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Hypoelliptic and Gevrey Hypoelliptic Invariant Differential Operators on Certain Symmetric Spaces.

PAUL GODIN

0. – Summary.

In this paper we find a necessary condition for the hypoellipticity of invariant differential operators on a Riemannian symmetric space G/K of the noncompact type. We prove that this necessary condition is also sufficient in the following cases: when G has a complex structure, when G is the product of real rank one groups, or when a transversality condition is satisfied. We obtain analogous results with hypoellipticity replaced by Gevrey hypoellipticity.

1. – Introduction.

A differential operator P on a C^∞ paracompact manifold X is called hypoelliptic if for each distribution $u \in \mathcal{D}'(X)$, u and Pu have the same singular support. If X is analytic and $s \geq 1$, P is said to be Gevrey hypoelliptic of class s if u and Pu have the same Gevrey singular support of class s for each $u \in \mathcal{D}'(X)$. Here the Gevrey singular support of class s of $v \in \mathcal{D}'(X)$ is the complement of the largest open set where v belongs to the s -th Gevrey class \mathcal{G}_s .

If X is an open subset of \mathbb{R}^n , Hörmander has characterized hypoelliptic and Gevrey hypoelliptic differential operators with constant coefficients on X in terms of their symbol (see [12]; [13], chapter IV). The symbol $p(\xi)$, $\xi \in \mathbb{R}^n$, of a differential operator $P(D)$ with constant coefficients on $X \subset \mathbb{R}^n$ is defined by $P(D_x)(e^{i\langle x, \xi \rangle}) = p(\xi)e^{i\langle x, \xi \rangle}$, where $x \in X$, $D = (D_1, \dots, D_n)$,

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$iD_j = \partial/\partial x_j$, and $\langle x, \xi \rangle = \sum x_j \xi_j$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, let us write $|\alpha| = \sum_{1 \leq j \leq n} \alpha_j$ and $p^{(\alpha)}(\xi) = \partial^{|\alpha|} p(\xi) / \partial \xi_1^{\alpha_1} \dots \partial \xi_n^{\alpha_n}$. Then Hörmander's condition for hypoellipticity can be written

$$(A) \quad p^{(\alpha)}(\xi)/p(\xi) \rightarrow 0 \quad \text{if } 0 \neq \alpha \in \mathbb{Z}_+^n \text{ and } \xi \rightarrow \infty,$$

and Hörmander's condition for Gevrey hypoellipticity of class s is

$$(A_s) \quad |p^{(\alpha)}(\xi)/p(\xi)| \leq C|\xi|^{-|\alpha|/s}$$

for some $C > 0$ independent of ξ , when ξ is large enough.

In this paper we shall study regularity properties of solutions of invariant differential equations on Riemannian symmetric spaces of the noncompact type, that is on coset spaces $X = G/K$, where G is a connected non compact semisimple Lie group with finite center and K is a maximal compact subgroup of G . There is an action τ of G on X defined by the formula $\tau(g)(hK) = (gh)K$ if $g, h \in G$. By «invariant differential operators» we mean those who have complex coefficients and are τ -invariant. As shown in [4], they form a commutative algebra which we shall denote by $\mathbf{D}(X)$. Helgason [6] (resp. [11]) has proved that any non zero $P \in \mathbf{D}(X)$ is locally solvable (resp. is surjective from $C^\infty(X)$ to $C^\infty(X)$).

A function $\varphi \in C^\infty(X)$ is called spherical if

- (i) $\varphi(K) = 1$,
- (ii) $\varphi(kxK) = \varphi(xK)$ for all $k \in K$ and $x \in G$,
- (iii) for each $Q \in \mathbf{D}(X)$, φ is an eigenvector of Q .

Harish-Chandra [3] determined all the spherical functions of X . If $G = KAN$ is an Iwasawa decomposition of G , he proved that they can be parametrized by \mathfrak{a}_c^* (the complexified space of the dual \mathfrak{a}^* of the Lie algebra \mathfrak{a} of A) when one associates with $\zeta \in \mathfrak{a}_c^*$ the spherical function

$$\varphi_\zeta(gK) = \int_k e^{\langle i\zeta - \rho, H(\sigma k) \rangle} dk.$$

Here we have to explain the notation in the right-hand side: dk is the Haar measure on K with total measure equal to 1, $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ is the half sum of the positive restricted roots α (counted with their multiplicity m_α) relative to the choice of a positive Weyl chamber, and $H: G \rightarrow \mathfrak{a}$ is defined by $\exp H(g) = a$ if $g = kan$ is an Iwasawa decomposition of G . Two spher-

rical functions φ_ζ and $\varphi_{\zeta'}$ are equal if and only if $\zeta' = s\zeta$ for some element s of the Weyl group W (see [3] and [4] chapter X).

With $P \in \mathcal{D}(X)$ we may associate a complex polynomial p on \mathfrak{a} , W -invariant, by the formula $P\varphi_\xi = p(\xi)\varphi(\xi)$ (see for example [4], chapter X , where $p(\xi)$ is denoted by $\Gamma(P)(i\xi)$).

The purpose of our paper is to study relations between the hypoellipticity (resp. Gevrey hypoellipticity of class s) of P and condition (A) (resp. (A_s)) imposed on p . This paper is divided into three sections.

In section I we prove some useful properties of the polynomial p .

Section II is devoted to the study of hypoellipticity. In II.1, we prove that p satisfies (A) if P is hypoelliptic.

To study the sufficiency of (A) for hypoellipticity, we construct in II.2 a parametrix \mathcal{T} of P , which is the « convolution » by some $T \in \mathcal{D}'(X)$. When G is complex, we show in II.3 that (A) implies hypoellipticity. In II.4 it is proved that T is smooth in X' when (A) is fulfilled. Here X' is the set of regular points of X . The method of II.4 is then used in II.5 to show that (A) implies hypoellipticity when G is a product of real rank one groups. In II.6, we consider T as the pullback of a distribution on \mathfrak{a} and introduce a transversality condition (condition (B)). We show that when (A) is fulfilled, T is smooth in a neighbourhood of each point where (B) is satisfied. Therefore P is hypoelliptic if (B) is satisfied at each point of $X \setminus (X' \cup \{K\})$. (Here K denotes the origin of G/K) ⁽¹⁾. We end II.6 by presenting some examples.

Section III, which parallels section II, is devoted to the study of Gevrey hypoellipticity. In III.1, we prove that (A_s) is a necessary condition for Gevrey hypoellipticity of class s . In III.2, we reduce the problem of sufficiency of (A_s) to the study of T . The sufficiency of (A_s) is proved in III.3 when G is complex and in III.5 when G is a product of real rank one groups. In III.4, we prove that $T \in G_s(X')$ if p satisfies (A_s) . In III.6, we introduce condition (B_s) . When it is fulfilled at each point of $X \setminus (X' \cup \{K\})$, P is Gevrey hypoelliptic of class s as soon as p satisfies (A_s) . Examples are given to conclude paragraph III.6.

This paper makes use of the theory of Riemannian symmetric spaces and of the theory of hypoelliptic differential equations with constant coefficients in \mathbb{R}^n . For all unexplained notions on the first topic we refer to [4], [5]; see also the beginning of [6], [8] and [10]. For the second topic we refer to [13].

⁽¹⁾ *Added in proof:* After this paper was written, Professor J. J. Duistermaat pointed out to me that the use of Abel transform allows to eliminate the transversality hypothesis.

I. - Some properties of the polynomial p .

In this section we collect some properties of p which we shall need in the sequel. First we recall some well known facts (see [4], chapter X).

Denote by $\mathbf{D}(G)$ the algebra of left invariant differential operators with complex coefficients on G . Let $\mathbf{D}_K(G)$ be the subalgebra consisting of those operators of $\mathbf{D}(G)$ which are also right invariant under K . If $\pi: G \rightarrow X = G/K$ is the canonical projection, there is a homomorphism $\mu: \mathbf{D}_K(G) \rightarrow \mathbf{D}(X)$ given by the formula $(\mu(Q)f) \circ \pi = Q(f \circ \pi)$ if $Q \in \mathbf{D}_K(G)$ and $f \in C^\infty(X)$. If \mathfrak{k} is the Lie algebra of K , the kernel of μ is equal to $\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k}$. On the other hand, if $I(\mathfrak{a}_\mathfrak{c})$ is the algebra of complex polynomials on \mathfrak{a} which are invariant under the Weyl group, there is a canonical homomorphism $\nu: \mathbf{D}_K(G) \rightarrow I(\mathfrak{a}_\mathfrak{c})$ with kernel $\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k}$. From μ and ν we get a canonical isomorphism of algebras $\Gamma: \mathbf{D}(X) \rightarrow I(\mathfrak{a}_\mathfrak{c})$ such that $P\varphi_\xi = (\Gamma(P)(i\xi))\varphi_\xi$ if $P \in \mathbf{D}(X)$. As said in the introduction, we put $p(\xi) = \Gamma(P)(i\xi)$.

If $P \in \mathbf{D}(X)$, denote by $\text{ord } P$ the order of P (as a differential operator) and by $\text{deg } p$ the degree of p (as a polynomial). Then the following holds:

LEMMA I.1. $\text{Ord } P = \text{deg } p$.

PROOF. (a) Let A^+ be a positive Weyl chamber in A . We can define the radial part $\text{rad}(P)$ of P in $A^+ \cdot o$. (We shall often denote by $g \cdot o$ the point $g\{K\}$ of $X = G/K$). $\text{rad}(P)$ is the unique differential operator on $A^+ \cdot o$ such that $Pf|_{A^+ \cdot o} = \text{rad}(P)(f|_{A^+ \cdot o})$ if $f \in C^\infty(X)$ is K -invariant, where $|_{A^+ \cdot o}$ means restriction to $A^+ \cdot o$ (see [3] or [9] chapter II).

Furthermore $\text{rad}(P) = e^{-\varrho} \Gamma(P)e^\varrho + \text{lower order terms}$, ([9], chapter II, prop. 1.5), where ϱ is as in the introduction half the sum of the positive restricted roots with multiplicity. (Here of course we have identified the polynomial $e^{-\varrho} \Gamma(P)e^\varrho$ with the differential operator on A it defines). This equality shows that $\text{deg } p = \text{ord } \text{rad}(P)$, which in turn is not larger than $\text{ord } P$.

(b) Let \mathfrak{g} be the Lie algebra of G and \mathfrak{p} the orthogonal complement of \mathfrak{k} in \mathfrak{g} , with respect to the Killing form, so that \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Denote by $I(\mathfrak{p}_\mathfrak{c})$ the algebra of complex $Ad_G(K)$ -invariant polynomials on \mathfrak{p} . Using the Killing form which allows to identify \mathfrak{a} and \mathfrak{a}^* (resp. \mathfrak{p} and \mathfrak{p}^*), we may consider elements of $I(\mathfrak{a}_\mathfrak{c})$ (resp. $I(\mathfrak{p}_\mathfrak{c})$) as polynomial functions on \mathfrak{a} (resp. \mathfrak{p}). Then the Chevalley isomorphism theorem ([3]; [4], chapter X) implies that any $p \in I(\mathfrak{a}_\mathfrak{c})$ has a unique extension to an element $\tilde{p} \in I(\mathfrak{p}_\mathfrak{c})$. Denote by $\lambda: I(\mathfrak{p}_\mathfrak{c}) \rightarrow \mathbf{D}_K(G)$ the symmetrization map

defined by $\lambda(Y_1 \dots Y_r) = \sum_{\sigma} Y_{\sigma(1)} \dots Y_{\sigma(r)} / r!$, where σ runs over the group of all permutations of the set $\{1, \dots, r\}$. We are going to show the following:

- (1) $\Gamma^{-1}(p)$ and $\mu\lambda(\tilde{p})$ have the same order equal to $\deg p$ and the same principal part.

We prove (1) by induction on $\deg p$ which we denote by d . If $d = 0$, (1) is clear, so we suppose that (1) is true for $d \leq s$ and we prove it when $d = s + 1$. By Lemma 6.12 of [4], chapter X, $\nu\lambda(\tilde{p}) - p$ is of degree less than or equal to s ; hence the induction hypothesis shows that $\text{ord } \Gamma^{-1}(\nu\lambda(\tilde{p}) - p) \leq s$. But $\Gamma^{-1}(\nu\lambda(\tilde{p}) - p) = \mu\lambda(\tilde{p}) - \Gamma^{-1}(p)$; furthermore $\text{ord } \mu\lambda(\tilde{p}) \leq \deg \tilde{p} = \deg p = s + 1$; and $\text{ord } \Gamma^{-1}(p) \geq \deg p$ by part (a) of the lemma. This proves (1) and completes the proof of the lemma.

In fact we can be more precise. Let $P \in \mathbf{D}(X)$ be of order m and write $p = \sum_{0 \leq j \leq m} p_{m-j}$, where $p_{m-j} \in I(\mathfrak{a}_0)$ is homogeneous of degree $m - j$. Denote by $\sigma(P)$ the principal symbol of P in the usual sense of differential operators theory; this means that $\sigma(P)$ is the function from $T^*X \setminus O$ to \mathbf{C} defined by $\sigma(P)(x, \xi) = i^m P(f^m)(x) / m!$ if $f \in C^\infty(X)$ vanishes at x and has a differential at x equal to ξ . Denote by Exp the exponential map $\mathfrak{p} \rightarrow X$ (that is the composition of the exponential map of \mathfrak{g} restricted to \mathfrak{p} with the canonical projection from G to X) and by \tilde{p}_m the canonical extension of p_m to an element of $I(\mathfrak{p}_0)$. Then we have:

LEMMA I.2. *If $x = g \circ o \in X$, we have $\sigma(P)(x, \xi) = \tilde{p}_m(\xi \circ d\tau(g)_0 \circ d \text{Exp}_o)$ where O denotes the origin of \mathfrak{p} .*

PROOF. As noticed in the proof of lemma I.1, P and $R = \mu\lambda(\widetilde{\Gamma(P)})$ have the same principal part. We are going to compute $i^m R(f^m)(x) / m!$ when $f \in C^\infty(X)$, $f(x) = 0$, $df_x = \xi$. Applying theorem 2.7 of [4], chapter X, we see that $R(f^m)(x) = R'(f^m \circ \tau(g) \circ \text{Exp})(O)$, where R' is the differential operator on \mathfrak{p} defined as follows. Choose any basis Y_1, \dots, Y_l of \mathfrak{p} and associate to $Y = \sum_{1 \leq j \leq l} y_j Y_j \in \mathfrak{p}$ the linear coordinates $y = (y_1, \dots, y_l) \in \mathbf{R}^l$. If $\widetilde{\Gamma(P)} = \sum_{1 \leq \alpha \leq l} a_\alpha Y^\alpha$ (with the usual multi-indices notation) and $\varphi \in C^\infty(\mathfrak{p})$, R' is defined by $(R'\varphi)(Y) = \sum_{1 \leq \alpha \leq l} a_\alpha \partial_y^\alpha \Phi(y)$ if $\Phi(y) = \varphi(\sum_{1 \leq j \leq l} y_j Y_j)$. Note that $\text{ord } R' = \deg \widetilde{\Gamma(P)}$, which by (1) is equal to m . Therefore $\sigma(R)(x, \xi) = \sigma(R')(\xi \circ d\tau(g)_0 \circ d \text{Exp}_o)$. The last expression is readily seen to be equal to $\tilde{p}_m(\xi \circ d\tau(g)_0 \circ d \text{Exp}_o)$. The proof is complete.

In section III the following result will be useful:

LEMMA I.3. *For each $\zeta \in \mathfrak{a}^*$, there exists $p \in I(\mathfrak{a}_0)$ homogeneous and not constant, such that $p(\zeta) \neq 0$.*

PROOF. Denote by s_1, \dots, s_r the elements of the Weyl group of \mathfrak{a} . They act on \mathfrak{a}^* by $\langle s_j \xi, H \rangle = \langle \xi, s_j^{-1} H \rangle$, $\xi \in \mathfrak{a}^*$, $H \in \mathfrak{a}$, and on \mathfrak{a}_c^* by complexification. If σ_j is the j -th elementary symmetric function in r indeterminates, define $f_j \in I(\mathfrak{a}_c)$ by the formula $f_j(\lambda) = \sigma_j(\langle s_1 \lambda, N \rangle, \dots, \langle s_r \lambda, N \rangle)$, $\lambda \in \mathfrak{a}_c^*$, where $N \in \mathfrak{a}$ is such that $\langle \zeta, N \rangle \neq 0$.

If $z_1, \dots, z_r \in \mathbf{C}$, z_1, \dots, z_r are the r complex roots w of the equation $w^r + \sum_{1 \leq j \leq r} (-1)^j \sigma_j(z_1, \dots, z_r) w^{r-j} = 0$. Therefore there must be some j such that $f_j(\zeta) \neq 0$.

As explained in [4], chapter X, § 6, $I(\mathfrak{a}_c)$ is finitely generated. Denote by p^1, \dots, p^l a set of not constant homogeneous elements of $I(\mathfrak{a}_c)$ which, together with 1, generates $I(\mathfrak{a}_c)$. Then we have the following obvious consequences of lemma I.3.

COROLLARY I.1. *If $\zeta \in \mathfrak{a}_c^*$, there exists some j , $1 \leq j \leq l$, such that $p^j(\zeta) \neq 0$.*

COROLLARY I.2. *Put $d_j = \deg p^j$ and denote by $|\zeta|$ any fixed norm in \mathfrak{a}_c^* . Then there exists a strictly positive constant C such that, for all $\zeta \in \mathfrak{a}_c^*$:*

$$\sum_{1 \leq j \leq l} |p^j(\zeta)|^{1/d_j} > C|\zeta|.$$

II. - Hypoellipticity.

II.1. - The necessity of (A).

In this section we are going to study operators $P \in \mathbf{D}(X)$ for which the corresponding $p \in I(\mathfrak{a}_c)$ satisfies condition (A). Since p can be viewed as a function on \mathfrak{a}^* , the precise meaning of (A) is of course that $p^{(\alpha)}(\xi)/p(\xi) \rightarrow 0$ when $\mathbf{Z}_+^l \ni \alpha \neq 0$ and $\xi \rightarrow \infty$ ($l = \dim \mathfrak{a}$), where $p^{(\alpha)}$ is computed in any linear coordinate system (ξ_1, \dots, ξ_l) of \mathfrak{a}^* .

Using the well known characterizations of polynomials satisfying (A) (see [13], chapter IV and [17], chapter 7), it suffices to prove the following theorem to show that p satisfies (A) if $P \in \mathbf{D}(X)$ is hypoelliptic.

THEOREM II.1.1. *Let S be a not empty relatively compact open subset of X . Assume that for each $v \in \mathcal{D}'(S)$, $Pv = 0$ in S implies that $v \in C^\infty(S)$. Then the following holds: if $\zeta \in \mathfrak{a}_c^*$ tends to ∞ and satisfies $p(\zeta) = 0$, then $\text{Im } \zeta$ tends to ∞ . (We write $\zeta = \text{Re } \zeta + i \text{Im } \zeta$; $\text{Re } \zeta, \text{Im } \zeta \in \mathfrak{a}^*$).*

PROOF. The proof is similar to that of the corresponding theorem for differential operators with constant coefficients in \mathbf{R}^n . We equip $N(S) = \{v \in L_{\text{loc}}^\infty(S), Pv = 0\}$ with the topology induced by the usual topology of $L_{\text{loc}}^\infty(S)$. We have $N(S) \subset C^\infty(S)$. Denoting by S' an open subset of S

such that $\bar{S}' \subset S$ and by $|_S$, the restriction to S' , the mapping $v \mapsto Lv|_S$, from $N(S)$ to $L^\infty(S')$ has a closed graph if L denotes the Laplace operator of X for the metric defined by the Killing form. Assuming, as we may, that $o \in S'$, the closed graph theorem implies that for some $F \in C_0^\infty(S)$ and some positive constant C :

$$(2) \quad |Lv(o)| \leq C \sup_{x \in S} |Fv(x)|$$

if $v \in N(S)$.

Now $\varphi_\zeta \in N(S)$ if $\zeta \in \mathfrak{a}_c^*$ satisfies $p(\zeta) = 0$. Furthermore corollary 2 of [3] implies that

$$(3) \quad L\varphi_\zeta = -\langle \zeta, \zeta \rangle + \langle \varrho, \varrho \rangle \varphi_\zeta,$$

where $\varrho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, as recalled in the introduction. We are using the following notation: if $\lambda, \mu \in \mathfrak{a}_c^*$, let $H_\lambda \in \mathfrak{a}_c$ be determined by $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}$, where B is the Killing form; then we put $\langle \lambda, \mu \rangle = B(H_\lambda, H_\mu)$. On the other hand,

$$(4) \quad \varphi_\zeta(g \cdot o) = \int_K e^{i\langle \zeta - \varrho, H(\sigma k) \rangle} dk.$$

Introducing (3) (resp. (4)) in the left-hand (resp. right-hand) side of (2), we see that when $p(\zeta) = 0$, $\text{Re } \zeta$ must remain bounded if $\text{Im } \zeta$ is bounded. The proof is complete.

From theorem II.1.1 we can get some information about the real characteristic points of P :

THEOREM II.1.2. *If $P \in \mathbf{D}(X)$ is hypoelliptic and $(x, \xi) \in T^*X \setminus O$, then $\sigma(P)(x, \xi) = 0$ implies that $d\sigma(P)(x, \xi) = 0$.*

PROOF. By lemma I.2, we have $\sigma(P)(g \cdot o, \xi) = \tilde{p}_m(\eta)$, where $\eta = \xi \circ d\tau(g)_o \circ d \text{Exp}_o$. So if $\sigma(P)(g \cdot o, \xi) = 0$, we have $\tilde{p}_m(\eta) = 0$. We are going to show that $d\tilde{p}_m(\eta) = 0$, which of course will imply that $d\sigma(P)(g \cdot o, \xi) = 0$.

Denote by M the centralizer of \mathfrak{a} in K and by \mathfrak{a}^+ a positive Weyl chamber in \mathfrak{a} . Let $\varphi: K/M \times \mathfrak{a}^+ \rightarrow \mathfrak{p}$ be defined by $\varphi(kM, H) = \text{Ad}_G(k)H$ for $k \in K$ and $H \in \mathfrak{a}^+$. φ is a diffeomorphism onto a dense subset \mathfrak{p}' of \mathfrak{p} (see [4], chapter X). Put $\varphi^{-1}(Y) = (x'(Y), x''(Y)) \in K/M \times \mathfrak{a}^+$ for $Y \in \mathfrak{p}'$, and $\mathfrak{a}' = \mathfrak{p}' \cap \mathfrak{a}$.

(a) Assume first that $\eta \in \mathfrak{p}'$. We can find $k \in K$ such that $\eta' = \text{Ad}_G(k)\eta \in \mathfrak{a}$. Since $\tilde{p}_m(\eta') = 0$, we have $p_m(\eta') = 0$ and so $dp_m(\eta') = 0$ because p_m satisfies condition (A) by theorem II.1.1. Since in the coordinate system (x', x'') , \tilde{p}_m is independent of x' , it is clear that $d\tilde{p}_m(\eta') = 0$, whence $d\tilde{p}_m(\eta) = 0$.

(b) Assume now that $\eta \in \mathfrak{p} \setminus \mathfrak{p}'$. If $dp_m(\eta) \neq 0$, the set $E = \{\eta' \in \mathfrak{p}, \tilde{p}_m(\eta') = 0 \neq d\tilde{p}_m(\eta')\}$ is a manifold through η of dimension equal to $\dim \mathfrak{p} - 1$. Furthermore $E \subset \mathfrak{p} \setminus \mathfrak{p}'$, as we have seen in (a). But $\dim(\mathfrak{p} \setminus \mathfrak{p}') \leq \dim \mathfrak{p} - 2$ (see [4], chapter X). This contradiction proves the theorem.

REMARK II.1.1. If $X = G/K$ is a rank one symmetric space of the noncompact type, equipped with the metric defined by the Killing form, then every nonzero $P \in \mathcal{D}(X)$ is a nonzero polynomial $\sum a_j L^j$ with complex coefficients a_j in the Laplace operator L of X ([4], chapter X). Formula (3) shows that the corresponding p satisfies condition (A_1) , hence (A_s) for all $s \geq 1$ and (A) . On the other hand, P is an elliptic operator with analytic coefficients on the analytic manifold X , hence it is hypoelliptic and Gevrey hypoelliptic of class s for each $s \geq 1$. Therefore when rank X is equal to 1, (A) (resp. (A_s)) is necessary and sufficient for hypoellipticity (resp. Gevrey hypoellipticity of class s).

REMARK II.1.2. When rank ≥ 2 , we can always find non elliptic operators $P \in \mathcal{D}(X)$ such that the corresponding p satisfies (A) . In fact let $r \in I(\alpha_c)$ be the polynomial of degree 2 associated with the Killing form B . Let $q \in I(\alpha_c)$ be real and not elliptic. For example, we may take $q = \prod_{\alpha \in \Sigma^+} H_\alpha^2$. Theorem 4.1.9 of [13] shows that the polynomial $p(\xi) = q(\xi)^{2k} + r(\xi)^{km-k+1}$ satisfies (A) if $m = \deg q$ and k is an integer larger than or equal to 2. It suffices then to take P such that $\Gamma(P)(i\xi) = p(\xi)$. Since p_{2mk} vanishes at some nonzero vector of α^* , lemma I.2 shows that P cannot be elliptic.

II.2. - Construction of a parametrix of P when condition (A) is satisfied.

We shall need suitably normalized measures, the definition of which we recall now. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra \mathfrak{g} of G , with Cartan involution θ . Let $G = KAN$ be a corresponding Iwasawa decomposition and denote by M the centralizer of \mathfrak{a} in K .

We define Haar measures dk, dm, da, dn, dg on K, M, A, N, G by the following conditions: $\int_K dk = 1, \int_M dm = 1, (2\pi)^{\text{rank } X/2} da$ is the Euclidean measure induced by the Killing form, $\theta(dn)$ is the Haar measure $d\bar{n}$ on $\bar{N} = \theta(N)$ normalized by

$$\int_{\bar{N}} e^{-2\langle e, H(\bar{n}) \rangle} d\bar{n} = 1, \quad \text{and} \quad \int_G f(g) dg = \int_{K \times A \times N} f(kan) e^{2\langle e, \log a \rangle} dk da dn$$

for each $f \in C_0^\infty(G)$ if $g = kan$ is the Iwasawa decomposition of g defined

above. (If Y is a C^∞ manifold, we denote by $C_0^\infty(Y)$ the space of C^∞ functions with compact support on Y). The G -invariant measure on G/K induced by dg and dk will be denoted by dx . Finally, the quotient $B = K/M$ will be equipped with the K -invariant measure db of total measure one induced by dk and dm .

Since G (resp. X) is equipped with a canonical positive C^∞ density dg (resp. dx), we may view $\mathcal{D}'(G)$ (resp. $\mathcal{D}'(X)$) as the dual of $C_0^\infty(G)$ (resp. $C_0^\infty(X)$) in a canonical way.

Then if $P \in \mathcal{D}(X)$ and $T \in \mathcal{D}'(X)$, PT is simply the map $C_0^\infty(X) \ni f \rightarrow \langle T, {}^tPf \rangle \in \mathbb{C}$, where tP denotes the adjoint of P with respect to dx .

We shall need some more notations (see [11]). If $f \in C_0^\infty(G)$ and $S \in \mathcal{D}'(X)$, we put $f_\pi(xK) = \int_K f(xk) dk$ and $\tilde{S}(f) = S(f_\pi)$. Then $f_\pi \in C_0^\infty(X)$ and $\tilde{S} \in \mathcal{D}'(G)$. If $v \in \mathcal{E}'(X)$ and $T \in \mathcal{D}'(X)$, we define $v \times T \in \mathcal{D}'(X)$ by $\langle v \times T, F \rangle = \langle \tilde{v} * \tilde{T}, \tilde{F} \rangle$ if $F \in C_0^\infty(X)$, where $*$ denotes the convolution on G .

To prove the hypoellipticity of P when condition (A) is satisfied, we shall try to construct a suitable $T \in \mathcal{D}'(X)$ such that

$$(5) \quad PT - \delta \in C^\infty(X),$$

where δ is the Dirac mass at o .

With T we associate the continuous linear operator $\mathfrak{C}: C_0^\infty(X) \rightarrow C^\infty(X)$ defined by $\mathfrak{C}v = v \times T$. \mathfrak{C} has an extension to a continuous linear operator from $\mathcal{E}'(X)$ to $\mathcal{D}'(X)$ (with their weak topologies, say). Since $P(v \times T) = v \times PT = Pv \times T$ (see [11]), (5) implies that $P\mathfrak{C}v = \mathfrak{C}Pv = v + v \times h$, where $h = PT - \delta \in C^\infty(X)$. Hence \mathfrak{C} is a twosided parametrix of P .

The action of \mathfrak{C} on the singularities is given in the following lemma:

LEMMA II.2.1. *Assume that $\text{sing supp } T \subset \{o\}$. Then $\text{sing supp } \mathfrak{C}f \subset \text{sing supp } f$ for each $f \in \mathcal{E}'(X)$.*

PROOF. Assume that f is in C^∞ in a neighbourhood of $g \cdot o$. If $\varphi \in C_0^\infty(X)$ is equal to 1 close to $g \cdot o$ and if $\psi \in C_0^\infty(X)$, we may write:

$$\varphi \mathfrak{C}f = \varphi \mathfrak{C}\psi f + \varphi \mathfrak{C}(1 - \psi)f.$$

We choose ψ with a small support; then $\psi f \in C_0^\infty(X)$ and $\varphi \mathfrak{C}\psi f \in C^\infty(X)$. We take $\text{supp } \varphi$ small and ψ such that $\psi = 1$ in a neighbourhood of $\text{supp } \varphi$. Then $\tilde{\varphi}(g)(1 - \tilde{\psi})(g\gamma^{-1}) = 0$ if $\gamma \in G$ and $\pi(\gamma)$ belongs to some narrow neighbourhood of the origin. Therefore $\varphi \mathfrak{C}(1 - \psi)f \in C^\infty(X)$, which shows that $\mathfrak{C}f$ is in C^∞ in some neighbourhood of $g \cdot o$. The proof is complete.

Classical arguments give easily:

COROLLARY II.2.1. *If T satisfies (5) and if $\text{sing supp } T \subset \{o\}$, then P is hypoelliptic.*

To construct T satisfying (5), we shall use the Fourier transform on X , for the study of which we refer for example to [10].

If $x = g \cdot o \in X$ and $b = kM \in B$, we put $A(x, b) = -H(g^{-1}k)$. Then the Fourier transform $\mathcal{F}u$ of $u \in C_0^\infty(X)$ is defined for $\xi \in \mathfrak{a}^*$ and $b \in B$ by

$$(6) \quad \mathcal{F}u(\xi, b) = \int_{\bar{X}} u(x) e^{\langle -i\xi + e, A(x,b) \rangle} dx.$$

(In [10], $\mathcal{F}u$ is denoted by \tilde{u} , but we want to avoid a possible confusion with the \sim operation introduced above).

There is a Fourier inversion formula:

$$(7) \quad u(x) = |W|^{-1} \int_{\mathfrak{a}^*} \int_B \mathcal{F}u(\xi, b) e^{\langle i\xi + e, A(x,b) \rangle} |c(\xi)|^{-2} d\xi db,$$

where $|W|$ is the order of the Weyl group W , $c(\xi)$ is the Harish-Chandra c function, and $(2\pi)^{\text{rank } X/2} d\xi$ is the Euclidean measure induced by the Killing form on \mathfrak{a}^* .

In the remainder of this paper we shall write $|\xi|^2$ for $\langle \xi, \xi \rangle$ if $\xi \in \mathfrak{a}^*$. Assume that p satisfies condition (A). Then for some $R > 0$, we have $|p(\xi)| \geq 1$ if $|\xi| \geq R$. Choose $\chi \in C^\infty(\mathfrak{a}^*)$, W -invariant, equal to 0 when $|\xi| < R$ and equal to 1 when $|\xi| \geq 2R$.

$$P_x(e^{\langle i\xi + e, A(x,b) \rangle}) = p(\xi) e^{\langle i\xi + e, A(x,b) \rangle} \quad (\text{see [8], p. 94}).$$

Hence (6) gives that $\mathcal{F}({}^t P u)(\xi, b) = p(-\xi) \mathcal{F}u(\xi, b)$. Therefore, an easy computation using (6) and (7) shows that the distribution T , defined for $u \in C_0^\infty(X)$ by

$$(8) \quad \langle T, u \rangle = |W|^{-1} \int_{\mathfrak{a}^*} \int_B \frac{\chi(\xi)}{p(-\xi)} \mathcal{F}u(\xi, b) \frac{d\xi db}{|c(\xi)|^2},$$

satisfies (5). The integral exists since, when $\xi \rightarrow \infty$, $\mathcal{F}u(\xi, b)$ is rapidly decreasing in ξ uniformly in b and $|c(\xi)|^{-2}$ has polynomial growth (see e.g. [10]).

II.3. - Study of T when G is complex.

If $G = K\bar{A}^+K$ is a Cartan decomposition of G , ([5], chapter IX), each $g \in G$ can be written $g = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \bar{A}^+ \cdot a$ is uniquely determined by g and we denote it by $\bar{A}^+(g)$.

If $\zeta \in \mathfrak{a}^*$, we put $\pi(\zeta) = \prod_{\alpha \in \Sigma^+} \langle \alpha, H_\zeta \rangle$.

Finally the mapping \mathfrak{B} defined on \mathfrak{a} by $\mathfrak{B}(H) = B(H, H)$ is a polynomial function on \mathfrak{a} , that is a polynomial on \mathfrak{a}^* . We have $\mathfrak{B} = \sum_{1 \leq j \leq l} (e^j)^2$ if e^1, \dots, e^l is the basis of \mathfrak{a}^* dual to some B -orthonormal basis of \mathfrak{a} . If $f \in C^\infty(\mathfrak{a}^*)$ we put

$$e^j(D_\xi) f(\xi) = \frac{1}{i} \frac{d}{dt} f(\xi + te^j)|_{t=0} \quad \text{and} \quad \mathfrak{B}(D_\xi) = \sum (e^j(D_\xi))^2.$$

Then we have:

LEMMA II.3.1. *If G is complex, the following holds:*

$$(9) \quad \mathfrak{B}(D_\xi)(\pi(-i\xi) \varphi_{-\xi}(g \cdot o)) = \mathfrak{B}(\log \bar{A}^+(g)) \pi(-i\xi) \varphi_{-\xi}(g \cdot o),$$

where $\log: A \rightarrow \mathfrak{a}$ is the inverse of the exponential map of \mathfrak{a} .

PROOF. Since G is complex, we have the following simple expression for φ_ξ (see [9], chapter II):

$$p_\xi(a \cdot o) = F(a, i\xi) / F(a, \rho)$$

when $a \in A$, where $F(a, \eta) = \sum_{s \in W} (\det s)^{\langle s\eta, \log a \rangle} / \pi(\eta)$. Since φ_ξ is K -invariant, an easy computation gives (9).

Let us fix some $g_0 \cdot o \in X$ and choose ξ such that $\pi(-i\xi) \varphi_{-\xi}(g \cdot o) \neq 0$ for $g \cdot o$ close to $g_0 \cdot o$. Then (9) shows that $g \cdot o \mapsto \mathfrak{B}(\log \bar{A}^+(g))$ is analytic close to $g_0 \cdot o$, and since $g_0 \cdot o$ is arbitrary, the function is analytic everywhere on X . It is positive and vanishes only at o . If $d(g \cdot o)$ denotes the Riemannian distance from $g \cdot o$ to o , when X is equipped with the Riemannian metric induced by the Killing form of \mathfrak{g} , one has:

$$(10) \quad d^2(g \cdot o) = \mathfrak{B}(\log \bar{A}^+(g)).$$

To show (10) it suffices to show that $d^2(a \cdot o) = \mathfrak{B}(\log a)$ for $a \in A$, since $d(ka \cdot o) = d(a \cdot o)$ if $k \in K$. Put $\log a = H$. Then $\{\text{Exp } tH, t \in \mathbb{R}\}$ is the geodesic through o and $a \cdot o$ (see [5], chapter IV), and since

$$\langle d \text{Exp}_{tH}, H \rangle = \frac{d}{ds} \text{Exp}(tH + sH)|_{s=0} = \langle d\tau(\text{exp } tH)_0 \circ d \text{Exp}_o, H \rangle,$$

where τ denotes the action of G on X , we get that $|\langle d \text{Exp}_{tH}, H \rangle|_{tH}^2$ is independent of t if $|\cdot|_{tH}^2$ denotes the norm in $T_{\text{Exp } tH}(X)$ defined by the metric of X . Since $|\langle d \text{Exp}_o, H \rangle|_o^2 = \mathfrak{B}(H)$, it is clear that $d^2(a \cdot o) = \mathfrak{B}(\log a)$.

We are now able to prove the following converse of theorem II.1.1:

THEOREM II.3.1. *If G is complex and T is defined by (8), one has $\text{sing supp } T \subset \{o\}$ as soon as p satisfies (A). In this case P is hypoelliptic.*

PROOF. Since $\varphi_\xi(x) = \int_B e^{\langle i\xi + e, A(x,b) \rangle} db$ (see [8], p. 94), one has

$$\int_B \mathcal{F}(d^{2N}u)(\xi, b) db = \int_X d^{2N}(x)u(x)\varphi_{-\xi}(x) dx$$

if $N \in \mathbf{Z}_+$ and $u \in C_0^\infty(X)$. Since G is complex, we have $c(\xi) = \pi(\rho)/\pi(i\xi)$ ([9], chapter II); therefore (9) shows that

$$\langle d^{2N}T, u \rangle = \int_{\mathfrak{a}^*} V(\xi) \mathcal{B}^N(D_\xi) U(\xi) d\xi = \int_{\mathfrak{a}^*} U(\xi)^t \mathcal{B}^N(D_\xi) V(\xi) d\xi,$$

where $U(\xi) = \pi(-i\xi) \int_X u(x)\varphi_{-\xi}(x) dx$ and $V(\xi) = \chi(\xi)\pi(i\xi)/(|W|\pi(\rho)^2 p(-\xi))$; t denotes transposition with respect to $d\xi$.

Since p satisfies (A), we have for some $C > 0$ and some $s \geq 1$: $|p^{(\alpha)}(\xi)| \leq C|\xi|^{-|\alpha|/s}|p(\xi)|$ if $|\xi| \geq R$. Hence for some $r \in \mathbf{R}$, we have $|^t \mathcal{B}^N(D_\xi) V(\xi)| \leq C_N(1 + |\xi|)^{r-2N/s}$, where $C_N > 0$ depends on N . Therefore, when N is large enough, $d^{2N}T$ is a continuous function and we have:

$$d^{2N}T(x) = \int_{\mathfrak{a}^*} p_{-\xi}(x)\pi(-i\xi)^t \mathcal{B}^N(D_\xi) V(\xi) d\xi,$$

which shows that for some fixed $\mu_0 \in \mathbf{R}$, $d^{2N}T \in C^\mu(X)$ if $\mu < \mu_0 + 2N/s$. Hence $T \in C^\infty(X \setminus \{o\})$. This completes the proof in view of corollary II.2.1.

II.4. - Study of T in the regular part of X .

When G is not complex, we have no simple expression for the spherical functions and therefore we are not able to prove an identity as (9). However let X' be the regular part of X , that is the image of KA^+K by the projection $G \rightarrow X = G/K$. It is known that X' is open and dense in X and that $\dim(X \setminus X') \leq \dim X - 2$ (see chapter X of [4]). The following theorem shows that T behaves well in X' even if G is not complex:

THEOREM II.4.1. *Assume that p satisfies condition (A) and let T be given by (8). Then $\text{sing supp } T \cap X' = \emptyset$.*

Before starting the proof, we have to recall some properties of the spherical functions, the c function and the polynomial p .

In A^+ , we have Harish-Chandra asymptotic expansion of $\tilde{\varphi}_{-\xi}$ (see [3], [10]), which implies that

$$(11) \quad \tilde{\varphi}_{-\xi}(a)|c(\xi)|^{-2} = e^{-\langle e, \log a \rangle} \sum_{s \in W} c(s\xi)^{-1} \Phi(a, -s\xi),$$

with

$$(12) \quad \Phi(a, \eta) = e^{i\langle \eta, \log a \rangle} \sum_{\mu \in L} \Gamma_{\mu}(\eta) e^{-\langle \mu, \log a \rangle}$$

for $a \in A^+$, $\xi \in \mathfrak{a}^*$, $\eta \in \mathfrak{a}^*$.

Here we are using the following notations (see [10]): L is the set of all linear combinations of the form $\sum_{1 \leq j \leq l} n_j \alpha_j$, $n_j \in \mathbb{Z}_+$, where the α_j are the simple restricted roots. The Γ_{μ} are rational functions and c is the Harish-Chandra c function.

We recall now a number of facts about the function

$$U(a, \xi) = e^{-\langle e, \log a \rangle} \sum_{\mu \in L} \Gamma_{\mu}(-\xi) e^{-\langle \mu, \log a \rangle}$$

(see [3], [7], [8]). If $r > 0$, put $\mathfrak{a}_r = \{\zeta \in \mathfrak{a}_c^*, \text{Im} \langle \alpha_j, \zeta \rangle < r \text{ for } j = 1, \dots, l\}$. Since the map $\zeta \mapsto \Gamma_{\mu}(-\zeta)$ is holomorphic outside $\{\zeta \in \mathfrak{a}_c^*, \langle \mu, \zeta \rangle = \frac{1}{2}i \langle \mu, \mu \rangle \text{ for some } \mu \in L \setminus \{0\}\}$, it is holomorphic in U_{ε} if $\varepsilon > 0$ is small enough. Furthermore, if $\varepsilon > 0$ is small, the proof of theorem 2.4 of [8] (see also lemma 2 of [1]) shows that for each $H_0 \in \mathfrak{a}^+$, there exists a constant K_{H_0} such that for each $\mu \in L$:

$$(13) \quad \sup_{\zeta \in U_{\varepsilon}} |\Gamma_{\mu}(-\zeta)| \leq K_{H_0} e^{\langle \mu, H_0 \rangle}.$$

Let S be a compact subset of \mathfrak{a}^+ . Choosing H_0 such that $\alpha_j(H - H_0) > 0$ for $1 \leq j \leq l$ and all $H \in S$, and taking ε small enough, we see, using (13), that the series $\sum_{\mu \in L} |\Gamma_{\mu}(-\zeta) e^{-\langle \mu, H' \rangle}|$ is uniformly convergent if $\zeta \in U_{\varepsilon}$ and H' belongs to a suitable neighbourhood of S in the complexified space \mathfrak{a}_c of \mathfrak{a} . This shows that $e^{-\langle e, H' \rangle} \sum_{\mu \in L} \Gamma_{\mu}(-\zeta) e^{-\langle \mu, H' \rangle}$ is holomorphic there. Also one sees easily that for each translation invariant differential operator Q on A , there exists a constant C such that

$$(14) \quad \sup_{a \in \exp S, \zeta \in U_{\varepsilon}} |Q_a U(a, \zeta)| \leq C,$$

if $U(a, \zeta) = e^{-\langle e, \log a \rangle} \sum_{\mu \in L} \Gamma_{\mu}(-\zeta) e^{-\langle \mu, \log a \rangle}$.

As shown in [7], there exist constants C and m such that

$$(15) \quad |e^{-1}(\zeta)| \leq C(1 + |\zeta|)^m,$$

for all $\zeta \in \mathfrak{a}_c^*$ satisfying $\text{Im} \langle \alpha_j, \zeta \rangle \leq 0, j = 1, \dots, l; e^{-1}(\zeta)$ is holomorphic in a neighbourhood of that set.

On the other hand, since p satisfies (A), there exists some $s \geq 1$ such that $|\zeta| \leq C'(1 + |\text{Im} \zeta|)^s$ if $p(\zeta) = 0$ and $\zeta \in \mathfrak{a}_c^*$, where C' is independent of ζ . Hence we may use the following lemma, which is part of lemma 2.1 of [2]:

LEMMA II.4.1. *Put $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ when $\xi \in \mathfrak{a}^*$. Let $v \in \mathfrak{a}^*$ be fixed. Then there are positive constants C_1, R, δ, λ such that*

$$|p(\xi + i\tau v)| > \lambda \langle \xi \rangle^{-\delta}$$

when $|\xi| \geq R$ and $0 \leq \tau \leq C_1 \langle \xi \rangle^{1/s}$.

Since this is no restriction, we shall assume that the R of the lemma is equal to the R introduced in the definition of χ (§ II.2).

We are now ready to give the

PROOF OF THEOREM II.4.1. We shall consider the functions

$$(16) \quad T_n(x) = |W|^{-1} \int_{\mathfrak{a}^*} \frac{\chi(\xi) \theta(\xi/n)}{p(-\xi) |c(\xi)|^2} \varphi_{-\xi}(x) d\xi,$$

$x \in X'$, where $n \in \mathbb{Z}_+$ tends to ∞ and $\theta(\xi) = \int_{\mathfrak{a}} e^{-i\langle \xi, H \rangle} \omega(H) dH$. Here $(2\pi)^{\text{rank } X/2} dH$ is the Euclidean measure on \mathfrak{a} induced by the Killing form; $\omega \in C_0^\infty(\mathfrak{a})$ is W -invariant and $\int_{\mathfrak{a}} \omega(H) dH = 1$.

It is clear that $T_n \in C^\infty(X)$ and that $\langle T_n, \psi \rangle \rightarrow \langle T, \psi \rangle$ for any $\psi \in C_0^\infty(X)$ when $n \rightarrow \infty$. Hence theorem II.4.1 will be proved if we show that T_n is a Cauchy sequence in $C^\infty(X')$ as $n \rightarrow \infty$.

Now the mapping $\varphi: K/M \times \mathfrak{a}^+ \rightarrow X'$ defined by

$$\varphi(kM, H) = \exp(Ad_{\mathfrak{g}}(k)H) \cdot o, \quad k \in K, H \in \mathfrak{a}^+,$$

is a diffeomorphism ([4], chapter X). Hence φ^{-1} defines a chart of X' in which T_n is independent of the first variable. Therefore to show that theorem II.4.1 holds, it suffices to prove the following result:

THEOREM II.4.2. $A^+ \ni a \mapsto \tilde{T}_n(a)$ is a Cauchy sequence in $C^\infty(A^+)$ as $n \rightarrow \infty$.

PROOF. We are going to show the following, which obviously implies theorem II.4.2:

(17) *If $a_0 \in A^+$, there exists an open neighbourhood V of a_0 in A^+ such that $V \ni a \mapsto \tilde{T}_n(a)$ is a Cauchy sequence in $C^\infty(V)$ as $n \rightarrow \infty$.*

If $a_0 \in A^+$, lemma 35 of [3] implies that there exists some j , $1 \leq j \leq l$, such that $B(e_j, \log a_0) > 0$. Relabeling the roots if necessary, we may as well assume that $j = 1$. To simplify notations, we introduce suitable linear coordinates in \mathfrak{a}^* and \mathfrak{a}_c^* (see [7]). Let e_1, \dots, e_l be the basis of \mathfrak{a}^* dual to the basis $\alpha_1, \dots, \alpha_l$ of \mathfrak{a} . If $\xi \in \mathfrak{a}^*$, we write $H_\xi = \sum_{1 \leq j \leq l} \xi_j e_j$. Then (ξ_1, \dots, ξ_l) are linear coordinates of \mathfrak{a}^* . If $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$, we put $\zeta_j = \xi_j + i\eta_j$. Then $(\zeta_1, \dots, \zeta_l)$ are linear coordinates of \mathfrak{a}_c^* . We shall write $\xi' = (\xi_2, \dots, \xi_l)$ and identify functions on subsets of \mathfrak{a}^* or \mathfrak{a}_c^* with their expression in the linear coordinates just described. Let $d\xi_1$ (resp. $d\xi'$) be the Lebesgue measure in ξ_1 -space (resp. ξ' -space). Then $d\xi = \gamma d\xi_1 d\xi'$ for some $\gamma \in \mathbb{R}_+ \setminus \{0\}$.

An easy computation, using (11), (12), (16), shows that:

$$(18) \quad \tilde{T}_n(a) = \int_{\mathfrak{a}^*} \frac{\chi(\xi) \theta(\xi/n) U(a, \xi)}{p(-\xi) c(\xi)} e^{-i\langle \xi, \log a \rangle} d\xi,$$

if $a \in A^+$.

Let $\nu \in \mathfrak{a}^*$ be such that $H_\nu = -e_1$. We are going to let ξ take complex values in the direction of ν . Put

$$\tau(\xi) = C_1 \langle \xi \rangle^{1/s} \quad \text{and} \quad U'(a, \zeta) = \frac{U(a, \zeta) e^{-i\langle \zeta, \log a \rangle}}{p(-\zeta) c(\zeta)}.$$

Assume that for some constant $C > 0$, $B(e_1, \log a) \geq C$; this is certainly true if a belongs to a small relatively compact neighbourhood V' of a_0 with closure contained in A^+ .

Apply Stokes formula for fixed ξ' to the domain $\{\zeta_1, |\operatorname{Re} \zeta_1| < w, -\tau(\xi) < \operatorname{Im} \zeta_1 < 0\}$ and let w tend to ∞ . If n is large enough, say $n \geq n_0$, (14), (15), and lemma II.4.1 show that for all $a \in V'$: $\tilde{T}_n(a) = E_n(a) + R_n(a)$, where

$$E_n(a) = \gamma \int d\xi' \int \chi(\xi) \theta(\zeta_1/n, \xi'/n) U'(a, \zeta_1, \xi') d\zeta_1,$$

$$R_n(a) = i\gamma \int d\xi' \int \frac{\partial \chi}{\partial \xi_1}(\xi) \theta(\zeta_1/n, \xi'/n) U'(a, \zeta_1, \xi') d\xi_1 d\eta_1,$$

where the integration with respect to $d\zeta_1$ (resp. $d\xi_1 d\eta_1$) is performed on the curve $\zeta_1 = \xi_1 - i\tau(\xi)$ (resp. on $\{(\xi_1, \eta_1), -\tau(\xi) < \eta_1 < 0\}$).

In fact we have also, when $n \geq n_0$:

$$(19) \quad E_n(a) = \gamma \iint \chi(\xi) \theta(\zeta_1/n, \xi'/n) U'(a, \zeta_1, \xi') d\zeta_1 d\xi',$$

$$(20) \quad R_n(a) = i\gamma \iiint \frac{\partial \chi}{\partial \xi_1}(\xi) \theta(\zeta_1/n, \xi'/n) U'(a, \zeta_1, \xi') d\xi_1 d\xi' d\eta_1,$$

by Fubini theorem. Using (14) we may differentiate (19) and (20) any number of times under the integral signs and the results are Cauchy sequences in $C(V')$ when $n \rightarrow \infty$.

Hence (17) is proved. At the same time, we get the formula

$$\begin{aligned} \tilde{T}(a) = \gamma \iint \chi(\xi) U'(a, \zeta_1, \xi') d\zeta_1 d\xi' + i\gamma \iiint \frac{\partial \chi}{\partial \xi_1}(\xi) \cdot \\ \cdot U'(a, \zeta_1, \xi') d\xi_1 d\xi' d\eta_1, \quad \text{for } a \in V'. \end{aligned}$$

REMARK II.4.1. If $\text{rank } X = 1$, then $X' = X \setminus \{o\}$. Hence in this case theorem II.4.1 implies that any nonzero element P of $D(X)$ is hypoelliptic. In fact P is even elliptic: see lemma I.2 and remark II.1.1.

II.5. – Products of rank one spaces.

On the singular set $X \setminus X'$, expansion (11) is not valid. We describe now a very special situation where it is however possible to use the expansion of spherical functions to show that condition (A) implies that $T \in C^\infty(X \setminus \{o\})$.

We assume the following:

- (21) G is the Cartesian product $G_1 \times \dots \times G_q$ (with the natural product law $(g_1, \dots, g_q)(g'_1, \dots, g'_q) = (g_1 g'_1, \dots, g_q g'_q)$ and product manifold structure) where $G_j, 1 \leq j \leq q$, is a real rank one connected noncompact semisimple Lie group with finite center.

Then of course G is itself a connected noncompact semisimple Lie group with finite center. Let K (resp. K'_j) be a maximal compact subgroup of G (resp. G_j). Since $K' = K'_1 \times \dots \times K'_q$ is a maximal compact subgroup of G , K and K' are conjugate under an inner automorphism of G ([5], chapter VI); hence K must be of the form $K_1 \times \dots \times K_q$, where K_j is a maximal compact

subgroup of G_j . Let \mathfrak{k}_j (resp. \mathfrak{g}_j) be the Lie algebra of K_j (resp. G_j) and let $\mathfrak{g}_j = \mathfrak{k}_j + \mathfrak{p}_j$ be the corresponding Cartan decomposition. Then $\mathfrak{a} = \mathfrak{a}_1 \times \dots \times \mathfrak{a}_q$ is a maximal abelian subspace of $\mathfrak{p}_1 \times \dots \times \mathfrak{p}_q$ if \mathfrak{a}_j is maximal abelian in \mathfrak{p}_j ; correspondingly $A = A_1 \times \dots \times A_q$ if $A_j = \exp \mathfrak{a}_j$. If $\xi_j \in (\mathfrak{a}_j^*)_c$, denote by $(\xi_1, \dots, \xi_q) \in \mathfrak{a}_c^*$ the map $\mathfrak{a} \ni (H_1, \dots, H_q) \mapsto \sum \xi_j H_j$. Then the restricted roots of \mathfrak{a} are of the form $(\beta_1, \dots, \beta_q)$ with $\beta_m = \delta_{mj} \alpha_j$ for all m and some j , $1 \leq m, j \leq q$. Here $\delta_{mj} = 1$ if $m = j$ and 0 otherwise, and α_j runs over the restricted roots of \mathfrak{a}_j . For \mathfrak{a}^+ we choose $\mathfrak{a}_1^+ \times \dots \times \mathfrak{a}_q^+$. Harish-Chandra integral formula $\tilde{\varphi}_\xi(g) = \int_K e^{\langle i\xi - \rho, H(gk) \rangle} dk$ shows that $\tilde{\varphi}_\xi(g) = \prod_{1 \leq j \leq q} \tilde{\varphi}_{\xi_j}(g_j)$ if $\xi = (\xi_1, \dots, \xi_q)$ and $g = (g_1, \dots, g_q)$.

We are going to prove the following result:

THEOREM II.5.1. *Let G be as in (21). Put $X = G/K$ where K is a maximal compact subgroup of G . Then $P \in D(X)$ is hypoelliptic if the corresponding polynomial p satisfies condition (A).*

PROOF. We shall show that for each $m \in \mathbb{Z}_+$, $L^m T_n$ is a Cauchy sequence in $C(X \setminus \{o\})$ as $n \rightarrow \infty$, if L is the Laplace operator on X corresponding to the metric defined by the Killing form. As a first step we are going to prove the following:

$$(22) \quad \widehat{L^m T_n}|_A \text{ is a Cauchy sequence in } C(A \setminus \{1\}) \text{ as } n \rightarrow \infty.$$

We know already by the proof of theorem II.4.1 that $\widehat{L^m T_n}|_A$ is a Cauchy sequence in $C(A^+)$. Hence it suffices to study what happens in a neighbourhood of $\partial A^+ \setminus \{1\}$ in $A \setminus \{1\}$ where ∂ means boundary in A , or in a later in this paper. (Recall that $\widehat{L^m T_n}|_A$ is W -invariant). If $a' = (a'_1, \dots, a'_q) \in \partial A^+ \setminus \{1\}$ we have $\alpha_j(\log a'_j) > 0$ for some j , $1 \leq j \leq q$, where $\alpha_j \in \mathfrak{a}_j^*$ is the simple root corresponding to the choice of A_j^+ . We may assume that $j = 1$. Using (3) and (16) we get an integral formula for $\widehat{L^m T_n}|_A$. When a belongs to a small neighbourhood of a' in $A \setminus \{1\}$, we may use (11) to expand $\tilde{\varphi}_{-\xi_1}(a_1)$ and give complex values ζ_1 to ξ_1 , with $\text{Im} \langle \alpha_1, \zeta_1 \rangle < 0$. ($\langle \alpha_1, \zeta_1 \rangle$ is computed in \mathfrak{a}_1^* .) The proof of (22) is then a repetition of that of theorem II.4.2.

Now the mapping $\text{Exp}: \mathfrak{p} \rightarrow X$ defines a chart of X in which $L^m T_n|_{\mathfrak{a}}$ is a Cauchy sequence in $C(\mathfrak{a} \setminus \{o\})$, because of (22). In this chart, $L^m T_n$ is $Ad_g(K)$ -invariant. Therefore it is clear that theorem II.5.1 will be proved as soon as we know that the following lemma is true:

LEMMA II.5.1. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a real semi-simple Lie algebra \mathfrak{g} and \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Let G and K correspond to \mathfrak{g} and \mathfrak{k} as usual. For each $n \in \mathbb{Z}_+$, assume that $f_n \in C(\mathfrak{p} \setminus \{0\})$*

is $Ad_G(K)$ -invariant. If for $n \rightarrow \infty$, $f_n|_{\mathfrak{a}}$ is a Cauchy sequence in $C(\mathfrak{a} \setminus \{0\})$, then f_n is a Cauchy sequence in $C(\mathfrak{p} \setminus \{0\})$.

PROOF. Let us fix some notation: for each $Z \in \mathfrak{p}$, there exists one and only one point of $\bar{\mathfrak{a}}^+$, which we shall denote by Z' throughout all this proof, such that $Z' = Ad_G(k_0)Z$ for some $k_0 \in K$ (see [5], chapter IX). We are going to prove that:

$$(23) \quad Y' \rightarrow Y'_0 \quad \text{if} \quad Y \rightarrow Y_0; \quad Y, Y_0 \in \mathfrak{p}.$$

Since $(Ad_G(k)Z)'$ is independent of $k \in K$, we may as well assume that $Y_0 \in \bar{\mathfrak{a}}^+$.

As we have already used before, the map $\varphi: K/M \times \mathfrak{a}^+ \rightarrow Ad_G(K)\mathfrak{a}^+$ defined by $\varphi(kM, H) = Ad_G(k)H$ for $k \in K$ and $H \in \mathfrak{a}^+$ is a diffeomorphism. φ^{-1} gives a chart of $Ad_G(K)\mathfrak{a}^+$ in which the map $Y \mapsto Y'$ is just the projection on the second component. Hence (23) is clear if $Y_0 \in \mathfrak{a}^+$.

If $Y_0 \in \partial\mathfrak{a}^+ \setminus \{0\}$, we have:

$$(24) \quad d(Y', \{\lambda Y_0, \lambda > 0\}) \rightarrow 0 \quad \text{if} \quad Y \rightarrow Y_0,$$

where d denotes the distance on \mathfrak{p} defined by the Killing form. In fact, if (24) is false, we can find $C > 0$ and a sequence $Y_n \rightarrow Y_0$ such that $d(Y'_n, \{\lambda Y_0, \lambda > 0\}) \geq C$ for all n . We have $Y'_n = Ad_G(k_n)Y_n$ with $k_n \in K$ and we may assume after perhaps taking a subsequence that $k_n \rightarrow k_0 \in K$ as $n \rightarrow \infty$. Hence $Y'_n \rightarrow Ad_G(k_0)Y_0$ which must belong to $\bar{\mathfrak{a}}^+$ since Y'_n does. But $Ad_G(k_0)Y_0 \notin \{\lambda Y_0, \lambda > 0\}$. Hence Y_0 and $Ad_G(k_0)Y_0$ are different, and both in $\bar{\mathfrak{a}}^+$, which is impossible ([5], chapter IX). Therefore (24) is proved.

On the other hand $B(Y', Y') = B(Y, Y) \rightarrow B(Y_0, Y_0)$ if B is the Killing form. This, together with (24), proves (23). Now let S be a compact subset of $\mathfrak{p} \setminus \{0\}$ and S' its image by the continuous map $Y \mapsto Y'$. Since f_n is $Ad_G(K)$ -invariant, we have:

$$(25) \quad \sup_{Y \in S} |f_n(Y) - f_r(Y)| = \sup_{Y' \in S'} |f_n(Y') - f_r(Y')|,$$

for each n, r , and the right-hand side of (25) is arbitrarily small when n and r are large enough, since S' is compact. This proves lemma II.5.1.

II.6. – The transversality condition (B).

We are now going to introduce a condition on $X \setminus (X' \cup \{o\})$ which allows to prove that T (given by (8)) has no singularities there when p satisfies condition (A).

Using pseudo-differential notation, it is natural to write $|c(D)|^{-2}\delta$ for the distribution on \mathfrak{a} defined by $\int_{\mathfrak{a}^*} e^{i\langle \xi, H \rangle} |c(\xi)|^{-2} d\xi$. Since $|c(\xi)|^{-2}$ has polynomial growth, $|c(D)|^{-2}\delta$ is well defined as an element of $\mathcal{S}'(\mathfrak{a})$. Denote by E the pseudo-differential operator on \mathfrak{a} defined (with χ, p as above, p satisfying condition (A)) by

$$Ez(H) = |W|^{-1} \int_{\mathfrak{a}^*} e^{i\langle \xi, H \rangle} \frac{\chi(-\xi)}{p(\xi)} \hat{z}(\xi) d\xi,$$

$z \in C_0^\infty(\mathfrak{a})$, $H \in \mathfrak{a}$, where $\hat{z}(\xi) = \int_{\mathfrak{a}} e^{-i\langle \xi, H \rangle} z(H) dH$ is the Euclidean Fourier transform of z . Then for some $\sigma > 0$, E is a pseudo-differential operator of type $S_{\sigma,0}^0$ in the sense of [14]. So if we denote by $WF(f)$ the wave front set of $f \in \mathcal{D}'(\mathfrak{a})$, the results of [14] show that $WF(Ef) = WF(f)$ if $f \in \mathcal{S}'(\mathfrak{a})$. Clearly the same equality holds also if $f \in \mathcal{S}'(\mathfrak{a})$ since E is the convolution by a distribution which belongs to $\mathcal{S}(\mathfrak{a})$ outside the origin. Therefore $WF(E|c(D)|^{-2}\delta) = WF(|c(D)|^{-2}\delta)$.

Let us introduce the following transversality condition if $x \in X \setminus \{o\}$. We shall say that *condition (B) is satisfied at x if the mapping*

$$B \ni b \mapsto A(x, b) \in \mathfrak{a}$$

has no normal contained in $WF(|c(D)|^{-2}\delta)$.

Here « normal » is used according to the terminology of [14]: a cotangent vector $(H, \xi) \in T^*\mathfrak{a}$ is called normal to the mapping $f: B \rightarrow \mathfrak{a}$ if, for some $b \in B$, $f(b) = H$ and $\langle df(b), \xi \rangle = 0$.

Notice that condition (B) at x is equivalent to the following, which we call (B') at x :

If $x = k_0 a_0 \cdot o$ with $k_0 \in K$ and $a_0 \in A \setminus \{1\}$, the mapping $K \ni k \mapsto H(a_0 k) \in \mathfrak{a}$ has no normal contained in $WF(|c(D)|^{-2}\delta)$.

To see the equivalence, note that $A(k_0 a_0 \cdot o, kM) = H(a_0^{-1} k_0^{-1} k)$, which is equal to $H(a_0 k(a_0^{-1} k_0^{-1} k))$ ([3], page 294). Here $k(a_0^{-1} k_0^{-1} k)$ denotes the element of K in the Iwasawa decomposition of $a_0^{-1} k_0^{-1} k$. Since the mapping $K \ni k \mapsto k(a_0^{-1} k_0^{-1} k)$ is a diffeomorphism of K ([3], page 294), the mappings $k \mapsto H(a_0 k(a_0^{-1} k_0^{-1} k))$ and $k \mapsto H(a_0 k)$ have the same normals, which shows the equivalence of (B) and (B').

The reason for introducing condition (B) is the following theorem, in which we keep the notations of the above paragraphs:

THEOREM II.6.1. *Assume that p satisfies (A). If (B) is satisfied at $x \in X \setminus \{o\}$, then $x \notin \text{sing supp } T$.*

PROOF. Put $F_y(b) = A(y, b)$ if $y \in X$ and $b \in B$. Since condition (B) is satisfied, theorem 2.5.11' of [14] shows that we can define the pullback

$$F_y^* e^{\langle e, \cdot \rangle} E|c(D)|^{-2} \delta \in \mathcal{D}'(B) \quad \text{of} \quad e^{\langle e, \cdot \rangle} E|c(D)|^{-2} \delta \in \mathcal{D}'(a)$$

by F_y when $y \in X$ belongs to a small neighbourhood V of x . If $u_n(y, b)$ is the smooth function

$$e^{\langle e, A(y, b) \rangle} |W|^{-1} \int_{\mathfrak{a}^*} e^{i \langle \xi, A(x, b) \rangle} \frac{\chi(-\xi)}{p(\xi)} \frac{\theta(-\xi/n)}{|c(\xi)|^2} d\xi$$

(with the notations of § II.4), the proof of theorem 2.5.11' of [14] shows that we have for any $\psi \in C^\infty(B)$:

$$(26) \quad \langle F_y^* e^{\langle e, \cdot \rangle} E|c(D)|^{-2} \delta, \psi \rangle = \lim_{n \rightarrow \infty} \langle u_n, \psi \rangle$$

in the $C^\infty(V)$ topology. (Using the results of [5], chapter X, pp. 369 and 380, we see that db is induced by a K -invariant Riemannian metric on B . Hence db is a C^∞ density which allows to identify $\mathcal{D}'(B)$ with the dual of $C^\infty(B)$).

If we take $\psi(b) = 1$ for all $b \in B$, then $\langle u_n, \psi \rangle$ is the function T_n defined by (16), computed at y . So (26) shows that the sequence T_n converges in the $C^\infty(V)$ topology when $n \rightarrow \infty$. The proof is complete.

On the other hand, we know by theorem II.4.1 that $\text{sing supp } T \cap X' = \emptyset$. Therefore, using corollary II.2.1, the following result is clear:

THEOREM II.6.2. *Assume that condition (B) is satisfied at each $x \in X \setminus (X' \cup \{o\})$. Then $P \in \mathcal{D}(X)$ is hypoelliptic if the corresponding p satisfies condition (A).*

Below we shall give examples. We shall see that condition (B) may be satisfied at each $x \in X \setminus (X' \cup \{o\})$ but violated at some $y \in X'$. In this case theorem II.4.1 is really needed to get the conclusion of theorem II.6.2. We shall have to compute $|c(D)|^{-2} \delta$ and therefore use the formula (see [7]):

$$(27) \quad c(\xi) = I(i\xi)/I(\varrho) \quad \text{where} \quad I(\mu) = \prod_{\alpha \in \Sigma^+} B\left(\frac{m_\alpha}{2}, \frac{m_\alpha/2}{4} + \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right),$$

for $\mu \in \mathfrak{a}_c^*$, B being Euler beta function.

Since we shall also have to make computations involving Iwasawa decompositions, let us recall briefly how these decompositions arise (see [5], chapter VI). One starts with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra \mathfrak{g} of G , where \mathfrak{k} (resp. \mathfrak{p}) is the eigenspace corresponding to the eigenvalue $+1$ (resp. -1) of a Cartan involution.

For α one takes a maximal abelian subspace of \mathfrak{p} , and one puts $\mathfrak{n} = \sum_{-\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ for all restricted roots α . Then $G = KAN$, where K, A, N are analytic subgroups of G with Lie algebra $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ respectively. If $g \in G$, its Iwasawa decomposition will be written $g = k(g) \exp H(g)n(g)$; we have already met some of this notation.

If R is a $p \times q$ matrix, we shall denote by R_{jk} the entry which lies on the j -th row and the k -th column. We shall write R_j for R_{jj} .

One can simplify the computations somewhat if one remarks that (B') is satisfied at each point $x \in X \setminus (X' \cup \{o\})$ if and only if the following holds, where $A' = \bigcup_{s \in W} sA^+$:

(B'') *There exists an open subset U of $\partial A' \setminus \{1\}$ such that*

(i) $\bigcup_{s \in W} sU \supset \partial A^+ \setminus \{1\}$,

(ii) *WF $(|c(D)|^{-2}\delta)$ does not contain any normal to the map $K \ni k \mapsto H(ak) \in \mathfrak{a}$ if $a \in U$.*

Indeed if $s \in W$, there exists $k_1 \in K$ such that $s(a) = k_1 a k_1^{-1}$. Therefore the two functions of $k, H(ak)$ and $H(s(a)k)$, have the same normals.

We are now ready to give examples.

EXAMPLE II.6.1. The rank 2 space X of type A I in Elie Cartan's terminology, that is $X = SL(3, \mathbb{R})/SO(3)$.

$SL(3, \mathbb{R})$ is the group of 3×3 real matrices with determinant equal to 1, and $SO(3)$ is the compact subgroup of orthogonal matrices with determinant equal to 1.

We are going to show that condition (B'') is satisfied. Hence theorem II.6.2 holds for this space.

As shown in [5], chapter X, we may take $\theta(Y) = -{}^tY$ as a Cartan involution for the Lie algebra of $SL(3, \mathbb{R})$, and correspondingly for \mathfrak{a} we choose the set of diagonal 3×3 matrices H whose trace is equal to 0. We may choose the maps $\alpha_{12}(H) = H_1 - H_2, \alpha_{13}(H) = 2H_1 + H_2, \alpha_{23}(H) = H_1 + 2H_2$ as the set Σ^+ of positive restricted roots. We have $m_\alpha = 1$ if $\alpha \in \Sigma^+$ and $\mathfrak{a}^+ = \{H \in \mathfrak{a}, -H_1/2 < H_2 < H_1\}$.

Using (27), we get for $\xi \in \mathfrak{a}^*$: $|c(\xi)|^{-2} = C \prod_{1 \leq j \leq 3} \eta_j \cdot \text{th}(\pi\eta_j/2)$, if $\eta_1 = 2\xi_1 - \xi_2, \eta_2 = -\xi_1 + 2\xi_2, \eta_3 = \xi_1 + \xi_2$, where now ξ_1, ξ_2 are the coordinates of ξ in the basis α_{12}, α_{23} , C is a strictly positive constant and th means hyperbolic tangent.

If $f(t)$ is the inverse Fourier transform $\int_{\mathbb{R}} e^{it\lambda} \text{th } \lambda \, d\lambda$ of th (in the distribution sense), we get $f \in \mathcal{S}'(\mathbb{R})$ and $f(t) = 2i$ p.v. $1/t + f_1(t)$, where p.v. denotes principal value and f_1 is analytic on \mathbb{R} .

If Q denotes a certain third order differential operator with constant coefficients in the variables H_1, H_2 , we have $|c(D)|^{-2} \delta = Qf_2$, where

$$f_2(H_1, H_2) = f((H_1 - H_2)/\pi) \delta(H_1 + H_2) + \\ + f((2H_2 + H_1)/\pi) \delta(H_1) - f(2(H_1 - H_2)/\pi) \delta(H_2).$$

In the last formula, all pullbacks and products are well defined (see for example chapter II of [14]). We get by an easy wave front set computation:

$$WF(|c(D)|^{-2} \delta) \subset (T_0^* \alpha \setminus 0) \cup N(H_1 = 0) \cup N(H_2 = 0) \cup N(H_1 + H_2 = 0),$$

where $T_0^* \alpha$ is the fiber of $T^* \alpha$ at $0 \in \alpha$ and $N(\psi(H_1, H_2) = 0)$ denotes the conormal bundle of $\psi(H_1, H_2) = 0$.

We want to show that the map $SO(3) \ni k \mapsto H(ak)$ has no normal contained in $WF(|c(D)|^{-2} \delta)$ when $a \in U$, for some U as in condition (B'').

First we have to compute $H(ak)$. Denote also by θ the isomorphism induced on $SL(3, \mathbb{R})$ by the Cartan involution $\theta(X) = -{}^t X$ of its Lie algebra (see [5], chapter VI).

We have $\theta(g) = {}^t g^{-1}$ if $g \in SL(3, \mathbb{R})$. To compute $H(ak)$, we write the Iwasawa decomposition of ak , that is $ak = k(ak) \exp H(ak) n(ak)$; then $\theta(ak) = k(ak) \exp(-H(ak)) \theta(n(ak))$, whence

$$(28) \quad k^{-1} a^2 k = \theta(n(ak))^{-1} \exp 2H(ak) n(ak).$$

Since $k \in SO(3)$, we have $k^{-1} = {}^t k$. On the other hand one checks easily that $n_j(ak) = 1$ and $n_{jl}(ak) = 0$ if $j > l$ since $n(ak) \in N$. Then (28) gives:

$$(29) \quad H_1(ak) = \frac{1}{2} \log A_{11}, \quad H_3(ak) = -\frac{1}{2} \log (A_{11} A_{22} - A_{12}^2),$$

where $A_{rs} = \sum_{1 \leq j \leq 3} k_{jr} k_{js} a_j^2$.

We take $U = \{a \in A, a_1 = a_2 \neq 1\} = \{\exp H, H \in \alpha, H_1 = H_2 \neq 0\}$.

If $s \in W$ is the reflection with respect to the root α_{13} , s transforms $\{H \in \alpha, H_1 = H_2 < 0\}$ into $\{H \in \alpha, N_1 = -2H_2 > 0\}$. Hence $U \cup sU \supset \partial A^+ \setminus \{1\}$.

To show that (B'') holds, it suffices to show that, when $a \in U$, we have:

- (i) $\text{rank } d_k(H(ak)) = 2$ if $H(ak) = 0$,
- (ii) $d_k(H_1(ak)) \neq 0$ if $H_1(ak) = 0$,
- (iii) $d_k(H_3(ak)) \neq 0$ if $H_3(ak) = 0$,
- (iv) $d_k((H_1 + H_3)(ak)) \neq 0$ if $(H_1 + H_3)(ak) = 0$,

where d_k denotes the differential with respect to k when a is fixed.

Using (29), we get when $a = \exp H \in U$:

$$H_1(ak) = \frac{1}{2} \log (e^{2H_1} + (e^{-4H_1} - e^{2H_1})k_{31}^2)$$

$$H_3(ak) = -\frac{1}{2} \log (e^{-2H_1} + (e^{4H_1} - e^{-2H_1})k_{33}^2),$$

from which (i), (ii), (iii), (iv) follow easily.

Remark that on the other hand if $H_1 = 0 \neq H_2$, (ii) is violated for $a = \exp H$, as a simple computation shows. Note that $a \cdot o \in X'$. So we are in the situation referred to immediately after theorem II.6.2.

EXAMPLE II.6.2. The rank 2 space X of type A II in Elie Cartan's terminology, that is $X = SU^*(6)/Sp(3)$.

$SU^*(6)$ is the group of 6×6 complex matrices with determinant equal to 1, of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$, where A and B are 3×3 complex matrices. $Sp(3) = SU^*(6) \cap U(6)$, where $U(6)$ is the group of unitary 6×6 matrices.

We are going to show that condition (B'') is satisfied. Hence theorem II.6.2 holds for this space.

As shown in [5], chapter X, we may take $\theta(Y) = -{}^t\bar{Y}$ as a Cartan involution for the Lie algebra of $SU^*(6)$.

Accordingly, for a we take the set of 6×6 matrices of the form $H = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}$, where Z is a real diagonal 3×3 matrix with trace equal to 0.

We may choose the maps $\alpha_{ij}(H) = Z_i - Z_j$, $1 \leq i < j \leq 3$, as the set Σ^+ of positive restricted roots. Their multiplicity is equal to 4 and $\alpha^+ = \{H : -Z_1/2 < Z_2 < Z_1\}$. One sees easily that N is the set of complex 6×6 matrices of the form $\begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix}$, where U, V are complex 3×3 matrices with $U_j = 1, V_j = 0, U_{jk} = V_{jk} = 0$ for $k < j$.

Since m_α is even for each $\alpha \in \Sigma^+$, $|c(\xi)|^{-2}$ is a nonzero polynomial function and $WF(|c(D)|^{-2}\delta) = T_0^*\alpha \setminus 0$.

Reasoning as in example II.6.1, we see that, to check condition (B'') , we may take $U = \exp \{H \in \alpha, Z_1 = Z_2 \neq 0\}$. Using (28), where now $\theta(g) = ({}^t\bar{g})^{-1}$ since $g \in SU^*(6)$, and $k^{-1} = {}^t\bar{k}$ since $k \in U(6)$, we get, if $H(ak) = \begin{pmatrix} Z(ak) & 0 \\ 0 & Z(ak) \end{pmatrix}$:

$$Z_1(ak) = \frac{1}{2} \log (e^{2Z_1} + (e^{-4Z_1} - e^{2Z_1})(|U_{31}|^2 + |W_{31}|^2)),$$

$$Z_3(ak) = -\frac{1}{2} \log (e^{-2Z_1} + (e^{4Z_1} - e^{-2Z_1})(|U_{33}|^2 + |W_{33}|^2)),$$

when $a = \exp \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \in U$.

From these formulas, one sees easily that the map $Sp(3) \ni k \mapsto H(ak)$ is of rank two when $H(ak) = 0$ and $a \in U$, so that condition (B'') holds.

REMARK II.6.1. $P \in \mathcal{D}(X)$ may be hypoelliptic as soon as p satisfies condition (A) , even when condition (B) is violated on $X \setminus (X' \cup \{o\})$. Take for example $X = (SU^*(4) \times SU^*(4)) / (Sp(2) \times Sp(2))$, with obvious definitions in view of example II.6.2. On the products we put the product group law and the product manifold structure. One can check that $WF(|c(D)|^{-2}\delta) = T_0^* \mathfrak{a} \setminus 0$, and that condition (B') is violated at all points $a \cdot o$ with $a = (a_1, 1) \in A$, $a_1 \neq 1$, if 1 denotes the neutral element of $SU^*(4)$. However theorem II.5.1 applies to the space X .

III. - Gevrey hypoellipticity.

We are now going to make a study of Gevrey hypoellipticity which is parallel to our study of hypoellipticity in section II.

We shall consider operators $P \in \mathcal{D}(X)$ for which the corresponding p satisfies condition (A_s) . Since p can be viewed as a function on \mathfrak{a}^* , the precise meaning of (A_s) is that $|p^{(\alpha)}(\xi)|/|p(\xi)| \leq C|\xi|^{-|\alpha|/s}$ for some constant C , when $\xi \rightarrow \infty$, where $p^{(\alpha)}$ is computed in some linear coordinate system (ξ_1, \dots, ξ_i) of \mathfrak{a}^* .

Recall also the definition of the Gevrey class G_s : if U is an open subset of \mathbb{R}^n and $u \in C^\infty(U)$, u is said to belong to the Gevrey class $G_s(U)$ ($s \geq 1$) if for every compact subset K of U , there is a constant C_K such that $|D^\alpha u(x)| \leq C_K^{|\alpha|+1} (|\alpha|!)^s$, $x \in K$, for all multi-indices α . The definition is invariant under an analytic change of variables, so now if Y is an analytic manifold, $G_s(Y)$ is defined by means of local coordinates. $G_1(Y)$ is the set of analytic functions on Y .

If $u \in \mathcal{D}'(Y)$, we shall denote by $\text{sing supp } G_s(u)$ the complement of the largest open subset of Y where $u \in G_s$, and by $WFG_s(u)$ its G_s wave front set in the sense of Hörmander (see [15], where WFG_s is denoted by WFL with $L_k = (1 + k)^s$).

A differential operator will be called G_s hypoelliptic if $\text{sing supp } G_s(Pu) = \text{sing supp } G_s(u)$ for all $u \in \mathcal{D}'(Y)$. We shall write G_s^{comp} instead of $G_s \cap C_0^\infty$.

III.1. - The necessity of (A_s) .

As in section I we shall sometimes use the symbol $|\cdot|$ to denote some fixed norm on \mathfrak{a}_c^* . We have the following result if $P \in \mathcal{D}(X)$:

THEOREM III.1.1. *Let S be a not empty open relatively compact subset of X . Assume that for any $v \in \mathcal{D}'(S)$ such that $Pv = 0$ in S , one has $v \in G_s(S)$. Then p satisfies condition (A_s) .*

PROOF. We shall follow closely the proof of the corresponding \mathbb{R}^n theorem given in theorem 7.3 of [17], with modifications due to the structure of $\mathbf{D}(X)$.

If $l = \text{rank } X$, there exist $Q_j \in \mathbf{D}(X)$, $1 \leq j \leq l$, corresponding to algebraically independent homogeneous polynomials $q_j \in I(\alpha_0)$ of degree $m_j > 0$, such that Q_1, \dots, Q_l and I (the identity operator) generate the commutative algebra $\mathbf{D}(X)$ (see [3]; [4], chapter X). If $\alpha \in \mathbb{Z}_+^l$, we write $Q^\alpha = Q_1^{\alpha_1} \dots Q_l^{\alpha_l}$ if $\alpha \neq 0$ and $Q^0 = I$. Note that m_j is the order of Q_j and put $|m, \alpha| = \sum_{1 \leq j \leq l} m_j \alpha_j$.

We topologize the space $N(S) = \{v \in \mathcal{D}'(S), Pv = 0\}$ with the semi-norms

$$\Phi_{V,r}(f) = \sup_{x \in V} \sum_{\alpha \in \mathbb{Z}_+^l} (1/|m, \alpha|)^{s+1/r} |Q^\alpha f(x)|,$$

where V runs over the compact subsets of S , and $r \in \mathbb{Z}_+ \setminus \{0\}$. Since $N(S) \subset G_s(S)$, it follows that $\Phi_{V,r}(f) < \infty$ if $f \in N(S)$. One checks easily that with the topology defined by those semi-norms, $N(S)$ becomes a Fréchet space whose topology is finer than that induced by $L_{\text{loc}}^\infty(S)$ (equipped with its usual topology).

In view of Banach theorem, both topologies coincide. So if we fix V , then, for any r , there exists a constant C_r and a compact subset S_r of S such that for any $v \in N(S)$:

$$(30) \quad \Phi_{K,r}(v) \leq C_r \sup_{x \in S_r} |v(x)|.$$

There is no restriction to assume that $o \in S$, since P is G -invariant. If we take $v = \varphi_\zeta$, $\zeta \in \alpha^*$, such that $p(\zeta) = 0$, we deduce from (30):

$$(31) \quad \sum_{k \in \mathbb{Z}_+} (1/(m, k)!)^{s+1/r} |q_j(\zeta)|^k \leq e^{B_r(1+|\text{Im } \zeta|)},$$

with some constant B_r depending on r ; here q_j is the polynomial such that $Q_j \varphi_\zeta = q_j(\zeta) \varphi_\zeta$, and we write $\text{Im } \zeta$ for η if $\zeta = \xi + i\eta$ with $\xi, \eta \in \alpha^*$.

Adapting the proof of lemma 7.4 of [17], one shows easily that there exist two positive constants C' and C'' such that

$$(32) \quad \sum_{k \in \mathbb{Z}_+} (C'R)^{mk} / (mk)! \leq C'' \sum_{k \in \mathbb{Z}_+} (R^{mk} / (mk)!)^{s+1/r},$$

for all $R \geq 0$.

From (31) and (32) one deduces that $|q_j(\zeta)|^{1/m_j} \leq C_r(1 + |\operatorname{Im} \zeta|)^{s+1/r}$ for all $\zeta \in \alpha_c^*$ satisfying $p(\zeta) = 0$.

Here C_r is a constant depending on r . Hence corollary I.2 gives the existence of a constant C'_r depending on r such that

$$|\zeta| \leq C'_r(1 + |\operatorname{Im} \zeta|)^{s+1/r},$$

for all $\zeta \in \alpha_c^*$ such that $p(\zeta) = 0$.

This means that p satisfies condition $(A_{s+1/r})$ for any $r \in \mathbb{Z}_+ \setminus \{0\}$, hence condition (A_s) since $\{t, p$ satisfies condition $(A_t)\}$ is a closed half-line (see e.g. theorem 7.1 of [17]). The proof is complete.

From theorem III.1.1 we get obviously:

COROLLARY III.1.1. *If $P \in \mathbf{D}(X)$ is G_s hypoelliptic, the corresponding p satisfies condition (A_s) .*

We are now able to characterize the analytic hypoelliptic elements of $\mathbf{D}(X)$:

THEOREM III.1.2. *$P \in \mathbf{D}(X)$ is analytic hypoelliptic if and only if it is elliptic.*

PROOF. (a) The manifold X is analytic and one can easily see that the elements of $\mathbf{D}(X)$ are differential operators with analytic coefficients because this is true for the elements of $\mathbf{D}_K(G)$. Hence if $P \in \mathbf{D}(X)$ is elliptic, it must be analytic hypoelliptic (see [13], chapter VII).

(b) If $P \in \mathbf{D}(X)$ is analytic hypoelliptic, theorem III.1.1 implies that $p_m(\xi) \neq 0$ for $0 \neq \xi \in \alpha^*$, if $p = \sum_{0 \leq j \leq m} p_{m-j}$ with the notation of section I. Then lemma I.2 implies that P is elliptic.

Thus (A_s) is a necessary and sufficient condition for G_s hypoellipticity on X when $s = 1$. So the problem to describe G_s hypoellipticity in terms of condition (A_s) is solved when $s = 1$. Therefore we may assume that $s > 1$ in what follows, although this is not necessary for the validity of the results we are going to prove. But if $s > 1$, G_s is not quasi-analytic so it contains functions with compact support. This will simplify proofs in paragraphs III.2 and III.6.

III.2. – Study of the parametrix \mathcal{C} .

As in II.2 we consider the distribution T given by (8) and the corresponding operator \mathcal{C} . Recall that, due to the existence of canonical C^∞

densities dg and dx on G and X , we may identify $\mathcal{D}'(G)$ with the dual of $C_0^\infty(G)$ and $\mathcal{D}'(X)$ with the dual of $C_0^\infty(X)$.

We are going to prove the following result:

THEOREM III.2.1. *Assume that $\text{sing supp } G_s(T) \subset \{o\}$. Then*

$$\text{sing supp } G_s(\mathcal{C}f) \subset \text{sing supp } G_s(f) \quad \text{for all } f \in \mathcal{E}'(X).$$

PROOF. As said earlier, we assume that $s > 1$. Let f belong to G_s in a neighbourhood of $g \cdot o$, where $g \in G$.

Take $\varphi, \psi \in G_s^{\text{comp}}(X)$ with φ equal to 1 close to $g \cdot o$; then

$$\varphi \mathcal{C}f = \varphi \mathcal{C}\psi f + \varphi \mathcal{C}(1 - \psi)f.$$

Choose ψ with support contained in a small neighbourhood of $g \cdot o$; then $\psi f \in G_s^{\text{comp}}(X)$. Since for each $x, y \in G$, the map $y \mapsto y^{-1}x$ is a diffeomorphism of G , we may apply theorem 4.1 of [15] to $\widetilde{\varphi \mathcal{C}\psi f} = \widetilde{\varphi}(\widetilde{\psi f} * \widetilde{T})$, which gives that $\widetilde{\varphi \mathcal{C}\psi f} \in G_s(G)$. Hence $\varphi \mathcal{C}\psi f \in G_s(X)$.

Assume that $\psi = 1$ in an open neighbourhood V of $g \cdot o$, and that $\text{supp } \varphi \subset V$. If S is a compact neighbourhood of $g \cdot o$, contained in V , one sees easily that there exists a neighbourhood V' of o such that, for any $x, y \in G$, one has $\pi(y) \in V$ as soon as $\pi(x) \in S$ and $\pi(y^{-1}x) \in V'$. This implies that $\varphi((1 - \psi)f \times FT) = 0$ if $F \in C_0^\infty(X)$ has support contained in a small neighbourhood of o .

But then $\varphi \mathcal{C}(1 - \psi)f = \varphi((1 - \psi)f \times (1 - F)T)$. We choose such a F in $G_s^{\text{comp}}(X)$, equal to 1 in some neighbourhood of o ; then $(1 - F)T$ belongs also to $G_s(X)$.

So another application of theorem 4.1 of [15] shows that

$$\widetilde{\varphi((1 - \psi)f \times (1 - F)T)} \in G_s(G),$$

whence $\varphi \mathcal{C}(1 - \psi)f \in G_s(X)$.

We conclude that $\varphi \mathcal{C}f \in G_s(X)$. The proof is complete.

We have seen in II.2 that $PT - \delta \in C^\infty(X)$. A simple computation shows that in fact $PT - \delta \in G_1(X)$; therefore $f \times (PT - \delta) \in G_1(X)$ for any $f \in \mathcal{E}'(X)$. Combining this with theorem III.2.1, one gets immediately the

COROLLARY III.2.1. *If T given by (8) satisfies $\text{sing supp } G_s(T) \subset \{o\}$, then P is G_s hypoelliptic.*

III.3. – Study of T when G is complex.

When G is complex we are going to prove the converse of corollary III.1.1. We keep the same notations as above.

THEOREM III.3.1. *Assume that G is complex. If p satisfies condition (A_s) , then P is G_s hypoelliptic.*

PROOF. Put $R = -L - \langle \varrho, \varrho \rangle$, where L and $\langle \varrho, \varrho \rangle$ have the same meaning as in the proof of theorem II.1.1. By theorem II.3.1, we already know that $\text{sing supp } T \subset \{o\}$. We are going to show the following:

(33) *For any compact subset S of $X \setminus \{o\}$, there exists a constant $C > 0$ such that*

$$\sup_{x \in S} |R^m T(x)| \leq C^{m+1} (2m!)^s, \quad \text{for all } m \in \mathbf{Z}_+.$$

In fact, since R is an elliptic differential operator with analytic coefficients, the theorem of elliptic iterates ([16], p. 55) shows that (33) implies that $\text{sing supp } G_s(T) \subset \{o\}$.

The proof of (33) is just a combination of the classical corresponding proof of the \mathbf{R}^n case (see [17], theorem 7.4) with arguments already developed in II.3, so we may be rather brief. Clearly (33) is a consequence of the following:

(34) *For any compact subset S of X , there exist positive constants C, C_1, C_2 such that the following holds: for each $m \in \mathbf{Z}_+$ there exists $N \in \mathbf{Z}_+$ such that $N \leq C_1 m + C_2$ and $|\langle d^{2N} R^m T, u \rangle| \leq C^{m+1} (2m!)^s \int |u(x)| dx$, for all $u \in C_0^\infty(S)$.*

Here d is the distance to o as in II.3.

Let us prove (34). Arguing as in the proof of theorem II.3.1 and using the same notations as there, we find that

$$(35) \quad \langle d^{2N} R^m T, u \rangle = \int_{\mathfrak{a}^*} \langle \xi, \xi \rangle^m V(\xi) \mathcal{B}^N(D_\xi) U(\xi) d\xi.$$

Let e^1, \dots, e^l be the basis of \mathfrak{a}^* defined in the beginning of II.3. It is clearly orthonormal for the Killing form. If $\xi \in \mathfrak{a}^*$, we write $\xi = \sum \xi_j e^j$ and identify \mathfrak{a}^* with \mathbf{R}^l via the map $\xi \mapsto (\xi_1, \dots, \xi_l)$. We also identify func-

tions on α^* with the corresponding functions on \mathbb{R}^l . Then $\langle \xi, \xi \rangle = \sum_{1 \leq j \leq l} \xi_j^2$ and $\mathcal{B}(D_\xi) = - \sum_{1 \leq j \leq l} (\partial^2 / \partial \xi_j^2)$.

With the identification described above, the right-hand side of (35) is the sum of l^{m+N} terms of the kind

$$(36) \quad (-1)^N \int_{\mathbb{R}^l} \chi(\xi) q(\xi) (\partial^{2N} U(\xi) / \partial \xi_{j_1}^2 \dots \partial \xi_{j_N}^2) d\xi_1 \dots d\xi_l,$$

with $q(\xi) = \pi(i\xi)(\xi_{k_1} \dots \xi_{k_m})^2 / (|W|\pi(\rho)^2 p(-\xi))$ where $j_1, \dots, j_N, k_1, \dots, k_m$ are strictly positive integers smaller than or equal to l .

We integrate by parts. Using formula 7.5 of [17], p. 404, we see that each term of type (36) can be written as a sum $T_0 + \sum_{1 \leq j \leq 2N} (-1)^{j+1} T_j$, with

$$T_0 = (2\pi)^{-l/2} \int_{\mathbb{R}^l} (D_0 \dots D_{2N+1} q(\xi)) \chi(\xi) U(\xi) d\xi_1 \dots d\xi_l,$$

$$T_j = (2\pi)^{-l/2} \int_{\mathbb{R}^l} (D_0 \dots D_{2N-j} q(\xi)) D_{2N-j+1} \chi(\xi) (D_{2N-j+2} \dots D_{2N+1} U(\xi)) d\xi_1 \dots d\xi_l,$$

where $D_0 = D_{2N+1} = I$ (the identity operator) and $D_t = i^{-1} \partial / \partial \xi_r$ for some r if $1 < t \leq 2N$.

With the usual multi-indices notation in the ξ_1, \dots, ξ_l variables, we put $\xi^\beta = \pi(i\xi) \xi_{k_1}^{\beta_1} \dots \xi_{k_m}^{\beta_m}$, where $\beta \in \mathbb{Z}_+^l$. Since p satisfies condition (A_s) , estimate (7.8) of [17], p. 406, gives the existence of a constant C_1 such that for all $\alpha \in \mathbb{Z}_+^l$ and all $\xi \in \text{supp } \chi$:

$$|D_\xi^\alpha q(\xi)| \leq \alpha! (2|\xi|)^{|\beta|} \sum (2C_1)^{|\alpha'|} |\xi|^{-|\alpha'| - |\alpha''|/s},$$

where the summation is extended to all α', α'' such that $\alpha' + \alpha'' = \alpha$ and $\alpha'_j \leq \beta_j$ for $j = 1, \dots, l$, $D_\xi^\alpha = i^{-|\alpha|} \partial^{|\alpha|} / \partial \xi_1^{\alpha_1} \dots \partial \xi_l^{\alpha_l}$ and $|\xi|$ is the Euclidean norm in \mathbb{R}^l .

Also, with some constant C_2 independent of α and β , we have $|D_\xi^\alpha q(\xi)| \leq 2^{|\beta|} C_2^{|\alpha|+1} |\alpha|!$ if $\xi \in \text{supp } d\chi$ (see [17], p. 406). Therefore one checks easily that for some fixed constant C :

$$(37) \quad |T_0| \leq C^{N+m+1} (2N)! \left(\int_{\mathbb{R}^l} |\xi|^\theta |\chi(\xi)| d\xi_1 \dots d\xi_l \right) \left(\int_{\mathbb{X}} |u(x)| dx \right),$$

where $\theta = 2m - 2N/s + 2 \deg \pi$,

$$(38) \quad \sum_{1 \leq j \leq 2N} |T_j| \leq C^{N+m+1} (2N)! \int_{\mathbb{X}} |u(x)| dx.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbf{Z}_+$ such that

$$(39) \quad 0 \leq N - \frac{s}{2}(2m + 2 \deg \pi + l + \varepsilon) < 1.$$

This implies that for some fixed constant C_3 , one has $\int_{\mathbf{R}^l} |\xi|^0 \chi(\xi) d\xi_1 \dots d\xi_l \leq C_3$ and $(2N)! \leq C_3^{m+1}((2m)!)^s$. Therefore (34) is a consequence of (37) and (38). The proof is complete.

III.4. – Study of T in the set X' of regular points.

We give now a result similar to theorem II.4.1.

THEOREM III.4.1. *If p satisfies condition (A_s) and T is given by (8), one has $\text{sing supp } G_s(T) \cap X' = \emptyset$.*

PROOF. We know already by theorem II.4.1 that $\text{sing supp } T \cap X' = \emptyset$. In view of the theorem of elliptic iterates ([17], p. 55), it is sufficient to show the following, where $R = -L - \langle \varrho, \varrho \rangle$ as in the proof of theorem III.3.1:

(40) *For any compact subset S' of X' , there exists a constant C such that*

$$\sup_{x \in S'} |R^m T(x)| \leq C^{m+1}((2m)!)^s, \quad \text{for all } m \in \mathbf{Z}_+.$$

Using once more the diffeomorphism from $K/M \times a^+$ onto X' which sends (kM, H) to $\text{Exp } Ad_G(k)H$, we see that (40) is a consequence of the following:

(41) *For any compact subset S of A^+ there exists a constant C such that*

$$\sup_{a \in S} |\widetilde{R^m T}(a)| \leq C^{m+1}((2m)!)^s, \quad \text{for all } m \in \mathbf{Z}_+.$$

Of course, to prove (41), it suffices to show that any point a_0 of A^+ has a compact neighbourhood V' in A^+ such that (41) holds with S replaced by V' . Using the notations of the proof of theorem II.4.2, we may assume that with some strictly positive constant C , $B(e_1, \log a) \geq C$ when $a \in V'$. Since p satisfies condition (A_s) , we may apply lemma II.4.1 with the same s as in condition (A_s) . Repeating the proof of theorem II.4.2 for $\widetilde{R^m T}$, we find

easily that when $a \in V'$, $\widetilde{R^m T}(a) = F_1(a) + F_2(a)$, with

$$F_1(a) = \gamma \int \chi(\xi) U'(a, \zeta_1, \xi') \langle (\zeta_1, \xi'), (\zeta_1, \xi') \rangle^m d\zeta_1 d\xi',$$

$$F_2(a) = i\gamma \int \int \frac{\partial \chi}{\partial \xi_1}(\xi) U'(a, \zeta_1, \xi') \langle (\zeta_1, \xi'), (\zeta_1, \xi') \rangle^m d\xi_1 d\xi' d\eta_1,$$

where (ζ_1, ξ') denotes the vector $\zeta_1 e_1 + \sum_{2 \leq j \leq l} \xi_j e_j$.

Denote by Γ Euler gamma function. Then we get with various positive constants C independent of m and some positive constant R' , also independent of m :

$$|F_1(a)| \leq C^{m+1} \int_{|\xi| \geq R'} e^{-|\xi|^{1/\mu}} |\xi|^{M+2m} d\xi \leq C^{m+1} \Gamma(2sm + C) \leq C^{m+1} (2m)!^s, \quad |F_2(a)| \leq C^{m+1}, \quad \text{when } a \in V'.$$

This proves (41) and completes the proof of the theorem.

III.5. – Product of rank one spaces.

In this paragraph we assume that G satisfies condition (21). Then the following holds:

THEOREM III.5.1. *Let X be as in theorem II.5.1. Then $P \in \mathbf{D}(X)$ is G_s hypoelliptic if the corresponding polynomial p satisfies condition (A_s) .*

PROOF. In view of corollary III.2.1, it suffices to show that

$$\text{sing supp } G_s(T) \subset \{o\}$$

if T is given by (8).

We already know by the proof of theorem II.5.1 that $\text{sing supp } T \subset \{o\}$, and we are going to show that (33) holds.

First we prove that the following is true:

(42) *For any compact subset S' of $A \setminus \{1\}$, there exists a constant C such that*

$$\sup_{a \in S'} |\widetilde{R^m T}(a)| \leq C^{m+1} (2m)!^s, \quad \text{for all } m \in \mathbf{Z}_+.$$

To show that (42) holds, we may, in view of (41), assume that S' is a small neighbourhood of a point of $\partial A^+ \setminus \{1\}$. Then an obvious refinement

of the proof of theorem II.5.1 (that is estimates similar to the ones of F_1 and F_2 in the proof of theorem III.4.1) gives (42). We omit the details.

Now if $Y \in \mathfrak{p}$, we define $Y' \in \bar{\mathfrak{a}}^+$ as in the proof of lemma II.5.1. In view of the continuity of the map $Y \mapsto Y'$ proved in the course of that lemma, (42) implies the following:

(43) For any compact subset S'' of $\mathfrak{p} \setminus \{0\}$, there exists a constant C such that

$$\sup_{Y \in S''} |(R^m T \circ \text{Exp})(Y)| \leq C^{m+1} (2m)!^s, \quad \text{for all } m \in \mathbb{Z}_+.$$

Clearly, (33) is a consequence of (43). The proof is complete.

III.6. – The transversality condition (B_s).

Now we are going to make a study similar to that of II.6.

Using the canonical C^∞ density dH on \mathfrak{a} , we identify $\mathcal{D}'(\mathfrak{a})$ with the dual of $C_0^\infty(\mathfrak{a})$.

Define

$$f \in \mathcal{S}'(\mathfrak{a}) \quad \text{by} \quad \langle f, z \rangle = |W|^{-1} \int_{\mathfrak{a}^*} \frac{\chi(\xi)}{p(-\xi)} \hat{z}(\xi) d\xi$$

if $z \in \mathcal{S}(\mathfrak{a})$. Then if E is as in II.6, we have $Ez = f * z$, where $*$ denotes convolution on \mathfrak{a} .

First we prove the following lemma:

LEMMA III.6.1. Assume that p satisfies condition (A_s). Then $WFG_s(Eu) = WFG_s(u)$ if $u \in \mathcal{S}'(\mathfrak{a})$.

PROOF. If $p(D)u(H) = |W| \int_{\mathfrak{a}^*} e^{i\langle \xi, H \rangle} p(\xi) \hat{u}(\xi) d\xi$, one has $p(D)Eu - u \in G_1(\mathfrak{a})$ when $u \in \mathcal{S}'(\mathfrak{a})$. Hence $WFG_s(u) \subset WFG_s(Eu)$. Note that if $s = 1$, the lemma is an immediate corollary of theorem 5.4 of [15], so we assume that $s > 1$.

For each $H \in \mathfrak{a}$, we may identify $T_H^*(\mathfrak{a})$ with \mathfrak{a}^* in a canonical way. Assume that $(H_0, \xi^0) \notin WFG_s(u)$. Then if $\varphi \in G_s^{\text{comp}}(\mathfrak{a})$ has its support contained in a small neighbourhood of H_0 , there exist a conic neighbourhood V of ξ^0 and a constant $C > 0$ such that for all $N \in \mathbb{Z}_+$:

$$\sup_{\xi \in V} |\xi|^N |\widehat{\varphi u}(\xi)| \leq C^{N+1} (N!)^s.$$

If $\psi \in G_s^{\text{comp}}(\mathfrak{a})$, we have $\widehat{\psi E \varphi u} = \hat{\psi} * (\hat{\psi} \widehat{\varphi u})$, where $*$ now denotes convolution on \mathfrak{a}^* . Hence estimates as in the proof of lemma 3.3 of [15] show

that

$$(44) \quad \sup_{\xi \in V'} \widehat{|\psi E \varphi u(\xi)|} |\xi|^N \leq C^{N+1} (N!)^s$$

for some $C > 0$ and all $N \in \mathbb{Z}_+$, if V' is a closed cone contained in V .

Choose ψ such that $\varphi = 1$ in a neighbourhood of $\text{supp } \psi$ and denote by F the distribution kernel of $\psi E(1 - \varphi)$.

It is easy to see that $F \in \mathcal{S}(\alpha \times \alpha)$. Hence we may write, when $u \in \mathcal{S}'(\alpha)$, $\alpha \in \mathbb{Z}_+^l$ and $H \in \alpha$:

$$(45) \quad |D_H^\alpha (\psi E(1 - \varphi) u)(H)| \leq C_1 \sup_{H' \in \alpha, |\beta| \leq r} (1 + |H'|)^t |D_H^\alpha D_{H'}^\beta F(H, H')|,$$

for some $r, t \in \mathbb{Z}_+$ and $C_1 > 0$. Here of course $\beta \in \mathbb{Z}_+^l$; D_H^α means $i^{-|\alpha|} \partial^{|\alpha|} / \partial H_1^{\alpha_1} \dots \partial H_l^{\alpha_l}$ where (H_1, \dots, H_l) are linear coordinates of α ; similar definition for $D_{H'}^\beta$.

Now there exists a constant C such that the right-hand side of (45) is bounded by $C^{|\alpha|+1} (|\alpha|!)^s$ for all $\alpha \in \mathbb{Z}_+^l$ and $H \in \alpha$: one way to see it is to deform the domain of integration in the definition of f as we did for T in II.4 and III.4. Therefore $\psi E(1 - \varphi) u \in G_s(\alpha)$.

Together with (44), this shows that $(H_0, \xi^0) \notin WFG_s(Eu)$. The proof is complete.

We introduce now the following condition, similar to condition (B) of II.6.

We shall say that *condition (B_s) is satisfied at $x \in X \setminus \{o\}$ if the mapping*

$$B \ni b \mapsto A(x, b) \in \alpha$$

has no normal contained in $WFG_s(|c(D)|^{-2}\delta)$.

Arguing as in II.6 one sees that condition (B_s) is satisfied at each point $x \in X \setminus (X' \cup \{o\})$ if and only if the following holds:

(B_s') *There exists an open subset U of $\partial A \setminus \{1\}$ such that*

$$(i) \quad \bigcup_{s \in W} sU \supset \partial A^+ \setminus \{1\},$$

$$(ii) \quad WFG_s(|c(D)|^{-2}\delta) \text{ does not contain any normal to the map } K \ni k \mapsto H(ak) \in \alpha \text{ if } a \in U.$$

We have the following result:

THEOREM III.6.1. *Assume that p satisfies (A_s). If (B_s) is satisfied at $x \in X \setminus \{o\}$, then $x \notin \text{sing supp } G_s(T)$.*

PROOF. By theorem II.6.1, we know already that $x \notin \text{sing supp } T$. Then it suffices to combine theorems 3.10 and 4.1 of [15] and to use lemma III.6.1. We omit the details.

Using theorems III.4.1 and III.6.1, and corollary III.2.1, we get the following result:

THEOREM III.6.2. *Assume that condition (B_s) is satisfied at each $x \in X \setminus (X' \cup \{o\})$. Then $P \in \mathbf{D}(X)$ is G_s hypoelliptic if the corresponding p satisfies (A_s) .*

EXAMPLES. The spaces $SL(3, \mathbf{R})/SO(3)$ and $SU^*(6)/Sp(3)$ (see II.6) satisfy condition (B_s') for any $s \geq 1$. In fact, if $s \geq 1$, $WFG_s(|c(D)|^{-2}\delta) \subset C(T_0^* \alpha \setminus 0) \cup N(H_1 = 0) \cup N(H_2 = 0) \cup N(H_1 + H_2 = 0)$ for the first space and $WFG_s(|c(D)|^{-2}\delta) \subset T_0^* \alpha \setminus 0$ for the second one. Hence theorem III.6.2 applies to those spaces.

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