .$$

Property (A) should be thought of as the analog of the Harish-Chandra class condition (see [8]) in the reductive case.

2a. The representations.

Let G have co-compact radical and satisfy Condition (A). We begin with the definition of the ingredients that go into the parameterization of $\hat{G}$.

DEFINITION 2.1. $\mathfrak{A}(G) = \{\varphi \in \mathfrak{g}^*: \exists \chi = \chi_\varphi$ a unitary character of G_φ^0 such that $d\chi = i\varphi|_{\mathfrak{g}_\varphi}\}$,

$$\mathfrak{X}(G_\varphi) = \{\tau \in \hat{G}_\varphi: \tau|_{G_\varphi^0} = (\dim \tau)\, \chi_\varphi\} .$$

The elements of $\mathfrak{A}(G)$ are called *allowable* linear functionals, and $\mathfrak{X}(G_\varphi)$ is defined whenever $\varphi \in \mathfrak{A}(G)$. It is known that for every $\varphi \in \mathfrak{A}(G)$, there exists an invariant positive polarization $\mathfrak{h}$ for φ which satisfies the Pukanszky condition and is strongly admissible [4]. We write $\pi(\varphi, \tau, \mathfrak{h}) = \mathfrak{h} - \mathrm{Ind}_{G_\varphi}^G \tau$ for the holomorphically induced representation determined by the data $(\varphi, \tau, \mathfrak{h})$—see [15, § 6] for the definition. The representation $\pi(\varphi, \tau, \mathfrak{h})$ can also be realized as a group extension representation as follows. Let $\theta = \varphi|_{\mathfrak{n}}$, $\xi = \varphi|_{\mathfrak{g}_\theta}$, $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{n}_c$, $\mathfrak{h}_2 = \mathfrak{h} \cap (\mathfrak{g}_\theta)_c$. Then $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ and $\mathfrak{h}_1$ is a positive G_θ-invariant polarization for θ, $\mathfrak{h}_2$ is a positive $(G_\theta)_\xi$-invariant polarization for ξ [15]. The following fact is taken as obvious in [6] and [11] but it is instructive to write down the proof.

LEMMA 2.2. $\mathfrak{A}(G)|_{\mathfrak{n}} = \mathfrak{A}(N)$, *i.e. every* $\varphi \in \mathfrak{A}(G)$ *restricts to an element* $\theta \in \mathfrak{A}(N)$ *and conversely every* $\theta \in \mathfrak{A}(N)$ *is the restriction of some* $\varphi \in \mathfrak{A}(G)$.

PROOF. Let $\varphi \in \mathfrak{A}(G)$. Then

$$\varphi[[\mathfrak{n}_\theta, \mathfrak{n}_\theta]\mathfrak{g}] \subseteqq \theta[[\mathfrak{g}, \mathfrak{n}_\theta], \mathfrak{n}_\theta] = \{0\} .$$

So $[\mathfrak{n}_\theta, \mathfrak{n}_\theta] \subseteqq \mathfrak{n}_\varphi$. Therefore N_φ is a normal subgroup of N_θ and N_θ/N_φ is abelian. But one also knows that $N_\theta \cdot \varphi = \varphi + (\mathfrak{g}_\theta + \mathfrak{n})^\perp$ (see [14, p. 271] or [11, Lemma 2]). Thus $N_\theta/N_\varphi \cong \mathbb{R}^n$. Furthermore $\varphi[\mathfrak{n}_\theta, \mathfrak{n}_\theta] = 0$ implies that $[\mathfrak{n}_\theta, \mathfrak{n}_\theta] \subseteqq \mathrm{Ker}\,(\varphi|_{\mathfrak{n}_\varphi})$. Hence $[N_\theta, N_\theta] \subseteqq \mathrm{Ker}\,(\chi_\varphi|_{N_\varphi})$. If we put $C = \mathrm{Ker}\,(\chi_\varphi|_{N_\varphi})$, $A = N_\theta/C$, $B = N_\varphi/C$, then we are in the situation of a connected abelian Lie group A, a closed connected subgroup B such that $A/B \simeq \mathbb{R}^n$, and a *faithful* unitary character $\chi \in \hat{B}$. It is obvious that χ can be extended to A so that $d\chi$ is *any* prescribed linear functional ψ on $\mathfrak{a}$ which extends $(1/i)\, d\chi \in \mathfrak{b}^*$. Hence χ_φ extends to a character χ_θ of N_θ satisfying $d\chi_\theta = i\theta|_{\mathfrak{n}_\theta}$.

Now for the converse. Let $\theta \in \mathfrak{A}(N)$. We set

$$\mathfrak{A}_\theta(G_\theta) = \{\xi \in \mathfrak{A}(G_\theta) \colon \xi|_{\mathfrak{n}_\theta} = \theta|_{\mathfrak{n}_\theta}\}\,.$$

It is enough to show that $\mathfrak{A}_\theta(G_\theta)$ is non-empty. Indeed if so, let $\varphi \in \mathfrak{g}^*$ be *any* extension of both $\theta \in \mathfrak{A}(N)$ and some $\xi \in \mathfrak{A}_\theta(G_\theta)$. One knows that

$$(G_\theta)_\xi^0 = G_\varphi^0 N_\theta$$

[16, § 3], [6, IV, Lemme 5]. The existence of a unitary character χ_φ of G_φ^0 such that $d\chi_\varphi = i\varphi|_{\mathfrak{g}_\varphi}$ is then obvious. Now the non-emptiness of $\mathfrak{A}_\theta(G_\theta)$ is not immediately evident—it requires that we invoke some of the structural assumptions we have made on G. Let $\mathfrak{q}_\theta = \operatorname{Ker}\,(\theta|_{\mathfrak{n}_\theta})$, and let Q_θ be the corresponding connected Lie subgroup. Then G_θ^0/Q_θ is an almost direct product of a compact connected semisimple Lie group and an at most 2-step nilpotent connected Lie group (see [11, I, Lemme 2] where the word « almost » is inadvertently omitted) in which N_θ/Q_θ is central. The non-emptiness is clear then from known results on co-compact nilradical groups [15].

REMARK. It is not clear if Lemma 2.2 is true in the most general situation—e.g. without reductivity of G_θ^0/N_θ.

Now we complete the realization of $\pi(\theta, \tau, \mathfrak{h})$ as a group extension representation. Set $\gamma = \gamma(\theta) \in \hat{N}$, where $\theta \to \gamma(\theta)$ is the Kirillov map of N. (Actually $\gamma \simeq \mathfrak{h}_1 - \operatorname{Ind}_{N_\theta}^N \chi_\theta$.) Then γ lifts *canonically* to a representation $\tilde{\gamma}$ of G_θ satisfying $\tilde{\gamma}(n_\theta) = \chi_\theta(n_\theta)^{-1}\gamma(n_\theta)$, $n_\theta \in N_\theta$ [5]. We know that $(G_\theta)_\xi = G_\varphi N_\theta$ [16, Cor. 3.5, Formula A], and that the representation τ extends canonically to $(G_\theta)_\xi$. The group G_θ may have infinitely many components, but we may still consider

$$\nu(\xi, \tau, \mathfrak{h}_2) = \mathfrak{h}_2 - \operatorname{Ind}_{(G_\theta)_\xi}^{G_\theta} \tau\,.$$

According to [15, Prop. 3.9], it is irreducible, its class is independent of $\mathfrak{h}_2$ and $\nu(\xi, \tau, \mathfrak{h}_2)|_{N_\theta} = (\dim \nu)\, \chi_\theta$. Therefore $(\nu \otimes \tilde{\gamma}) \otimes \gamma$ defines a unitary representation of $G_\gamma = G_\theta N$, and

$$\pi(\varphi, \tau, \mathfrak{h}) \simeq \operatorname{Ind}_{G_\gamma}^G (\nu \otimes \tilde{\gamma}) \otimes \gamma$$

[16, Prop. 3.10], [11, Th. IV, 4.4]. This proves irreducibility of $\pi(\varphi, \tau, \mathfrak{h})$ (because of the Mackey machine) as well as independence of $\mathfrak{h}$. We write $\pi_{\varphi,\tau}$ for the class. Finally one knows that for $\varphi, \varphi' \in \mathfrak{A}(G)$, $\tau \in \mathfrak{X}(G_\varphi)$, $\tau' \in \mathfrak{X}(G_{\varphi'})$ we have

$$\pi_{\varphi,\tau} \cong \pi_{\varphi',\tau'} \Leftrightarrow \exists g \in G, \quad g\cdot\varphi = \varphi', \quad g\cdot\tau \cong \tau'.$$

(It is easy to remove the splitting and simply connected assumptions from the results in [16].) Thus if we set

$$\mathfrak{B}(G) = \{(\varphi, \tau) : \varphi \in \mathfrak{A}(G),\ \tau \in \mathfrak{X}(G_\varphi)\},$$

we get an injective map

$$\varkappa : \mathfrak{B}(G)/G \to \hat{G}.$$

I shall refer to $\varkappa$ as the Kirillov map. In the type I situation—which we characterize in this section—the Kirillov map $\varkappa$ is a bijection. We shall find it convenient in Section 3 to depict this bijection by the fiber diagram:

$$(2.1) \qquad \begin{array}{ccc} \mathfrak{X}(G_\varphi) \to \hat{G} & & \\ & \downarrow & \\ & \mathfrak{A}(G)/G. & \end{array}$$

2b. *The connected case.*

It is evident from the last subsection that the type I condition on G must be tied to properties of the objects $\mathfrak{A}(G)/G$ and $\mathfrak{X}(G_\varphi)$. Since G being type I amounts to « well-behavedness » of $\hat{G}$, we expect it to be equivalent to « well-behavedness » of $\mathfrak{A}(G)/G$ and $\mathfrak{X}(G_\varphi)$. Part of the interest of a theorem asserting such an equivalence would be the exact formulation of « well-behavedness » of these sets. For the former, it amounts to the G-orbits of $\varphi \in \mathfrak{A}(G)$ being locally closed in $\mathfrak{g}^*$. For the latter, it must be that $\mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi$ is type I. But because of the structural assumptions on G we shall be able to turn that into an equivalent statement on the structure of G_φ. All this has already been done by Pukanszky in the case that G is connected and simply connected. To state his result we need

DEFINITION 2.3. Let G be connected with co-compact radical. Let $\varphi \in \mathfrak{A}(G)$, $Q_\varphi = \mathrm{Ker}\, \chi_\varphi \subseteq G_\varphi^0$. The reduced stabilizer is $\bar{G}_\varphi = \{g \in G_\varphi : [g, G_\varphi] \subseteq Q_\varphi\}$.

If we denote by p_φ the canonical projection $p_\varphi: G_\varphi \to G_\varphi/Q_\varphi$, then $\bar{G}_\varphi = p_\varphi^{-1}(\mathrm{Cent}\,(G_\varphi/Q_\varphi))$. (Note the difference in the definitions of Q_θ and Q_φ —Q_θ is by definition connected, whereas Q_φ is in general not.)

THEOREM 2.4 (Pukanszky [19, Th. 1]). *Let G be simply connected with co-compact radical. Then G is type* I *if and only if for every $\varphi \in \mathfrak{A}(G)$ we have*

(i) *$G\cdot\varphi$ is locally closed in $\mathfrak{g}^*$; and*

(ii) $[G_\varphi : \bar{G}_\varphi] < \infty$.

REMARKS. (1) We observed in [16] that in the type I case, G_φ/Q_φ is a central group (indeed its center $\bar{G}_\varphi/Q_\varphi$ is co-finite). In fact $[G_\varphi : \bar{G}_\varphi] < \infty$ is equivalent to G_φ/Q_φ being a central group which—by [20, Chp. I]—is equivalent to $\mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi$ being type I. In addition, $\mathfrak{X}(G_\varphi)$ is a compact subset of $(G_\varphi/Q_\varphi)^\wedge$ equal to: all of it if $Q_\varphi = G_\varphi^0$; or the class one portion if $G_\varphi^0/Q_\varphi \cong \boldsymbol{T}$ [16, Prop. 4.2].

(2) It is proven in [16] that $\varkappa$ is a bijection when G is simply connected type I.

Now we extend Theorem 2.4 to the case of connected but non-simply connected groups. Let G be connected with co-compact radical. Let $\mathfrak{G}$ be the universal covering group of G, $p: \mathfrak{G} \to G$ the canonical projection, $\Gamma = \mathrm{Ker}\, p$. Then p generates a dual mapping $p: \hat{G} \to \hat{\mathfrak{G}}_\Gamma$ which is a topological homeomorphism. G is type I if and only if $\mathfrak{G}_\Gamma$ is type I. The following lemma consists of a sequence of easily-verified facts. I omit the details.

LEMMA 2.5. (i) $p(\mathfrak{G}_\varphi^0) = G_\varphi^0$, $p(\mathfrak{G}_\varphi) = G_\varphi$, *and* $\mathfrak{G}_\varphi/\mathfrak{G}_\varphi^0\Gamma \cong G_\varphi/G_\varphi^0$;

(ii) $\mathfrak{A}(G) = \mathfrak{A}_\Gamma(\mathfrak{G}) = \{\varphi \in \mathfrak{A}(\mathfrak{G}) : \chi_\varphi|_{\mathfrak{G}_\varphi \cap \Gamma} = 1\}$;

(iii) *$\mathfrak{X}(G_\varphi)$ is identified to* $\mathfrak{X}_\Gamma(\mathfrak{G}_\varphi) = \{\tau \in \mathfrak{X}(\mathfrak{G}_\varphi) : \tau|_\Gamma = 1\}$;

(iv) $\mathrm{Ind}_{\mathfrak{G}_\varphi^0\Gamma}^{\mathfrak{G}_\varphi} \chi_\varphi = p \circ \mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi$, *where $\chi_\varphi|_\Gamma$ is understood to be* 1;

(v) $\bar{\mathfrak{G}}_\varphi = p^{-1}(\bar{G}_\varphi)$ *and* $[\mathfrak{G}_\varphi : \bar{\mathfrak{G}}_\varphi] = [G_\varphi : \bar{G}_\varphi]$.

Taking these properties into consideration and changing notation, we see that to extend Theorem 2.4 to the connected case we must prove

THEOREM 2.6. *Let G be simply connected with co-compact radical, $\Gamma \subseteq \mathrm{Cent}\, G$ a discrete subgroup. Then G_Γ is type* I *if and only if for every*

$\varphi \in \mathfrak{A}_\Gamma(G)$ *we have*

(i) $G \cdot \varphi$ *is locally closed in* $\mathfrak{g}^*$; *and*

(ii) $[G_\varphi : \bar{G}_\varphi] < \infty$.

PROOF. We make critical use of the results and of the terminology of [19]. All unexplained terminology and notation are as in [19].

We first prove necessity. Suppose G_Γ is type I. Let $\varphi \in \mathfrak{A}_\Gamma(G)$. Choose any $\tau \in \mathfrak{X}_\Gamma(G)$ and form $\pi = \pi_{\varphi,\tau}$. Then $\pi \in \hat{G}_\Gamma$. If we let $J = \operatorname{Ker} \pi$ (in the sense of primitive ideals in the C^*-algebra $C^*(G)$), then J is type I (by the hypothesis). It suffices (by [19, Lemma 20 and Prop. 2]) to show that $\varphi \in \Omega(J)$ (see [19, p. 31]). Let $L = [G, G]N$ the nilradicalisé of G. L is simply connected and its nilradical N is co-compact. We write η for the canonical surjection $\eta : \hat{L} \to \mathfrak{A}(L)/L$. We make use of the fact—obvious from the co-compact nilradical theory [15]—that for $\gamma \in \hat{N}$ fixed, we have $\omega \in \hat{L}$ lies over γ iff any $\psi \in \eta(\omega)$ satisfies $\psi|_\mathfrak{n} \in L \cdot \theta$. Now we know [18] that $\pi|_L$ is concentrated in a G-orbit, say $G \cdot \omega$. More importantly, we also know that $\pi|_N$ is concentrated in a G-orbit $G \cdot \gamma$, where $\gamma = \gamma(\theta)$ and $\theta = \varphi|_\mathfrak{n}$. This is because of the group extension realization of π. It follows that ω must lie over $g \cdot \gamma$ for some $g \in G$. But then any $\psi_1 \in \eta(\omega)$ satisfies $\psi_1|_\mathfrak{n} \in L \cdot (g \cdot \theta)$. Thus

$$\eta(G \cdot \omega) = \{\psi_2 \in \mathfrak{A}(L) : \psi_2|_\mathfrak{n} \in G \cdot \theta\}.$$

Now $\psi = \varphi|_\mathfrak{l} \in \mathfrak{A}(L)$ [19; Lemma 7]. Therefore $\psi|_\mathfrak{n} = \varphi|_\mathfrak{n} = \theta$. Finally $\Omega(J) = \{\varphi_1 \in g^* : \varphi_1|_\mathfrak{l} \in \eta(\tilde{G} \cdot \omega)\}$, $\tilde{G} = \text{alg hull } G$. Therefore $\varphi \in \Omega(J)$.

The argument for sufficiency is somewhat more subtle. Let J be a primitive ideal which is the kernel of an irreducible representation trivial on Γ. By [18], $J = J(\varrho)$ for some ϱ obtained as follows. There exists $\omega \in \hat{L}$, a (maximal) closed subgroup $K = K_\omega \subseteqq G$ and $\varrho \in \hat{K}$ such that $\varrho|_L = \omega$, $\pi_\varrho = \operatorname{Ind}_K^G \varrho$ is primary, and $J(\varrho) = \operatorname{Ker} \pi_\varrho$. It suffices (by [19]) to demonstrate that $\Omega(J) \cap \mathfrak{A}_\Gamma(G) \neq \emptyset$. So let $\varphi \in \Omega(J)$, $\varphi|_\mathfrak{l} = \psi \in \eta(\omega)$. The set $\Omega(J)$ is invariant under translation by elements $\nu \in \mathfrak{g}^*$, $\nu|_\mathfrak{l} = 0$. So the question is: can we modify φ by such a ν so that the resulting character $\chi_{\varphi+\nu} = 1$ on $G_\varphi^0 \cap \Gamma$.

First consider $G_\varphi^0 \cap \Gamma \cap L$. By [19, p. 23], $G_\varphi^0 \cap L = L_\varphi^0$. Obviously $L_\varphi^0 \subseteqq L_\psi^0$, therefore $G_\varphi^0 \cap \Gamma \cap L \subseteqq L_\psi^0 \cap \Gamma$. Next we have $\omega = \omega(\psi, \tau)$ for some $\tau \in \mathfrak{X}(L_\psi)$ [15]. Therefore

$$\pi_\varrho|_\Gamma = 1 \Rightarrow \varrho|_{\Gamma \cap K} = 1 \Rightarrow \omega|_{L \cap \Gamma} = 1 \Rightarrow \chi_\psi|_{L_\psi^0 \cap \Gamma} = 1 \Rightarrow \chi_\varphi|_{G_\varphi^0 \cap \Gamma \cap L} = 1.$$

The character χ_φ passes from $G_\varphi^0 \cap \Gamma$ to the quotient $(G_\varphi^0 \cap \Gamma)/(G_\varphi^0 \cap \Gamma \cap L) \cong$ $\cong (G_\varphi^0 \cap \Gamma)L/L$. The latter is a discrete subgroup of the vector group G/L.

Let ν be any linear functional in $(\mathfrak{g}/\mathfrak{l})^*$ such that $e^{i\nu} \in (G/L)^\wedge$ is an extension of χ_φ on $(G_\varphi^0 \cap \Gamma)/(G_\varphi^0 \cap \Gamma \cap L)$. We lift ν back to $\mathfrak{g}$ and consider $\varphi - \nu \in \Omega(J)$. It is obvious that $\chi_{\varphi-\nu} \equiv 1$ on $G_\varphi^0 \cap \Gamma$. To finish, we must show $\varphi - \nu \in \mathfrak{A}_\Gamma(G)$, and for that it is enough to demonstrate that $G_\varphi = G_{\varphi-\nu}$, i.e. $G_\nu = G$. But that is obvious since $\nu|_{\mathfrak{l}} = 0$ and G/L is abelian.

We are now in a position to state

THEOREM 2.7. *Let G be connected with co-compact radical. Then G is type* I *if and only if for every $\varphi \in \mathfrak{A}(G)$ we have*

(i) *$G \cdot \varphi$ is locally closed in $\mathfrak{g}^*$; and*

(ii) *G_φ/Q_φ is a central group.*

This theorem follows from Theorem 2.6 and the reasoning preceding it once we observe that the equivalence of $[G_\varphi : \overline{G}_\varphi] < \infty$ and G_φ/Q_φ central is valid whether G is simply connected or not. Finally we note that the exhaustion proof of [16] is also valid—simple connectivity is not required. This is because Propositions 5.2 and 5.3 of [16] are true in the connected case (see Proposition 3.3 and [6]). Therefore we have that in the type I situation the Kirillov map $\varkappa \colon \mathfrak{B}(G)/G \to \hat{G}$ is a bijection. Alternately the fiber diagram (2.1) obtains.

REMARK. We conclude this subsection with a remark—which on the surface is frivolous, but—which will play an important role in the next subsection.

Namely: for G connected, G_φ/Q_φ is a central group if and only if it is a finite extension of an abelian group. In fact in the latter case, G_φ/Q_φ must be type I which implies $\mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi$ is type I. We already commented (after Theorem 2.4) that this forces $G_\varphi/\overline{G}_\varphi$ to be finite.

2c. *The disconnected case.*

Now we assume only that G is a co-compact radical Lie group (not necessarily satisfying Condition (A)). We extend Theorem 2.7 to this situation, and then consider the map $\varkappa$ when G is type I and satisfies Condition (A). The main result is

THEOREM 2.8. *Let G have co-compact radical. Then G is type* I *if and only if for every $\varphi \in \mathfrak{A}(G)$ we have*

(i) *$G \cdot \varphi$ is locally closed in $\mathfrak{g}^*$; and*

(ii) *G_φ/Q_φ is a finite extension of an abelian group.*

PROOF. Let $H = G^0$ so that $[G:H] < \infty$. We first suppose that G is type I. Then the open subgroup H is also type I [10, Prop. 2.4]. Let $\varphi \in \mathfrak{A}(G) = \mathfrak{A}(H)$. Then by Theorem 2.7, $H \cdot \varphi$ is locally closed in $\mathfrak{h}^* = \mathfrak{g}^*$ and $[H_\varphi : \bar{H}_\varphi] < \infty$. But $G \cdot \varphi = U_{G/H} g \cdot (H \cdot \varphi)$ is clearly also locally closed in $\mathfrak{g}^*$. Furthermore $\bar{H}_\varphi / Q_\varphi$ is a co-finite abelian normal subgroup of G_φ / Q_φ. Indeed it is abelian by definition, normal since H_φ and Q_φ are normal in G_φ, and co-finite since $G_\varphi / H_\varphi = G_\varphi / (H \cap G_\varphi) \cong G_\varphi H / H \subseteq G/H$ is finite.

NOTE. The group $\bar{H}_\varphi$ may not be a central subgroup of G_φ; and the group $\bar{G}_\varphi$ may be no bigger than H^0_φ in the disconnected case. We shall illustrate this phenomenon with an example at the end of this subsection.

Now we prove the converse. Suppose that for every $\varphi \in \mathfrak{A}(G)$, we have properties (i) and (ii). It is trivial to check that, since $H \cdot \varphi$ is an open subset of $G \cdot \varphi$, $H \cdot \varphi$ is also locally closed in $\mathfrak{g}^*$. Next we see that H_φ / Q_φ must also be a finite extension of an abelian group (since G_φ / H_φ is finite). Therefore by the remark after Theorem 2.7, $[H_\varphi : \bar{H}_\varphi] < \infty$. Hence by Theorem 2.6 H is type I. Finally, since G is a finite extension of H, it too must be type I [9, Th. 1].

REMARKS. (1) By Theorem 2.8, if G is type I, every $\tau \in \mathfrak{X}(G_\varphi)$ is finite-dimensional.

(2) The results of [6] suggest that Theorem 2.8 might be true without assuming G/G^0 finite, but only G/G^0 amenable. That remains to be seen.

We next indicate (without details) why in the disconnected case, the Kirillov map $\varkappa\colon \mathfrak{B}(G)/G \to \hat{G}$ is still a bijection when G is type I. Of course, it is enough to show surjectivity. We also assume Condition (A). Let $\pi \in \hat{G}$, and let $\pi^0 \in \hat{H}$ be a representation over which π lies. Then there exists $\varphi \in \mathfrak{A}(H) = \mathfrak{A}(G)$ and $\sigma \in \mathfrak{X}(H_\varphi)$ such that $\pi^0 = \pi_{\varphi,\sigma}$. We may choose a positive polarization $\mathfrak{b}$ for φ which is G_φ-invariant. Then we have $\pi^0 \cong \pi_H(\varphi, \sigma, \mathfrak{b})$. Now the stabilizer of π^0 in G is contained in $G_\varphi H$. In fact it is exactly $(G_\varphi)_\sigma H$. Therefore (by the Mackey machine) there is an irreducible representation ν of $(G_\varphi)_\sigma H$ whose restriction to H is a multiple of π^0 such that $\pi = \mathrm{Ind}^G_{(G_\varphi)_\sigma H} \nu$. What does ν look like? The representation π^0 extends canonically to a representation $\tilde{\pi}^0$ of $(G_\varphi)_\sigma$ with the property $\tilde{\pi}^0(h_\varphi) = \sigma(h_\varphi)^{-1} \pi^0(h_\varphi)$. Everything in sight is type I and all extensions are regular. Hence there exists an irreducible representation ω of $(G_\varphi)_\sigma$, with restriction to H_φ a multiple of σ, such that

$$\nu = (\omega \otimes \tilde{\pi}^0) \otimes \pi^0 .$$

But then $\tau = \mathrm{Ind}_{(G_\varphi)_\sigma}^G \omega$ is irreducible, belongs to $\mathfrak{X}(G_\varphi)$, and it is routine to check that

$$\pi(\varphi, \tau, \mathfrak{b}) \cong \mathrm{Ind}_{(G_\varphi)_\sigma H}^G \nu .$$

That completes our discussion of surjectivity.

EXAMPLE. Consider the Lie algebra $\mathfrak{g}$ of the motion group of the plane. $\mathfrak{g}$ has generators H, P, Q satisfying commutation relations $[H, P] = Q$, $[H, Q] = -P$. Let $H = G^0$ be the simply connected solvable Lie group having $\mathfrak{g}$ as Lie algebra. The two element group $\mathbb{Z}_2$ acts on $\mathfrak{g}$ by $\varepsilon(H) = -H$, $\varepsilon(P) = Q$, $\varepsilon(Q) = P$, $\varepsilon^2 = 1$, and therefore gives rise to a group G which is the corresponding semi-direct product of $\mathbb{Z}_2$ and H. The solvradical R of G is H and the nilradical N is $\exp(\mathbb{R}P + \mathbb{R}Q)$. If we take $g = \varepsilon^j \exp tH \cdot \cdot \exp xP \exp yQ$ and $\varphi = \tau H^* + \xi P^* + \eta Q^*$, then we may compute that for $\tau \neq 0$ and $\xi^2 + \eta^2 = 0$, then $H_\varphi = H$ and $G_\varphi = H$; but for $\xi^2 + \eta^2 \neq 0$, we have

$$\begin{aligned} H_\varphi &= \{h = (t, x, y) : t \in 2\pi\mathbb{Z}, \xi y = \eta x\} \\ H_\varphi^0 &= \{(0, x, y) : \xi y = \eta x\} \\ \bar{H}_\varphi &= H_\varphi \quad \text{is abelian} \end{aligned}$$

while

$$\begin{aligned} G_\varphi &= \{g = (\varepsilon^j, t, x, y) : g \cdot \varphi = \varphi\} = \\ &= H_\varphi \cup \left\{(\varepsilon, t, x, y) : \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, 2\tau = \xi y - \eta x \right\} \\ G_\varphi^0 &= H_\varphi^0 \\ \bar{G}_\varphi &= \{g \in G : [g, G_\varphi] \subseteq Q_\varphi\} = G_\varphi^0 . \end{aligned}$$

The point is that G_φ / G_φ^0 is a semidirect product of $\mathbb{Z}_2$ and $\mathbb{Z}$, and therefore has no center. The group G does not satisfy Condition (A). I do not know an example of a type I group which satisfies condition (A) for which G_φ / Q_φ is not central.

3. – The Plancherel formula.

In this section we describe the Plancherel measure in the type I situation. It is well-known that even if G is type I, the group N may fail to be regularly embedded in G. Examples are found in [1] and [2]. However we shall make the following assumption:

$$(B) \qquad \hat{N}/G \quad \text{is countably separated a.e.} ;$$

that is, there is a G-invariant (Plancherel) co-null set $\mathcal{N} \subseteq \hat{N}$ such that $\mathcal{N}/G$ is countably separated. I know of no type I group G that does not satisfy Condition (B).

3a. *The Plancherel formula and semi-invariants.*

We give here the statement of the abstract Plancherel formula. Since the groups we deal with may not be unimodular, we have to use the non-unimodular version of the Plancherel Theorem [7], [12]. We shall work with the non-unimodular Plancherel formula that contains the unbounded semi-invariants in infinitesimal form. This of course avoids the domain problems encountered (e.g. in [17]) when the semi-invariants are realized globally. Actually, thus far all orbital presentations of the non-unimodular Plancherel formula have been derived with the semi-invariants in infinitesimal form. It is an open and interesting problem to give an orbital interpretation of the global semi-invariant.

Let G be locally compact separable and type I, with a choice dg of right Haar measure. We denote the modular function by $\delta = \delta_G$ so that

$$\delta(g)\int_G f(gx)\,dx = \int_G f(x)\,dx\,.$$

If π is an irreducible unitary representation of G acting on $\mathcal{H}_\pi$, then by a semi-invariant of weight δ^r, $r > 0$, we mean a positive self-adjoint closed unbounded operator D_π on $\mathcal{H}_\pi$ satisfying

$$\pi(g)D_\pi = \delta(g)^r D_\pi \pi(g)\,, \qquad g \in G\,.$$

The Plancherel Theorem asserts (among other things) the existence of a positive Borel measure μ_G on $\hat{G}$ and semi-invariants D_π on $\mathcal{H}_\pi$ $(\pi \in \hat{G})$ of weight δ_G such that for $f \in C_C^\infty(G)$.

$$(3.1) \qquad f(e) = \int_{\hat{G}} \mathrm{Tr}\,[D_\pi^{\frac12}\pi(f)D_\pi^{\frac12}]\,d\mu_G(\pi)\,, \qquad \int_G |f(g)|^2 dg = \int_{\hat{G}} \|D_\pi^{\frac12}\pi(f)\|_2^2\, d\mu_G(\pi)\,.$$

(See [7]). The operator-valued measure $D_\pi^{\frac12} d\mu_G(\pi)$ is uniquely determined in the sense that if $D_\pi'^{\frac12} d\mu_G'(\pi)$ is another satisfying (3.1), then there exists a positive measurable numerical-valued function $c(\pi)$ which obeys the relations

$$\frac{d\mu_G}{d\mu_G'} = c(\pi)\,, \qquad D_\pi' = c(\pi)D_\pi\,.$$

We now construct the semi-invariants which will enter into the Plancherel formula for co-compact radical groups. Let G be a Lie group, H a closed subgroup. Fix right Haar measures dg, dh on G, H. We also fix a positive continuous function q on G satisfying

$$q(e) = 1\,, \qquad q(hg) = \delta_H(h)\delta_G(h)^{-1}q(g)\,, \qquad h \in H,\ g \in G\,.$$

Then q defines a quasi-invariant measure $d\bar{g}$ on G/H by the formula

$$\int_G f(x)q(x)\,dx = \int_{G/H}\int_H f(hx)\,dh\,d\bar{x}\,.$$

The measure $d\bar{g}$ is relatively invariant iff $(\delta_G\delta_H^{-1})|_H$ extends to a positive character ω of G. Then we may choose $q = \omega^{-1}$ and ω is the modulus of the relatively invariant measure.

Now let σ be a unitary representation of H. We may realize the induced representation $\pi = \mathrm{Ind}_H^G\sigma$ (as in [12, pp. 467 ff]) in the space

$$\mathscr{H}_\pi = \Big\{f\colon G \to \mathscr{H}_\sigma \text{ measurable};\ f(hg) = \sigma(h)f(g),\ h \in H,\ g \in G;\ \int_{G/H}\|f\|_\sigma^2\,d\bar{g} < \infty\Big\}\,.$$

The action is

$$\pi(g)f(x) = f(xg)[q(xg)/q(x)]^{\frac{1}{2}}\,, \qquad g \in G,\ f \in \mathscr{H}_\pi\,.$$

Let $\omega\colon G \to \mathbb{R}_+^\times$ be a positive character. Suppose there exists $\psi\colon G \to \mathbb{R}_+^\times$, a positive measurable function satisfying $\psi(hg) = \psi(g)$ and $\psi(xg) = \omega(g)\psi(x)$, $h \in H$, $x, g \in G$. The operator $D = D_\psi$ defined by

$$(D_\psi f)(g) = \psi(g)f(g)\,, \qquad g \in G$$

is well-defined on $\mathscr{H}_\pi$, and it is trivial to check that D is semi-invariant of weight ω. In particular, the above will be applied when $\psi = \omega$ is a character that is trivial on H.

LEMMA 3.1. *Let $f \in C_C^\infty(G)$. Then*

$$\mathrm{Tr}\,D\pi(f)D = \int_{G/H}\delta_G(g)^{-1}q(g)^{-1}\psi(g)^2\,\mathrm{Tr}\int_H f(g^{-1}hg)\sigma(h)\delta_H(h)^{-\frac{1}{2}}\delta_G(h)^{\frac{1}{2}}\,dh\,d\bar{g}\,.$$

The case $\psi \equiv 1$ is exactly [12, Th. 3.2]. We shall omit the proof of this lemma. The proof is basically the same as that of [12, Th. 3.2] except that the presence of D introduces the function ψ into the kernel. The resulting diagonal integration brings the factor ψ^2 into the integrand.

Now we return to the case of a co-compact radical group satisfying Condition (A). We have seen that the irreducible representations $\pi(\varphi, \tau, \mathfrak{h})$ —which are defined by holomorphic induction—can also be realized as ordinary induced representations by inducing from $G_\gamma = G_\theta N$, $\theta = \varphi|_\mathfrak{n}$, $\gamma = \gamma(\theta) \in \hat{N}$. To apply Lemma 3.1 to these representations therefore, we need to show that $\delta_G|_{G_\gamma} \equiv 1$—at least for generic γ. We do that now. Since N is normal in G, we have $\delta_G(g) = \delta_{G|N}(g)\delta_{G/N}(Ng)$, where $\delta_{G|N}$ is the modulus of the outer automorphism of N determined by g—i.e. for any choice dn of Haar measure on N we have

$$\delta_{G|N}(g)\int_N f(gng^{-1})dg = \int_N f(n)\,dn\,.$$

But G/N is unimodular (e.g. because it contains the co-compact central subgroup R/N). Thus

$$\delta_G(g) = \delta_{G|N}(g)\,.$$

That is dn is relatively invariant (under the action of G) with modulus δ_G. If μ_N is the Plancherel measure of N determined by dn, i.e.

$$\int_N |f(n)|^2 dn = \int_{\hat{N}} \|\gamma(f)\|_2^2 d\mu_N(\gamma)\,,$$

then μ_N is relatively invariant under the action of G with modulus δ_G^{-1}. Hence [12, § 2] for μ_N-a.a. γ, the homogeneous space $G\cdot\gamma \approx G/G_\gamma$ has a relatively invariant measure of modulus δ_G^{-1}. We write $\mathcal{N}_1$ for a G-invariant μ_N-co-null subset of $\hat{N}$ such that

$$G\cdot\gamma \text{ has a relatively invariant measure of modulus } \delta_G^{-1},\ \forall\gamma\in\mathcal{N}_1\,.$$

Then

$$(\delta_G\delta_{G_\gamma}^{-1})|_{G_\gamma} = \delta_G^{-1}|_{G_\gamma}\,, \qquad \gamma\in\mathcal{N}_1\,.$$

From this we deduce that $G_\gamma \subseteq \operatorname{Ker}\delta_G$ as follows. Since G/N is central, all its closed subgroups are unimodular. Thus G_γ/N is unimodular for every $\gamma \in \hat{N}$. But then

$$\delta_G^{-1}|_{G_\gamma} = \frac{\delta_G|_{G_\gamma}}{\delta_{G_\gamma}} = \frac{(\delta_{G|N}\,\delta_{G/N})|_{G_\gamma}}{\delta_{G_\gamma|N}\,\delta_{G_\gamma/N}} = \frac{\delta_{G|N}|_{G_\gamma}}{\delta_{G_\gamma|N}} = 1\,, \qquad \gamma\in\mathcal{N}_1\,.$$

Hence $\delta_G^{-1}|_{G_\gamma} = 1$. Therefore $\delta_{G_\gamma} = \delta_G^2|_{G_\gamma} = 1$, which implies G_γ is unimodular and $G_\gamma \subseteq \operatorname{Ker} \delta_G$ for any $\gamma \in \mathcal{N}_1$. We summarize in

LEMMA 3.2. *There exists a G-invariant co-null set $\mathcal{N}_1 \subseteqq \hat{N}$ such that for every $\gamma \in \mathcal{N}_1$, G_γ is unimodular and contained in* $\operatorname{Ker} \delta_G$.

Finally, we construct the semi-invariant operators. Consider the map

$$\begin{gathered} \varphi \to \theta = \varphi|_{\mathfrak{n}} \to \gamma(\theta) \\ \mathfrak{A}(G) \to \mathfrak{A}(N) \to \hat{N}\,. \end{gathered}$$

By Lemma 2.2, this is surjective. Let us set $\mathfrak{A}_1(G)$ to be the inverse image of $\mathcal{N}_1$. In subsection 3c we shall define a measure on $\mathfrak{A}(G)/G$ and we shall see that $\mathfrak{A}_1(G)/G$ is of full measure. Now for $\varphi \in \mathfrak{A}_1(G)$, $\tau \in \mathfrak{X}(G_\varphi)$, we have $\pi_{\varphi,\tau}$ realized by

$$\operatorname{Ind}_{G_\gamma}^G (\nu_{\xi,\tau} \otimes \tilde{\gamma}) \otimes \gamma$$

(see § 2*a*). Thus for $\varphi \in \mathfrak{A}_1(G)$, we may set

$$D_{\pi_{\varphi,\tau}} = D_{\delta_G}$$

to obtain a semi-invariant of weight δ_G on the space of $\pi_{\varphi,\tau}$.

3*b*. *Some results on co-compact nilradical groups.*

Exactly as the *representation theory* of co-compact radical groups used representation-theoretic facts on co-compact nilradical groups (viz. G_θ^0/Q_θ), so does the derivation of the *Plancherel formula* of the former require that of the latter. In fact we need results on the Plancherel formula of a co-compact nilradical group with prescribed central character. These are a straightforward generalization of the results of [15]. I state here only the results—the proofs involve nothing new beyond [15]. But first I recall the unimodular projective Plancherel theorem.

Let G be locally compact separable and Z a closed subgroup of Cent G. Fix $\chi \in \hat{Z}$. Suppose G/Z is unimodular. Suppose also that $\operatorname{Ind}_Z^G \chi$ is type I. We fix a Haar measure $d\bar{g}$ on G/Z and put

$$\hat{G}_\chi = \{\pi \in \hat{G} : \pi|_Z = (\dim \pi)\chi\}\,.$$

Then there is a positive Borel measure $\mu = \mu_{G,\chi}$ on $\hat{G}_\chi$ such that

$$\int_{G/Z} |f(g)|^2 d\bar{g} = \int_{\hat{G}_\chi} \|\pi(f)\|_2^2 d\mu_{G,\chi}(\pi)\,.$$

Here f is C^∞, satisfies $f(zg) = \chi(z)^{-1}f(g)$, $|f|$ is compactly supported on G/Z, and $\pi(f)$ denotes $\int_{G/Z} \pi(g) f(g)\, d\bar{g}$.

EXAMPLE. Let G have co-compact radical, satisfy Condition (A), and be of type I. Let $\varphi \in \mathfrak{A}(G)$. Then $\mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi$ is type I. Moreover G_φ^0/Q_φ is a central subgroup of G_φ/Q_φ. The discrete group G_φ/G_φ^0 has a counting Haar measure, and thus there is uniquely determined a positive (finite) Borel measure $d\tau$ on the compact space $\mathfrak{X}(G_\varphi)$ such that

$$\mathrm{Ind}_{G_\varphi^0}^{G_\varphi} \chi_\varphi \cong \int_{\mathfrak{X}(G_\varphi)}^{\oplus} (\dim \tau)\, \tau\, d\tau$$

$$f(1) = \int_{\mathfrak{X}(G_\varphi)} \mathrm{Tr}\, \tau(f)\, d\tau\, .$$

Now let G be a Lie group, N its nilradical (not required to be simply connected as in [15]) and suppose G/N is compact. G is type I. Let Z be a closed subgroup of Cent G, $\chi \in \hat{Z}$. Of course G/Z is unimodular. We set

$$\mathfrak{A}_\chi(G) = \{\varphi \in \mathfrak{A}(G)\colon \chi_\varphi|_{Z\cap G_\varphi^0} = \chi|_{Z\cap G_\varphi}\}\, .$$

$$\mathfrak{X}_\chi(G_\varphi) = \{\tau \in \mathfrak{X}(G_\varphi)\colon \tau|_Z = (\dim \tau)\, \chi\}\, .$$

In this case every $\tau \in \mathfrak{X}_\chi(G)$ is finite-dimensional. For every $\varphi \in \mathfrak{A}_\chi(G)$, there exists an invariant, strongly admissible positive polarization $\mathfrak{h}$; the holomorphically induced representation $\pi(\varphi, \tau, \mathfrak{h})$ is irreducible; its class $\pi_{\varphi,\tau}$ is independent of $\mathfrak{h}$ and is in $\hat{G}_\chi$. If we set

$$\mathfrak{B}_\chi(G) = \{(\varphi, \tau)\colon \varphi \in \mathfrak{A}_\chi(G),\ \tau \in \mathfrak{X}_\chi(G_\varphi)\}$$

then $(\varphi, \tau) \to \pi_{\varphi,\tau}$ generates a bijection $\mathfrak{B}_\chi(G)/G \to \hat{G}_\chi$. We may also write

$$\begin{array}{ccc} \mathfrak{X}_\chi(G) & \to & \hat{G}_\chi \\ & & \downarrow \\ & & \mathfrak{A}_\chi(G)/G\, . \end{array} \tag{3.2}$$

The representations $\pi_{\varphi,\tau}$ also have a group extension realization. For $\varphi \in \mathfrak{A}_\chi(G)$, we have $\theta = \varphi|_{\mathfrak{n}} \in \mathfrak{A}_{\chi_1}(N)$, $\chi_1 = \chi|_{Z\cap N}$. The representation $\gamma = \gamma(\theta)$ extends canonically to a representation $\tilde{\gamma}$ of G_θ. Set $\xi = \varphi|_{\mathfrak{g}_\theta} \in \mathfrak{A}_{\theta,\chi}(G_\theta)$, where

$$\mathfrak{A}_{\theta,\chi}(G_\theta) = \{\xi \in \mathfrak{A}_\theta(G_\theta) : \chi_\xi|_{Z\cap (G_\theta)^0_\xi} = \chi|_{Z \cap (G\ )^0_\xi}\}$$

is a discrete set. We form

$$\nu_{\xi,\tau} = [\mathfrak{h} \cap (\mathfrak{g}_\theta)_c] - \operatorname{Ind}^G_{(G_\theta)_\xi} \tau$$

(using $(G_\theta)_\xi = G_\varphi N_\theta$ as usual), and then

$$\pi(\varphi, \tau, \mathfrak{h}) \cong \operatorname{Ind}^G_{G_\theta N}(\nu_{\xi,\tau} \otimes \tilde{\gamma}) \otimes \gamma .$$

The projective Plancherel measure can be described as follows. Fix Haar measure on G/Z. If we choose normalized Haar measure on G/NZ, then a unique Haar measure is specified on $N/N \cap Z$. We have the fiber diagram (see [15, § 7])

$$\begin{array}{ccc} \mathfrak{A}_{\theta,\chi}(G_\theta)/G_\theta & \rightarrow & \mathfrak{A}_\chi(G)/G \\ & & \downarrow \\ & & \hat{N}_{\chi_1}/G . \end{array} \tag{3.3}$$

Let μ_{N,χ_1} denote the projective Plancherel measure on $\hat{N}_{\chi_1}$ corresponding to the Haar measure on $N/N \cap Z$ already determined. We write $\bar{\mu}_{N,\chi_1}$ for its *image* on $\hat{N}_{\chi_1}/G = \hat{N}_{\chi_1}/(G/N)$. The space $\mathfrak{A}_{\theta,\chi}(G_\theta)/G_\theta$ also carries a natural discrete measure. Indeed exactly as in [15] we have a canonical projection

$$(G^0_\theta)\hat{}_{\chi,\chi_\theta} \cong \mathfrak{A}_{\theta,\chi}(G_\theta)/G^0_\theta \rightarrow \mathfrak{A}_{\theta,\chi}(G_\theta)/G_\theta$$

where $(G^0_\theta)\hat{}_{\chi,\chi_\theta}$ consists of the irreducibles of G^0_θ that restrict to multiples of χ on Z, χ_θ on N_θ. $G_\theta/N_\theta Z$ is compact and has a normalized Haar measure. This uniquely specifies a discrete projective Plancherel measure on $(G^0_\theta)\hat{}_{\chi,\chi_\theta}$. We take the image of that measure. Thus a natural measure is defined (via (3.3)) on $\mathfrak{A}_\chi(G)/G$. We place the measure $\dim \tau$ on the finite set $\mathfrak{X}(G_\varphi)$ and the resulting measured defined (via (3.2)) on $\hat{G}_\chi$ is the projective Plancherel measure.

Actually in what follows we shall only need the above results when Z is connected and contained in N. In that case the presentation of the above results simplifies considerably—e.g. $Z \cap G^0_\theta = Z$, $Z \cap N = Z$, $\chi_1 = \chi$ and $Z \cap (G_\theta)^\theta_\xi = Z$. Nevertheless, for potential future use, it is worthwhile to state the most general results.

3c. Definition of the Plancherel measure.

Let G have co-compact radical, satisfy Condition (A) and be of type I. We recall the fiber diagram

$$\text{(3.4)} \qquad \begin{array}{ccc} \mathfrak{X}(G_\varphi) & \to & \hat{G} \\ & & \downarrow \\ & & \mathfrak{A}(G)/G\,. \end{array}$$

It is necessary to define a canonical measure on the base $\mathfrak{A}(G)/G$. That is the main content of this subsection.

We have a surjective map

$$\begin{gathered} \mathfrak{A}(G) \to \mathfrak{A}(N) \to \hat{N} \\ \varphi \to \theta = \varphi|_{\mathfrak{n}} \to \gamma = \gamma(\theta)\,. \end{gathered}$$

It is obviously G-equivariant, so we get a map

$$\mathscr{F}: \mathfrak{A}(G)/G \to \hat{N}/G\,.$$

The fibers of this map are computed in [16].

PROPOSITION 3.3. *Fix $\theta \in \mathfrak{A}(N)$. Then $\mathfrak{A}_\theta(G_\theta)$ is non-empty. Let $\xi \in \mathfrak{A}_\theta(G_\theta)$. Suppose $\varphi \in \mathfrak{g}^*$ extends both θ and ξ. Then $\varphi \in \mathfrak{A}(G)$. Moreover any two such extensions are in the same G-orbit. Thus $\xi \to G\cdot\varphi$, $\mathfrak{A}_\theta(G_\theta) \to \mathfrak{A}(G)/G$ is well-defined. It is G_0-equivariant and factors to a continuous injection*

$$\mathfrak{A}_\theta(G_\theta)/G_\theta \to \mathfrak{A}(G)/G\,.$$

The image is precisely $\mathscr{F}^{-1}(G\cdot\gamma(\theta))$.

The first statement is Lemma 2.2. Everything else, except the last statement, is found in [16, Prop. 5.2]. The last statement is clear once we observe that if $\varphi \in \mathfrak{A}(G)$, $\theta = \varphi|_{\mathfrak{n}}$ and $\xi = \varphi|_{\mathfrak{g}_\theta}$, then $\xi \in \mathfrak{A}_\theta(G_\theta)$. This is because $\theta \in \mathfrak{A}(N)$, $(G_\theta)^\theta_\xi = G^0_\varphi N_\theta$, and χ_θ and χ_φ agree on N_φ.

COROLLARY 3.4. *$\mathfrak{A}_\theta(G_\theta)/G_\theta$ is countably separated.*

Thus we have a Borel fiber diagram

$$(3.5)\qquad \begin{array}{ccc} \mathfrak{A}_\theta(G_\theta)/G_\theta & \to & \mathfrak{A}(G)/G \\ & & \downarrow \\ & & \hat{N}/G\,. \end{array}$$

We utilize (3.5) to define a measure on $\mathfrak{A}(G)/G$. As usual we fix right Haar measures dg, dn on G, N respectively. Then Plancherel measure μ_N on $\hat{N}$ is determined as usual. Now we use Condition (B). We form $\mathscr{N}_2 = \mathscr{N} \cap \mathscr{N}_1$, a co-null, G-invariant subset of $\hat{N}$ such that $\mathscr{N}_2/G$ is countably separated and $G\cdot\gamma$ has a relatively invariant measure of modulus δ_G^{-1} for every $\gamma \in \mathscr{N}_2$. In particular G_γ is unimodular and contained in $\operatorname{Ker}\delta_G$ for every $\gamma \in \mathscr{N}_2$ (Lemma 3.2). Let $\mathscr{S}$ be a Borel cross-section for $\mathscr{N}_2/G$. Choose a pseudo-image $\bar{\mu}_N$ of μ_N on $\mathscr{S}$. It will take a while, but we shall show that this choice uniquely determines a Borel measure $\bar{\mu}_\theta$ on $\mathfrak{A}_\theta(G_\theta)/G_\theta$, for every $\gamma = \gamma(\theta) \in \mathscr{N}_2$.

First, the choice of $\bar{\mu}_N$ uniquely specifies a choice of relatively invariant measure μ_γ on $G\cdot\gamma \approx G/G_\gamma$ such that

$$(3.6)\qquad \int_{\hat{N}} f(\gamma)\,d\mu_N(\gamma) = \int_{\mathscr{S}} \int_{G/G_\gamma} f(\gamma\cdot g)\,d\mu_\gamma(\bar{g})\,d\bar{\mu}_N(\bar{\gamma})\,.$$

This in turn uniquely specifies Haar measures dg_γ on G_γ so that

$$\int_G f(g)\,dg = \int_{G/G_\gamma} \int_{G_\gamma} f(g_\gamma g)\,dg_\gamma\,d\mu_\gamma(\bar{g})\,.$$

This further uniquely specifies Haar measures $d\bar{g}_\gamma$ on G_γ/N so that

$$\int_{G_\gamma} f(g_\gamma)\,dg_\gamma = \int_{G_\gamma/N} \int_N f(ng_\gamma)\,dn\,d\bar{g}_\gamma\,.$$

But $G_\gamma/N \cong G_\theta/N_\theta$ and so Haar measures on G_θ/N_θ are also determined by the previous choices. Going on, we have uniquely specified Haar measures on the open subgroups G_θ^0/N_θ. Then we invoke the results of subsection 3b to see that projective Plancherel measures are uniquely specified on $(G_\theta^0/Q_\theta)\hat{}_{\chi_\theta}$. In fact since $(G_\theta)_\xi^0$ is connected (see [16, Lemma 3.4]), it is easy to see that

$$(G_\theta^0/Q_\theta)\hat{}_{\chi_\theta} \approx \mathfrak{A}_\theta(G_\theta)/G_\theta^0;$$

and the projective Plancherel measures of $(G_\theta/Q_\theta)^\wedge_{\chi_\theta}$ give canonical measures μ_{θ,χ_θ} on $\mathfrak{A}_\theta(G_\theta)/G_\theta^0$. We disintegrate μ_{θ,χ_θ} under the action of G_θ/G_θ^0. We get the fiber diagram

$$\begin{array}{ccc} G_\theta/G_\varphi G_\theta^0 & \to & \mathfrak{A}_\theta(G_\theta)/G_\theta^0 \\ & & \downarrow \\ & & \mathfrak{A}_\theta(G_\theta)/G_\theta, \end{array}$$

because the stability group of $G_\theta^0\cdot\xi$ is $(G_\theta)_\xi G_\theta^0$. Each of the fibers has counting measure, and thus there is uniquely determined a pseudo-image $\bar\mu_\theta$ of μ_{θ,χ_θ} on the quotient $\mathfrak{A}_\theta(G_\theta)/G_\theta$. This is the promised measure on $\mathfrak{A}_\theta(G_\theta)/G_\theta$. It has a further property which we will use in the next subsection. It is described as follows. The projective dual $(G_\theta/Q_\theta)^\wedge_{\chi_\theta}$ satisfies a fiber diagram

$$\begin{array}{ccc} \mathfrak{X}(G_\varphi) = \mathfrak{X}\big((G_\theta)_\xi\big) & \to & (G_\theta/Q_\theta)^\wedge_{\chi_\theta} \\ & & \downarrow \\ & & \mathfrak{A}_\theta(G_\theta)/G_\theta. \end{array}$$

We place the measures $d\tau$ on the fibers, $\bar\mu_\theta$ on the base. Then it is routine to verify (using [16, Lemmas 3.7, 3.8]) that the resulting measure on $(G_\theta/Q_\theta)^\wedge_{\chi_\theta}$ is *the* projective Plancherel measure corresponding to the above-determined Haar measure on G_θ/N_θ.

Next we take the pseudo-image $\bar\mu_N$ on $\mathscr{S}$, the canonical measure $\bar\mu_\theta$ on $\mathfrak{A}_\theta(G_\theta)/G_\theta$ and use (3.5) to get a canonical measure $\bar\mu$ on $\mathfrak{A}(G)/G$. It is clear that the set $\mathfrak{A}_1(G)$ (subsection 3*b*) is of full measure. Finally we take $d\tau$ on the fiber, $d\bar\mu$ on the base in (3.4) to a get measure μ on $\hat G$. We shall verify in the next subsection that $D^{\frac{1}{2}}_{\pi_\varphi,\tau}d\mu$ is the Plancherel measure on G.

3*d*. *The computation.*

Let G have co-compact radical, satisfy Conditions (A) and (B), and be of type I. We have the diagram

$$\begin{array}{ccc} \mathfrak{X}(G_\varphi) & \to & \hat G \\ & & \downarrow \\ & & \mathfrak{A}(G)/G. \end{array}$$

By means of this diagram we have defined a measure μ on $\hat G$ in Subsection 3*c*.

Furthermore the semi-invariants $D_{\pi_{\varphi,\tau}}$ have been defined in Subsection 3*a*. Then we have

THEOREM 3.5 (Plancherel formula). *Let* $f \in C_C^\infty(G)$. *Then*

$$f(1_G) = \int_{\hat{G}} \operatorname{Tr} D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} \pi_{\varphi,\tau}(f) D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} d\mu(\pi)$$

$$= \int_{\mathfrak{A}(G)/G} \int_{\mathfrak{X}(G_\varphi)} \operatorname{Tr} D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} \pi_{\varphi,\tau}(f) D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} d\tau\, d\bar{\mu}(\bar{\varphi}) .$$

PROOF. The computation is modelled after those of [13, [17]. On the one hand we have

$$(3.7) \qquad f(1_G) = f(1_N) = \int_{\hat{N}} \operatorname{Tr} \gamma(f)\, d\mu_N(\gamma) \qquad \text{(Plancherel formula of } N)$$

$$= \int_{\mathscr{S}} \int_{G/G_\gamma} \operatorname{Tr} (\gamma \cdot g)(f)\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma}) . \qquad \big(\text{formula (3.6)}\big) .$$

On the other hand we have

$$\int_{\mathfrak{A}(G)/G} \int_{\mathfrak{X}(G_\varphi)} \operatorname{Tr} D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} \pi_{\tau,\varphi}(f) D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} d\tau\, d\bar{\mu}(\bar{\varphi})$$

$$= \int_{\mathscr{S}} \int_{\mathfrak{A}_\theta(G_\theta)/G_\theta} \int_{\mathfrak{X}(G_\varphi)} \operatorname{Tr} D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} \pi_{\varphi,\tau}(f) D^{\frac{1}{2}}_{\pi_{\varphi,\tau}} d\tau\, d\bar{\mu}_\theta\, d\bar{\mu}_N(\bar{\gamma}) \qquad (\text{definition of } \bar{\mu})$$

$$= \int_{\mathscr{S}} \int_{\mathfrak{A}_\theta(G_\theta)/G_\theta} \int_{\mathfrak{X}(G_\varphi)} \operatorname{Tr} D^{\frac{1}{2}}_{\delta_G} \operatorname{Tr} \operatorname{Ind}^G_{G_{\gamma(\theta)}} \big[\big(\nu_{\xi,\tau} \otimes \tilde{\gamma}(\theta)\big) \otimes \gamma\big](f) D^{\frac{1}{2}}_{\delta_G} d\tau\, d\bar{\mu}_\theta\, d\bar{\mu}_N(\bar{\gamma})$$

$$= \int_{\mathscr{S}} \int_{\mathfrak{A}_\theta(G_\theta)/G_\theta} \int_{\mathfrak{X}(G_\varphi)} \int_{G/G_\gamma} \delta_G(g)^{-1} q_{G/G_\gamma}(g)^{-1} \operatorname{Tr} \int_{G_\gamma} f(g^{-1} g_\gamma g)(\nu_{\xi,\tau} \otimes \tilde{\gamma} \otimes \gamma)(g_\gamma) \cdot$$

$$\cdot\, \delta^{\frac{1}{2}}_G(g_\gamma)\, \delta^{-\frac{1}{2}}_{G_\gamma}(g_\gamma)\, dg_\gamma\, d\mu_\gamma(g)\, d\tau\, d\bar{\mu}_\theta\, d\bar{\mu}_N(\bar{\gamma}) \qquad \text{(Lemma 3.1)}$$

$$= \int_{\mathscr{S}} \int_{G/G_\gamma} \int_{\mathfrak{A}_\theta(G_\theta)/G_\theta} \int_{\mathfrak{X}(G_\varphi)} \delta_G(g)^{-1} \sum_{i,j} \int_{G_\gamma} f(g^{-1} g_\gamma g) \langle \nu_{\xi,\tau}(g_\gamma) \zeta_i, \zeta_i \rangle \langle \tilde{\gamma} \otimes \gamma(g_\gamma) \eta_j, \eta_j \rangle \cdot$$

$$\cdot\, dg_\gamma\, d\tau\, d\bar{\mu}_\theta\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})$$

where $\{\zeta_i\}$, $\{\eta_j\}$ are orthonormal bases of $\mathscr{H}_\nu$, $\mathscr{H}_\gamma$ respectively. Now let

$$\Omega_{g,j}(g_\gamma) = \int_N f(g^{-1} n g_\gamma g) \langle \tilde{\gamma} \otimes \gamma(n g_\gamma) \eta_j, \eta_j \rangle\, dn .$$

$\Omega_{\sigma,j}$ is of course a function on $G_\gamma/N \cong G_\theta/N_\theta$, and we have

$$(3.8)\qquad \int_{\mathfrak{A}(G)/G}\int_{\mathfrak{X}(G_\varphi)} \operatorname{Tr} D^{\frac{1}{2}}_{\pi_{\varphi,\tau}}\pi_{\varphi,\tau}(f)D^{\frac{1}{2}}_{\pi_{\varphi,\tau}}d\tau\, d\bar{\mu}(\bar{\varphi})$$

$$=\int_{\mathscr{S}}\int_{G/G_\gamma}\int_{\mathfrak{A}_\theta(G_\theta)/G_\theta}\int_{\mathfrak{X}(G_\varphi)}\delta_G(g)^{-1}\sum_{i,j}\int_{G_\gamma/N}\Omega_{\sigma,j}(g_\gamma)\langle\nu_{\xi,\tau}(g_\gamma)\zeta_i,\zeta_i\rangle\, d\bar{g}_\gamma\, d\tau\, d\bar{\mu}_\theta\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})$$

$$=\int_{\mathscr{S}}\int_{G/G_\gamma}\delta_G(g)^{-1}\sum_j\Omega_{\sigma,j}(1)\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})$$

(projective Plancherel formula for $(G_\theta/Q_\theta)_{\chi_\theta}$)

$$=\int_{\mathscr{S}}\int_{G/G_\gamma}\delta_G(g)^{-1}\sum_j\int_N f(g^{-1}ng)\langle\gamma(n)\eta_j,\eta_j\rangle\, dn\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})$$

$$=\int_{\mathscr{S}}\int_{G/G_\gamma}\delta_G(g)^{-1}\operatorname{Tr}\int_N f(g^{-1}ng)\gamma(n)\, dn\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})$$

$$=\int_{\mathscr{S}}\int_{G/G_\gamma}\operatorname{Tr}\int_N f(n)\gamma(gng^{-1})\, dn\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})=\int_{\mathscr{S}}\int_{G/G_\gamma}\operatorname{Tr}(\gamma\cdot g)(f)\, d\mu_\gamma(g)\, d\bar{\mu}_N(\bar{\gamma})\,.$$

Comparing (3.7) and (3.8), we see that the Plancherel formula is proven.

REMARKS. (1) In this paper, we have constructed the Plancherel measure of a type I co-compact radical Lie group (which satisfies Conditions (A) and (B)). We used the formerly known structure theory and Plancherel measure of co-compact nilradical groups to construct the basic ingredient —namely, the canonical measure on $\mathfrak{A}(G)/G$. One would like to have an intrinsic scheme for constructing this measure. Duflo has indicated how this might be done [6, Appendix]. His scheme is in terms of admissible orbits. I also plan to change to admissible orbits in a later publication on harmonically induced representations of non-amenable groups.

(2) It is interesting to speculate if the canonical measure on $\mathfrak{A}(G)/G$ is the pseudo image of a canonical measure on $\mathfrak{A}(G)$. Of course $\mathfrak{A}(G)$ is in general of lower dimension than $\mathfrak{g}^*$. Nevertheless Vergne [21] has demonstrated the existence of a measure on $\mathfrak{A}(G)$ which is in duality with Lebesgue measure on $\mathfrak{g}$ by means of a generalized Poisson Summation Formula. It would be interesting to relate Vergne's measure to the canonical measure.

(3) The results of this paper are of course valid for connected solvable groups—the case treated by Charbonnel [3]. But the proofs here are dra-

matically more simple than those of [3]. Condition (B) removes the need for the algebraic hull. So it is important to determine if Condition (B) always holds, or if there are counterexamples.

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