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Antilocality of complex powers of elliptic differential operators with analytic coefficients

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1. – Introduction.

1. Consider X a real analytic, connected, n-dimensional manifold, and let $T\colon C_0^\infty(X) \to C_0^\infty(X)$ be a linear operator. We say that T is antilocal, if $\operatorname{supp} f \cup \operatorname{supp} Tf = X$, for every $f \in C_0^\infty(X)$, $f \neq 0$. Interesting examples of antilocal operators appear in theoretical physics. Thus H. Reeh and S. Schlieder have proved in [10] that the operator $(m^2 I - \Delta)^{1/2}$, Δ the Laplacian, is antilocal. This result has been extended by R. W. Goodman-I. E. Segal [4], Murata and K. Masuda [8], who showed that arbitrary nonintegral powers of $m^2 I - \Delta$ and square roots of very general second order elliptic operators are antilocal.

In this paper we prove analoguous results for complex powers of general elliptic operators with analytic coefficients. The main result of the paper is theorem 1.3, and its variants.

2. Complex powers of elliptic differential operators are defined in two, not completely equivalent, fashions, and before we can state the results of this paper, we must briefly recall these definitions.

Thus consider $p(x, D)$ an even order elliptic linear partial differential operator with real analytic coefficients defined on X. In a local coordinate system, $p(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$, $i = \sqrt{-1}$, for some, locally defined, functions a_α, which are real analytic. Real analytic functions may have, if not otherwise specified, complex values. The fact that p is elliptic means that $p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$, the principal symbol, which is a function defined on T^*X, is $\neq 0$ for $\xi \neq 0$. In a fixed coordinate neighborhood, we will also use the notation $p(x, \xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \xi^\alpha$.

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In order to speak about complex powers of the operator p, we must suppose that the range of $p_m(x, \xi)\colon T^*X \to C$ avoids some halfray in the complex plane C. Without loss of generality, we will assume that $p_m(x, \xi)$ avoids R_-, the negative part of the real axis. It is then possible to define for every $z \in C$ a C^∞ function $p_m^z(x, \xi)$, $\xi \neq 0$, by

$$p_m^z(x, \xi) = \exp\left(z \ln |p_m(x, \xi)| + iz \arg p_m(x, \xi)\right),$$

where $-\pi < \arg t \leqslant \pi$ if $t \in C$. Moreover, for every x_0 there is a coordinate neighborhood V and $M \geqslant 0$ such that $p(x, \xi) \notin R_-$ for $x \in V$ and $|\xi| > M$. It follows that $p^z(x, \xi)$, the principal value of the z-th complex power of p, defined in a similar way with p_m^z, is a C^∞ function for $x \in V$ and $|\xi| > M$.

3. The first method of defining complex powers of $p(x, D)$ has been studied by R. T. Seeley [14] and T. Burak [3]. Suppose that X is a real analytic compact manifold, and consider $p(x, D)$ an elliptic differential operator defined on X, which has real analytic coefficients (for the moment, we may suppose that X, and the coefficients of p are just C^∞). Suppose that $p_m(x, \xi)$ does not have real negative values, and suppose that 0 is not in the spectrum of the closure $\overline{p}$ of p in $\mathcal{H}^0(X)$. In this case it is possible to choose a constant ϱ such that the set $(-\infty, -1) \cup \{\lambda \in C;\ |\lambda| \leqslant 1\}$ is not in the spectrum of $\varrho\overline{p}$, and such that ϱp_m still does not take real negative values (cf. e.g., [14]). We assume that $\varrho = 1$, and denote by Λ the contour $(-\infty, -1 - iO) \cup \cup S \cup (-1 + iO, -\infty)$, where S is the unit circle in C, with initial point at -1 and anticlockwise orientation. Finally we set for $\operatorname{Re} z < 0$,

$$p^z(x, D) = (i/2\pi)\int \lambda^z(\overline{p} - \lambda)^{-1}\, d\lambda. \tag{1}$$

$(\overline{p} - \lambda)^{-1}$ is here the resolvent of $\overline{p}$. The integral makes sense in view of the estimate $\|(\overline{p} - \lambda)^{-1}\| < C/|\lambda|$ (cf. [14]). $p^z(x, D)$ is called the z-th complex power of p. It is a remarkable fact, proved in [3], [14], that the operators $p^z(x, D)$ are pseudo-differential operators. By functional calculus, it follows that $p^{z_1}(x, D)\, p^{z_2}(x, D) = p^{z_1+z_2}(x, D)$ and that $p^{-1}(x, D) = (p(x, D))^{-1}$, such that we may extend the definition of $p^z(x, D)$ to all z by setting $p^z(x, D) = p^k(x, D)\, p^{z-k}(x, D)$, where the natural number k is chosen, for $\operatorname{Re} z \geqslant 0$, such that $-1 \leqslant \operatorname{Re} z - k < 0$.

4. Another approach to complex powers has been used by M. Nagase-K. Shinkai, [9]. Using only symbolic calculus, they proved the following result:

PROPOSITION 1.1. *Consider $U \subset R^n$, an open domain, and let* $p(x, D) = \sum_{|\alpha| \leq m} a_\alpha(D) D^\alpha$ *be an elliptic partial differential operator (p may have C^∞ coefficients, for the moment), defined on U, such that $p_m(x, \xi) \notin R_-$, for $\xi \neq 0$. Then there are entire analytic functions $z \to C_{j,k}$ and symbols $p_{j,k} \in S_{1,0}^{-j+mk}(U, R^n)$, such that the symbols*

$$p(z; x, \xi) \sim p^z(x, \xi)\Big(1 + \sum_{j=1}^{\infty} \sum_{k=1} C_{j,k}(z) p^{-k}(x, \xi) p_{j,k}(x, \xi)\Big) \tag{2}$$

have the following properties

$$p(z_1; x, \xi) \circ p(z_2; x, \xi) \sim p(z_1 + z_2; x, \xi)\,, \tag{3}$$

$$p(1; x, \xi) \sim p(x, \xi)\,, \tag{4}$$

$$p(0; x, \xi) \sim 1\,, \tag{5}$$

$$p(z; x, \xi) - p^z(x, \xi) \in S_{1,0}^{m\operatorname{Re} z - 1}(U, R^n)\,. \tag{6}$$

Moreover, the $p(z; x, \xi)$ are uniquely defined by the properties (3), (4), (5), (6), *if we suppose them of form*

$$p(z; x, \xi) \sim p^z(x, \xi)\Big(1 + \sum_{j=1}^{\infty} \sum_{k=1}^{N(j)} C_{j,k}(z) p_{j,k}(x, \xi)\Big)$$

for some finite numbers $N(j)$, some entire $C_{j,k}$ and some $p_{j,k} \in S_{1,0}^{-j}(U, R^n)$.

Here $S_{1,0}^{\mu}(U, R^n)$ is the standard space of symbols of pseudo-differential operators, and all relations from (3) to (6) are in the symbol algebra.

We may now introduce pseudodifferential operators associated with the symbols $p(z; x, \xi)$, and call the resulting operators $p(z; x, D)$ again complex powers of $p(x, D)$. In view of the unicity in proposition 1.1, the definition of $p(z; x, D)$ can be extended to the case of C^∞ varieties (we always assume varieties to be paracompact.) No global conditions are needed this time, but instead of $p(z_1; x, D) \circ p(z_2; x, D) = p(z_1 + z_2; x, D)$, we only obtain, in the general case, $p(z_1; x, D) \circ p(z_2; x, D) = p(z_1 + z_2; x, D) + K(z_1, z_2)$, where $K(z_1, z_2)$ is an integral operator with C^∞ kernel.

5. We cannot expect that an operator T which is defined only up to integral operators with C^∞ kernel is antilocal. Therefore we must restrict our attention in the sequel to the category of analytic pseudodifferential operators, introduced in [2], [13]. For this purpose we need the following proposition:

PROPOSITION 1.2. *Suppose that $p(x, D)$ from proposition 1.1 has analytic coefficients, and let $p(z; x, \xi)$ be the formal sum from (2). Then, $p(z; x, \xi)$ is a formal analytic symbol on $(U, R^n\setminus\{0\})$, in the sense of L. Boutet de Monvel.*

Proposition 1.2 will be proved in § 3. In view of this proposition, we can define for every $z \in C$ an analytic pseudodifferential operator $p(z; x, D)$: $C_0^\infty(U) \to C_0^\infty(U)$, which is associated with the formal analytic symbol $p(z; x, \xi)$. The definition of formal analytic symbols and of the analytic pseudodifferential operators associated with such symbols, will be recalled in § 2. Let us note however, that when p has constant coefficients, when p is elliptic, and when $p_m(\xi) \notin R_-$ then we can define $p(z; x, D)f$ by

$$(1/2\pi)^n \int\limits_{|\xi| \geqslant M} \exp i\langle x, \xi\rangle\, p^z(\xi) \hat{f}(\xi)\, d\xi$$

and get an analytic pseudodifferential operators associated with $p(z; x, \xi) = p^z(\xi)$, if $M \geqslant 0$ is suitably chosen.

6. We can now state the main result from this paper.

THEOREM 1.3. *Consider $U \subset R^n$ an open domain, and let $p(x, D) = \sum_{|\xi| \geqslant M} a_\alpha(x) D^\alpha$ be an elliptic partial differential operator, with coefficients a_α which are real analytic functions in U, and such that $p_m(x, \xi) \notin R_-$ for $x \in U$ and $\xi \in R^n\setminus\{0\}$. Consider $z \in C$ such that mz is not an even integer, and let $p(z; x, D)$ be an analytic pseudodifferential operator associated with the formal analytic symbol $p(z; x, \xi)$. Then $p(z; x, D)$: $C_0^\infty(U) \to C^\infty(U)$ is antilocal. Moreover, if $f \in C_0^\infty(U)$ is such that $\operatorname{supp} f \subset \{x;\, x_n \geqslant x_n^0\}$ for some $x^0 \in U$, and if there is $\varepsilon > 0$ and a real analytic function h, defined on $|x - x^0| < \varepsilon$ such that $h = p(z; x, D)f$ on $\{x;\, |x - x^0| < \varepsilon,\, x_n \leqslant x_n^0\}$, then it follows that $x^0 \notin \operatorname{supp} f$.*

Theorem 1.3 will be proved in § 5. All other results on antilocality which will be considered in this paper, are consequences of theorem 1.3.

THEOREM 1.4. *Consider X a connected real analytic manifold, and $p(x, D)$ an elliptic partial differential operator of order m on X with real analytic coefficients. Suppose that $p_m(x, \xi) \notin R_-$ and let $z \in C$ be such that mz is not an even integer. Suppose further that $p(z; x, D)$: $C_0^\infty(X) \to C^\infty(X)$ is a given linear operator which is an analytic pseudodifferential operator associated with the formal analytic symbol $p(z; x, \xi)$ (cf. definition 2.9 below). Then $p(z; x, D)$ is antilocal. Moreover, if g is a real analytic function on X and $f \in C_0^\infty(X)$, then*

$$\operatorname{supp} f \cup \operatorname{supp}\,(g + p(z; x, D))f = X\,.$$

Theorem 1.4 will follow from results proved in § 5.

Finally, we want to transfer theorem 1.4 to the case of exact complex powers, studied by R. T. Seeley and T. Burak. This is possible if we use the following result:

THEOREM 1.5. *Consider X a compact, real analytic, manifold and $p(x, D)$ an elliptic partial differential operator of order m, with real analytic coefficients, such that $p_m(T^*X\setminus\{0\}) \cap R_- = \emptyset$, and such that 0 is not in the spectrum of the closure $\overline{p}$ of p in $\mathcal{H}^0(X)$. Denote $p^z(x, D)\colon C^\infty(X) \to C^\infty(X)$, the operator associated by formula (1) with p. Further let U' be a neighborhood of some point $x^0 \in X$, $\varkappa\colon U' \to R^n$ an analytic coordinate system on X, which maps U' on R^n and denote $p(z; x, D)$ an analytic pseudodifferential operator, defined on $C_0^\infty(R^n)$, associated with the formal analytic symbol $p(z; x, \xi)$, which one obtains from proposition 1.2, applied to the differential operator $\tilde{p}(x, D)f = (p(x, D)f\circ\varkappa)\circ\varkappa^{-1}$. Then for every $f \in C_0^\infty(U')$ there is a real analytic function g on R^n such that $(p^z(x, D)f)\circ\varkappa^{-1} - p(z; x, D)(f\circ\varkappa^{-1}) = g$.*

Thus, if X is connected, and if mz is not an even integer, then $p^z(x, D)$ is antilocal.

Roughly speaking, theorem 1.5 is the analogue in the analytic case of some results proved in [3], [14] for the C_0^∞ case. Since we have not found this result in the literature, we will briefly sketch a proof for it in § 6 below.

7. In concluding this introduction, we now want to explain, considering a very simple example, some of the ideas involved in the proofs from this paper. Thus we take $n = 1$ and consider the operator $p(D) = D^2 = -(d/dx)^2$. We want to show that the square root $|D|$ of $p(D)$, defined on $C_0^\infty(R)$ by $|D|f = (1/2\pi)\int \exp(ix\xi)|\xi|\hat{f}(\xi)\,d\xi$ is antilocal. In order to do so, we do not try to give the shortest proof, but rather show how one can prove this fact with the arguments used below. Note that $|D|$ is an operator of the type considered in theorem 1.3.

We start with some elementary remarks.

Let us then choose $f \in C_0^\infty(R)$, $f \neq 0$. Our first remark is that $|D|f$ is real-analytic outside $\operatorname{supp} f$. Thus $\operatorname{supp} |D|f$ contains any component U of $R\setminus\operatorname{supp} f$ on which $|D|f$ does not vanish identically. In particular, $\operatorname{supp} |D|f$ therefore contains at least one unbounded component from $R\setminus\operatorname{supp} f$. In fact, otherwise $|\xi|\hat{f}(\xi)$ were an entire analytic function, which it is not.

Let us now show that both unbounded components from $R\setminus\operatorname{supp} f$ belong to $\operatorname{supp}|D|f$ (I learned the following simple argument from prof. W. Littman). To make a choice, assume for example that $\operatorname{supp} f \subset \{x \in R;\ x \geqslant 0\}$, that $0 \in \operatorname{supp} f$ and that we want to show that R_- is in $\operatorname{supp} |D|f$. Assume then, by contradiction (cf. the « first remark ») that $|D|f = 0$ for $x < 0$.

Then however $|\xi|\hat{f}(\xi)$, defined on R, would have an analytic extension to the complex upper half plane, which it has not.

We have now proved that $\operatorname{supp}|D|f$ contains both unbounded components from $R\setminus\operatorname{supp} f$. The same argument also almost shows that $\operatorname{supp}|D|f$ contains any bounded component U of $R\setminus\operatorname{supp} f$. For, let, e.g., U be of form $U=\{x\in R;\ |x|<1\}$ and write $f=f_1+f_2$ with $\operatorname{supp} f_1\subset\{x\in R;\ x\leqslant -1\}$, $\operatorname{supp} f_2\subset\{x\in R;\ x\geqslant 1\}$, $1\in\operatorname{supp} f_2$. Then $|D|f=|D|f_1+|D|f_2$, and both $|D|f_1$ and $|D|f_2$ are nonvanishing analytic functions on $|x|<1$. However, we cannot exclude a priori that $|D|f_1\equiv-|D|f_2$! Therefore, in order to prove the antilocality of $|D|$ we must prove a little more than just that $|D|f_2\not\equiv 0$ for $x<1$, namely that the restriction of $|D|f_2$ to $x<1$ has no real analytic extension accross $x=1$. After a translation it thus suffices to prove:

LEMMA 1.6. *Consider* $g\in C_0^\infty(R)$, $\operatorname{supp} g\subset\{x\in R;\ x\geqslant 0\}$. *Suppose that there is* $\varepsilon>0$ *and a real analytic function* h *defined on* $|x|<\varepsilon$ *such that* $h=|D|g$ *on* $-\varepsilon<x<0$. *Then* $0\notin\operatorname{supp} g$.

PROOF OF LEMMA 1.6. Let g, ε, h be as in the statement of lemma 1.6 and denote by $g'=|D|g-Dg-h$. Thus g' is defined for $|x|<\varepsilon$ and $\operatorname{supp} g'\subset\subset\{x\in R;\ x\geqslant 0\}$. Now

$$|D|g-Dg=(1/\pi)\int_{-\infty}^{0}\exp(ix\xi)\hat{g}(\xi)\,d\xi\,,$$

so $|D|g-Dg$, and therefore also g' has an analytic extension to a set of form $\{x\in C;\ \operatorname{Im} x\geqslant 0,\ |x|<\varepsilon'\}$, for some $\varepsilon'\leqslant\varepsilon$. This is only possible if $g'\equiv 0$ for $|x|<\varepsilon'$. Therefore

$$Dg=Dg-(1/2)(|D|g-Dg-h)=(1/2\pi)\int_0^\infty\exp(ix\xi)\,\xi\hat{g}(\xi)\,d\xi+h/2\,.$$

We conclude that Dg has an analytic extension to $\{x\in C;\ \operatorname{Im} x\leqslant 0,\ |x|<\varepsilon'\}$, which implies, since $\operatorname{supp} g\subset\{x\in R;\ x\geqslant 0\}$, that $Dg\equiv 0$ for $|x|<\varepsilon'$. This gives $g\equiv 0$ for $|x|<\varepsilon'$, whence the lemma.

We have now proved the lemma, and this also brings the proof of the fact that $|D|$ is antilocal to an end.

The arguments used to prove antilocality for general complex powers $p(z;x,D)$ will be parallel to the above ones. Again we must know that $p(z;x,D)f$ is real analytic outside $\operatorname{supp} f$. This will follow if we show that $p(z;x,D)$ is an analytic pseudodifferential operator. Once we have proved this fact, the type of antilocality which we study in this paper (antilocality

« modulo real analytic functions ») is a local property of just one point from the boundary of $\operatorname{supp} f$ for every component of $R^n \setminus \operatorname{supp} f$. This allows us to transform the case of bounded components from $R^n \setminus \operatorname{supp} f$ to the case of an (or, « the », if $n > 1$), unbounded component from $R^n \setminus \operatorname{supp} f$ by means of an analytic change of coordinates. In this way we can avoid some difficulties which come from the fact that the methods of Fourier analysis are hard to apply in the case of holes. The various extensions to the complex upper or lower half plane, used in the proof of lemma 1.6, will then be replaced by wave front set arguments. Finally let us recall the fact that during the proof of lemma 1.6 we have twice tested the operator $|D|$ against the operator D. In the general case, when we consider $p(z; x, D)$ we must at first introduce an operator which can replace D.

2. – Preliminaries about analytic pseudodifferential operators.

1. We briefly recall here some facts concerning analytic pseudodifferential operators. This is necessary since we cannot use them in their most standard form as presented e.g. in [13].

First we need some notation. If $K \subset R^n$ is a compact, $\varepsilon > 0$, $M > 0$, and $\Gamma \subset R^n \setminus \{0\}$ is an open cone, then we denote by $K_{\varepsilon,M,\Gamma} = \{(z, \zeta) \in C^n \times C^n;$ distance $(z, K) < \varepsilon$, $|\zeta| > M$, $\operatorname{Re} \zeta \in \Gamma$, $|\operatorname{Im} \zeta| < \varepsilon |\operatorname{Re} \zeta|\}$.

DEFINITION 2.1. Consider $U \subset R^n$ a domain and $\Gamma \subset R^n \setminus \{0\}$ an open cone.

a) $S^\mu_A(U, \Gamma)$ is the set of all functions $a(x, \xi)$ from $C^\infty(U \times \Gamma)$ such that for every compact $K \subset U$ and for every open cone $\Gamma' \subset\subset \Gamma$ (i.e. $\bar{\Gamma}' \subset \Gamma$) there are $\varepsilon > 0$, $M > 0$ and $c > 0$ for which $a(x, \xi)$ extends to an analytic function on $K_{\varepsilon,M,\Gamma'}$, which satisfies $|a(z, \zeta)| \leqslant c(1 + |\zeta|)^\mu$ on $K_{\varepsilon,M,\Gamma'}$.

The elements from $S^\mu_A(U, \Gamma)$ will be called analytic symbols of order μ defined on (U, Γ).

b) $SF^\mu_A(U, \Gamma)$ is the set of all formal sums $\sum\limits_{j \geqslant 0} a_j(x, \xi)$, $a_j \in S^{\mu-j}_A(U, \Gamma)$, such that for every compact $K \subset U$ and every open cone $\Gamma' \subset\subset \Gamma$ there are constants $\varepsilon > 0$, $M > 0$, $c > 0$ and $A > 0$, such that all $a_k(x, \xi)$ extend to analytic functions on $K_{\varepsilon,M,\Gamma'}$ and satisfy $|a_k(z, \zeta)| \leqslant cA^k k!(1 + |\zeta|)^{\mu-k}$ on $K_{\varepsilon,M,\Gamma'}$. $\sum a_j$ will be called a formal analytic symbol of order μ defined on (U, Γ). Further, $\sum a_j$ will be called a formal analytic symbol on (U, Γ), if it is a formal analytic symbol of some order μ, defined on (U, Γ).

c) If $\sum a_j$, $\sum b_j \in SF^\mu_A(U, \Gamma)$, then we say that $\sum a_j$ is equivalent with $\sum b_j$, and write $\sum a_j \sim \sum b_j$ in $SF^\mu_A(U, \Gamma)$, if for every compact $K \subset U$ and every open cone $\Gamma' \subset\subset \Gamma$, there are constants $\varepsilon > 0$, $M > 0$, $c > 0$, $A > 0$, such that for all natural numbers s

$$\Big|\sum_{j<s} \big(a_j(z, \zeta) - b_j(z, \zeta)\big)\Big| \leqslant cA^s s!(1 + |\zeta|)^{\mu - s} \quad \text{on } K_{\varepsilon, M, \Gamma'}.$$

Note that when $\Gamma = R^n \setminus \{0\}$, then definition 2.1 reduces to wellknown definitions from [2].

PROPOSITION 2.2 (cf. [2]). *Consider* $\sum a_j \in SF^\mu_A(U, \Gamma)$, U' *a relatively compact open subset in* U *and* $\Gamma' \subset\subset \Gamma$. *Then there is* $a \in S^\mu_A(U', \Gamma')$ *such that* $a \sim \sum a_j$ *in* $SF^\mu_A(U', \Gamma')$. (*An element* a *from* $S^\mu_A(U, \Gamma)$ *defines an element*

$$\sum a_j \in SF^\mu_A(U, \Gamma) \text{ by setting } a_0 = a,\ a_j = 0, \text{ for } j > 0).$$

PROPOSITION 2.3 (cf. [2]). *a) if* $\sum a_j \in SF^\mu_A(U, \Gamma)$ *and* $\sum b_j \in SF^{\mu'}_A(U, \Gamma)$, *then* $\sum c_k$, *defined by*

$$c_k(x, \xi) = \sum_{j + j' + |\beta| = k} (i^{-|\beta|}/\beta!)(\partial/\partial\xi)^\beta a_j(x, \xi)(\partial/\partial x)^\beta b_{j'}(x, \xi)$$

is in $SF^{\mu+\mu'}_A(U, \Gamma)$. *We will denote* $\sum c_k = (\sum a_j)\circ(\sum b_j)$.

b) If $\sum a_j \in SF^\mu_A(U, \Gamma)$, *then* $\sum_k \Big(\sum_{j+|\beta|=k} (i^{|\beta|}/\beta!)(\partial/\partial\xi)^\beta(\partial/\partial x)^\beta a_j\Big)$ *is in* $SF^\mu_A(U, \Gamma)$.

2. DEFINITION 2.4. *We denote* $S^\mu_A(U, R^n, \Gamma)$ *the space of symbols* $a(x, \xi)$ *from* $S^\mu_{1,0}(U, R^n)$, *such that the restriction of* $a(x, \xi)$ *to* $U \times \Gamma$ *is in* $S^\mu_A(U, \Gamma)$, *and such that for every compact* $K \subset U$ *there is* c *and* A *such that*

$$|D^\alpha_x a(x, \xi)| \leqslant cA^{|\alpha|}\alpha!(1 + |\xi|)^\mu, \qquad \forall x \in K,\ \forall \xi \in R^n.$$

For $a \in S^\mu_A(U, R^n, \Gamma)$, we denote $a(x, D)\colon C^\infty_0(U) \to C^\infty(U)$ the pseudo-differential operator

$$a(x, D)f = (2\pi)^{-n}\int \exp i\langle x, \xi\rangle\, a(x, \xi)\hat{f}(\xi)\, d\xi\,.$$

$a(x, D)$ is called the Γ-analytic pseudodifferential operator associated with $a(x, \xi)$.

Concerning the case $\Gamma = R^n \setminus \{0\}$, we will use the following result:

PROPOSITION 2.5. *Consider* $a \in S_A^\mu(U, R^n, R^n \setminus \{0\})$, K *compact in* U, $f \in C_0^\infty(K)$ *and* $x^0 \in \complement \operatorname{supp} f$. *Suppose that* δ *is such that* $\{x; |x-x^0| \leqslant \delta\} \cap K = \emptyset$. *Then there are constants* c, A, *which depend only on* K, δ, *and the estimates of* a, *such that*

$$|D_x^\alpha a(x, D) f(x)| \leqslant c A^{|\alpha|} \alpha! \Big(\int |f(x)|^2 \, dx\Big)^{1/2} \quad \text{if } |x - x^0| \leqslant \delta/2.$$

Proposition 2.5 follows essentially from the fact that the distribution kernel of $a(x, D)$ in $\mathcal{D}'(U \times U)$ is an analytic function outside the diagonal. Cf. anyway [7].

A remarkable property of $a(x, D)$ is given in the following

THEOREM 2.6. *If* $a \in S_A^\mu(U, R^n, \Gamma)$, *then*

$$WF_A \, a(x, D) f \cap (U \times \Gamma) \subset WF_A f .$$

Here $WF_A u$ denotes the analytic wave front set (also called analytic singular spectrum, essential spectrum, etc.) of u, introduced by M. Sato [12] and L. Hörmander [5]. For the equivalence of the definitions from [12] and [5], cf. [1].

3. Suppose now $\sum a_j \in SF_A^\mu(U, \Gamma)$. If U' is relatively compact and open in U, and if $\Gamma' \subset\subset \Gamma$, then we can find a^1, $a^2 \in S_A^\mu(U', R^n, \Gamma')$ such that $a^1 \sim \sum a_j$ in $SF_A^\mu(U', \Gamma')$, respectively

$$a^2 \sim \sum_j \Big(\sum_{k+|\beta|=j} (i^{|\beta|}/\beta!)(\partial/\partial\xi)^\beta (\partial/\partial x)^\beta a_k \Big)$$

in $SF_A^\mu(U', \Gamma')$. This is an easy consequence of the propositions 2.2 and 2.3.

We now introduce the operators

$$a^1(x, D) f = (2\pi)^{-n} \int \exp i\langle x, \xi \rangle \, a^1(x, \xi) \hat{f}(\xi) \, d\xi ,$$

$$a^2(x; D) f = (2\pi)^{-n} \iint \exp i\langle x - y, \xi \rangle \, a^2(y, \xi) f(y) \, dy \, d\xi ,$$

where the second expression is an oscillatory integral (in fact, we integrate in y first).

$a^1(x, D)$ and $a^2(x; D)$ are defined on $C_0^\infty(U')$ with values in $C^\infty(U')$, but it should be remarked that $a^2(x; D)$ is in a natural way defined with values in $S'(R^n)$.

$a^1(x, D)$ and $a^2(x; D)$ will both be called Γ'-analytic pseudodifferential operators associated with $\sum a_j$. The definition is somewhat improper since $a^1(x, D)$, $a^2(x; D)$ are not uniquely associated with $\sum a_j$, and depend in particular on U'. The operators $a^1(x, D)$, $a^2(x; D)$ are however wellsuited for microlocal analysis at points from $U' \times \Gamma'$. The following result is then useful:

THEOREM 2.7. *$(x^0, \xi^0) \notin WF_A\big(a^1(x, D)f - a^2(x; D)f\big)$, for any $x^0 \in U'$ and $\xi^0 \in \Gamma'$.*

We also mention the following result:

THEOREM 2.8. (M. Sato, L. Hörmander). *Consider $a \in S_A^\mu(U, R^n, \Gamma)$ and suppose there is $b \in S_A^{-\mu}(U, \Gamma)$ such that the restriction a of a to $U \times \Gamma$ satisfies $\sum_j \Big(\sum_{|\beta|=j} (i^{-|\beta|}/\beta!)(\partial/\partial\xi)^\beta a (\partial/\partial x)^\beta b\Big) \sim 1$ in $SF_A^0(U, \Gamma)$. Suppose further that $(x^0, \xi^0) \notin WF_A a(x, D)f$. Then it follows that $(x^0, \xi^0) \notin WF_A f$.*

For proofs of the theorems 2.6 and 2.8, we refer the reader to [13] (where similar statements appear) and to [7], where there is also a proof of theorem 2.7. In particular, [7], was written essentially having the necessities of this (and another) paper in mind.

4. We conclude this paragraph with a few remarks concerning global definitions of analytic pseudodifferential operators.

DEFINITION 2.9. *Let X be a real analytic manifold and $T: C_0^\infty(X) \to C^\infty(X)$ a linear operator such that $WF_A Tg \subset WF_A g$ for all $g \in C_0^\infty(X)$. We will say that T is an analytic pseudodifferential operator of order μ on X, if for every $x \in X$ there are*

$$\begin{cases} U \text{ an open set in } R^n, \\ \varkappa\colon U \to X \text{ a real analytic map which maps } U \text{ diffeomorphically on} \\ \quad \text{a neighborhood of } x, \\ a(x, \xi) \in S_A^\mu(U, R^n, R^n \setminus \{0\}), \end{cases}$$

with the following property: for every $g \in C_0^\infty(U)$, the difference $T(g \circ \varkappa^{-1}) \circ \varkappa - a(x, D) g$ is real analytic on U.

Further if there is given a covering of X with open sets U', $\varkappa\colon U' \to U \subset R^n$ real analytic diffeomorphisms of U' onto U, $\sum a_j(x, \xi) \in SF_A^\mu(U, R^n \setminus \{0\})$, and if $T: C_0^\infty(X) \to C^\infty(X)$ is a given analytic pseudodifferential operator on X, then we will say (by abuse) that T is an analytic pseudodifferential operator associated with the $\sum a_j$ if the following happens: for every $\varkappa: U' \to U$, every V, relatively compact subset in U, every $a \sim \sum a_j$, $a(x, \xi) \in S^\mu_A(V, R^n, R^n \setminus \{0\})$ and every $g \in C_0^\infty(V)$, we have that $T(g \circ \varkappa) \circ \varkappa^{-1} - a(x, D) g$ is real analytic on V. Clearly, if the $\varkappa: U' \to U$ and $\sum a_j \in SF^\mu_A(U, R^n \setminus \{0\})$ are given, then an analytic pseudodifferential operator associated with the $\sum a_j$ can only exist if the different $\sum a_j$ are related by some (obvious) compatibility conditions (which come out from possible coordinate transformations), but even if these conditions are satisfied, for the $\varkappa: U' \to U$, $\sum a_j$, it is not clear how to associate an analytic pseudodifferential operator with the $\sum a_j$. A notable exception from this occurs when $X = U$ is an open set in R^n and $\sum a_j \in SF^\mu_A(U, R^n \setminus \{0\})$ is given. We want to show how one can then construct an analytic pseudodifferential operator $\big(\sum a_j\big)(x, D)$: $C_0^\infty(U) \to C^\infty(U)$ associated with $\sum a_j$.

To do this, let $\varphi_k \in C_0^\infty(U)$, $k = 1, 2, \ldots$, be a partition of unity for U and choose U_k relatively compact open subsets in U such that $\operatorname{supp} \varphi_k \subset U_k$. For every k we consider $b_k \in S^\mu_A(U_k, R^n, R^n \setminus \{0\})$ such that $b_k \sim \sum (i^{|\beta|}/\beta!) \cdot (\partial/\partial\xi)^\beta (\partial/\partial x)^\beta (a)$ in $SF^\mu_A(U_k, R^n \setminus \{0\})$, and define T_k: $C_0^\infty(U_k) \to \mathcal{S}'(R^n) \cap C^\infty(R^n)$ by

$$T_k(x, D) f = (2\pi)^{-n} \iint \exp i\langle x - y, \xi\rangle\, b_k(y, \xi) f(y)\, dy\, d\xi\ .$$

The sum $\sum T_k \varphi_k f$ is finite for every f, and it is easy to see, using theorem 2.7, that $\big(\sum a_j\big)(x, D) f = \sum T_k \varphi_k f$ is an analytic pseudodifferential operator associated with $\sum a_j$ (when co-restricted to U).

3. – Proof of proposition 1.2.

1. The $C_{j,k}$ and $p_{j,k}$ from proposition 1.2 are unique, and we recall their precise form fiom [9]. With the notation $p^{(\gamma)}_{(\delta)}(x, \xi) = (\partial/\partial x)^\delta (\partial/\partial \xi)^\gamma p(x, \xi)$, we have

$$\text{(1)} \quad p_{j,k}(x, \xi) = \big(i^{-j}/(\alpha_2^1!\,\alpha_3^1! \ldots \alpha_k^{k-1}!)\big)\, p^{(\beta_1)}(x, \xi)\, p^{(\beta_2)}_{(\alpha_2^1)}(x, \xi) \cdot \\ \cdot p^{(\beta_3)}_{(\alpha_3^1 + \alpha_3^2)}(x, \xi) \ldots p^{(\beta_{k-1})}_{(\alpha_{k-1}^1 + \ldots + \alpha_{k-1}^{k-2})}(x, \xi)\, p_{(\alpha_k + \ldots + \alpha_k^{k-1})}(x, \xi)\ ,$$

where $\beta_i = \alpha_{i+1}^i + \ldots + \alpha_k^i$, $i = 1, \ldots, k-1$, and where the summation is for all α_s^r such that $|\beta_1 + \ldots + \beta_{k-1}| = j$, $|\beta_1| \neq 0$, $|\beta_i| + |\alpha_i^1 + \ldots + \alpha_i^{i-1}| \neq 0$, $i = 2, \ldots, k-1$ and $|\alpha_k^1 + \ldots + \alpha_k^{k-1}| \neq 0$.

Further $C_{j,1} = 0$ and $C_{j,k}(z) = z(z-1) \ldots (z - k + 1)/k!$ for $k \geqslant 2$, such that if we denote

$$p_j(z;\, x,\, \xi) = p^z(x,\, \xi) \sum_{k=2}^{2j} \big(z(z-1)\, \dots\, (z-k+1)/k!\big)\, p_{j,k}(x,\, \xi)\, p^{-k}(x,\, \xi)$$

then $p(z;\, x,\, \xi) = \sum_{j\geqslant 0} p_j(z;\, x,\, \xi)$.

We want to show that $p(z;\, x,\, \xi)$ is a formal analytic symbol on $\big(U,\, R^n\diagdown\{0\}\big)$ such that now $\Gamma = R^n\diagdown\{0\}$. If $K \subset U$ is a compact set, then it follows from Cauchy's inequalities that there is $\varepsilon > 0$ and $M > 0$ such that

$$|p^{(\gamma)}_{(\delta)}(x,\, \xi)| \leqslant C(1 + |\xi|)^{m-|\gamma|} A^{|\delta+\gamma|}\delta!$$

on $K_{\varepsilon,M,R^n\diagdown\{0\}}$ for $|\gamma| \leqslant m$ and $p^{(\gamma)}_{(\delta)}(x,\, \xi) = 0$ for $|\gamma| > m$.

Let us also introduce the following notations:

$$I^1(\alpha) = \alpha^1_2 + \alpha^1_3 + \dots + \alpha^1_k\,,$$

$$I^i(\alpha) = \alpha^i_{i+1} + \dots + \alpha^i_k + \alpha^1_i + \dots + \alpha^{i-1}_i\,, \qquad i = 2,\, \dots,\, k-1,$$

$$I^k(\alpha) = \alpha^1_k + \dots + \alpha^{k-1}_k\,,$$

$$A^j_k(\alpha) = \{\alpha = (\alpha^1_2,\, \alpha^1_3,\, \dots,\, \alpha^{k-1}_k);$$

$$\alpha^i_r \text{ multiindices, } I^i(\alpha) \neq 0,\ i = 1,\, \dots,\, k,\ \sum_{i,r} \alpha^i_r = j\}\ .$$

(Note that I^s depends also on k, but k will be clear from the context).

In the sequel we will think about $\alpha = (\alpha^i_r)$ as part of a matrix in which the lines are indexed by i and the columns by r.

It is clear that proposition 1.2 follows, if we can prove the following

LEMMA 3.1. *There is a constant C, which depends only on n, such that*

$$S_{j,k} = \sum_{\alpha\in A^j_k} \alpha^1_2!|\alpha^1_3 + \alpha^2_3|!\, \dots\, |\alpha^1_{k-1} + \dots + \alpha^{k-2}_{k-1}|!|\alpha^1_k + \dots + \alpha^{k-1}_k|!/[\alpha^1_2!\,\alpha^1_3!\, \dots\, \alpha^{k-1}_k!]$$

satisfies the estimate $S_{j,k} \leqslant C^j j!$.

In the sequel we will call $\alpha^1_2!|\alpha^1_3 + \alpha^2_3|!\, \dots\, |\alpha^1_k + \dots + \alpha^{k-1}_k|!/[\alpha^1_2!\alpha^1_3!\, \dots\, \alpha^{k-1}_k!]$ the weight of α.

It is possible to estimate $S_{j,k}$ starting from the remark that $p(-1;\, x,\, \xi)$, which is the inverse (in the symbol algebra) of $p(x,\, \xi)$, is known to be a formal analytic symbol (cf. also lemma 6.1 below). One may also study the construction of $p(z;\, x,\, \xi)$ from [9], and is then reduced to an estimate from [5].

Here we prefer a proof which is more direct, and which takes into account the combinatorial aspects of the problem.

PROOF OF LEMMA 3.1. The proof will be by doublestep induction, for increasing j, and if j is fixed, for decreasing k.

Let us then suppose that we have found some C such that $S_{r,k} \leqslant C^r r!$ for all $r < j$ and all k, and we want to show that also $S_{j,k} \leqslant C^j j!$, $\forall k$, if C is chosen suitably.

a) It is easy to see that $A_k^j = \emptyset$ if $k > 2j$. In fact, an α_r^i enters in at most two conditions $I^s(\alpha) \neq 0$ and there are k conditions $I^r(\alpha) \neq 0$ which must be satisfied in order to have $\alpha \in A_k^j$.

Further, if $\alpha \in A_{2j}^j$, the same argument shows that $|\alpha_r^i| \leqslant 1$, for all i and r and an easy induction in j (to be effectuated independently from the other inductions), shows that $S_{j,2j} = (2j-1)!! \leqslant 2^j j!$ for $n = 1$. For an arbitrary n this gives $S_{j,2j} \leqslant n^j 2^j j!$.

b) We now want to handle the case $k < 2j$ by lowering j or increasing k. With obvious notations we have

$$S_{j,k} = \sum_{\alpha \in A_k^j,\, |I^k(\alpha)|=1} + \sum_{\alpha \in A_k^j,\, |I^k(\alpha)|>1} = S'_{j,k} + S''_{j,k} \,.$$

c) At first we will estimate $S'_{j,k}$.

Let then $\alpha \in A_k^j$ be given with $|I^k(\alpha)| = 1$. Then there is only one i such that $\alpha_k^i \neq 0$. This element contributes also to $I^i(\alpha) \neq 0$. If $|I^i(\alpha)| > 1$, then we can cancel the column with index k from α and get $\alpha' \in A_{k-1}^{j-1}$. If $|I^i(\alpha)| = 1$, then after cancellation of the last column, $I^i(\alpha') = 0$. Cancellation of all elements which come in in I^i and relabelling will however produce some $\alpha'' \in A_{k-2}^{j-1}$. Since the weights of α' and α, respectively of α'' and α are the same, we obtain

$$S'_{j,k} \leqslant nk(S_{j-1,k-1} + S_{j-1,k-2}) \leqslant 2kn\, C^{j-1}(j-1)!\,,$$

and therefore

$$S'_{j,k} \leqslant 4n\, C^{j-1} j!\,, \tag{1}$$

since $k \leqslant 2j$.

d) It remains to estimate $S''_{j,k}$, and this we will do by reducing ourselves to sums of type $S'_{j,\tilde{k}}$.

For convenience, we will assume $n = 1$, for the moment.

Let then $\alpha \in A_k^i$ be given with $I^k(\alpha) > 1$, and denote $\alpha_k^{i_1}, \alpha_k^{i_2}, \ldots, \alpha_k^{i_r}, 1 \leqslant r \leqslant k$, the nonvanishing elements of the last column. For every fixed s, $1 \leqslant s \leqslant r$, we will construct an element $\gamma(s) \in A_{k+1}$ in the following way:

$$\begin{aligned}
\gamma(s)^i_j &= \alpha^i_j && \text{if } j \leqslant k-1,\\
\gamma(s)^i_k &= \alpha^i_k && \text{if } i \neq s,\ \gamma(s)^s_k = \alpha^s_k - 1,\\
\gamma(s)^i_{k+1} &= 0 && \text{if } i \neq s,\ \gamma(s)^s_{k+1} = 1.
\end{aligned}$$

It is clear from this definition that $\gamma(s) \neq \gamma(s')$ if $s \neq s'$ or if $\alpha \neq \tilde{\alpha}$, $I^k(\alpha) > 1$, $I^k(\tilde{\alpha}) > 1$. Further the sum of the weights of the different $\gamma(s)$, $1 \leqslant s \leqslant r$ for a fixed α, gives the weight of α. This shows that $S''_{j,k} \leqslant S'_{j,k+1}$. Together with (1) this leads to $S_{j,k} \leqslant 8C^{j-1} j!$. The lemma now follows, at least for $n = 1$. It is however clear that the construction of $\gamma(s)$ can be adaptated for $n > 1$, with only notational complications, and the argument then proceeds as before also for $n > 1$.

4. – A technical preparation.

1. Consider $U \subset R^n$ an open domain, $x^0 \in U$, $\Gamma \subset R^n \setminus \{0\}$ an open cone, $\xi^0 \in \Gamma$. To simplify notations, we will suppose that $x^0_n = 0$ and that $\xi^0 = (0, \ldots, 0, 1)$. Let also $K \subset U$ be a compact which contains x_0 in its interior and $b \in S^\mu_A(U, R^n, \Gamma \cup -\Gamma)$.

The main step in the proof of theorem 1.3 is the following technical result:

PROPOSITION 4.1. *Suppose there are $a_j \in S^{\mu-j}_A(U, R^n, \Gamma \cup -\Gamma)$ and constants ε, c, A, M, for which*

a) All functions a_j and b are analytic on $K_{\varepsilon, M, \Gamma \cup -\Gamma}$.

b) $\left|b(z, \zeta) - \sum_{j<k} a_j(z, \zeta)\right| \leqslant cA^k k!(1 + |\zeta|)^{\mu-k}$ for (z, ζ) in $K_{\varepsilon, M, \Gamma \cup -\Gamma}$.

c) For every $(x, \xi') \in K \times R^{n-1}$ the functions $\xi_n \to a_j(x, (\xi', \xi_n))$ can be extended analytically for $|\zeta_n| \geqslant M(1 + |\xi'|)$, $\operatorname{Im} \zeta_n \leqslant 0$, and satisfy the estimate $|a_j(x, (\xi', \zeta_n))| \leqslant cA^j j!(1 + |(\xi', \zeta_n)|)^{\mu-j}$ on that set.

Further choose $f \in C^\infty_0(U)$, $\operatorname{supp} f \subset \{x;\ x_n \geqslant 0\} \cap K$.

Then there are constants γ and C such that for every $g \in S(R^{n-1})$ which satisfies $|\hat{g}(\xi')| \leqslant \exp - \gamma|\xi'|$, it follows that

$$(1)\qquad \left|D^\alpha_x \iint \exp i\langle x - y, \xi\rangle \hat{g}(\xi')\, b(y, \xi) f(y)\, dy\, d\xi\right| \leqslant C(C|\alpha|)^{|\alpha|},$$

if $x \in K$ and $x_n < 0$.

The integral here has the meaning of an oscillatory integral. When $n = 1$, then we replace g with 1 in (1).

PROOF. We denote $G(x) = \iint \exp i\langle x-y, \xi\rangle \hat{g}(\xi') b(y, \xi) f(y)\, dy\, d\xi$. As an oscillatory integral, $D_x^\alpha G(x) = \iint \exp i\langle x-y, \xi\rangle \xi^\alpha \hat{g}(\xi') b(y, \xi) f(y)\, dy\, d\xi$.

To estimate this, we choose M sufficiently great such that $\xi_n \geqslant M(1 + |\xi'|)$ implies $\xi \in \Gamma$, and consider $\varkappa$, $\varkappa'$ in $S^0_{1,0}(R^n, R^n)$, which depend only on ξ and satisfy:

$$\varkappa(\xi) + \varkappa'(\xi) \equiv 1\,, \quad \varkappa(\xi) \geqslant 0\,, \quad \varkappa'(\xi) \geqslant 0\,,$$

$$\varkappa(\xi) = 1 \quad \text{when} \quad |\xi_n| \geqslant 2M(1 + |\xi'|)\,,$$

$$\varkappa'(\xi) = 1 \quad \text{when} \quad |\xi_n| \leqslant 2M(1 + |\xi'|) - M/2\,.$$

With the notation $F(x, y, \xi) = \xi^\alpha f(y) \hat{g}(\xi') \exp i\langle x-y, \xi\rangle$, we now have

$$D_x^\alpha G(x) = \iint F(x, y, \xi) \varkappa'(\xi) b(y, \xi)\, dy\, d\xi + \iint F(x, y, \xi) \varkappa(\xi)\big(b(y, \xi) - \\ - \sum_{<} a_j(y, \xi)\big)\, dy\, d\xi + \iint F(x, y, \xi) \varkappa(\xi) \sum_{j<k} a_j(y, \xi)\, dy\, d\xi\,.$$

We will separately estimate the three integrals from the preceding equality for $k = |\alpha| + |\mu| + n + 1$. (We may suppose that μ is an integer.).

I) $\left|\iint F(x, y, \xi) \varkappa'(\xi) b(y, \xi)\, dy\, d\xi\right| \leqslant C(C|\alpha|)^{|\alpha|}$ for $x \in K$.

In fact the support of $\varkappa'(\xi)$ avoids a conic neighborhood of the points $\pm\, \xi^0$, such that it follows from the estimate of $\hat{g}$ that $(1 + |\xi|)^{|\mu|+n+1} |\xi|^\alpha |\hat{g}(\xi')| \varkappa'(\xi) \leqslant \leqslant C'(C'|\alpha|)^{|\alpha|}$ for some C'.

This gives I) immediately.

II) $\left|\iint F(x, y, \xi) \varkappa(\xi)\big(b(y, \xi) - \sum_{j<k} a_j(y, \xi)\big) dy\, d\xi\right| \leqslant C(C|\alpha|)^{|\alpha|}$ for $x \in K$.

Here we use the information that, on the support of $\varkappa(\xi)$, and for $y \in K$

$$\Big|b(y, \xi) - \sum_{j<k} a_j(y, \xi)\Big| \leqslant cA^k k!(1 + |\xi|)^{\mu-k} \leqslant C'(C'|\alpha|)^{|\alpha|}(1 + |\xi|)^{-|\alpha-n-1}\,.$$

To obtain the second inequality, one uses Stirlings formula.

III) $\left|\iint F(x, y, \xi) \varkappa(\xi) \sum_{j<k} a_j(y, \xi)\, dy\, d\xi\right| \leqslant C(C|\alpha|)^{|\alpha|}$ for $x \in K$, $x_n < 0$.

This estimate is based on the fact that $x_n - y_n < 0$ when $x_n < 0$ and

$y \in \operatorname{supp} f$. If we denote $\Lambda_{\xi'}$ the contour

$$\Lambda_{\xi'} = \big(-\infty, -2M(1+|\xi'|)\big) \cup \{\zeta_n;\ |\zeta_n| = \\ = 2M(1+|\xi'|), \operatorname{Im}\zeta_n < 0\} \cup \big(2M(1+|\xi'|), +\infty\big)$$

$\Lambda_{\xi'}$ being considered with the natural orientation, then it follows from standard contour deformation that

$$\int_{\xi'} \int_{\Lambda_{\xi'}} F(x, y, (\xi', \zeta_n)) \sum_{j<k} a_j(y, (\xi', \zeta_n))\, dy\, d\zeta_n\, d\xi' = 0 .$$

We conclude that

$$\iint F(x, y, \xi)\, \varkappa(\xi) \sum_{j<k} a_j(y, \xi)\, dy\, d\xi = \\ = \int_{\xi'} \int_{-2M(1+|\xi'|)}^{2M(1+|\xi'|)} \int F(x, y, \xi)\, \varkappa(\xi) \sum_{j<k} a_j(y, \xi)\, dy\, d\xi_n\, d\xi' + \\ + \int_{\xi'} \int_{\substack{|\zeta_n| = 2M(1+|\xi'|) \\ \operatorname{Im}\zeta_n < 0}} \int F(x, y, (\xi', \zeta_n)) \sum_{<k} a_j(y, (\xi', \zeta_n))\, dy\, d\zeta_n\, d\xi' .$$

In the last two integrals, the integral in y is over a fixed compact, and for F we have, in view of $|\hat{g}(\xi')| \leqslant \exp - \gamma|\xi'|$ the estimate $|F(x, y, \zeta)| \leqslant$ $\leqslant C'(C'|\alpha|)^{|\alpha|}(1+|\zeta|)^{-\mu-n-1}$ for the (x, y, ζ) which come in here. Since there are k terms in each integral, this gives III), and therefore also the proposition.

2. Remark 4.2. Suppose there is given $a \in S_A^\mu(U, R^n, \Gamma \cup -\Gamma)$ such that $b \sim \sum (i^{|\beta|}/\beta!)(\partial/\partial x)^\beta (\partial/\partial\xi)^\beta a$ in SF_A^μ, $(U, \Gamma \cup -\Gamma)$, and let $a(x, D)f =$ $= (2\pi)^{-n}\int \exp i\langle x, \xi\rangle\, a(x, \xi) \hat{f}(\xi)\, d\xi$ be the analytic pseudodifferential operator associated with a. In view of theorem 2.7, the proposition 4.1 states, in the case of $n = 1$, precisely that $a(x, D)f$ is a real analytic function on $x < 0$, which has an analytic extension to the set $|x| < \delta$ for some $\delta > 0$. We need a similar result also in the case $n > 1$, but now, instead of knowing that $a(x, D)f$ is real analytic for $x_n < 0$ (near x^0), we only get $(x^0, \xi) \notin WF_A \cdot$ $\cdot a(x, D)f$ for $x_n < 0$ and $\xi \in \Gamma \cup -\Gamma$. For this reason, we have first smoothed out analytically in the variables $x' = (x_1, \ldots, x_{n-1})$. In fact, if $g \in \mathcal{S}(R^{n-1})$ satisfies $|\hat{g}(\xi')| \leqslant \exp - \gamma|\xi'|$, then g is real analytic, with estimates which depend only on γ, and $G(x) = g *' \iint \exp i\langle x - y, \xi\rangle\, b(y, \xi) f(y)\, dy\, d\xi$ with $*'$ denoting convolution in x'. The conclusion of proposition 4.1 implies then that G has a real analytic extension to the set $\{x;\ |x - x^0| < \delta\}$.

5. – Proof of the theorems 1.3, 1.4.

1. We will reduce the proof of theorem 1.3 to the following

PROPOSITION 5.1. *Suppose with the notations from theorem* 1.3 (*and* § 2) *that* $U = R^n$, *and let* z, $p(x, D)$ *and* $p(z; x, D)$ *be as in theorem* 1.3. *Suppose further that* $\operatorname{supp} f \subset \left\{x \in R^n;\ x_n \geqslant \sum_{i=1}^{n-1} x_i^2\right\}$ *and that there is* ε *and a real analytic function* h *defined on* $\{x;\ |x| < \varepsilon\}$, *such that* h *coincides with* $p(z; x, D)f$ *for* $|x| < \varepsilon$, $x_n < 0$. *Then it follows that* $0 \notin \operatorname{supp} f$.

In the proof of proposition 5.1 we rely on proposition 4.1, but before we can do so, we need some preparations.

First denote ${}^tp(x, \xi)$ the symbol of the formal adjoint of p. The principal symbol of ${}^tp$ is $p_m(x, -\xi)$, such that ${}^tp$ is also elliptic. Moreover, it follows from the hypothesis on p, that the principal symbol of ${}^tp$ does not take values in R_- for $\xi \in R^n \setminus \{0\}$. Using the ellipticity, we can find for every compact $K \subset R^n$ constants $c > 0$ and $M > 0$ such that ${}^tp(x, \zeta) \neq 0$ if $x \in K$ and $\zeta \in C^n$, $|\zeta| \geqslant M$, $|\zeta'| \leqslant c|\zeta_n|$, where $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$. K will be in the sequel a sphere.

We now consider the restriction of the function ${}^tp^z(x, \xi)$, (which function is again defined with the principal value of the function ln), to the set $\{(x, \xi);\ x \in K,\ \xi \in R^n,\ |\xi'| \leqslant c\xi_n,\ |\xi| \geqslant M\}$. In particular, $\xi_n \geqslant 0$ then. If we denote

$$W = \{\zeta \in R^{n-1} \times C;\ |\zeta| \geqslant M,\ |\zeta'| \leqslant c|\zeta_n|,\ \operatorname{Im} \zeta_n \leqslant 0\}\ ,$$

then we can use analytic continuation along curves in $K \times W$, to define a continuous function $r(z; x, \xi) \colon K \times W \to C$ which has the following three properties *a*), *b*), *c*):

a) $r(z; x, \xi) = {}^tp^z(x, \xi)$ for $x \in K$, $\xi \in R^n$, $|\xi'| \leqslant c\xi_n$, $|\xi| \geqslant M$.

b) For every $(x, \xi') \in K \times R^{n-1}$ the function $\zeta_n \to r(z; x, (\xi', \zeta_n))$ is an analytic function on the set $|\zeta| \geqslant c(1 + |\xi'|)$, $\operatorname{Im} \zeta_n \leqslant 0$, $|\zeta| \geqslant M$.

To show that this is possible, we use the fact that ${}^tp(x, \zeta)$ does not vanish on $K \times W$ and that $K \times W$ is simply connected.

We also claim that, from the hypothesis that mz is not an even integer, it follows that there is $\theta \in C$, $\theta \neq 1$ such that

c) $r(z; x, \xi) = \theta\, {}^tp^z(x, \xi)$ for $\xi \in R^n$, $|\xi'| \leqslant -c\xi_n$, $|\xi| \geqslant M$ and $x \in K$.

In fact, since $r(z; x, \xi)$ and ${}^tp^z$ are continuous functions, it follows that there is θ of form $\theta = \exp ik\pi z$, k some integer, such that $r(z; x, \xi) = \theta\, {}^tp^z(x, \xi)$

for $\xi \in R^n$, $|\xi'| \leqslant -c\xi_n$, $|\xi| \geqslant M$ and $x \in K$. To show that $\theta \neq 1$, it remains to find some (x, ξ), $x \in K$, $\xi \in R^n$, $|\xi'| \leqslant -c\xi_n$, $|\xi| \geqslant M$, such that $r(z; x, \xi) \neq \neq {}^tp^z(x, \xi)$. Here we may fix $x = x^0$, take $\xi' = 0$, and reduce the situation to the case when ${}^tp(x, \xi)$ coincides with its principal part. The latter is possible since

$$ {}^tp(x, \xi) = {}^tp_m\big(1 + ({}^tp - {}^tp_m)\, {}^tp_m\big) \quad \text{and} \quad \big(1 + ({}^tp(x, \zeta) - {}^tp_m(x, \zeta))/{}^tp_m(x, \zeta)\big)^z $$

is an analytic function in $\zeta = (0, \zeta_n)$, if $|\zeta|$ is great and x remains in K. Now ${}^tp_m(x^0, (0, \xi_n)) = d\xi_n^m$, for some constant d, such that our claim $\theta \neq 1$ follows from the fact that $(\xi_n^m)^z$ cannot be extended from $R \setminus \{0\}$ to an analytic function on the lower half plane, if mz is not an even integer.

From c) it follows in particular that $r(z; x, \xi) \in S_A^{m\,\mathrm{Re}\,z}(V, \Gamma')$ where V is the interior of K and Γ' is some open cone in $R^n \setminus \{0\}$ which contains $-\xi^0$.

Further, denote $q(z; x, \xi)$ an analytic symbol in $S_A^0(V, R^n \setminus \{0\})$ such that $q \sim \sum \big(z(z-1) \dots (z-k+1)/k!\big)\, {}^tp^{-k}({}^tp)_{j,k}$ in $SF_A^0\big(V, R^n \setminus \{0\}\big)$. Here $({}^tp)_{j,k}$ is the expression which one obtains when one computes formula (1), § 3, for ${}^tp$ instead of p. Now we choose $\chi(\xi) \in S_{1,0}^0(R^n \times R^n)$ a symbol which is a function which depends only on ξ, which has support in some suitable conic neighborhood of $\{\xi^0\} \cup \{-\xi^0\}$, and which is identically one if ξ lies in another conic neighborhood of $\{\xi^0\} \cup \{-\xi^0\}$ and is great. We may choose χ with these properties such that

$$ r(x; x, \xi)\, q(z; x, \xi)\, \chi(\xi) \in S_A^{m\,\mathrm{Re}\,z}(V, R^n, \Gamma \cup -\Gamma) $$

for some conic neighborhood Γ of ξ^0. Also $r(z; x, \xi) q(z; x, \xi) \sim ({}^tp)(z; x, \xi)$ in $SF_A^{m\,\mathrm{Re}\,z}(V, \Gamma)$ if Γ is small enough. Here $({}^tp)(z; x, \xi)$ is the symbol associated in proposition 1.1 with ${}^tp$. It follows from the unicity in proposition 1.1 that

$$ (1) \qquad ({}^tp)(z; x, \xi) \sim {}^t\big(p(z; x, \xi)\big) \quad \text{in } SF_A^{m\,\mathrm{Re}\,z}(V, \Gamma) $$

where

$$ {}^t\big(p(z; x, \xi)\big) = \sum i^{-|\beta|}/\beta!(\partial/\partial\xi)^\beta(\partial/\partial x)^\beta p(z; x, -\xi)\,. $$

(Another more significant explanation of (1) follows from lemma 6.1 below).

Let us now denote

$$ p_0(z; x, D) f(x) = (2\pi)^{-n} \iint \exp i\langle x - y, \xi\rangle\, r(z; y, \xi)\, q(z; y, \xi)\, \chi(\xi) f(y)\, dy\, d\xi\,. $$

2. Proof of proposition 5.1. We choose in the preceding discussion for K the compact $|x| \leqslant 2$. It is clear that the conditions of proposition 4.1

are satisfied for $b(x, \xi) = r(z; x, \xi)\, q(z; x, \xi)\, \chi(\xi)$. It follows that there are constants γ, C, δ such that

$$|D^\alpha g *' p_0(z; x, D) f(x)| \leqslant C\big(C|\alpha|\big)^{|\alpha|} \quad \text{for } x_n < 0,\ |x| < \delta$$

for any $g \in \mathcal{S}(R^{n-1})$ with $|\hat{g}(\xi')| \leqslant \exp - \gamma|\xi'|$.

If h is the function from the statement of the proposition, then it is clear from $WF_A p(z; x, D) f \subset WF_A f \subset \operatorname{supp} f$, that $p(z; x, D) f - h$ vanishes if $|x| < \varepsilon$ and $x_n < \sum_{i \leqslant n-1} x_i^2$. Moreover, it is clear that h must be defined, as a real analytic function for $x_n < \sum_{i \leqslant n-1} x_i^2$ and we must have $p(z; x, D) f - h = 0$, if $|x_n| < \varepsilon$ and $x_n < \sum_{i \leqslant n-1} x_i^2$. Therefore $g *' \big(p(z; x, D) f - h\big)$ makes sense for $|x_n| < \varepsilon$ and vanishes for $-\varepsilon < x_n \leqslant 0$.

It follows trivially that

$$(2) \qquad |D^\alpha g *' \big(p_0(z; x, D) f - p(z; x, D) f + h\big)| \leqslant C\big(C|\alpha|\big)^{|\alpha|} \quad \text{for } |x| < \varepsilon',\ x_n < 0\,.$$

It is also easy to verify that

$$(3) \qquad (x, \xi^0) \notin WF_A\big(p_0(z; x, D) f - p(z; x, D) f\big)\,, \qquad \forall x \in K.$$

In fact, since we are here interested only in the wave front set of $p(z; x, D) f$, and not directly in $p(z; x, D) f$, we may use (cf. § 2) for $p(z; x, D) f$ the representation

$$p(z; x, D) f = (2\pi)^{-n} \iint \exp i\langle x - y, \xi\rangle\, {}^t p^z(y, \xi)\, q(z; y, \xi)\, w(\xi) f(y)\, dy\, d\xi\,,$$

where $w(\xi) \in C^\infty(R^n)$ is identically one for $|\xi|$ great, and vanishes there where ${}^t p^z q$ is not C^∞. In this case $p(z; x, D) f$, as well as $p_0(z; x, D) f$ are in $\mathcal{S}'(R^n)$, and

$$\mathcal{F}\big(p_0(z; x, D) f - p(z; x, D) f\big) = \\ = \int \exp - i\langle y, \xi\rangle \big(r(z; y, \xi)\, q(z; y, \xi)\, \chi(\xi) - {}^t p^z(y, \xi)\, q(z; y, \xi)\, w(\xi)\big) f(y)\, dy$$

vanishes in a conic neighborhood of ξ^0 for large $|\xi|$.

We want to show that (2) and (3) lead to a contradiction if we assume that $0 \in \operatorname{supp} f$. To do so, we apply the following result

THEOREM 5.2 (cf., e.g., [6]). *Consider* $v \in \mathcal{D}'(R^n)$, *suppose that* $(0, \xi^0) \notin WF_A v$ *and that* $\operatorname{supp} v \subset \{x; x_n \geqslant 0\}$. *Then it follows that* $0 \notin \operatorname{supp} v$.

To use this theorem, we introduce a function $u(=u_g)$ such that

$$|D_\alpha u(x)| \leqslant C(C|\alpha|)^{|\alpha|} \quad \text{for } |x| < \varepsilon', \tag{4}$$

and such that

$$u(x) = g *' \big(p_0(z; x, D)f - p(z; x, D)f + h\big) \quad \text{for } x_n < 0,\ |x| < \varepsilon' . \tag{5}$$

Clearly then $v = u - g *' \big(p_0(z; x, D)f - p(z; x, D)f + h\big)$ vanishes for $x_n \leqslant 0$ and also satisfies $(0, \xi^0) \notin WF_A v$, such that v vanishes in a neighborhood of zero, in view of theorem 5.2.

We obtain from (4) that

$$|D^\alpha g *' \big(p_0(z; x, D)f - p(z; x, D)f + h\big)| \leqslant C(C|\alpha|)^{|\alpha|} \tag{6}$$

for $|x| < \varepsilon'$ and for any $g \in \mathcal{S}(R^{n-1})$ with $|g(\xi')| \leqslant \exp - \gamma|\xi'|$.

We claim that (6) implies

$$(0, -\xi^0) \notin WF_A\big(p_0(z; x, D)f - p(z; x, D)f\big) . \tag{7}$$

Supposing for the moment that (7) is proved, it is easy to conclude the proof of proposition 5.1. In fact, in view of c) from above, we can apply the regularity theorem of M. Sato-L. Hörmander from § 2 for $p_0 - p$, and obtain that $(0, -\xi^0) \notin WF_A f$ (the inverse for $p_0 - p$ near $(0, -\xi^0)$ is $p(-z; x, \xi)/(\theta - 1)\big)$. Using again theorem 5.2, it follows that f must vanish near zero. This completes the proof of proposition 5.1, modulo the proof of of the fact that (6) implies (7).

3. Proof that (6) implies (7). Consider $g \in \mathcal{S}(R^{n-1})$ with $g(0) \neq 0$ and $|\hat{g}(\xi')| \leqslant \exp - \gamma|\xi'|$. The function $g_{\eta'} = g(x') \exp i\langle x', \eta'\rangle$, $\eta' \in R^{n-1}$ is also in $\mathcal{S}(R^{n-1})$ and satisfies $|\hat{g}_{\eta'}(\xi')| \leqslant \exp - \gamma|\xi'| + \gamma|\eta'|$. We denote again with $g_{\eta'}$ the function $x \to g_{\eta'}(x_1, \ldots, x_{n-1})$ defined on R^n. We want to show that $(0, -\xi^0) \notin WF_A g(x')\big(p_0(z; x, D)f - p(z; x, D)f\big)$, using condition (6) for the functions $g_{\eta'}$. In fact now we have

$$|D^\alpha g_{\eta'} *' \big(p_0(z; x, D)f - p(z; x, D)f + h| \leqslant C(C|\alpha|)^{|\alpha|} \exp \gamma|\eta'| , \qquad |x| < \varepsilon',$$

which gives in particular

$$|(d/dx_n)^j \int \exp i\langle x', \eta'\rangle g(x')\big(p_0(z; x, D)f - p(z; x, D)f + h\big)\, dx'| \leqslant$$
$$\leqslant C(Cj)^j \exp \gamma|\eta'| \qquad \text{for } -\varepsilon' \leqslant x_n \leqslant \varepsilon'.$$

Therefore, if $\varphi \in C_0^\infty(-\varepsilon', \varepsilon')$ satisfies the estimate

$$|(d/dx_n)^k \varphi(x_n)| \leqslant C'(C' j_0)^k \quad \text{for } k \leqslant j_0,$$

then

$$\left|\mathcal{F}\Big(\varphi(x_n)\, g(x')\big(p_0(z; x, D) f - p(z; x, D) f + h\big)\Big)(\eta', \eta_n)\right| \leqslant C''(C'' j_0/|\eta_n|)^j \exp \gamma |\eta'| .$$

The last expression is majorized by $C_1(C_1 j_0/|\eta|)^j$ if $|\eta'| < d|\eta_n|$ for some suitable d. Using L. Hörmander's definition for WF_A, we have now proved (7).

4. Proof of theorem 1.3. *a*) We start with the proof of the second statement from the theorem.

This is a quite easy consequence of proposition 5.1. First we write $f = f_1 + f_2$ with $f_1, f_2 \in C_0^\infty(U)$, both concentrated in $x_n \geqslant x_n^0$, such that f_1 is concentrated near x^0 and such that the support of f_2 avoids x^0. In view of the pseudolocality of $p(z; x, D)$ we obtain that $p(z; x, D) f_2$ is real analytic near x_0, such that the hypothesis of f is also satisfied for f_1. Now we choose a small neighborhood V of x^0 for which there is a real analytic diffeomorphism χ which maps V onto R^n, $x_n \geqslant x_n^0$ in $x_n \geqslant \sum_{i \leqslant n-1} x_i^2$ and x^0 to 0. $\Big($Note that $(x_1, \ldots, x_n) \to \Big(x_1, \ldots, x_{n-1}, x_n + \sum_{i \leqslant n-1} x_i^2\Big)$ is a diffeomorphism from R^n to R^n which maps $x_n \geqslant 0$ on $x_n \geqslant \sum_{i \leqslant n-1} x_i^2\Big)$. We may suppose that $\operatorname{supp} f_1 \subset V$ and can therefore use χ to reduce ourselves to the case $U = R^n$. In view of the unicity in proposition 1.1, the operator which corresponds to $p(z; x, D)$ (restricted to $C_0^\infty(V)$), under this diffeomorphism is of type $p(z; x, D)$ in the new coordinates. Therefore we are in the conditions from proposition 5.1. Application of this proposition gives the second part of theorem 1.3.

b) The proof of the fact that $p(z; x, D)$ is antilocal is now immediate. In fact, if W is a connected component of $U \setminus \operatorname{supp} f$, then $p(z; x, D) f$ is real analytic on W, and it suffices to find some point $x^0 \in \partial \operatorname{supp} f \cap \overline{W}$, such that the restriction h of $p(z; x, D) f$ to W cannot be extended to a real analytic function near $x^0 \cdot h$ must then be $\not\equiv 0$, and since it is real analytic, it follows that $\operatorname{supp} f \supset W$.

To find such a point x^0, consider $\bar{x} \in W$ and let S be a sphere with center in $\bar{x}$ which is contained in W. If we perform an inversion with respect to S with center in $\bar{x}$, we arrive at the situation considered in the second part of the theorem, and therefore our claim about h follows for a suitable x^0 from the part of the theorem proved already. This concludes the proof of theorem 1.3, and it is also clear that theorem 1.4 follows from arguments similar to those used in part *b*) of the above proof.

6. – Proof of theorem 1.5.

1. In the proof, we use ideas from [14]. In fact,

$$p^z(x, D)f = (i/2\pi)\int_\Lambda \lambda^z(\bar{p}-\lambda)^{-1} f\,d\lambda \tag{1}$$

with notations and conventions from § 1.3, and we will try to approximate $(\bar{p}-\lambda)^{-1}$ suitably with analytic pseudodifferential operators on relatively compact subsets of coordinate neighborhoods.

LEMMA 6.1. *Consider $p(x, D)$ an elliptic operator as in proposition 1.2, let z be with* $\operatorname{Re} z < -1$, *and let V be an open, relatively compact subset in U. For some c, we denote with Λ' the contour*

$$\Lambda' = (-\infty, -c) \cup \{\lambda \in C;\ |\lambda| = c\} \cup (-c, -\infty)$$

(*natural orientation*). *Then we have, if c is small enough*

$$p(z; x, \xi) = (i/2\pi)\int_{\Lambda'} \lambda^z s(\lambda; x, \xi)\,d\lambda$$

where $s(\lambda; x, \xi) \in SF_A^{-m}(U, R^n\setminus\{0\})$ satisfies $s(\lambda; x, \xi)\circ(p(x, \xi)-\lambda) \sim 1$ in $SF_A^0(U, R^n\setminus\{0\})$.

PROOF. To obtain the expression of $s(\lambda; x, \xi)$, we may just compute with the aid of proposition 1.1, the -1-th power in the symbol algebra, of $p(x, \xi)-\lambda$. What we get is, in view of $p_{(\beta)}^{(\alpha)}(x, \xi) = (p(x, \xi)-\lambda)_{(\beta)}^{(\alpha)}$ if $|\alpha+\beta| > 0$

$$s(\lambda; x, \xi) = 1/(p(x, \xi)-\lambda) + \sum_{j=1}^{\infty}\sum_{k=2}^{2j}(-1)^k p_{j,k}(x, \xi)(p(x, \xi)-\lambda)^{-k-1}\,.$$

For suitable c, c', M, ε, we now have that $|p(x, \xi)-\lambda| \geqslant c'(1+|\lambda|+|\xi|^m)$ on $K_{\varepsilon, M, R^n\setminus\{0\}}$ if $\lambda \in \Lambda'$ and the map $\lambda \to s(\lambda; x, \xi)$ which is defined on Λ' and takes values in $SF_A^{-m}(V, R^n\setminus\{0\})$ is continuous, if we endow the latter space with an obvious topology (for $|\xi| > M$). It follows easily that the integral in the statement makes sense, and it remains to take into account that (cf. [4])

$$(i/2\pi)\int_{\Lambda'} \lambda^z/(p(x, \xi)-\lambda)^{k+1}\,d\lambda = z(z-1)\ldots(z-k+1)p^{z-k}(x, \xi)/k!\,.$$

We now apply lemma 6.1 for the situation which appears in theorem 1.5, with $U = R^n$ and for the operator $\tilde{p}(x, D)\colon C_0^\infty(R^n) \to C_0^\infty(R^n)$ defined by $\tilde{p}(x, D)v = (p(x, D)v\circ\chi)\circ\chi^{-1}$ (p is the operator from the statement of theorem 1.5). For V we take an arbitrary relatively compact open set in R^n which contains $\operatorname{supp} f\circ\chi^{-1}$. We can then choose for every $\lambda \in \Lambda'$ some

$$\tilde{s}(\lambda; x, \xi) \in S_\Lambda^{-m}(V, R^n\setminus\{0\}) , \qquad \tilde{s}(\lambda; x, \xi) \sim s(\lambda; x, \xi) \qquad \text{in } SF_\Lambda^{-m}(V, R^n\setminus\{0\})$$

(in order to be able to apply proposition 2.2 directly, and not only its proof, we apply lemma 6.1 for some V' in which V is relatively compact). Moreover (this time however, we must look at the proof of proposition 2.2 given in [2]), it is easy to see that we may choose $\tilde{s}(\lambda; x, \xi)$ in such a way that the function $\lambda \to \tilde{s}(\lambda; x, \xi)$ is continuous from Λ' to $S_\Lambda^{-m}(V, R^n\setminus\{0\})$, if we endow $S_\Lambda^{-m}(V, R^n\setminus\{0\})$ with some natural topology (for $|\xi|$ great). Let us also choose $h(\xi) \in C^\infty(R^n)$ a function which is identically one if $|\xi|$ is sufficiently great, but which vanishes in such a neighborhood of the origin that $h(\xi)\,\tilde{s}(\lambda; x, \xi) \in S_\Lambda^{-m}(V, R^n, R^n\setminus\{0\})$ with uniform estimates and constants for $\lambda \in \Lambda'$ (by this we mean that if $K \subset V$ is a compact, then there are $c > 0$, $\varepsilon > 0$, $M > 0$, which do not depend on $\lambda \in \Lambda'$ such that all functions $\tilde{s}(\lambda; x, \xi)$ can be extended to analytic functions on $K_{\varepsilon, M, R^n\setminus\{0\}}$ and satisfy the estimate

$$|\tilde{s}(\lambda; x, \xi)| \leqslant c(1 + |\xi|)^{-m}$$

there. If we prefer, we may also assume that c, ε and M do not depend on K).

Let us now define for $\operatorname{Re} z < -1$ and $x \in V$

$$(2) \qquad q(z; x, D)f(x) = (i/2\pi)(2\pi)^{-n}\int_{\Lambda'}\int_{R^n} \exp i\langle x, \xi\rangle\, \lambda^z \tilde{s}(\lambda; x, \xi)\, h(\xi)\hat{f}(\xi)\, d\xi\, d\lambda$$

and

$$(3) \qquad \tilde{s}(\lambda; x, D)\, v(x) = (2\pi)^{-n}\int \exp i\langle x, \xi\rangle\, \tilde{s}(\lambda; x, \xi)\, h(\xi)\, \hat{v}(\xi)\, d\xi$$

such that

$$q(z; x, D)f(x) = (i/2\pi)\int_{\Lambda'} \tilde{s}(\lambda; x, D) f\, d\lambda .$$

Further it follows from lemma 6.1 that

$$p(z; x, \xi) \sim (i/2\pi)\int_{\Lambda'} \lambda^z \tilde{s}(\lambda; x, \xi)\, d\lambda \qquad \text{in } SF_\Lambda^{m \operatorname{Re} z}(V, R^n\setminus\{0\})$$

such that $q(z; x, D)$ is an analytic pseudodifferential operator associated with $p(z; x, \xi)$ on V. It follows that $p(z; x, D)v - q(z; x, D)v$ is real analytic on V for any $v \in C_0^\infty(V)$.

It is now clear that the proof of theorem 1.5 comes, for $\operatorname{Re} z < -1$, to an end with the following lemma:

LEMMA 6.2. *$(p^z(x, D)f)\circ\chi^{-1} - q(z; x, D)(f\circ\chi^{-1})$ is real analytic on V.*

PROOF. Let W be an open relatively compact subset in V and choose $\psi \in C_0^\infty(U')$ to be identically one on $\chi^{-1}(W)$. Since W is arbitrary, it suffices to show that $(p^z(x, D)f)\circ\chi^{-1} - q(z; x, D)(f\circ\chi^{-1})$ is real analytic for $x \in W$.

Now we decompose f in the form

$$f = (p - \lambda)(\bar{p} - \lambda)^{-1} f =$$

$$= (p - \lambda)\psi(\bar{p} - \lambda)^{-1} f + (p - \lambda)(1 - \psi)(\bar{p} - \lambda)^{-1} f = (p - \lambda)\psi g + u$$

with $g = (\bar{p} - \lambda)^{-1} f$, $u = (p - \lambda)(1 - \psi) g$. It follows that

$$(p^z(x, D)f)\circ\chi^{-1} - p(z; x, D)(f\circ\chi^{-1}) =$$

$$= (i/2\pi)\Big[\int_{\Lambda}(\lambda^z g\, d\lambda)\circ\chi^{-1} - \int_{\Lambda'}\lambda^z \tilde{s}(\lambda; x, D) f\circ\chi^{-1}\, d\lambda\Big] =$$

$$= (i/2\pi)\Big[\int_{\Lambda}(\lambda^z(1 - \psi) g\, d\lambda)\circ\chi^{-1} + \int_{\Lambda'}(\lambda^z \psi g\, d\lambda)\circ\chi^{-1} -$$

$$- \int_{\Lambda'}\lambda^z \tilde{s}(\lambda; x, D)(p - \lambda)\psi g\circ\chi^{-1}\, d\lambda - \int_{\Lambda'}\lambda^z \tilde{s}(\lambda; x, D) u\circ\chi^{-1}\, d\lambda\Big].$$

When $x \in W$, the first integral in the last expression vanishes and the last integral is a real analytic functon on W. In fact, $u\circ\chi^{-1}$ vanishes on W and is C_0^∞, such that we can apply proposition 2.5 and conclude that $\tilde{s}(\lambda; x, D) u\circ\chi^{-1}$ is real analytic on W, with estimates for the real analyticity which are uniform for $\lambda \in \Lambda'$ (in an obvious sense). To see this it suffices to note that the norm in the Sobolev space $\mathcal{H}^{-m}$ of $u\circ\chi^{-1}$ has an estimate $\|u\|_{(-m)} \leqslant C'$ with C' independent of λ (since $\|(\bar{p} - \lambda)^{-1}\| < C/|\lambda|$), and that in theorem 2.5 we can replace the L_2 norm with the norm in any given Sobolev space (essentially this amounts to applying proposition 2.5 to the operator $p(x, D)\Delta^k$, instead of $p(x, D)$, where Δ is the Laplacian and k is an integer which depends on what Sobolev space we are working with.). We can now integrate over Λ' and conclude that $\int \lambda^z \tilde{s}(\lambda; x, D) u\circ\chi^{-1}\, d\lambda$ is real analytic on W.

To conclude the proof of the lemma, it remains to show that

$$\Big(\int_{\Lambda'}\lambda^z\psi g\,d\lambda\Big)\circ\chi^{-1}-\int_{\Lambda'}\lambda^z\tilde{s}(\lambda;x,D)(p-\lambda)(\psi g)\circ\chi^{-1}d\lambda=$$
$$=\int_{\Lambda'}\lambda^z[I-\tilde{s}(\lambda;x,D)(\tilde{p}(x,D)-\lambda)](\psi g)\circ\chi^{-1}d\lambda$$

is real analytic on W. Here $I-\tilde{s}(\lambda;x,D)(\tilde{p}(x,D)-\lambda)$ is an analytic pseudo-differential operator, and its symbol is easily seen to be ~ 0 in $SF^0_A(V,R^n\setminus\{0\})$, uniformly in λ. Therefore $I-\tilde{s}(\lambda;x,D)(p(x,D)-\lambda)$ must be an integral operator with analytic kernel. Inspection of the proofs which give the last result, also shows that the estimates for the real-analyticity of $\big(I-\tilde{s}(\lambda;x,D)\cdot\cdot(p(x,D)-\lambda)(\psi g)\circ\chi^{-1}\big)$only depend on the L_2 norm (say) of $(\psi g)\circ\chi^{-1}$. Since this norm can be estimated by $C'/|\lambda|$ (in view of $\|(\overline{p}-\lambda)^{-1}\|\leqslant C/|\lambda|$), it now follows that $\int\lambda^z\big(I-\tilde{s}(\lambda;x,D)(p(x,D)-\lambda)\big)(\psi g)\circ\chi^{-1}d\lambda$ is real analytic.

We have now proved theorem 1.5 in the case when $\operatorname{Re} z<-1$. The case of a general z can be reduced to this one in view of (3), § 1.

7. – Comments.

1. A functional analytic approach to complex powers of elliptic operators, parallel to the approaches from [3], [14], has also been considered in [15].

2. One may compute the number of elements from the sets A^j_k in § 3 explicitely, using results from [11]. The author is indepted to Laura Liess, who drew his attention to these results, and also helped him to exploit them on a computer.

3. Starting point for this paper, was the following, unpublished, result of the author: consider $\sigma(\xi)\colon R\to C$ a function which is piecewise algebraic, but which is not a polynomial. Let also $\sigma(D)f$ be the operator $\sigma(D)f(x)=$ $=\int\exp ix\xi\sigma(\xi)\hat{f}(\xi)\,d\xi$, $\sigma(D)\colon C_0^\infty(R)\to C^\infty(R)$. Then $\operatorname{supp}\sigma(D)f$ contains, for $f\neq 0$, all bounded, and at least one unbounded component of $R\setminus\operatorname{supp} f$.

4. It is clear from the proof of theorem 1.3 that the antilocality which we have studied in this paper is a local phenomenon. In the result which we have mentioned in nr. 3 from before, there do also appear weaker forms of antilocality, which are not of local nature. These weaker forms of antilocality lead to results of antilocality also for some cases of complex powers of elliptic operators, where the results of this paper do not apply (i.e. when mz is not an even integer).

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