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Irregular Obstacles and Quasi-Variational Inequalities of Stochastic Impulse Control.

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0. – Introduction.

In this paper we study variational inequalities (v.i.) for second order nonlinear elliptic operators and *irregular obstacles*, that is, obstacles which are not continuous functions and do not belong to Sobolev spaces.

Our study has two main motivations. The first one is very classical and comes from *potential theory*. It is well known that capacitary potentials associated with elliptic operators are weak solutions of unilateral Dirichlet problems involving obstacles of the form $\psi = \chi_E - \infty \cdot \chi_{\boldsymbol{R}^n - E}$, where E is some subset of $\boldsymbol{R}^n$ and χ_S denotes the characteristic function of S, if $S \subset \boldsymbol{R}^n$. That was indeed the starting point of the whole theory of variational inequalities, as developed in the sixties by G. Stampacchia [30], J. L. Lions and G. Stampacchia [24] and H. Lewy and G. Stampacchia [22], [23]. These obstacles are discontinuous accross the boundary Γ of E and the regularity of the corresponding potentials depends on the (local) behaviour of Γ, as precised by the classical conditions for regular points.

The second motivation to irregular obstacles is of more recent origin and arises in *stochastic control theory*. The problem is that of the optimal impulse and continuous control of a system that evolves in time according to a stochastic Ito differential equation in $\boldsymbol{R}^n$. As shown by A. Bensoussan and J. L. Lions [1], [2], here a dynamic programming approach leads to characterize the Hamilton-Jacobi function of the problem as a solution of a quasi-variational inequality (q.v.i.) of obstacle type, that still involves a second order (nonlinear) elliptic or parabolic differential operator. The implicit obstacle $\psi = M(u)$ of such a q.v.i. depends on the (weak) solution u *via* a map M that behaves irregularly on Sobolev spaces. Here again we are thus confronted with v.i. whose obstacles are not, « a priori », continuous functions and do not have finite energy.

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Despite the irregular behaviour of the obstacle in both problems above, however, we still expect the solutions to share a certain degree of smoothness, under fairly general assumptions on the remaining data.

We are thus led to investigate the weakest requirements to be demanded to an obstacle ψ in order to ensure the *continuity* or *Hölder continuity* of the weak solutions of the corresponding variational inequalities. It is of great significance, even if not entirely surprising, that both for potentials and Hamilton-Jacobi functions of stochastic control the regularity we are looking for can be obtained under the same mild regularity assumptions on ψ. These can be indeed formulated in a unified manner, as suitable *unilateral regularity conditions of Wiener type*, as those described in Section 3 below.

To the proof of such continuity results are devoted the first four sections of the paper, where we consider a v.i. of the following type:

$$\begin{cases} u \in H_0^1(\mathcal{O}), \quad u \leq \psi \\ \sum_{i=1}^{n} \big(a_i(x, u, \nabla u), D_i u - D_i v\big) + \big(a_0(x, u, \nabla u), u - v\big) \leq 0 \\ \text{for all } v \in H_0^1(\mathcal{O}),\ v \leq \psi,\ v - u \in L^\infty(\mathcal{O}) . \end{cases} \tag{0.1}$$

Here $\mathcal{O}$ is a bounded open subset of $\boldsymbol{R}^n$ and

$$A(u) = -\sum_{i=1}^{n} D_i a_i(x, u, \nabla u) \tag{0.2}$$

is an elliptic operator in $\mathcal{O}$, the functions $a_i\colon \mathcal{O} \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$, $i = 1, \ldots, n$, being measurable functions that satisfy natural growth and coerciveness conditions. The lower order term is assumed to have *quadratic growth* in ∇u, i.e.,

$$|a_0(x, r, p)| \leq c + c|p|^2 \tag{0.3}$$

on every region $\mathcal{O} \times \{|r| \leq C\} \times \boldsymbol{R}^n$, as is natural in optimal control problems. Here and in the following we denote by $H_0^1(\mathcal{O})$, $H^1(\mathcal{O})$, $W^{2,q}(\mathcal{O})$ for $q \geq 2$, the usual Sobolev spaces in $\mathcal{O}$ and we write $(w, z) = \int wz\, dx$ for arbitrary $w \in L^q(\mathcal{O})$, $z \in L^{q'}(\mathcal{O})$, $q \geq 1$, $q' = q/(q-1)$, where $\int$ denotes integration over $\mathcal{O}$.

The obstacle ψ in (0.1) is taken to be a measurable function bounded from below in $\mathcal{O}$ and satisfying unilateral Wiener conditions as (3.1) or (3.6) of Section 3.

We then prove the continuity and Hölder continuity of every bounded solution of (0.1), first in the interior of $\mathcal{O}$ (Theorems 3.1 and 3.2) and then, under some additional regularity assumptions on the boundary data, up to the boundary Γ of $\mathcal{O}$ (Theorems 4.1 and 4.2). The proofs are based on suitable *integral estimates* of Morrey type, as in [8], [9], leading to inequalities such as

$$\int_R |\nabla u|^2 |x-z|^{2-n}\,dx \le c \int_{mR}^{*} |\nabla u|^2 |x-z|^{2-n}\,dx + cR^{\beta}, \tag{0.4}$$

for given $z \in \mathcal{O}$ and all $R \in]0, R_0]$, with $c > 0$, $\beta \in]0, 1[$ and $m \geqslant 2$ suitable constants. Here $\int_R$ denotes the integration over $B_R(z) = \{x \in \boldsymbol{R}^n \,|\, |x-z| < R\}$ and $\int_{mR}^{*}$ the one over $B_{mR}(z) - B_R(z)$. From estimates of this kind the so called « hole filling » technique from Widman [33] and Hildebrandt-Widman [17] yields, for instance, the following local behaviour of ∇u

$$\int_R |\nabla u|^2 |x-z|^{2-n}\,dx \le cR^{2\alpha}, \qquad R < R_0, \qquad \alpha \in]0, 1[, \tag{0.5}$$

from which the Hölder continuity of u with exponent α follows by a classical lemma of Morrey.

As a further step in our study of regularity, we then look in Section 5 for conditions ensuring that a solution belongs to the Sobolev spaces $W^{2,q}(\mathcal{O})$ for all $q \ge 2$ and has therefore Hölder continuous first order derivatives in $\overline{\mathcal{O}}$.

Our approach here is that of reducing the regularity problem for v.i. to the analogous problem for *equations*, for which classical results are available. This is achieved by relying on *dual estimates* of the form

$$0 \ge A(u) + a_0(x, u, \nabla u) \ge 0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi)] \tag{0.6}$$

for a solution u of (0.1). The inequalities above and the infimum $\wedge$ have to be intended in the sense of *measures* in $\mathcal{O}$. Such an estimate is also of potential theoretic nature, the solution u being connected with the notion of *réduite*, and was first obtained by Lewy and Stampacchia [23], in the framework of a classical Perron approach to the unilateral Dirichlet problem. The interest of establishing an estimate such as (0.6) for *weak* solutions of v.i., as done in [26] for a linear operator, is that it clearly implies that u satisfies an equation like

$$A(u) + a_0(x, u, \nabla u) = g \in L^q(\mathcal{O}) \tag{0.7}$$

in the distribution sense in $\mathcal{O}$, provided a function $g' \in L^q$ exists which is a lower bound of $0 \wedge [A(\psi) + a_0(x, \psi, \nabla x)]$ in $\mathcal{O}$, in the sense of measures (notice that again a one-sided condition on ψ comes in). From (0.7) the $W^{2,q}$ regularity of u can then be obtained by relying, for instance, on Ladyzenskaya-Uralt'zeva [21] or Tomi [32].

Our proof of (0.6) is based, as in Hanouzet-Joly [14] and [27], on the natural lattice structure of $H^1(\mathcal{O})$ as a Dirichlet space and the corresponding lattice properties of the operator A.

In the same framework we also prove another estimate from potential theory, namely

$$A(\wedge v_\alpha) \geq \wedge A(v_\alpha) \tag{0.8}$$

again in the sense of measures in $\mathcal{O}$, where $\{v_\alpha\}$ is a family of functions in $H^1(\mathcal{O})$ and the infimum $\wedge$ is extended over all indexes α. This estimate is used in Section 7 in our application to control problems.

Let us point out that our study of regularity is done by assuming that a *bounded* solution of (0.1) actually *exists*. Sufficient conditions for boundedness have been given in [8] for general v.i. as (0.1). Existence results for v.i. and q.v.i. connected with stochastic control problems are the subject of a joint work with A. Bensoussan [4].

The remainder of the paper, Sections 6, 7 and 8, is devoted to the study of Bensoussan-Lions q.v.i. of stochastic impulse control. This q.v.i. can be formulated in the stationary case as follows

$$\begin{cases} u \in H_0^1(\mathcal{O}), \quad u \leq M(u) \quad \text{a.e. in } \mathcal{O} \\ \sum\limits_{i=1}^{n} (a_{i,k}(x) D_k u, D_i u - D_i v) \leq (H(x, u, \nabla u), u - v) \\ \text{for every } v \in H_0^1(\mathcal{O}), \ v \leq M(u), \ v - u \in L^\infty(\mathcal{O}) . \end{cases} \tag{0.9}$$

Here

$$A(u) = - \sum_{i,k=1,n}^{n} D_i(a_{ik} D_k u) \tag{0.10}$$

is a linear elliptic operator and $H(x, r, p)$, $x \in \mathcal{O}$, $r \in \boldsymbol{R}$, $p \in \boldsymbol{R}^n$, is a *Hamiltonian function* of the form

$$H(x, r, p) = \inf \{f(x, r, d) + p \cdot g(x, r, d) | d \in U\} . \tag{0.11}$$

Here $U \subset \boldsymbol{R}^m$, $m \geq 1$, is a given set of *admissible controls*, that we do not require to be bounded. Therefore, it is natural to allow the function H to grow at infinity with $|p|$, and we assume that the growth of H is *quadratic* in $|p|$ on every region of the form $\mathcal{O} \times \{|r| \leq C\} \times \boldsymbol{R}^n$.

The *implicit obstacle*

$$\psi := M(u) \tag{0.12}$$

in (0.9) is given by

$$M(u) = k + \operatorname{ess\,inf}\{c(x, \xi) + u(x + \xi) | \xi \geq 0, x + \xi \in \mathcal{O}\}, \quad x \in \overline{\mathcal{O}}, \tag{0.13}$$

where $c(x, \xi)$ is a given function that represents the *variable cost* of carrying the system under control from the state x to the state $x + \xi$, and the constant $k \geq 0$ is the *fixed cost* associated with any stopping of the system.

We assume (Section 6) that a solution u of (0.9) exists and we prove that it is bounded. We then consider (0.9) as a v.i. of the form (0.1), with the obstacle ψ given by (0.12), and u as a bounded solution of it. This allows us to apply all the results of the previous sections by simply verifying that the assumptions made on ψ are now actually satisfied.

Under suitable assumptions on the function $c(x, \xi)$ and for $k \geq 0$, it is not difficult to verify the unilateral Wiener condition (3.6) and this yields the Hölder continuity of u up to the boundary (again Section 6).

Somewhat more cumbersome is to obtain an estimate from below of the right hand side of (0.6), when ψ is the implicit obstacle (0.12). Under additional assumption on $c(x, \xi)$ and for $k > 0$, this is done in Section 7 by using the previous continuity result, the inequality (0.8) and estimation techniques from [19], [7] and [29]. Then the $W^{2,q}$ and $C^{1,\alpha}$ regularity of u follows according to the remarks above.

The results of the present paper were announced in [11]. For previous results on Hölder regularity of solutions of v.i., see [8] and [5] for the case of Hölder continuous obstacles and [12] for irregular obstacles. For previous results concerning the continuity of the solution of the q.v.i. of impulse control, see [3], [25], where probabilistic methods are employed, and [15], [6] for analytic proofs. However these results do not yield Hölderianity and do not apply to quadratic Hamiltonians. For the $W^{2,q}$ regularity see [18], [19], [28], [7], [29].

1. – Notations and basic assumptions.

Throughout this paper we shall assume that

$$\mathcal{O} \textit{ is a bounded open subset of } \boldsymbol{R}^n, \tag{1.1}$$

and that

$$a_i\colon \mathcal{O} \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}, \quad i = 1, \dots, n, \tag{1.2}$$

are measurable functions with respect to $x \in \mathcal{O}$, *continuous with respect to* $(r, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$.

On each region $\mathcal{O} \times \{r \in \boldsymbol{R} | |r| \leq C\} \times \boldsymbol{R}^n$ where C is some given constant we assume the functions a_i to satisfy the growth condition

$$(1.3) \qquad |a_i(x, r, p)| \leq K + K|p|, \quad i = 1, \ldots, n,$$

with a suitable constant K *possibly depending on* C.

We shall also assume that

(1.4) *The partial derivatives*

$$(\partial/\partial p_k) a_i(x, r, p), \quad i, k = 1, \ldots, n,$$

exist for $x \in \mathcal{O}$, $r \in \boldsymbol{R}$, *and* $p \in \boldsymbol{R}^n$; *they are measurable with respect to* $x \in \mathcal{O}$ *and continuous with respect to* $(r, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$ *and satisfy*

$$(1.5) \qquad |(\partial/\partial p_k) a_i(x, r, p)| \leq K, \quad i, k = 1, \ldots, n,$$

$$(1.6) \qquad \sum_{i,k=1} (\partial/\partial p_k) a_i(x, r, p) \xi_i \xi_k \geq K_0 |\xi|^2$$

for $x \in \mathcal{O}$, $r \in \boldsymbol{R}$, $r \leq |C|$, $p \in \boldsymbol{R}^n$, $\xi \in \boldsymbol{R}^n$, *and* $K, K_0 > 0$ *suitable constants that may depend on* C.

Moreover, we suppose that

$$(1.7) \qquad a_0 \colon \mathcal{O} \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R} \quad \textit{is measurable with respect to } x \in \mathcal{O},$$

continuous with respect to $(r, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$ *and*

$$(1.8) \qquad |a_0(x, r, p)| \leqq K_1 + K_2 |p|^2, \quad x \in \mathcal{O}, \ |r| \leqq C, \ p \in \boldsymbol{R}^n,$$

for suitable constants K_1, K_2 *possibly depending on* C.

In the following sections, we shall consider a variational inequality of the type

$$(1.9) \qquad \begin{cases} u \in \boldsymbol{K} := \{v \in H_0^1(\mathcal{O}) | v \leqq \psi\} \\ \sum\limits_{i=}^{n} (a_i(x, u, \nabla u), D_i u - D_i v) + (a_0(x, u, \nabla u), u - v) \leqq 0 \\ \textit{for all } v \in \boldsymbol{K} \textit{ such that } u - v \in L^\infty(\mathcal{O}) \end{cases}$$

where

(1.10) $\psi\colon \overline{\mathcal{O}} \to \boldsymbol{R} \cup \{\infty\}$ *is a measurable function, essentially bounded from below*.

As already said, our study will concern the regularity properties of any given solution u of (1.9), which we assume to exist.

Moreover, we shall be concerned with *bounded* solutions u of (1.9). For results showing, under additional assumptions on the data, that any solution u of (1.9) is bounded, we refer, for instance, to [8]. For a given bounded solution u (1.9) we linearize the higher order terms in (1.9) as follows. We denote by a_{ik}, $i, k = 1, \ldots, n$, the (bounded measurable) functions on $\mathcal{O}$ given by

$$(1.11)\qquad \begin{cases} a_{ik}(x) = \displaystyle\int_0^1 (\partial/\partial p_k)\, a_i(x, u(x), t\nabla u(x))\, dt\,, & \text{if } x \in \mathcal{O}\,, \\ a_{ik}(x) = \delta_{ik}\,, & \text{if } x \in \boldsymbol{R}^n - \mathcal{O} \end{cases}$$

where $\delta_{ik} = 1$ if $i = k$, $\delta_{ik} = 0$ if $i \neq k$.

Clearly,

$$a_i(x, u(x), \nabla u(x)) = \sum_{k=1}^{n} a_{ik}(x) D_k u(x) + a_i(x, u(x), 0)$$

for every $i = 1, \ldots, n$, and $x \in \mathcal{O}$. Hence we can rewrite the inequality in (1.9) as follows

$$(1.12)\qquad \sum_{i,k=1}^{n} \big(a_{ik} D_k u, D_i(u - v)\big) \leq \\ \leq -\big(a_0(x, u, \nabla u), u - v\big) - \sum_{i=1}^{n} \big(a_i(x, u, 0), D_i(u - v)\big)\,.$$

Note that by (1.5), (1.6) the functions a_{ik} satisfy for all $x \in \mathcal{O}$

$$(1.13)\qquad |a_{ik}(x)| \leqq K_3\,, \qquad i, k = 1, \ldots, n$$

$$(1.14)\qquad \sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geqq K_0|\xi|^2\,.$$

A basic tool in the following will be the Green function, and the regularized Green function, associated with the (elliptic) operator

$$(1.15)\qquad -\sum_{i,k=1}^{n} D_i\big(a_{ik}(x) D_k\big)$$

on a fixed ball $Q \supset\supset \mathcal{O}$. For each $z \in \mathcal{O}$, the Green function $G = G^z$ is defined to be the solution

$$G \in H_0^{1,s}(Q), \quad 1 \leqq s < n/(n-1) \tag{1.16}$$

of the equation

$$\sum_{i,k=1}^{n} (a_{ik} D_k \varphi, D_i G)_Q = \varphi(z), \quad \varphi \in C_0^\infty(Q), \quad (w, v)_Q = \int_Q wv\,dx. \tag{1.17}$$

It is well known that such a function exists, see e.g. [31], [34], and satisfies the inequalities

$$\begin{cases} K_4|x-z|^{2-n} \leqq G(x) \leqq K_5|x-z|^{2-n}, & n \geqq 3 \\ K_4\big|\ln|x-z|\big| \leqq G(x) \leqq K_5\big|\ln|x-z|\big|, & n = 2 \end{cases} \tag{1.18}$$

for all x in a neighbourhood $Q_0 \subset\subset Q$ of z, with K_4 and K_5 suitable constants depending only on n, K_0, K_3 and K_4, also on Q_0. For each given $\varrho > 0$, the regularized Green function $G_\varrho = G_\varrho^z$ is defined to be the solution $G_\varrho \in H_0^1(Q)$ of the equation

$$\sum_{i,k=1}^{n} (a_{ik} D_k \varphi, D_i G_\varrho)_Q = |B_\varrho(z)|^{-1} \int \varphi(\zeta)\,d\zeta, \quad \varphi \in C_0^\infty(\mathcal{O}). \tag{1.19}$$

Again, the function G_ϱ exists and satisfies

$$G_\varrho \in L^\infty(\mathcal{O}), \quad G_\varrho \geqq 0, \quad G_\varrho(x) \to G(x) \quad \text{as } \varrho \to 0 \quad \text{if } x \neq z. \tag{1.20}$$

Moreover, as $\varrho \to 0$,

$$G_\varrho \to G \quad \text{strongly in } L^r(Q), \quad 1 \leqq r < n/(n-2), \tag{1.21}$$

$$\nabla G_\varrho \to \nabla G \text{ weakly in } L^t(Q), \quad 1 \leqq t < n/(n-1), \tag{1.22}$$

see [34].

We conclude the section by proving the following inequality

$$\int |\nabla G|^2 v^2\,dx \leqq c \int G^2 |\nabla v|^2\,dx, \tag{1.23}$$

with $G = G^z$, $c = c(K_0, K_3, n)$, that holds for every function $v \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$

such that $v \equiv 0$ in a neighbourhood of z. We have, in fact, by (1.17)

$$0 = \sum_{i,k=1}^{n} (a_{ik} D_k(v^2 G), D_i G) = \sum_{i,k=1}^{n} \left\{ \int a_{ik} D_k G D_i G v^2\, dx + 2\int a_{ik} G v D_k v D_i G\, dx \right\} \geqq$$

$$\geqq K_0 \int |\nabla G|^2 v^2\, dx - \varepsilon \int |\nabla G|^2 v^2\, dx - \varepsilon^{-1} n K_3^2 \int G^2 |\nabla v|^2\, dx\,.$$

Hence, inequality (1.23) follows by choosing $\varepsilon = K_0/2$.

2. – Some preliminary estimates.

In this section we establish some basic estimates which are satisfied by any solution $u \in L^\infty(\mathcal{O})$ of the variational inequality (1.9).
Let $\xi\colon \mathcal{O} \to \boldsymbol{R}$ and $\tau\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be functions with the following properties

(2.1) $\qquad \xi \in H^{1,s}(\mathcal{O}) \cap L^\infty(\mathcal{O}) \quad$ *for some* $s > n$,

(2.2) $\qquad \tau$ *is Lipschitz continuous on* $\boldsymbol{R}^n$ *and* $0 \leqq \tau \leqq 1$,

(2.3) $\qquad \xi\tau \in H_0^1(\mathcal{O})$ *and* $\xi \leqq \psi$ *on* $\operatorname{supp} \tau \cap \mathcal{O}$.

Let us remark that if $\Gamma = \partial\mathcal{O}$ is smooth, then by the Sobolev imbedding theorems $\xi \in H^{1,s}(\mathcal{O})$ implies $\xi \in C(\overline{\mathcal{O}})$; however, all estimates of the present section do not require any regularity of $\partial\mathcal{O}$ and will be used also to obtain local regularity results. The norm in $H^{1,s}$ will be denoted by $\|\cdot\|_s$.

We now consider $z \in \mathcal{O}$ and for every $0 < \varrho < \varrho_0$ the approximate Green function $G_\varrho = G_\varrho^z$ defined in the previous section. We shall prove that there exist suitable constants $c = c(K_0, K_2, K_3, \|u\|_\infty, \|\xi\|_\infty)$ and $\bar{c} = c(K, K_0, K_1, K_2, K_3, n, \|u\|_\infty, \|\xi\|_\infty)$, which are independent of $z \in \mathcal{O}$ and τ, such that the following estimate holds:

$$\text{(2.4)} \quad \int |\nabla u - \nabla\xi|^2 G_\varrho \tau^2\, dx + |B_\varrho|^{-1} \int |u - \xi|^2 \tau^2 \chi(B_\varrho(z))\, dx \leqq$$

$$\leqq c \int |u - \xi|^2 (G_\varrho |\nabla\tau|^2 + |\nabla G_\varrho| |\nabla\tau| \tau)\, dx + \bar{c}\sigma_\varrho$$

where

$$\text{(2.5)} \quad \sigma_\varrho = \int [(1 + |\nabla\xi|^2) G_\varrho \tau^2 + (1 + |\nabla\xi|)(G_\varrho |\nabla\tau| \tau + |\nabla G_\varrho| \tau^2)]\, dx\,,$$

for all $0 < \varrho < \varrho_0$, $B_{\varrho_0}(z) \subset \mathcal{O}$. Here $\chi(M)$ denotes the characteristic function of M.

The proof of (2.4) consists of the following steps (i)-(v):

(i) Let ξ, ξ_0 be arbitrary functions satisfying the following conditions:

$$\xi, \xi_0 \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}), \qquad \xi_0 \geqq 0, \tag{2.6}$$

$$\xi\xi_0 \in H_0^1(\mathcal{O}) \text{ and } \xi \leqq \psi \text{ on } \operatorname{supp} \xi_0 . \tag{2.7}$$

If $\varepsilon > 0$ is small enough the function

$$v := u - \varepsilon\xi_0(u - \xi) \tag{2.8}$$

is admissible in (1.9) and hence in (1.12). In fact, we have $v \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$; moreover, if $\varepsilon > 0$ is such that $\varepsilon\xi_0 \leqq 1$, we have by (2.7)

$$v = (1 - \varepsilon\xi_0)u + \varepsilon\xi_0\xi \leqq \psi \qquad \text{on supp } \xi_0$$

while

$$v = u \leqq \psi \qquad \text{on } \mathcal{O} - \text{supp } \xi_0$$

which together imply $v \in \boldsymbol{K} \cap L^\infty(\mathcal{O})$.

(ii) Let $z \in \mathcal{O}$ be fixed, $\varrho_0 > 0$ such that $B_{\varrho_0}(z) \subset \mathcal{O}$ and $0 < \varrho < \varrho_0$. Let ξ and τ be as in (2.1), (2.2), (2.3). We define in $\mathcal{O}$ the function

$$\xi_0 := |u - \xi|^{q-1} G_\varrho \tau^2 \tag{2.9}$$

where $G_\varrho = G_\varrho^z$ is the approximate Green function defined in Sec. 1 and $q \geqq 1$ is a real number to be chosen. The pair ξ, ξ_0 we consider clearly satisfies (2.6), (2.7), so that the function v given by (2.8) can be inserted in inequality (1.12). After cancelling the factor ε we obtain

$$\begin{aligned} \sum_{i,k=1}^{n} \big(a_{ik} D_k u, D_i[|u - \xi|^{q-1}(u - \xi) G_\varrho \tau^2]\big) \leqq \\ \leqq - \big(a_0(x, u, \nabla u), |u - \xi|^{q-1}(u - \xi) G_\varrho \tau^2\big) + \\ + - \sum_{i=1}^{n} \big(a_i(x, u, 0), D_i[|u - \xi|^{q-1}(u - \xi) G_\varrho \tau^2]\big) . \end{aligned} \tag{2.10}$$

We evaluate the derivatives appearing in (2.10) and rewrite it in the form

$$\begin{aligned} D_i[|u - \xi|^{q-1}(u - \xi) G_\varrho \tau^2] = q|u - \xi|^{q-1} G_\varrho \tau^2 D_i(u - \xi) + \\ + [|u - \xi|^{q-1}(u - \xi) \tau^2 D_i G_\varrho + |u - \xi|^{q-1}(u - \xi) G_\varrho 2\tau D_i \tau] , \end{aligned} \tag{2.11}$$

and

$$(2.12)\quad D_k u D_i[|u-\xi|^{q-1}(u-\xi)G_\varrho\tau^2] = q|u-\xi|^{q-1}G_\varrho\tau^2 D_k(u-\xi)D_i(u-\xi) +$$
$$+ (q+1)^{-1}D_k(|u-\xi|^{q+1}\tau^2)D_iG_\varrho +$$
$$+ [|u-\xi|^{q-1}(u-\xi)G_\varrho 2\tau D_i\tau D_k(u-\xi) - (q+1)^{-1}|u-\xi|^{q+1}2\tau D_k\tau D_i G_\varrho] +$$
$$+ \{q|u-\xi|^{q-1}G_\varrho\tau^2 D_k\xi D_i(u-\xi) + |u-\xi|^{q-1}(u-\xi)\tau^2 D_k\xi D_i G_\varrho +$$
$$+ |u-\xi|^{q-1}(u-\xi)G_\varrho 2\tau D_i\tau D_\varrho\xi\}$$

for every $i, k = 1, \ldots, n$.

We estimate the left hand side of (2.10) from below and the right hand side from above by evaluating separately each term that arises from the splitting (2.12) and (2.11).

(iii) *Estimate of the left side of* (2.10) *from below*: By the ellipticity condition (1.14) the first term arising via Leibniz' rule is bounded from below by

$$(2.13)\qquad qK_0\int|\nabla u-\nabla\xi|^2|u-\xi|^{q-1}G_\varrho\tau^2\,dx\,.$$

The second term in (2.12), by the definition of G_ϱ (1.19), is equal to

$$(2.14)\qquad (q+1)^{-1}|B_\varrho|^{-1}\int|u-\xi|^{q+1}\chi(B_\varrho(z))\,\tau^2\,dx\,.$$

Taking (1.13) into account we can apply Young's inequality to split the term in (2.12) which is written in brackets [...] and contains the derivative $D_k(u-\xi)$. Thus the term in [...], (2.12), contributes a quantity which is bounded from below by the sum of the following two terms:

$$(2.15)\qquad -\varepsilon\int|\nabla u-\nabla\xi|^2|u-\xi|^{q-1}G_\varrho\tau^2\,dx$$

and

$$(2.16)\qquad -(4\varepsilon)^{-1}K_3^2\int|u-\xi|^{q+1}G_\varrho|\nabla\tau|^2\,dx - K_3\int|u-\xi|^{q+1}|\nabla G_\varrho||\nabla\tau|\tau\,dx$$

(note that $2(q+1)^{-1}\leq 1$). Here $\varepsilon>0$ is a constant to be chosen later. As to the contribution from the term $\{\ldots\}$ in (2.12) (which vanishes if $\xi=$ constant) the integral containing $D_i(u-\xi)$ can be estimated using Young's inequality by the sum of a term like (2.15) and the term

$$(2.17)\qquad -(4\varepsilon)^{-1}q^2K_3^2\|u-\xi\|_\infty^{q-1}\int|\nabla\xi|^2G_\varrho\tau^2\,dx;$$

the remaining terms from $\{...\}$ are bounded by

$$(2.18)\qquad -K_3\|u-\xi\|_\infty^q\int|\nabla\xi|(|\nabla G_\varrho|\tau^2+2G_\varrho|\nabla\tau|\tau)\,dx\,.$$

Combining these estimates we conclude that the left hand side of (2.10) can be bounded from below by the sum of (2.13), (2.14), (2.16), (2.17), (2.18), together with twice the term (2.15).

(iv) *Estimate of the right hand side of* (2.10) *from above.* By the growth condition (1.8) the integral containing $a_0(x, u, \nabla u)$ is bounded by the sum of the term

$$(2.19)\qquad 2K_2\int|\nabla u-\nabla\xi|^2|u-\xi|^q G_\varrho\tau^2\,dx$$

and the term

$$(2.20)\qquad \|u-\xi\|_\infty^q\int(K_1+2K_2|\nabla\xi|^2)\,G_\varrho\tau^2\,dx\,.$$

We now estimate the contribution from the coefficients a_i. The first term according to the splitting (2.11) is of first order in $D_i(u-\xi)$ and can be estimated once again by Young's inequality. By this and (1.3) we obtain a bound which is the sum of

$$(2.21)\qquad \varepsilon\int|\nabla u-\nabla\xi|^2|u-\xi|^{q-1}G_\varrho\tau^2\,dx$$

and

$$(2.22)\qquad (4\varepsilon)^{-1}K^2q^2\|u-\xi\|_\infty^{q-1}\int G_\varrho\tau^2\,dx\,.$$

Finally, the contribution from the term $[...]$ in (2.11) can be estimated by

$$(2.23)\qquad nK\|u-\xi\|_\infty^q\int(|\nabla G_\varrho|\tau^2+2G_\varrho|\nabla\tau|\tau)\,dx\,.$$

Again combining the various estimates we see that the right hand side of (2.10) can be bounded from above by the sum of (2.19)-(2.23).

Note that (2.21) is a term similar to (2.15) and that the contribution of (2.20) can be assimilated by (2.17) and (2.22). Furthermore the term (2.23) is a term like (2.18) but with $|\nabla\xi|$ replaced by 1.

(v) Combining the estimates obtained in (iii) and (iv), we deduce from (2.10) the following inequality

$$\begin{aligned}(2.24)\quad &(qK_0-3\varepsilon)\int|\nabla u-\nabla\xi|^2|u-\xi|^{q-1}G_\varrho\tau^2\,dx+\\ &+(q+1)^{-1}|B_\varrho|^{-1}\int|u-\xi|^{q+1}\chi(B_\varrho(z))\,\tau^2\,dx\leq 2K_2\int|\nabla u-\nabla\xi|^2|u-\xi|^qG_\varrho\tau^2\,dx+\\ &+c_1\int|u-\xi|^{q+1}(G_\varrho|\nabla\tau|^2+|\nabla G_\varrho||\nabla\tau|\tau)\,dx+c_2\sigma_\varrho\end{aligned}$$

where

$$(2.25)\qquad \sigma_\varrho = \int [(1+|\nabla\xi|^2)\,G_\varrho\tau^2 + (1+|\nabla\xi|)(|\nabla G_\varrho|\tau^2 + G_\varrho|\nabla\tau|\tau)]\,dx$$

and

$$c_1 = c(\varepsilon, K_3)\,,\qquad c_2 = c(\varepsilon, q, K, K_1, K_2, K_3, n, \|u\|_\infty, \|\xi\|_\infty)\,.$$

We remark in passing that if (1.8) holds with $K_2 = 0$ (so that $a_0(x, r, p)$ is bounded) then we obtain (2.4) from (2.24) above by choosing

$$(2.26)\qquad \varepsilon = 12^{-1}K_0$$

and $q = 1$.

In order to handle the general case of a quadratic a_0, we use the same value of ε but leave the parameter $q \geqq 1$ free. We then use (2.24) *twice* for different values of q. When ε is given by (2.26) and $q = 1$, (2.24) yields

$$(2.27)\qquad \int |\nabla u - \nabla\xi|^2 G_\varrho\tau^2\,dx + |B_\varrho|^{-1}\int |u-\xi|^2\tau^2\chi(B_\varrho(z))\,dx \leqq$$
$$\leqq c_3\int |\nabla u - \nabla\xi|^2|u-\xi|G_\varrho\tau^2\,dx + c_4\int |u-\xi|^2(G_\varrho|\nabla\tau|^2 + |\nabla G_\varrho||\nabla\tau|\tau)\,dx + c_5\sigma_\varrho$$

where σ_ϱ is given by (2.25) and $c_3 = c(K_0, K_2)$, $c_4 = c(K_0, K_3)$, $c_5 = (K, K_0, K_1, K_2, K_3, n, \|u\|_\infty, \|\xi\|_\infty)$.

The additional term in (2.27), with respect to the case $K_2 = 0$, is the first integral at the right hand side of (2.27). In order to estimate this term, we introduce a new parameter $l > 0$ and we decompose the integration $\int = \int_l + \int^l$ over $\mathcal{O}$ into an integration $\int_l$ over the region

$$\mathcal{O}_l = \{x \in \mathcal{O}\,|\,|u-\xi| \geqq l\}$$

and an integration $\int^l$ over $\mathcal{O} - \mathcal{O}_l$, where $|u-\xi|$ is bounded by l. The second integral is easily estimated from above by

$$(2.28)\qquad \int^l \leqq l\int |\nabla u - \nabla\xi|^2 G_\varrho\tau^2\,dx\,.$$

Hence for l small enough this can be absorbed into the integral on the left side of (2.27). In order to obtain a bound uniformly in l for the integral over $\mathcal{O}_l$, we return to the estimate (2.24) with ε still given by (2.26), but where we now choose $q = \bar{q} = \bar{q}(K_0, K_2, \|u-\xi\|_\infty)$ so that

$$(\bar{q}-\tfrac{1}{4})K_0 - 2K_2\|u-\xi\|_\infty \geqq 1\,.$$

This yields the estimate

$$\int |\nabla u - \nabla\xi|^2 |u-\xi|^{\bar{q}-1} G_\varrho \tau^2\,dx + (\bar{q}+1)^{-1}|B_\varrho|^{-1}\int |u-\xi|^{\bar{q}+1}\tau^2\chi(B_\varrho(z))\,dx \leq$$
$$\leqq c_6\|u-\xi\|_\infty^{\bar{q}-1}\int |u-\xi|^2(G_\varrho|\nabla\tau|^2 + |\nabla G_\varrho||\nabla\tau|\tau)\,dx + c_7\sigma_\varrho$$

where

$$c_6 = c(K_0, K_2, K_3, \|u\|_\infty, \|\xi\|_\infty) \qquad c_7 = c(K, K_0, K_1, K_2, K_3, n, \|u\|_\infty, \|\xi\|_\infty).$$

Similarly, we obtain the estimate

$$\int_l |\nabla u - \nabla\xi|^2|u-\xi| G_\varrho\tau^2\,dx \leqq \tag{2.29}$$
$$\leqq c_6\|u-\xi\|_\infty^{\bar{q}-1} l^{2-\bar{q}}\int |u-\xi|^2(G_\varrho|\nabla\tau|^2 + |\nabla G_\varrho||\nabla\tau|\tau)\,dx + c_7 l^{2-\bar{q}}\sigma_\varrho\,.$$

Again the estimate is uniform in l.

By combining the two estimates (2.28) and (2.29), we obtain

$$\int |\nabla u - \nabla\xi|^2|u-\xi| G_\varrho\tau^2\,dx \leqq l\int |\nabla u - \nabla\xi|^2 G_\varrho\tau^2\,dx +$$
$$+ c_6\|u-\xi\|_\infty^{\bar{q}-1} l^{2-\bar{q}}\int |u-\xi|^2(G_\varrho|\nabla\tau|^2 + |\nabla G_\varrho||\nabla\tau|\tau)\,dx + c_7 l^{2-\bar{q}}\sigma_\varrho\,.$$

By substituting this inequality into (2.27) and choosing $l = \bar{l} = \bar{l}(K_0, K_2)$ small enough so that

$$1 - c_3\bar{l} \geqq \tfrac{1}{2}$$

we finally conclude the proof of (2.4).

From inequality (2.4) we now derive two estimates that will play a crucial role in the following.

LEMMA 2.1. *Let $u \in L^\infty(\mathcal{O})$ be a solution of* (1.9) *and let us suppose that there exist functions ξ and τ satisfying* (2.1), (2.2), (2.3), *with $\tau = 1$ on some open subset $\mathcal{O}_0$ of $\mathcal{O}$. Then*

$$\int |\nabla u|^2 G^z \tau\,dx \leqq c \qquad \textit{for all } z \in \mathcal{O}_0\,, \tag{2.30}$$

where c is some constant independent of $z \in \mathcal{O}_0$.

PROOF. Inequality (2.4) implies

$$(2.31)\qquad \tfrac{1}{2}\int |\nabla u|^2 \tau G_\varrho\, dx \leq c\int |u-\xi|^2 (G_\varrho |\nabla\tau|^2 + |\nabla G_\varrho||\nabla\tau|)\, dx + \bar{c}\sigma_\varrho + \int |\nabla\xi|^2 G_\varrho\, dx$$

for all $\varrho > 0$. As a consequence of (2.1) and (1.21), (1.22), (1.18), we have

$$(2.32)\qquad |\nabla\xi|^2 G_\varrho\,, \quad |\nabla\xi||\nabla G_\varrho| \in L^q(\mathcal{O})\,, \quad |\nabla\xi| G_\varrho \in L^{2q}(\mathcal{O})$$

for $1 \leq q < s/n$, uniformly for $\varrho \to 0$.

This shows that the last two terms at the right hand side of (2.31) stay bounded as $\varrho \to 0$. The first term at the right hand side of (2.31) also remains bounded as $\varrho \to 0$, because $\nabla\tau$ vanishes in a neighbourhood of the singularity of $G = G^z$. Therefore, (2.30) follows from (2.31) by virtue of Fatou's lemma. ∎

For fixed z and every $R > 0$, we shall denote by $\tau = \tau_R$ a function satisfying (2.2) such that in addition

$$(2.33)\qquad \tau = 1 \quad \text{on } B_R(z)\,, \qquad \tau = 0 \quad \text{on } \mathbf{R}^n - B_{2R}(z)$$

$$(2.34)\qquad |\nabla\tau| \leq R^{-1}\,.$$

LEMMA 2.2. *Let $u \in L^\infty(\mathcal{O})$ be a solution of (1.9) and $z \in \mathcal{O}$. Let ξ be an arbitrary function satisfying (2.1) and (2.3) with some $\tau = \tau_R$ as above. Then we have*

$$(2.35)\qquad \int_R |\nabla u|^2 G^z dx \leq c\overset{*}{\int_{8R}} |\nabla u|^2 G^z dx + cR^{-n}\overset{*}{\int_{8R}} |u-\xi|^2 dx + cR^\beta\,,$$

and if z is a Lebesgue point of u, we further have

$$(2.36)\qquad |u(z)-\xi(z)|^2 \leq c\overset{*}{\int_{8R}} |\nabla u|^2 G^z dx + cR^{-n}\overset{*}{\int_{8R}} |u-\xi|^2 dx + cR^\beta$$

$\beta = (s-n)/s$, where $\int_R$ denotes integration over $B_R(z) \cap \mathcal{O}$, and $\overset{*}{\int_{R'}} = \overset{z}{\int_{R'}}$ integration over $[B_{R'}(z) - B_R(z)] \cap \mathcal{O}$, $R' > R$. The constant c depends on $K, K_0, \ldots, \ldots, K_3, n, \|u\|_\infty, \|\xi\|_\infty, \|\nabla\xi\|_s$, but not on z and R.

REMARK. If $\xi \equiv$ constant, any real $\beta \in]0, 1[$ is admissible.

PROOF. We shall apply the estimate (2.4) for the case at hand. We first remark that by (2.32) the term σ_ϱ can be estimated from above, as $\varrho \to 0$, by cR^{β}, with $\beta = (s-n)/s$ and c a constant depending on $\|u\|_{\infty}$ and $\|\nabla\xi\|_s$, but independent of z and R. Moreover, the first term at the right hand side of (2.4) converges to the integral

$$\int |u-\xi|^2(|\nabla G||\nabla\tau|\tau + G|\nabla\tau|^2)\,dx \qquad (\varrho \to 0)\,.$$

This can be estimated from above by

$$\int\limits_{2R} |u-\xi|^2(2R^{2-n}|\nabla\tau|^2 + 2R^{n-2}|\nabla G|^2 + K_5R^{2-n}|\nabla\tau|^2)\,dx \leqq$$

$$\leqq (2+K_5)R^{-n}\overset{*}{\int\limits_{2R}}|u-\xi|^2\,dx + 2R^{n-2}\overset{*}{\int\limits_{2R}}|u-\xi|^2|\nabla G|^2\,dx\,.$$

Here we have used the estimate

$$G \leqq K_5|x-z|^{2-n} \leqq K_5R^{2-n}\,, \qquad n\geqq 3\,, \qquad \text{since } |x-z|\geqq R\,.$$

(For the case $n=2$ one has to use the « local » Green function $G \in H_0^1(B_{4R}(z)$, i.e. $G(x) = \log(4R/|x|)\big)$.

In turn, the last integral above can be estimated according to (1.23), by introducing a Lipschitz continuous function ω on $\boldsymbol{R}^n$, such that $0 \leqq \omega \leqq 1$, $\omega = 1$ on $B_{2R}(z) - B_R(z)$, $\omega = 0$ on $B_{R/2}(z) \cup \complement B_{4R}(z)$, $|\nabla\omega| \leqq 2R^{-1}$. This yields

$$\overset{*}{\int\limits_{2R}}|u-\xi|^2|\nabla G|^2\,dx = \overset{*}{\int\limits_{2R}}(\omega|u-\xi|)^2|\nabla G|^2\,dx \leqq$$

$$\leqq c\overset{**}{\int\limits_{4R}}G^2(\omega^2|\nabla u-\nabla\xi|^2 + |u-\xi|^2|\nabla\omega|^2)\,dx \leqq$$

$$\leqq cR^{2-n}\Big\{\overset{**}{\int\limits_{4R}}|\nabla u-\nabla\xi|^2G\,dx + R^{-n}\overset{**}{\int\limits_{4R}}|u-\xi|^2\,dx\Big\}$$

where c is some constant depending on K_0, K_3, n, and independent of z and R. Here $\overset{**}{\int\limits_{4R}}$ denotes integration over $B_{4R}(z) - B_{R/2}(z)$. (For $n=2$, use the above « local » Green function in the proof, since the latter is bounded on $B_{4R}(z) - B_{R/2}(z)\big)$.

We obtain from (2.4) and the above estimates

$$\int|\nabla u|^2G\,dx \leqq \int|\nabla u|^2G\tau\,dx \leqq c\overset{**}{\int\limits_{4R}}|\nabla u|^2G\,dx + cR^{-n}\int\limits_{4R}|u-\xi|^2\,dx + cR^{\beta}\,.$$

The passage to the limit $\varrho \to 0$ is justified by Fatou's lemma. Thus (2.35) follows by replacing R with $2R$. If z is a Lebesgue point of u, we obtain from (2.4) as $\varrho \to 0$

$$|u(z) - \xi(z)|^2 \leqq c\int_{4R}^{**} \nabla u^2 G\,dx + cR^{-n}\int_{4R}^{**} |u - \xi|^2\,dx + cR^{\beta}$$

from which (2.36) follows. ∎

3. – Interior regularity.

We suppose in this section that the obstacle ψ satisfies the following *unilateral condition of Wiener type*:

(3.1) *There exist constants $m \geqq 4$ and $c_0 > 0$ such that the following holds: For every $z \in \mathcal{O}_0 \subset\subset \mathcal{O}$ and every $\delta > 0$ there exists a constant $R_\delta = R_\delta(\mathcal{O}_0) > 0$, independent of $z \in \mathcal{O}_0$, such that $B(z, 2mR_\delta) \subset \mathcal{O}$ and such that for every $R \in]0, R_\delta]$ there exists a closed set:*

$$T_R(z) \subset B_{mR}(z) - B_{2R}(z) \tag{3.2}$$

with the property

(3.3) *The capacity of $T_R(z)$ with respect to $B_{2mR}(z)$ is larger than $c_0 R^{n-2}$, and*

$$\psi(x) \geqq \psi(y) - \delta \quad \textit{for all } x \in B_{2R}(z),\ y \in T_R(z)\,. \tag{3.4}$$

If $T_R(z)$ is an $(n-1)$-dimensional manifold we can replace condition (3.4) by

$$\psi(x) \geq \bar{\psi}_R(z) - \delta \quad \text{for all } x \in B_{2R}(z) \tag{3.5}$$

where $\bar{\psi}_R(z) = \fint_R \psi(s)\,ds$ and $\fint_R$ denotes the mean value taken over $T_R(z)$.

Furthermore, in the case (3.5), the sets $T_R(z)$, $R > 0$, $z \in \mathcal{O}_0$, have to satisfy uniform weak regularity assumptions such that Poincaré's inequality holds in the following form

$$\int^{*} |u - \bar{u}_R|^2\,dx \leq KR^2 \int_{mR}^{*} |\nabla u|^2\,dx\,, \qquad R > 0,\ z \in \mathcal{O}_0\,.$$

Here $\int\limits_{R'}^{*}$ denotes integration over $B_{R'}(z) - B_R(z)$ for $R' > R$, and $\bar{u}_R(z) = = \fint\limits_R u(s)\,ds$ with $\fint\limits_R$ defined as above.

THEOREM 3.1. *Under the assumptions* (1.1)-(1.8), *let us suppose that* ψ *satisfies* (1.10), (3.1)-(3.4), *and that* $u \in L^\infty(\mathcal{O})$ *is a solution of* (1.9). *Then* u *is continuous in* $\mathcal{O}$.

Let us assume that ψ satisfies the following unilateral Hölder condition

(3.6) *There exist constants* $m \geq 4$, $c_1, \alpha > 0$ *such that for every* $z \in \mathcal{O}$ *and every ball* $B(z, 2mR) \subset \mathcal{O}$, $0 < R < R_0$, *there exists a closed set* $T_R(z)$ *that satisfies* (3.2) *and* (3.3) *and is such that*

$$\psi(x) \geq \psi(y) - c_1|x-y|^\alpha \qquad \text{for all } x \in B_{2R}(z) \text{ and all } y \in T_R(z)\,. \tag{3.7}$$

We then shall prove

THEOREM 3.2. *Under the assumptions* (1.1)-(1.8) *let* ψ *satisfy* (1.10), (3.6), (3.7) *and* $u \in L^\infty(\mathcal{O})$ *be a solution of* (1.9). *Then* u *is Hölder continuous in* $\mathcal{O}$.

Moreover we shall prove that the following estimate holds

$$\int\limits_R |\nabla u|^2 G\,dx \leq cR^\lambda$$

for some $\lambda \in]0,1[$ where $G = G^z$, the constant c being independent of $0 < R < R_0$ and $z \in \mathcal{O}_0 \subset\subset \mathcal{O}$.

REMARK. For continuous obstacle ψ, cf. the results in [8] on the continuity of the solution u.

PROOF OF THEOREM 3.1. Let $u \in L^\infty(\Omega)$ be a solution of (1.9) and $\mathcal{O}_0 \subset\subset \mathcal{O}$.

1. We first apply lemma 2.1 in order to show that

$$\int\limits_{\mathcal{O}_0} |\nabla u|^2 G^z dx \leq c \tag{3.8}$$

for all $z \in \mathcal{O}_0$, where c is a constant independent of $z \in \mathcal{O}_0$.

The functions ξ and τ are chosen so that $\xi = m_0$, where m_0 is a lower bound of ψ on $\mathcal{O}_0$ and $\tau = \tau(\mathcal{O}_0)$ is any function in $C_0^1(\mathcal{O})$ such that $\tau = 1$ on $\mathcal{O}_0$.

2. We now apply Lemma 2.2 to show that the following estimate holds for given $z \in \mathcal{O}$, $\delta > 0$ and $R \in]0, R_\delta]$ such that $B(z, 2mR_\delta) \subset \mathcal{O}$ with m, R_δ, β the constants appearing in assumption (3.1) and in (2.35):

$$\int_R |\nabla u|^2 G^z dx \le c \overset{*}{\int_{2mR}} |\nabla u|^2 G^z dx + c(R^\beta + \delta^2) \tag{3.9}$$

and moreover, if $B(z, 4mR_\delta) \subset \mathcal{O}$ we also have

$$|u(\zeta) - d|^2 \le c \int_{B_{8R}(\zeta)} |\nabla u|^2 G^\zeta dx + c \int_{B_{4mR}(z)} |\nabla u|^2 G^z dx + cR^\beta + c\delta^2 \tag{3.10}$$

for every Lebesgue point $\zeta \in B_R(z)$ of u with $d = d(z, R)$ a constant to be chosen later, independently of ζ and such that, in addition

$$M_0 := c\text{-inf}\,\{u(y) | y \in T_{2R}(z)\} \le d \le M_1 := c\text{-sup}\,\{u(y) | y \in T_{2R}(z)\}\,. \tag{3.11}$$

Here $T_{2R}(z)$ is the set appearing in assumption (3.1); for the definition of the «capacity-essential» supremum and infimum, «c-sup» and «c-inf», see Appendix A.

In order to verify the assumptions of Lemma 2, we choose $\xi = \xi_{R,\delta}$ to be the constant function

$$\xi = d - \delta\,, \qquad M_0 \le d \le M_1\,.$$

By applying (3.1) with R replaced by $2R$, we conclude from (3.4) and the inequality $u \le \psi$ that for every $x \in B_{4R}(z)$ and $y \in T_{2R}(z)$

$$\psi(x) \ge u(y) - \delta\,.$$

Hence we have by (3.11)

$$\psi(x) \ge d - \delta \tag{3.12}$$

which is a fortiori true for every $x \in B_{2R}(\zeta)$ and arbitrary fixed $\zeta \in B_R(z)$.

We are now in a position to apply Lemma 2.2 at any point ζ as above and with the above choice of ξ. We notice that for $\tau = \tau_R$ satisfying (2.2) and (2.33) (where $z = \zeta$), we have $\xi\tau \in H_0^1(\mathcal{O})$ and by (3.12), $\xi \le \psi$ on supp τ, since supp $\tau \subset B_{2R}(\zeta)$; therefore assumption (2.3) of Lemma 2.2 is also satisfied. Thus, for $\zeta = z$ we have by (2.35)

$$\int_R |\nabla u|^2 G^z dx \le c \overset{*}{\int_{8R}} |\nabla u|^2 G^z dx + cR^{-n} \overset{*}{\int_{8R}} |u - d - \delta|^2 dx + cR^\beta$$

and for every Lebesgue point $\zeta \in B_R(z)$ we obtain via (2.36)

$$|u(\zeta) - d - \delta|^2 \le c\int\limits_{B_{8R}(\zeta)} |\nabla u|^2 G^\zeta dx + cR^{-n}\int\limits_{B_{8R}(\zeta)} |u - d - \delta|^2 dx + cR^\beta$$

which hold for $d \in [M_0, M_1]$. From the above inequalities we can derive both (3.9) and (3.10) easily, once we have estimated the term

$$A_1 := R^{-n}\overset{*}{\int\limits_{8R}} |u - d|^2 dx \quad \text{and} \quad A_2 := R^{-n}\int\limits_{B_{8R}(\zeta)} |u - d|^2 dx\,.$$

First, we choose $d \in [M_0, M_1]$ such that it realizes the minimum of the functional I defined by

$$I(d) = R^{-n}\overset{*}{\int\limits_{2mR}} |u - d|^2 dx\,.$$

Thus we obtain by Poincaré's inequality in the form as it is given in the appendix that

$$A_1 \le R^{-n}\overset{*}{\int\limits_{2mR}} |u - d|^2 dx \le cR^{2-n}\overset{*}{\int\limits_{2mR}} |\nabla u|^2 dx\,. \tag{3.13}$$

Here we have taken into account that $T_{2R}(z) \subset B_{2mR}(z) - B_{kR}(z)$ and that the relative capacity of T_{2R} is larger than $c_0 R^{n-2}$. From (3.13) and (1.18) we arrive at (3.9).

The term A_2 is first estimated by

$$A_2 \le R^{-n}\int\limits_{B_{8R}(\zeta)} |u - d|^2 dx \le R^{-n}\int\limits_{B_{4mR}(z)} |u - d|^2 dx =: J(d)\,.$$

We then choose $d \in [M_0, M_1]$ to minimize $J(d)$ on $[M_0, M_1]$ and apply Poincaré's inequality as above, but in the ball $B_{4mR}(z)$. Proceeding as before, we obtain (3.10).

3. Using the « hole-filling » technique we can now derive from (3.9) the estimate

$$\int\limits_R |\nabla u|^2 G^z dx \le c(R^\gamma + \delta^2) \tag{3.14}$$

for every $z \in \mathcal{O}$, $\delta > 0$, and all $R \in]0, R_\delta]$, $R \le 1$, such that $B_{2mR}(z) \subset \mathcal{O}$.

Here $\gamma \in]0, 1[$, $\gamma \leq \beta$, is a suitable constant, both c and γ being independent of z and R as above.

In fact, if we set

$$\eta(R) = \int_R |\nabla u|^2 G^z dx$$

we have $\eta(R) \leq c$ where c is a constant independent of z and R; moreover, by adding $c\eta(R)$ to both sides of (3.9) and dividing the resulting inequality by $1 + c$, we obtain for $\eta(R)$ a relation of the form

$$\eta(R) \leq \theta\eta(\nu R) + R^\beta + \delta^2 \tag{3.15}$$

where $\theta = c/(1 + c) < 1$, c being the constant of (3.9), and $\nu > 1$.

We now choose θ_0 such that $0 < \theta_0 < 1$, and $\gamma \in]0, 1[$ such that $\gamma \leq \beta$, $\nu^\gamma\theta = \theta_0$. For fixed $R_0 \leq \min(1, R_\delta)$ we set

$$M = \sup \{R^{-\gamma}\eta(R) | \nu^{-1}R_0 < R \leq R_0\} .$$

Therefore, for all R such that $\nu^{-2}R_0 < R \leq \nu^{-1}R_0$ we have

$$\eta(\nu R) \leq M\nu^\gamma R^\gamma$$

and by (3.15)

$$\eta(R) \leq (1 + \theta_0 M)R^\gamma + \delta^2 .$$

By iterating the above argument we obtain

$$\eta(R) \leq (1 + \theta_0 + \dots + \theta_0^{j-1} + \theta_0^j M)R^\gamma + (1 + \theta_0 + \dots + \theta_0^{j-1})\delta^2$$

for all $j = 1, 2, \dots,$ and $R \in]\nu^{-(j+1)}R_0, \nu^j R_0[$.

Hence

$$\eta(R) \leq [(1 - \theta_0)^{-1} + M]R^\gamma + (1 - \theta_0)^{-1}\delta^2$$

for all $R \in]0, R_0[$.

4. We are now in a position to conclude the proof of the theorem by showing that for every $\delta > 0$ we have

$$|u(x_1) - u(x_2)| \leq c\delta^2 \tag{3.16}$$

for all Lebesgue points of u such that $x_1, x_2 \in \mathcal{O}$, $|x_1 - x_2| \leq R \leq \bar{R}_\delta$,

$B(4m\bar{R}_\delta, x_1) \subset \mathcal{O}$. In fact, it follows from (3.10) and (3.14) that

$$|u(\zeta) - d|^2 \leq c(R^\gamma + \delta^2)$$

for all Lebesgue points $\zeta \in B_R(z)$. Here d is chosen independently of ζ as above. Setting $z = x_1$, $\zeta = x_1, x_2$ we obtain (3.16) with $\bar{R}_\delta = \min\{1, (2m)^{-1}R_\delta, \delta^{2/\gamma}\}$. ■

PROOF OF THEOREM 3.2. By assumption (3.6), the hypothesis (3.1) of theorem 3.1 is now satisfied with R_δ, for each $\delta > 0$, given by $R_\delta = c_1^{-1/\alpha}(m+2)^{-1}\delta^{1/\alpha}$. Therefore, for every given $R \in]0, 1[$ with $B_{2mR}(z) \subset \mathcal{O}$ we chose $\delta = R$ and we obtain from (3.14) that

$$\int_R |\nabla u|^2 G^z \, dx \leq c(R^\gamma + R^{2\alpha})$$

with c a constant as above, independent of z and R. Thus the conclusion of theorem 3.2 follows by a classical lemma of C. B. Morrey. We remark that it also follows directly from (3.16) that

$$|u(x_1) - u(x_2)| \leq c|x_1 - x_2|^\mu$$

where $\mu = \min(\alpha, \gamma/2)$ with c a constant independent of $x_1, x_2 \in \mathcal{O}_0 \subset\subset \mathcal{O}$. ■

4. – Boundary regularity.

In addition to the assumptions of Sec. 3 we now suppose that Γ is Lipschitz continuous or, more generally, that it satisfies a Wiener condition of the following type

(4.1) *There exist constants $c > 0$ and $R_1 > 0$ such that for all $x_0 \in \Gamma$, $0 < R < R_1$, we have*

$$\inf\left\{\int |\nabla\phi|^2 dx;\ \phi \in C_0^\infty(B_{2R}(x_0)),\ \phi \geq 1 \text{ on } B_R(x_0) \cap [\mathbf{R}^n - \mathcal{O}]\right\} \geq cR^{n-2}.$$

Moreover, we suppose that the obstacle ψ satisfies not only (3.1) but also the following condition at the boundary.

(4.2) *The convex set $\mathbf{K}$ of (1.9) contains a function*

$$u_0 \in H^{1,s}(\mathcal{O}) \cap C^\beta(\bar{\mathcal{O}})$$

where $s > n$ and $\beta \in]0, 1[$.

REMARK. Clearly such a u_0 exists if $\psi > 0$ in a neighbourhood of Γ. Its existence is also assured if ψ satisfies (3.7) with $\alpha \geq 1$ and Γ is Lipschitz continuous. Note that $\psi \geq 0$ on Γ since $u \in \boldsymbol{K}$.

THEOREM 4.1. *Suppose that, in addition to the assumptions of Th.* 3.1., *both* (4.1) *and* (4.2) *are satisfied and let* $u \in L^\infty(\mathcal{O})$ *be a solution of* (1.9). *Then* $u \in C(\overline{\mathcal{O}})$.

THEOREM 4.2. *Under the assumptions of Th.* 3.2 *and conditions* (4.1), (4.2) *we have* $u \in C^\alpha(\overline{\mathcal{O}})$ *for some* $\alpha \in]0, 1[$.

PROOF OF THEOREM 4.1. The proof below, which follows the same pattern as the proof of Theorem 3.1., will supply additional boundary estimates on regions $B_{2mR}(z)$ such that $B_{2mR}(z) \cap \mathcal{O} \neq \emptyset$ from which the regularity up to the boundary can be derived easily.

1. We first prove that the estimate (3.8) now holds uniformly with respect to all $z \in \overline{\mathcal{O}}$. For this, we apply Lemma 2.1 with $\mathcal{O}_0 = \mathcal{O}$, $\xi = u_0$, $\tau = 1$ on $\mathcal{O}$. Note, in particular that the hypothesis (2.3) of the lemma holds because $u_0 \in \boldsymbol{K}$ and $u_0 \in H_0^1(\Omega)$ with $u_0 \leq \psi$ on $\mathcal{O}$.

2. We now prove that the following estimates hold

$$\int_R |\nabla u|^2 G^z \leq c \overset{*}{\int_{4mR}} |\nabla u|^2 G^z dx + cR^\beta \tag{4.3}$$

$$|u(z) - u_0(z)|^2 \leq c \int_{4mR} |\nabla u^2 G^z dx + cR^\beta \tag{4.4}$$

for every $z \in \overline{\mathcal{O}}$ which is a Lebesgue point of u, and for every $R > 0$ such that

$$[B_{2mR}(z) - B_R(z)] \cap \complement \mathcal{O} \neq \emptyset,$$

c being a constant depending on $K, K_0, K_1, \ldots, n, \|u\|_\infty, \|u_0\|_\infty, \|\nabla u_0\|_s, m,$ but independent of $z \in \overline{\mathcal{O}}$ and $R > 0$, and $\beta = (s - n)/s$. We recall that $\int_R$ denotes integration over $B_R(z) \cap \mathcal{O}$ and $\overset{*}{\int_{4mR}}$ integration over $[B_{4mR}(z) - B_R(z)] \cap \mathcal{O}$.

In order to obtain (4.3) and (4.4) we apply Lemma 2.2 with $\xi = u_0$. Note again that the hypothesis (2.3) of the lemma is now satisfied for arbitrary $R > 0$ since $u_0 \in \boldsymbol{K}$.

We thus obtain

$$\int_R |\nabla u|^2 G^z dx \le c \int_{2mR}^{*} |\nabla u|^2 G^z dx + cR^{-n} \int_{2mR}^{*} |u - u_0|^2 dx + cR^\beta \tag{4.5}$$

$$|u(z) - u_0(z)|^2 \le c \int_{2mR} |\nabla u|^2 G^z dx + cR^{-n} \int_{2mR}^{*} |u - u_0|^2 dx + cR^\beta \tag{4.6}$$

for all $z \in \overline{\mathcal{O}}$ and $R > 0$, z being a Lebesgue point of u in (4.6).

We now restrict our attention to those $z \in \overline{\mathcal{O}}$ and $R > 0$ such that $[B_{2mR}(z) - B_R(z)] \cap \complement \mathcal{O} \neq \emptyset$. By our assumption on $\partial \mathcal{O}$ we have B_{8mR}-cap $[B_{4mR}(z) - B_R(z)] \cap \complement \mathcal{O} \ge c_0 R^{n-2}$, $c_0 > 0$, and since $u - u_0 \in H_0^1(\mathcal{O})$ we may apply Poincaré inequality to estimate

$$R^{-n} \int_{2mR}^{*} |u - u_0|^2 dx \le cR^{2-n} \int_{4mR}^{*} |\nabla u - \nabla u_0|^2 dx \le$$

$$\le c \int_{4mR}^{*} |\nabla u|^2 G^z dx + c \int_{4mR}^{*} |\nabla u_0|^2 G^z dx \le c \int_{4mR}^{*} |\nabla u|^2 G^z dx + cR^\beta .$$

The last inequality holds because (4.2) implies $|\nabla u_0|^2 G^z \in L^q(\mathcal{O})$ for $1 \le q < s/n$. This completes the proof of (4.3) and (4.4).

3. From (4.3) and (3.9), via the «hole filling» technique, we obtain the estimate

$$\int_R |\nabla u|^2 G^z dx \le c(R^\gamma + \delta^2) \tag{4.7}$$

with $\gamma \in]0, 1[$, and c constants independent of

$$z \in \overline{\mathcal{O}}, \quad \delta > 0, \quad R \in]0, R_\delta], \quad R < 1 .$$

Therefore, from (4.4) and (3.16)

$$|u(z) - u(x)| \le |u(z) - u_0(z)| + |u_0(z) - u_0(x)| +$$

$$+ |u(x) - u_0(x)| \le cR^\gamma + c\delta^2 + |u_0(z) - u_0(x)| .$$

This completes the proof of the theorem.

5. – Dual estimates.

We shall establish in this section some estimates of potential theoretic nature involving the operator

$$A(v) = -\sum_{i=1}^{n} D_i a_i(x, v, \nabla v) .$$

As in Sec. 1, we suppose that

(5.1) $\qquad \mathcal{O}$ *is a bounded open subset of* $\boldsymbol{R}^n$

and that

(5.2) $a_i\colon \boldsymbol{R}\times\boldsymbol{R}^n \to \boldsymbol{R}$, $i = 0, 1, \dots, n$, *are measurable functions with respect to* $x \in \mathcal{O}$, *continuous with respect to* $(r, p) \in \boldsymbol{R}\times\boldsymbol{R}^n$.

We suppose furthermore that the following growth conditions are satisfied

(5.3) $$|a_i(x, r, p)| \le K + K|p| , \qquad i = 1, \dots, n$$

(5.4) $$|a_0(x, r, p)| \le f \; + K|p|^2$$

for almost all $x \in \mathcal{O}$, all $r \in R$, $|r| \le C$, and all $p \in \boldsymbol{R}^n$, with $K > 0$ constant possibly depending on C, and some $f \in L^1(\mathcal{O})$. We also assume that the following monotonicity condition is satisfied

(5.5) $\sum_{i=1}^{n} [a_i(x, r, p) - a_i(x, r', p')](p_i - p_i') \ge c|p - p'|^2 - K|r - r'|^2$ *for almost all* $x \in \mathcal{O}$, *all* $r, r' \in \boldsymbol{R}$ *such that* $|r| \le C$, $|r'| \le C$, *all* $p, p' \in \boldsymbol{R}^n$, *with* $c, K > 0$ *suitable constants possibly depending on* C.

Let us remark that (5.5) follows from the coerciveness assumption (1.6) of Sec. 1 if simply

(5.6) $$(\partial/\partial r)\, a_i(x, r, p) = 0$$

or, more generally,

(5.7) $$|(\partial/\partial r)\, a_i(x, r, p)| \le c$$

for $|r| \le C$, $p \in \boldsymbol{R}^n$ and almost all $x \in \mathcal{O}$, with c a constant possibly depending on C.

In consequence of (5.2), (5.3) and (5.4), $A(v) + a_0(x, v, \nabla v)$ is well defined as a distribution in $\mathcal{O}$ for every function $v \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ by the identity

$$\langle A(v) + a_0(x, v, \nabla v), \phi\rangle =$$
$$= \sum_{i=1}^{n} \int_{\mathcal{O}} a_i(x, v, \nabla v) D_i\phi\, dx + \int_{\mathcal{O}} a_0(x, v, \nabla v)\phi\, dx \qquad \text{for every } \phi \in C_0^\infty(\mathcal{O}) .$$

Now we consider the variational inequality (1.9). We assume that the obstacle ψ appearing in (1.9) is such that

$$\psi \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}) \tag{5.8}$$

and we suppose, in addition, that the distribution $A(\psi)$ is indeed a *measure* in $\mathcal{O}$ satisfying the condition

$$0 \wedge A(\psi) \in H^{-1}(\mathcal{O}) \oplus L^1(\mathcal{O}) , \tag{5.9}$$

where the negative part has to be intended in the sense of measures.

Then, we can estimate any bounded solution of (1.9) according to the following

THEOREM 5.1. *Under the growth and monotonicity assumptions* (5.1) *up to* (5.5), *if the function* ψ *satisfies* (5.8) *and* (5.9) *then every solution* $u \in L^\infty(\mathcal{O})$ *of the variational inequality* (1.9) *satisfies*

$$0 \geq A(u) + a_0(x, u, \nabla u) \geq 0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi)] \tag{5.10}$$

in the sense of measures in $\mathcal{O}$.

In order to prove this theorem, we approximate the function a_0 by a sequence of functions a_{0m}, $m = 1, 2, \ldots$ that satisfy the conditions

$$a_{0m}(x, r, p) \to a_0(x, r, p) \qquad \text{as } m \to \infty \tag{5.11}$$

for almost all $x \in \mathcal{O}$, *all* $r \in \mathbf{R}$, $p \in \mathbf{R}^n$;

$$|a_{0m}(x, r, p)| \leq c_m \leq f + K|p|^2 \tag{5.12}$$

for almost all $x \in \mathcal{O}$, *all* $r \in R$, $|r| \leq C$ *and all* $p \in \mathbf{R}^n$;

$$|a_{0m}(x, r, p) - a_{0m}(x, r', p')| \leq c|r - r'| + c_m|p - p'| \tag{5.13}$$

for almost all $x \in \mathcal{O}$, *all* $r, r' \in \mathbf{R}$ *with* $|r| \leq C$, $|r'| \leq C$ *and all* $p, p' \in \mathbf{R}^n$.

In the above conditions the c_m are constants possibly depending on C and m, unlike the constant c which may depend on C but is independent of m.

A possible choice of the approximate a_{0m} is

$$a_{0m}(x, r, p) := \tau_m\big(a_0^m(x, r, \omega_m(p))\big)\,, \qquad x \in \mathcal{O},\ r \in \boldsymbol{R},\ p \in \boldsymbol{R}^n\,, \tag{5.14}$$

where $\tau_m(t) = t$ if $|t| \le m$, $\tau_m(t) = m$ if $|t| > m$ and $\omega_m(p) = \big(\tau_m(p_1), \dots, \tau_m(p_n)\big)$ for $p = (p_1, \dots, p_n) \in \boldsymbol{R}^n$ and a_0^m is a suitable regularization of a_0 with respect to the variables (r, p).

PROOF OF THEOREM 5.1. Let $u \in L^\infty(\mathcal{O})$ be a solution of (1.9). For each $m = 1, 2, \dots,$ we put

$$a_m(v, w) := \sum_{i=1}^{n} \big(a_i(x, v, \nabla v), D_i w\big) + \big(a_{0m}(x, v, \nabla v), w\big) + \lambda_m(v, w) \tag{5.15}$$

for every $v, w \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$,

$$f_m := a_{0m}(x, u, \nabla u) - a_0(x, u, \nabla u) + \lambda_m u\,, \tag{5.16}$$

where $\lambda_m > 0$ is some constant to be chosen conveniently large.

It follows from (1.9) that u satisfies the inequality

$$a_m(u, u - v) \le (f_m, u - v) \tag{5.17}$$

for all $v \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$, such that $v \le \psi$.

For each fixed m, we now define

$$T_m := A(\psi) + a_{0m}(x, \psi, \nabla\psi) + \lambda_m \psi \tag{5.18}$$

and we consider the following auxiliary variational inequality

$$\begin{cases} z \in Q \\ a_m(z, z - v) \le (f_m \wedge T_m, z - v) \qquad \text{for all } v \in Q\,, \end{cases} \tag{5.19}$$

where $f_m \wedge T_m \in H^{-1}(\mathcal{O}) \oplus L^1(\mathcal{O})$ and

$$Q := \{v \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O}) | u \le v \le u + 1\}\,. \tag{5.20}$$

The standard theory of monotone operators does not apply to problem (5.19) due to the term $f_m \wedge T_m$ that does not belong to the dual space of $H_0^1(\mathcal{O})$.

However, since the convex subset Q of $H_0^1(\mathcal{O})$ is also bounded in $L^\infty(\mathcal{O})$, suitable monotonicity arguments can still be used in order to prove that if the constant λ_m is large enough then the solution $z = z_m$ of the variational inequality (5.19) actually exists and is unique. For sake of completeness in Appendix B we give a general result that can be applied to the present case.

We go on with the proof of the theorem by showing that for every m we have

$$z = u \,. \tag{5.21}$$

Let us show first that it suffices to prove that

$$z \leq \psi \,. \tag{5.22}$$

In fact, then $v = z$ is allowed in (5.17), hence

$$a_m(u, u - z) \leq (f_m, u - z) \leq (f_m \wedge T_m, u - z) \,, \tag{5.23}$$

the last inequality being a consequence of $z \in Q$; on the other hand, $v = u$ can be obviously replaced in (5.19), hence

$$a_m(z, z - u) \leq (f_m \wedge T_m, z - u) \,. \tag{5.24}$$

Therefore, from (5.23) and (5.24) we obtain

$$a_m(u, u - z) - a_m(z, u - z) \leq 0$$

and this implies $u = z$ in consequence of the monotonicity assumption (5.5), provided λ_m is conveniently large.

For proving (5.22) we observe that the vector

$$v = z - (z - \psi)^+ = z \wedge \psi \tag{5.25}$$

belongs to Q. In fact, since $z \in Q$, $u \leq \psi$ and $\psi \in H^1(\mathcal{O})$, we have $v \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$; moreover, $u \leq z \leq u + 1$, therefore also $u \leq v \leq u + 1$.

By replacing v in (5.19) we obtain

$$a_m(z, (z - \psi)^+) \leq (f_m \wedge T_m, (z - \psi)^+) \tag{5.26}$$

and since

$$f_m \wedge T_m \leq T_m = A(\psi) + a_{0m}(x, \psi, \nabla\psi) + \lambda_m \psi \,,$$

we get from (5.26), by taking (5.8), (5.9) and (5.15) into account,

$$a_m(z, (z-\psi)^+) - a_m(\psi, (z-\psi)^+) \le 0 \,. \tag{5.27}$$

By (5.5) this implies $(z-\psi)^+ = 0$, hence (5.22) holds.

As a consequence of (5.21) we get from (5.19)

$$a_m(u, u-v) \le (f_m \wedge T_m, u-v) \quad \text{for all } v \in Q \,. \tag{5.28}$$

For arbitrary $\phi \in C_0^1(\mathcal{O})$, $\phi \ge 0$, we have $v := u + \varepsilon\phi \in Q$ if $\varepsilon > 0$ is small enough, therefore from (5.28)

$$a_m(u, \phi) \ge (f_m \wedge T_m, \phi) \,.$$

This means, by taking (5.15), (5.16) and (5.18) into account, that in the distribution sense we have

$$\begin{aligned} A(u) + a_{0m}(x, u, \nabla u) + \lambda_m u \ge \\ \ge [a_{0m}(x, u, \nabla u) - a_0(x, u, \nabla u) + \lambda_m u] \wedge [A(\psi) + a_{0m}(x, \psi, \nabla\psi) + \lambda_m \psi] \end{aligned}$$

which in view of $u \le \psi$ implies

$$A(u) + a_0(x, u, \nabla u) \ge 0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi) + \sigma_m] \tag{5.29}$$

where

$$\sigma_m := a_0(x, u, \nabla u) - a_{0m}(x, u, \nabla u) - a_0(x, \psi, \nabla\psi) + a_{0m}(x, \psi, \nabla\psi) \,.$$

By (5.11), $\sigma_m \to 0$ a.e. in $\mathcal{O}$ as $m \to \infty$. Therefore the second inequality in (5.10) follows from (5.29) by passing to the limit $m \to \infty$, by (5.4), (5.12) and the dominated convergence theorem, once we notice that

$$\{0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi) + \sigma_m]\}(\zeta) - \{0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi)]\}(\zeta) \le \int |\sigma_m| \zeta \, dx$$

for every non-negative continuous function ζ with compact support in $\mathcal{O}$.

As to the first inequality in (5.10), this follows directly from (1.9), once we replace $v := u - \phi$ for arbitrary $\phi \in C_0^1(\mathcal{O})$, $\phi \ge 0$.

REMARK 5.1. The theorem above extends to arbitrary obstacles $\psi \in H^1(\mathcal{O})$ satisfying (5.9), the proof being the same, provided one has growth conditions with respect to $|r|$, with exponent $s = n/(n-2)$ in (5.3) and $2s$ in (5.4) in case $n > 2$, and with exponents $s = 1$ and $2s = 2$ if $n = 2$.

Then, $a_i(x, v, \nabla v) \in L^2(\mathcal{O})$ and $a_0(x, v, \nabla v) \in L^1(\mathcal{O})$ for arbitrary $v \in H^1(\mathcal{O})$. Notice that ψ is always bounded from below, since $\psi \geq u \in L^\infty(\mathcal{O})$.

REMARK 5.2. By taking the estimate (5.10) into account, the problem of showing further regularity of bounded solutions of the variational inequality (1.9) is clearly reduced to the analogous problem for *equations*

$$A(u) + a_0(x, u, \nabla u) = g\,, \qquad |g| \leqslant |g'|\,, \tag{5.30}$$

for suitable regular g' such that $g' \leq 0 \wedge [A(\psi) + a_0(x, \psi, \nabla\psi)]$ in $\mathcal{O}$. For equations like (5.30) with a term a_0 of quadratic growth classical regularity results are indeed available as given for instance in Ladyzenskaya-Ural'zeva [21] and Tomi [32], cf. also [10].

REMARK 5.3. An alternative way of applying Theorem 5.1 to study the regularity of solutions of (1.9) follows from noticing that a solution $u \in L^\infty(\mathcal{O})$ of (1.9) can be also viewed, « a posteriori », as being a solution of the variational inequality

$$a(u, u - v) \leq (-a_0(x, u, \nabla u), u - v) \qquad \text{for all } v \in Q$$

where the term $-a_0(x, u, \nabla u) \in L^1(\mathcal{O})$ is now considered as given. Thus Theorem 5.1 yields

$$0 \geq A(u) + a_0(x, u, \nabla u) \geq 0 \wedge [A(\psi) + a_0(x, u, \nabla u)] \tag{5.31}$$

from which the estimate

$$|A(u)| \leq |a_0(x, u, \nabla u)| + |0 \wedge A(\psi)|$$

follows, as well as, by the growth condition (5.4)

$$|A(u)| \leq f + c + c|\nabla u|^2 + |0 \wedge A(\psi)|\,. \tag{5.32}$$

From differential inequalities such as (5.32) again further regularity of u can be derived, according to the regularity of f and $0 \wedge A(\psi)$, as also shown in references [32], [10].

The following result, which is of some interest in itself, will be used in Sec. 7 to obtain an estimate of the implicit obstacle of the quasi-variational inequality of impulse control theory.

THEOREM 5.2. *Under the monotonicity and growth assumptions* (5.1) *up to* (5.5), *let $\mathcal{O}'$ be an open subset of $\mathcal{O}$ and let v_α be a family of functions* *satisfying in* $\mathcal{O}'$ *the following properties*

$$v_\alpha \in H^1(\mathcal{O}') \cap L^\infty(\mathcal{O}') \qquad \textit{for every } \alpha \tag{5.33}$$

$$\wedge v_\alpha \in H^1(\mathcal{O}') \cap L^\infty(\mathcal{O}') \tag{5.34}$$

$$\wedge \big(A(v_\alpha) \in H^{-1}(\mathcal{O}') \oplus L^1(\mathcal{O}') \tag{5.35}$$

where the infimum $\wedge$ *is extended over all indeces and in* (5.35) *is intended in the sense of measures in* $\mathcal{O}'$.

Then, we have

$$A(\wedge v_\alpha) \geq \wedge A(v_\alpha) \tag{5.36}$$

in the sense of measures in $\mathcal{O}'$.

Proof. We put

$$\bar{v} = \wedge v_\alpha \quad \text{and} \quad T = \wedge A(v_\alpha) + \lambda \bar{v}$$

where the constant $\lambda > 0$ will be chosen conveniently large.

We then consider the following variational inequality

$$\begin{cases} \Phi \in Q \\ a(\bar{v} + \Phi, \Phi - w) + \lambda(\bar{v} + \Phi, \Phi - w) \leq (T, \Phi - w) \qquad \text{for all } w \in Q \end{cases} \tag{5.37}$$

where

$$Q := \{w \in H_0^1(\mathcal{O}') \cap L^\infty(\mathcal{O}') | 0 \leq w \leq 1 + \|z\|_\infty - z\}, \tag{5.38}$$

and

$$a(v, w) = \sum_{i=1}^{n} (a_i(x, v, \nabla v), w)$$

for every $v \in H^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$ and $w \in H_0^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$.

The existence and uniqueness of the solution Φ, for $\lambda > 0$ large enough, can again be shown by applying the theorem of Appendix B, as in the proof of the previous theorem.

For arbitrary α let us set

$$\Phi_\alpha = v_\alpha - \bar{v}.$$

We have $0 \leq \Phi_\alpha \in H^1(\mathcal{O}')$, hence $\Phi \wedge \Phi_\alpha \in H_0^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$, moreover $0 \leq \leq \Phi \wedge \Phi_\alpha \leq \Phi \leq 1 + \|z\| - z$. Therefore,

$$w := \Phi \wedge \Phi_\alpha \in Q$$

and from (5.37)

$$a(\bar{v}+\Phi, (\Phi-\Phi_\alpha)^+) + \lambda(\bar{v}+\Phi, (\Phi-\Phi_\alpha)^+) \le (T, (\Phi-\Phi_\alpha)^+) . \tag{5.39}$$

On the other hand, by (5.35), since $v_\alpha \ge \bar{v}$,

$$\begin{aligned} &a(v_\alpha, -(\Phi-\Phi_\alpha)^+) + \lambda(v_\alpha, -(\Phi-\Phi_\alpha)^+) \le \\ &\le (\wedge A(v_\alpha), -(\Phi-\Phi_\alpha)^+) + \lambda(\bar{v}, -(\Phi-\Phi_\alpha)^+) = (T, -(\Phi-\Phi_\alpha)^+) . \end{aligned} \tag{5.40}$$

Since $\bar{v}+\Phi-v_\alpha = \Phi-\Phi_\alpha$, it follows from (5.39) and (5.40)

$$a(\Phi-\Phi_\alpha, (\Phi-\Phi_\alpha)^+) \le 0$$

hence, by (5.5), $(\Phi-\Phi_\alpha)^+ = 0$, that is, $\Phi \le \Phi_\alpha$.

Since α was arbitrary, we get

$$0 \le \Phi \le \wedge \Phi_\alpha .$$

On the other hand,

$$\wedge \Phi_\alpha = \wedge (v_\alpha - \wedge v_\beta) = 0 ,$$

thus, $\Phi = 0$.

This shows that (5.37) reduces to

$$a(\bar{v}, w) + \lambda(\bar{v}, w) \le (T, w) \quad \text{for every } w \in Q .$$

By cancelling the term $\lambda(\bar{v}, w)$ from both sides, we finally get

$$a(\bar{v}, w) \le (\wedge A(v_\alpha), w) \quad \text{for every } w \in Q . \tag{5.41}$$

For arbitrary $\zeta \in C_0^1(\mathcal{O}')$, $\zeta \ge 0$, we have $w := t\zeta \in Q$ if $t > 0$ is small enough, therefore (5.41) implies

$$A(\bar{v}) \ge \wedge A(v_\alpha)$$

as measures in $\mathcal{O}'$ and the theorem has been proved.

REMARK 5.4. The assumption $v_\alpha \in L^\infty(\mathcal{O}')$ for every α can be dropped provided in (5.3) a growth condition in $|r|$ is satisfied as in Remark 5.1. However, by (5.34), all v_α are still required to be bounded from below.

6. – The control problem: Hölder regularity.

We consider now the quasi-variational inequality of impulse control theory, see A. Bensoussan and J. L. Lions [1], [2]:

$$
(6.1)\qquad \begin{cases} u \in H_0^1(\mathcal{O})\,, \quad u \le M(u) \\ \displaystyle\sum_{i,k=1}^{n} \big(a_{ik}(x) D_k u,\, D_i u - D_i v\big) \le \big(H(x, u, \nabla u),\, u - v\big) \\ \textit{for all } v \in H_0^1(\mathcal{O}),\ v \le M(u),\ v - u \in L^\infty(\mathcal{O}) \end{cases}
$$

where

(6.2) *$\mathcal{O}$ is a bounded open subset of $\boldsymbol{R}^n$*

(6.3) *a_{ik} are bounded measurable functions in $\mathcal{O}$, $i, k = 1, \ldots, n$.*

$$
(6.4)\qquad \sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \ge \alpha_0 |\xi|^2
$$

for almost all $x \in \mathcal{O}$ and all $\xi \in \boldsymbol{R}^n$ with some constant $\alpha_0 > 0$.

In the control case, the Hamiltonian $H(x, r, p)$ is of the form (0.11) of the introduction. We assume here only that

$$
H\colon \mathcal{O} \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n
$$

is a function satisfying

(6.5) *$H(x, r, p)$ is measurable in x and continuous in $(r, p) \in \boldsymbol{R}^{n+1}$*

$$
(6.6)\qquad |H(x, r, p)| \le K + K|p|^2
$$

for every $x \in \mathcal{O}$, $|r| \le O$, $p \in \boldsymbol{R}^n$, with K a suitable constant possibly depending on C.

For conditions on f and g in (0.11) implying the required property of H we refer for instance to [2].

The implicit obstacle $\psi = M(u)$ is given by

$$
(6.7)\qquad M(u)(x) = k + \operatorname{essinf}\{c(x, \xi) + u(x + \xi) | \xi \ge 0,\, x + \xi \in \mathcal{O}\}
$$

where k is some given constant

$$
(6.8)\qquad k \ge 0
$$

and $c(\cdot,\cdot)\colon \overline{\mathcal{O}}\times \boldsymbol{R}\colon \to \boldsymbol{R}_+$ is a given function satisfying

$$c(x, \xi+\eta) \leq c(x, \xi) + c(x+\xi, \eta) \tag{6.9}$$

for every $x \in \overline{\mathcal{O}}$, all $\xi \geq 0$ such that $x+\xi \in \overline{\mathcal{O}}$ and all $\eta \geq 0$, and, moreover, such that for every $x \in \mathcal{O}$ and given $\delta > 0$

$$0 \leq c(x, \xi) \leq \delta \quad \textit{for all } |\xi| \leq \sigma,\ \xi \geq 0 \tag{6.10}$$

with $\sigma = \sigma_\delta$ a suitable constant independent of $x \in \mathcal{O}_0 \subset\subset \mathcal{O}$.

We shall also consider the stronger condition

$$0 \leq c(x, \xi) \leq \bar{c}|\xi|^\alpha \tag{6.11}$$

for every $x \in \mathcal{O}$ and $|\xi| \leq \sigma_0$, $\xi \geq 0$, where $\bar{c} > 0$ and $\alpha \in]0, 1[$ are given constants, and $\sigma_0 > 0$ is some constant independent of $x \in \mathcal{O}_0 \subset\subset \mathcal{O}$.

We now introduce the functions $\gamma_\pm$ on $\overline{\mathcal{O}}$ by setting, for every $x \in \overline{\mathcal{O}}$

$$\gamma_-(x) = \inf\{c(y, x-y) | y \in \Gamma_-(x)\}$$

where

$$\Gamma_-(x) = \{y \in \Gamma | y = x - \eta,\ \eta > 0\}$$

and

$$\gamma_+(x) = \inf\{c(x, z-x) | z \in \Gamma_+(x)\}$$

where

$$\Gamma_+(x) = \{z \in \Gamma | z = x + \xi,\ \xi \geq 0\}.$$

We remark in passing that since $c(x, \xi)$ is interpreted as a *cost* of shifting the state x of the controlled system into the state $x+\xi$, we interpret $\gamma_-(x)$ as the minimum cost for attaining the state x from the boundary, and $\Gamma_+(x)$ as the minimum cost for attaining the boundary from x, only non-negative shifts being allowed.

For proving the boundary regularity of u, we also need the following assumptions

$$\Gamma \quad \textit{is Lipschitz} \tag{6.12}$$

$$\gamma_- \in \mathrm{Lip}\,(\mathcal{O}). \tag{6.13}$$

As it will become clear from the proof below, it is sufficient to assume that

$$\gamma_- \in H^{1,s}(\mathcal{O}) \quad \text{for some } s > n . \tag{6.14}$$

Using the results of the previous sections, we now prove the following theorems

THEOREM 6.1. *Let u be a solution of problem* (6.1). *Under the assumptions* (6.2), *up to* (6.10), *u is continuous in $\mathcal{O}$ and, if* (6.11) *also holds, then u is Hölder continuous in $\mathcal{O}$. Moreover, u is bounded and*

$$-k - \gamma_-(x) \leq u(x) \leq k + \gamma_+(x) \tag{6.15}$$

for all $x \in \mathcal{O}$.

THEOREM 6.2. *In addition to the assumptions of theorem* 6.1 *suppose that* (6.12) *and* (6.13) *are satisfied. Then, u is continuous up to the boundary and, if* (6.11) *also holds, then u is Hölder continuous on $\overline{\mathcal{O}}$.*

The proof of these theorems depends on some preliminary lemmas

LEMMA 6.1. *Let u be any solution of* (6.1). *Then, the inequalities* (6.15) *hold, for a.e. $x \in \mathcal{O}$.*

PROOF. Since $u = 0$ on Γ in H^1-sense, we have for almost all $x \in \mathcal{O}$

$$u(x) \leq M(u)(x) \leq k + c(x, z - x) \tag{6.16}$$

for all $z \in \Gamma_+(x)$ and the second inequality of (6.15) follows. On the other hand, for all $y \in \Gamma_-$ and almost all $x \in \mathcal{O}$

$$0 = u(y) \leq M(u)(y) \leq k + c(y, x - y) + u(x) \tag{6.17}$$

where we have used a suitable representative a.e. of u which vanishes everywhere on Γ. Note that $M(u)$ is defined pointwise on $\overline{\mathcal{O}}$ and does not depend on the representative a.e. of u. From (6.17), the left hand side inequality of (6.15) follows.

Since γ_- and γ_+ are bounded on $\mathcal{O}$, it follows from Lemma 6.1. the following

LEMMA 6.2. $u \in L^\infty(\mathcal{O})$.

We now prove

LEMMA 6.3. *Let $\psi(x) = M(u)(x)$, $x \in \overline{\mathcal{O}}$, where u is a solution of* (6.1). *Then*

$$\psi(y) \geq \psi(x) - c(x, y - x) \quad \textit{for all } y \geq x,\ x, y \in \overline{\mathcal{O}} . \tag{6.18}$$

Moreover,

$$-\gamma_-(x) \leq \psi(x) \leq k + \gamma_+(x) \quad \textit{for all } x \in \overline{\mathcal{O}} . \tag{6.19}$$

PROOF. For every $x \in \overline{\mathcal{O}}$, $y = x + \xi \in \overline{\mathcal{O}}$ and almost all $\eta \geq 0$ we have, in consequence of (6.9)

$$\begin{aligned} M(u)(x) &\leq k + c(x, \xi + \eta) + u(x + \xi + \eta) \leq \\ &\leq c(x, \xi) + k + c(y, \eta) + u(y + \eta) , \qquad \text{thus} \\ M(u)(x) &\leq c(x, \xi) + M(u)(y) ; \end{aligned}$$

hence (6.18) holds.

In turn, inequality (6.18) for all $x \in \overline{\mathcal{O}}$ and every $y \in \Gamma_-(x)$ implies

$$0 = u(y) \leq \psi(y) \leq c(y, x - y) + \psi(x)$$

where u is an a.e. representative of u vanishing everywhere on Γ. The above inequality clearly implies the left hand inequality in (6.19) while the right hand inequality follows from (6.16).

PROOF OF THEOREM 6.1. We apply theorem 3.1 to the variational inequality (6.1) with $\psi = M(u)$. We remark that, by Lemma 6.2, $u \in L^\infty(\mathcal{O})$, which also implies $\psi = M(u) \in L^\infty(\mathcal{O})$. Moreover, it is easy to check, by lemma 6.3, that ψ satisfies the hypothesis (3.1) of theorem 3.1. We do this by choosing for every $x_0 \in \mathcal{O}$ and $R > 0$:

$$T_R(x_0) = (x_0 - 2\,\mathrm{Re} - \boldsymbol{R}_+^n) \cap B_{4R}(x_0)$$

$e = (1, \ldots, 1) \in \boldsymbol{R}^n$. In fact, $x \in T_R(x_0)$ and $y \in B_{2R}(x_0)$ imply $x \leq y$, hence, by (6.18) and (6.10), for every $\delta > 0$:

$$\psi(y) \geq \psi(x) - c(x, y - x) \geq \psi(x) - \delta$$

provided $|y - x| \leq \sigma_\delta$.

This shows the continuity of u in $\mathcal{O}$. For proving the Hölder continuity of u, we apply theorem 3.2 again to (6.1) with $\psi = M(u)$, by checking that assumption (3.6) of that theorem is now a consequence of (6.11).

PROOF OF THEOREM 6.2. We apply theorem 4.1 and 4.2 to (6.1) with $\psi = M(u)$.

Assumption (4.2) can be satisfied, in consequence of (6.13), by taking

$u_0 = \gamma_-$ and noticing that $\gamma_-(x) = 0$ for all $x \in \Gamma$ due to the fact that $x \in \Gamma$ implies $x \in \Gamma_-(x)$ and hence $0 \leq \gamma_-(x) \leq c(x, 0) = 0$.

REMARK. The theorems above, due to the generality of the theorems 4.1 and 4.2 on which they rely, also hold if the linear operator

$$A(v) = -\sum_{i,k=1}^{n} D_i\big(a_{ik}(x) D_k v\big) \tag{6.20}$$

appearing in (6.1) is replaced by a non linear operator

$$A(v) = -\sum_{i,k=1}^{n} D_i a_i(x, v, \nabla v) \tag{6.21}$$

as in Sec. 1. The functions a_i are then assumed to satisfy the coerciveness and growth conditions (1.2), ..., (1.6), all other assumptions of Theorems 6.1 and 6.2 remaining unchanged.

7. – A dual estimate for the operator M.

We consider a second order operator $A(v)$ and a first order operator $a_0(x, v, \nabla v)$, satisfying all the assumptions of Sec. 5.

Our main goal here is to establish a one-sided dual estimate for the operator M introduced in Sec. 6, i.e.,

$$M(v)(x) = k + \operatorname{essinf}\{c(x, \xi) + u(x + \xi) | \xi \geq 0, x + \xi \in \mathcal{O}\}\,, \qquad x \in \overline{\mathcal{O}}\,.$$

Such an estimate plays a crucial role in our approach to the regularity control problem we deal with in Sec. 8.

Let u be a function of the space $H_0^1(\mathcal{O})$, which is continuous in $\overline{\mathcal{O}}$ and whose second order derivatives are locally bounded up to the boundary of $\mathcal{O}$ in the region $\mathcal{C}^u$ of $\overline{\mathcal{O}}$ where $u(x) < M(u)(x)$.

We shall prove then that $M(u)$ satisfies a Lipschitz condition on the whole of $\overline{\mathcal{O}}$ and, moreover, that $M(u)$ can be dually estimated from below by a constant, in the sense of Sec. 5, that is,

$$A\big(M(u)\big) + a_0\big(x, M(u), \nabla M(u)\big) \geq -c_0 \tag{7.1}$$

as a distribution in $\mathcal{O}$, where $c_0 > 0$ is some constant (that may depend on u). Therefore, the original regularity information about u is somehow transported over all $\mathcal{O}$. We assume that the given constant k and the given function $c(\cdot,\cdot)$ appearing in the expression of M satisfy the following conditions:

$$k > 0 \tag{7.2}$$

$$c(x, \xi + \eta) \leq c(x, \xi) + c(x + \xi, \eta) \quad \text{for every } x \in \mathcal{O}\,, \tag{7.3}$$

all ξ such that $x + \xi \in \overline{\mathcal{O}}$ and all $\eta \geq 0$.

$$c(x, \xi) \text{ is non-decreasing in } \xi \text{ for each } x \in \overline{\mathcal{O}} \tag{7.4}$$

$$c(\cdot, \xi) \in W^{2,\infty}(\mathcal{O}) \text{ for each } \xi, \text{ and } \|D_i D_j c(\cdot, \xi)\|_\infty \leq C \tag{7.5}$$

uniformly with respect to bounded sets of vectors ξ.

We also assume that

$$\gamma_+ \in \mathrm{Lip}\,(\overline{\mathcal{O}}) \quad \text{and} \quad A(\gamma_+ + d) \geq -c \quad \text{for } d \in [-d_0, 0]\,, \tag{7.6}$$

$d_0 > 0$ and for some constant $c \geq 0$, as a distribution in $\mathcal{O}$, where γ_+ is the function defined in Sec. 6. Notice that, because of the definition of γ_+ involving the boundary Γ of $\mathcal{O}$, some regularity of Γ (such as (6.12)) is implicitely admitted by (7.6).

We then have the following result, where

$$\mathcal{C}^u := \{x \in \overline{\mathcal{O}} | u(x) < M(u)(x)\}\,. \tag{7.7}$$

THEOREM 7.1. *Under the assumptions* (1.1), ..., (1.8), (5.5), *and* (7.2), ..., (7.6), *let u be any function satisfying*

$$u \in H^1_0(\mathcal{O}) \cap C(\mathcal{O}) \cap W^{2,\infty}(D) \tag{7.8}$$

for every open subset D, $\overline{D} \subset \mathcal{C}^u$. Then we have $M(u) \in \mathrm{Lip}\,(\overline{\mathcal{O}})$ and the estimate (7.1) *also holds.*

The proof relies on the dual estimate of Theorem 5.2 and comes from the following lemmas 7.1, 7.2, 7.3. Lemma 7.1 yields a local representation of $M(u)$ based on an idea from Caffarelli-Friedman [7], see also [29]. The proof of Lemma 7.3 adapts the dual estimation techniques from [19] and [29] to the present case.

LEMMA 7.1. *Let $u \in H^1_0(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ and $\hat{u}$ be the extension of u by zero over $\boldsymbol{R}^n - \{0\}$. Then, for every $z \in \overline{\mathcal{O}}$ there exists a ball $B_\varrho(z)$, $\varrho > 0$, a subset*

$T_z \subset \boldsymbol{R}^n_+$ and an open subset $V_z \subset \boldsymbol{R}^n$, such that

$$M(u)(x) = k + \inf\{c(x, \xi) + \hat{u}(x + \xi) | \xi \in T_z\} \tag{7.9}$$

for every $x \in B_\delta(z)$

$$B_\delta(z) + T_z \subset V_z \tag{7.10}$$

$$\overline{V_z \cap \mathcal{O}} \subset \mathcal{C}^u . \tag{7.11}$$

PROOF. For every $x \in \overline{\mathcal{O}}$ we denote by $\Sigma(x)$ the set of all $y \in \overline{\mathcal{O}}$ of the form $y = x + \eta$, $\eta \geq 0$ such that

$$c(x, \eta) + u(x + \eta) = \min\{c(x, \xi) + u(x + \xi) | \xi \geq 0, x + \xi \in \overline{\mathcal{O}}\} .$$

Then, we have

$$\Sigma(x) \subset \mathcal{C}^u . \tag{7.12}$$

In fact, if $y \in \Sigma(x)$, we have by (7.3)

$$k + c(x, \eta) + u(x + \eta) = M(u)(x) \leq c(x, \eta) + M(u)(y) .$$

Since $k > 0$, this implies $u(y) < M(u)(y)$, that is, $y \in \mathcal{C}^u$.

It follows from (7.12) that $\Sigma(z)$ is a compact subset of $\mathcal{C}^u$ therefore we can choose $\delta_0 > 0$ so that the open subset of $\boldsymbol{R}^n$

$$V_z = \Sigma(z) + B_{\delta_0} ,$$

where $B_\delta := B_\delta(0)$ for arbitrary δ, satisfies condition (7.11).

We now choose δ_1 so that $0 < \delta_1 < \delta_0$. Since the multivalued mapping $x \to \Sigma(x)$ is upper semicontinuous, there exists $\varrho_1 > 0$ such that

$$\Sigma(x) \subset \Sigma(z) + B_{\delta_1} \tag{7.13}$$

for all $x \subset B_{\varrho_1}(z) \cap \mathcal{O}$. Hence (7.13) holds for all $x \in B_\varrho(z) \cap \mathcal{O}$, where $\varrho > 0$ has been chosen such that $0 < \varrho < \varrho_1$ and, moreover,

$$2\varrho + \delta_1 < \delta_0 . \tag{7.14}$$

From (7.13) we obtain

$$\Sigma(x) - x \subset \Sigma(z) - z + B_{\varrho + \delta_1} .$$

Therefore, if we define

$$T_z = \{\Sigma(x) - x | x \in B_\varrho(z) \cap \mathcal{O}\} \tag{7.15}$$

we find

$$T_z \subset \Sigma(z) - z + B_{\varrho + \delta_1}.$$

Hence

$$B_\varrho(z) + T_z \subset \Sigma(z) + B_{2\varrho+\delta_1} \subset \Sigma(z) + B_{\delta_0},$$

the last inclusion being a consequence of (7.14).

Thus, condition (7.10) is also satisfied. Finally, since $\Sigma(x) - x \subset T_z$ for every $x \in B_\varrho(z) \cap \mathcal{O}$ and $c(x, \cdot)$ is non-decreasing, we have

$$M(u)(x) = k + c(x, \eta) + u(x + \eta) \leq k + c(x, \xi) + \hat{u}(x + \xi)$$

for every $\eta \in \Sigma(x) - x$ and all $\xi \in T_z$. Hence (7.9) also holds.

LEMMA 7.2. *Let $u \in H_0^1(\overline{\mathcal{O}}) \cap C(\overline{\mathcal{O}}) \cap \mathrm{Lip}\,(D)$ for every open subset D, $\overline{D} \subset \mathcal{C}^u$. Then*

$$M(u) \in \mathrm{Lip}\,(\overline{\mathcal{O}}). \tag{7.16}$$

PROOF. It suffices to prove that for every $z \in \mathcal{O}$ there exists $\varrho > 0$ such that

$$M(u) \in \mathrm{Lip}\,\left(\overline{B_\varrho(z) \cap \mathcal{O}}\right).$$

We choose $B_\varrho(z)$, T_z and V_z as in Lemma 7.1. By (7.11), we have

$$u \in \mathrm{Lip}\,(\overline{V_z \cap \mathcal{O}})$$

and hence the functions

$$v_\xi(x) := c(x, \xi) + \hat{u}(x + \xi), \tag{7.17}$$

in consequence of (7.20), satisfy a Lipschitz condition on $\overline{B_\varrho(z) \cap \mathcal{O}}$ which is uniform with respect to $\xi \in T_z$. Therefore, property (7.16) follows from the above representation, (7.9), of $M(u)$.

We have shown, in particular, that for u as in Lemma 7.2 we have $M(u) \in H^1(\mathcal{O})$. Therefore, $A(M(u))$ is well defined as a distribution in $\mathcal{O}$ and it belongs to the dual space $H^{-1}(\mathcal{O})$ of $H_0^1(\mathcal{O})$.

LEMMA 7.3. *Let $u \in H_1^0(\mathcal{O}) \cap C(\overline{\mathcal{O}}) \cap W^{2,\infty}(D)$ for every open subset D,*

$\overline{D} \subset \mathcal{C}^u$. *Then, there exists some constant* $c_0 > 0$ *such that*

$$A\big(M(u)\big) + a_0\big(x, M(u), \nabla(M(u))\big) \geq -c_0 \tag{7.18}$$

as a distribution in $\mathcal{O}$.

PROOF. It clearly suffices to prove that for every $z \in \overline{\mathcal{O}}$ there exists $\varrho > 0$ such that

$$A\big(M(u)\big) + a_0\big(x, M(u), \nabla(M(u))\big) \geq -c \tag{7.19}$$

as a distribution in $B_\varrho(z) \cap \mathcal{O}$, where $c > 0$ is some constant possibly depending on z.

We choose $B_\varrho(z)$, T_z, V_z again as in Lemma 7.1. Let γ_+ be the function introduced in Sec. 6. Since

$$M(u) \leq k + \gamma_+ ,$$

we can also write the local representation (7.9) of $M(u)$ as

$$M(u)(x) = k + \inf\{v_\xi(x) \wedge \gamma_+(x) | \xi \in T_z\} \tag{7.20}$$

for every $x \in B_\varrho(z)$, where v_ξ is given by (7.17).

For $\xi \in T_z$ and $\varepsilon > 0$, we define

$$w_{\xi,\varepsilon}(x) = v_\xi(x) \wedge [\gamma_+(x) - \varepsilon]$$

as a function on $B_\varrho(z) \cap \mathcal{O}$. For every fixed $\varepsilon > 0$, we have

$$B_\varrho(z) \cap \mathcal{O} = \mathcal{O}_1^{\xi,\varepsilon} \cup \mathcal{O}_2^{\xi,\varepsilon} \tag{7.21}$$

where $\mathcal{O}_i^{\xi,\varepsilon}$, $i = 1, 2$, are the open subsets

$$\mathcal{O}_1^{\xi,\varepsilon} = \left\{x \in B_\varrho(z) \cap \mathcal{O} | v_\xi(x) < \gamma_+(x) - \frac{\varepsilon}{2}\right\}$$
$$\mathcal{O}_2^{\xi,\varepsilon} = \{x \in B_\varrho(z) \cap \mathcal{O} | v_\xi(x) > \gamma_+(x) - \varepsilon\} .$$

We first estimate the distribution

$$A(w_{\xi,\varepsilon}) + a_0(x, w_{\xi,\varepsilon}, \nabla w_{\xi,\varepsilon})$$

from below in $\mathcal{O}_1^{\xi,\varepsilon}$.

By (7.11) we have

$$\xi + \mathcal{O}_1^{\xi,\varepsilon} \subset V_z \cap \mathcal{O}\,.$$

Therefore, by (7.11), all functions v_ξ, $\xi \in T_z$, are bounded in $\mathcal{O}_1^{\xi,\varepsilon}$, with their first and second order derivatives, by constants that depend neither on $\xi \in T_z$ nor $\varepsilon > 0$. It follows that the distribution

$$A(v_\xi) - a_0(x, v_\xi, \nabla v_\xi)$$

is also bounded from below in $\mathcal{O}_1^{\xi,\varepsilon}$, by a constant independent of both ξ and ε.

Therefore, we can apply Theorem 5.2 to the functions v_ξ and $(\gamma_+ - \varepsilon)$ in $\mathcal{O}' := \mathcal{O}_1^{\xi,\varepsilon}$ and we obtain, by taking (7.6) into account, that

$$A(w_{\xi,\varepsilon}) + a_0(x, w_{\xi,\varepsilon}, \nabla w_{\xi,\varepsilon}) \geq -c \tag{7.22}$$

as a distribution in $\mathcal{O}_1^{\xi,\varepsilon}$, for some constant $c > 0$ independent of $\xi \in T_z$ and $\varepsilon < \varepsilon_0$.

We now estimate the distribution above, this time over $\mathcal{O}_2^{\xi,\varepsilon}$.

Noticing that $v_\xi(x) > \gamma_+(x) - \varepsilon$ in $\mathcal{O}_2^{\xi,\varepsilon}$, we obtain from (7.6) that

$$A(w_{\xi,\varepsilon}) + a_0(x, w_{\xi,\varepsilon}, \nabla w_{\xi,\varepsilon}) \geq -c\,, \tag{7.23}$$

as a distribution in $\mathcal{O}_2^{\xi,\varepsilon}$, where the constant c is independent of $\xi \in T_z$ and $\varepsilon < \varepsilon_0$.

It follows from (7.21) and (7.23) that

$$A(w_{\xi,\varepsilon}) + a_0(x, w_{\xi,\varepsilon}, \nabla w_{\xi,\varepsilon}) \geq -c \tag{7.24}$$

as a distribution in $B_\varrho(z) \cap \mathcal{O}$, with c a constant independent of $\xi \in T_z$ and $\varepsilon < \varepsilon_0$.

By letting $\varepsilon \to 0$, we obtain from (7.24) that

$$A(v_\xi \wedge \gamma_+) + a_0\big(x, v_\xi \wedge \gamma_+, \nabla(v_\xi \wedge \gamma_+)\big) \geq -c \tag{7.25}$$

in $B_\varrho(z) \cap \mathcal{O}$, with c a constant independent of $\xi \in T_z$.

We now consider the family of functions

$$k + v_\xi \wedge \gamma_+\,, \qquad \xi \in T_z\,.$$

They satisfy a uniform Lipschitz condition on $\overline{\mathcal{O}}'$, where $\mathcal{O}' := B_\varrho(z) \cap \mathcal{O}$, and their infimum over $\xi \in T_z$, belongs to $H^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$, by (7.20) and (7.16). Moreover, they are uniformly estimated in $\mathcal{O}'$ according to (7.25). Therefore, the estimate (7.19) follows from Theorem 5.2 and the proof of the lemma is concluded.

Let us now consider the case that the operator A *commutes with translations*, by assuming that

$$\text{(7.26)} \qquad \textit{The functions } a_i \textit{ in } (5.2) \textit{ do not depend on } x\,.$$

Then, according to a similar estimate in [19], the following variant of Theorem 7.1 holds:

THEOREM 7.2. *In addition to* (1.1), ..., (1.8), (5.5) *and* (7.2), ..., (7.6), *let us suppose that* (7.26) *also holds. Let* u *be an arbitrary function satisfying*

$$\text{(7.27)} \qquad u \in H_0^1(\mathcal{O}) \cap C(\overline{\mathcal{O}})\,,$$

$$\text{(7.28)} \qquad u \in \operatorname{Lip}(\overline{D}) \quad \textit{and} \quad A(u) \geq -c$$

for every open subset D, $\overline{D} \subset \mathcal{C}^u$ *and a suitable constant* $c > 0$, *possibly depending on* D. *Then,* $M(u) \in \operatorname{Lip}(\overline{\mathcal{O}})$ *and the estimate* (7.1) *also holds.*

The proof of the present theorem is the same as that of Theorem 7.1 once we remark that in the proof of Lemma 7.3 above the distribution

$$A(v_\xi) - a_0(x, v_\xi, \nabla v_\xi)$$

can be now estimated in the open set $\mathcal{O}_1^{\xi,\varepsilon}$ by using (7.28) and noticing that $A(v_\xi)(x) = A(v)(x + \xi)$.

8. – The control problem: $W^{2,p}$ regularity.

We come back now to the control problem of Sec. 6, that is, to the quasi-variational inequality (6.1).

Under suitable assumptions on the data we shall prove that every solution u of (6.1) satisfies

$$\text{(8.1)} \qquad u \in W^{2,p}(\mathcal{O}) \quad \text{for every } p \geq 2\,,$$

so that we obtain, in particular, the following global regularity result

$$\text{(8.2)} \qquad u \in C^{1,\alpha}(\overline{\mathcal{O}}) \quad \text{for all } \alpha \in \left]0, 1\right[\,.$$

We shall prove indeed more, namely that

$$A(u) \in L^\infty(\mathcal{O}) \tag{8.3}$$

where A is the second order operator (6.20).

Our proof relies both on the continuity results of Sec. 6 and on the dual estimates of Sec. 5 and Sec. 7.

In order to estimate the implicit obstacle $M(u)$ of (6.1) according to Theorem 7.1, we must verify, in particular, that u has second order derivatives which are bounded on every open set D, $\overline{D} \subset \mathcal{C}^u$, $\mathcal{C}^u$ being the set (7.7). Since by (6.1) the solution u satisfies the equation

$$A(u) = H(x, u, \nabla u) \tag{8.4}$$

in the (open) region $\mathcal{C}^u \cap \mathcal{O}$ in the distribution sense, the above regularity information about u can be obtained from the existing regularity theory for bounded weak solutions of equation (8.4), see [21], [32].

What we actually need in this respect is the following property:

(∗) Let $\mathcal{O}'$ be an open subset of $\mathcal{O}$, Γ' a smooth part of $\partial\mathcal{O}$ and let $\tilde{u} \in H_0^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$ be such that

$$\sum_{i,k=1}^{n} \left(a_{i,k}(x) D_i \tilde{u}, D_k v\right) = \left(H(x, \tilde{u}, \nabla\tilde{u}), v\right) \tag{8.5}$$

for every $v \in H_0^1(\mathcal{O}') \cap L^\infty(\mathcal{O}')$. Then

$$\tilde{u} \in W^{2,\infty}(D) \tag{8.6}$$

for every open region D, $\overline{D} \subset \mathcal{O}' \cup \Gamma'$.

According to the references quoted above, sufficient conditions ensuring property (∗) are the following ones

$$\Gamma = \partial\mathcal{O} \quad \text{is of class } C^{2,\beta} \tag{8.7}$$

$$a_{i,k} \in C^{1,\beta}(\overline{\mathcal{O}}), \quad i, k = 1, \ldots, n \tag{8.8}$$

$$H \in C^{0,\beta}(\overline{\mathcal{O}}, \boldsymbol{R}, \boldsymbol{R}^n) \tag{8.9}$$

for some $\beta \in \,]0, 1]$, in addition to those of Sec. 6.

Then we have the following

THEOREM 8.1. *Suppose that the coerciveness and growth conditions* (6.2), ..., (6.6) *and the regularity conditions* (8.7), .., (8.9) *for* Γ, a_{ik} *and* H

are satisfied. Suppose also that the assumptions (6.10), (6.13) *and* (7.2), ..., (7.6) *on the operator M are satisfied. Then every solution u of the quasi-variational inequality* (6.1) *has the regularity* (8.1), ..., (8.3).

PROOF. By Theorem 6.2 u is continuous on $\overline{\mathcal{O}}$. Since in addition $u = 0$ on Γ, it is easily checked from the definition (6.7) that $M(u)$ also is continuous on $\overline{\mathcal{O}}$.

We now consider the region $\mathcal{C}^u$ defined by (7.7) and the open set $\mathcal{O}' := \mathcal{C}^u \cap \mathcal{O}$. It follows from (6.1) that the restriction $\tilde{u}$ of u to $\mathcal{O}'$ is a (bounded weak) solution of the equation (8.4).

Therefore, for any given open set D, $\overline{D} \subset \mathcal{C}^u$, by choosing Γ' to be a smooth part of Γ such that $\overline{D} \cap \Gamma \subset \Gamma' \subset \partial\mathcal{O}' \cap \Gamma$, we obtain from property $(*)$ that $u \in W^{2,\infty}(D)$.

Thus u satisfies all the assumptions of Theorem 7.1, so we obtain that $M(u)$ is Lipschitz on $\overline{\mathcal{O}}$ and, furthermore, it can be estimate from below according to (7.1).

Therefore, we can apply Theorem 5.1 to the variational inequality (6.1) and we get

$$0 \geq A(u) - H(x, u, \nabla u) \geq -c_0 \tag{8.10}$$

in the distribution sense in $\mathcal{O}$.

The known regularity results for equations like

$$A(u) = H(x, u, \nabla u) + g\,, \qquad \text{with } g \in L^\infty(\mathcal{O})\,, \tag{8.11}$$

(see again the references quoted above) imply the regularity (8.1) of u, which in turn, jointly with (8.10), immediately gives (8.3). ∎

If the operator A has constant coefficients a_{ik}, then the further regularity assumption (8.9) on H can be dropped. In fact, *via* property $(*)$, it ensures the $W^{2,\infty}$ regularity of u on D as above, as needed in Theorem 7.1 for the lower estimate of $A(M(u))$. In case of constant a_{ik} we can get the same estimate of $A(M(u))$ by relying on Theorem 7.2 instead than on Theorem 7.1 as in the proof above.

Therefore, by only assuming now the further regularity

$$\Gamma = \partial\mathcal{O} \quad \text{is of class } \mathcal{C}^2 \tag{8.12}$$

(which is enough for $W^{2,p}$ regularity), we obtain

THEOREM 8.2. *Suppose that the coefficients a_{ik}, $i, k = 1, ..., n$ of the operator A are constant and that the coerciveness and growth conditions* (6.2), (6.4), ..., (6.6) *and the regularity condition* (8.12) *are satisfied. Then, every solution* u *of the q.v.i.* (6.1) *has the regularity* (8.1), ..., (8.3).

PROOF. From the equation (8.4), which is satisfied on every open $D \subset\subset \mathcal{C}^u$, we still obtain under our present assumptions on A and H that,

$$A(u) \in L^\infty(D)$$

as it follows, from instance, from Tomi [32]. This also implies, in particular, that $u \in \operatorname{Lip}(\bar{D})$, so that we can apply Theorem 7.2 to prove that $M(u) \in \operatorname{Lip}(\bar{\mathcal{O}})$ and that $A(M(u))$ is estimated from below according to (7.1). The remaining of the proof is the same as above. ■

REMARK. Theorem 8.1 also holds if more general nonlinear operators A of the form (5.3) are considered in place of the linear A appearing above. All the results of previous sections used in the proof still hold in this general case (see also the remark at the end of Sec. 6). The only point to be checked again is the analogue of property (*) stated above. If the functions a_i appearing in the expression (5.3) of A are independent of ∇v, the required regularity result (8.6) follows from [21], [32], [10], under the assumptions of Sec. 5. For the general case one has to adapt the methods of [21], [32] and [8], see [10].

Appendix A. Poincaré's inequality.

In the following, we consider concentric balls B_{mR}, B_{kR} with $m > k \geq 0$ fixed numbers and the set $S_R := B_{mR} - B_{kR}$. The integration over S_R is denoted by $\int\limits_R$. We need the notion of the *relative capacity* of a set $T \subset B_{mR}$ which is defined by

$$B_{2mR}\text{-cap } T = \sup \{B_{2mR}\text{-cap } K | K \subset T,\ K \text{ compact}\}$$

and for compact sets K

$$B_{2mR}\text{-cap } K = \inf \Big\{ \int\limits_{2mR} |\nabla\phi|^2 dx \mid \phi \in C_0^\infty(B_{2mR}),\ \phi \geq 1 \text{ on } K \Big\}.$$

Here and in the following, H^1-functions are considered to be defined everywhere except a set of capacity 0. For $u \in H^1(B_{mR})$ we can define the « capacity-essential » maximum of u on T as the infimum of all « capacity-essential » upper bounds of u on T. A number k is called « capacity-essential » upper bound of u on T if $k \geq u$ on T except of a set of capacity 0. We shortly write c-sup $\{u(x)|x \in T\}$ or $c\text{-}\sup\limits_{T} u$.

The « capacity-essential » infimum c-inf $\{u(x)tx \in T\}$ is defined analogously. We need Poincaré's inequality in the following form.

THEOREM A1. *Let $T_R \subset S_R$ be sets whose capacity satisfies*

$$B_{2mR}\text{-cap } T_R \geq cR^{n-2}$$

with a constant $c > 0$ as $R \to 0$. Then there is a constant K independent of R such that

$$\inf\left\{\int_R |u-d|^2 dx | M_0 \leq d \leq M_1\right\} \leq KR^2 \int_R |\nabla u|^2 dx \tag{1}$$

for all $u \in H^1(B_{mR})$. Here

$$M_0 = c\text{-inf } \{u(x)|x \in T_R\}$$

$$M_1 = c\text{-sup } \{u(x)|x \in T_R\}.$$

PROOF. Because of homogeneity, we need to prove the theorem only for $R = 1$, $T = T_1$. We first show that there exists $d_0 \in I = [M_0, M_1]$ such that the B_{2m}-capacity of both the sets

$$\begin{cases} \{x \in T | u(x) - d_0 \geq 0\} \\ \{x \in T | u(x) - d_0 \leq 0\} \end{cases} \tag{2}$$

is larger than $c/4$. In fact, define

$$d_0 = \sup\left\{\theta \in I \middle| B_{2m}\text{-cap } \{x \in T | u(x) - \theta \geq 0\} \geq \frac{c}{4}\right\}.$$

The set whose supremum is d_0 is not empty since $\theta = M_0$ is admissible. Thus d_0 exists and satisfies

$$B_{2m}\text{-cap } \{x \in T | u(x) - d_0 \geq 0\} \geq \frac{c}{4} \tag{3}$$

on account of the upper semi-continuity properties of the capacity.

If $d_0 = M_1$ then

$$\{x \in T | u(x) - d_0 \leq 0\} = T \tag{4}$$

and the B_{2m}-capacity of both the sets (2) is larger than $c/4$ on account of (3) and (4). Recall that B_{2m}-cap $T \geq c$.

If $d_0 < M$ then

$$B_{2m}\text{-cap}\ \{x \in T|u(x) - (d_0 + \varepsilon) \geq 0\} \leq \frac{c}{4}\,.$$

for $\varepsilon \to 0$, $\varepsilon > 0$, and hence

$$B_{2m}\text{-cap}\ \{x \in T|u(x) - d_0 > 0\} \leq \frac{c}{4}\,. \tag{5}$$

Suppose that

$$B_{2m}\text{-cap}\ \{x \in T|u(x) - d_0 \leq 0\} < \frac{c}{4}\,. \tag{6}$$

Since

$$B_{2m}\text{-cap}\ T \leq 2B_{2m}\text{-cap}\ \{x \in T\big|u(x) - d_0 \leq 0\} + 2B_{2m}\text{-cap}\ \{x \in T\big|u(x) - d_0 > 0\}$$

we obtain from (5), (6) that

$$B_{2m}\text{-cap}\ T < c$$

which contradicts the hypothesis on T. Thus (6) cannot be true and the capacity of both the sets is larger than $c/4$.

We now split

$$\int_R |u - d_0|^2\, dx = \int_R (u - d_0)_+^2\, dx + \int_R (u - d_0)_-^2\, dx$$

and both the integrals at the right hand side can be estimated by $K\int_R |\nabla u|^2\, dx$ since the B_{2m}-capacity of the set of zeros of $(u - d_0)_+$ and $(u - d_0)_-$ is larger than $c/4$. This completes the proof of theorem A1 provided we prove

THEOREM A2. *Let $w \in H^1(B_m)$ and*

$$B_{2m}\text{-cap}\ \{x \in S_1|w(x) = 0\} \geq \frac{c}{4} > 0\,. \tag{7}$$

Then there exists a constant $K = K(c, m, k)$ such that

$$\int_1 w^2\, dx \leq K\int_1 |\nabla w|^2\, dx\,. \tag{8}$$

PROOF. Otherwise, there is a sequence of functions $w_j \in H^1$ satisfying (7) and

$$\int\limits_1 w_j^2\,dx = 1\,, \qquad \nabla w_j \to 0 \text{ in } L^2 \ (j \to \infty)\,.$$

It follows that $w_j \to \text{constant} = \overline{w} \neq 0$ in H^1 for a subsequence. Furthermore, it is well known that w_j converges uniformly except on a set of capacity smaller than ε. Since the limit of the w_j is constant $\neq 0$ we obtain that $w_j \neq 0$ except possibly on a set of capacity smaller than ε, for almost all w_j. This contradicts (7). The theorem is proved.

Appendix B.

The following general result is used in the proof of Theorems 5.1 and 5.2:

THEOREM B. *Let X be a reflexive Banach space, Q a nonempty convex subset of X and*

$$g\colon Q\times Q \to \boldsymbol{R}$$

a function satisfying for arbitrary $u, v \in Q$:

(i) $\quad g(v, v) \leq 0$

(ii) $\quad g(u, v) + g(v, u) \geq 0$

(iii) $\quad g(u + t(v-u), v)$ *is l.s.c. as* $t\downarrow 0^+$

(iv) $\quad g(u, \cdot)$ *is concave and u.s.c. on* Q.

Let us suppose, in addition, that there exists a bounded subset B of X and a vector $w_0 \in Q \cap B$, such that

(v) $$g(v, w_0) > 0 \quad \textit{for all } v \in Q\setminus B\,.$$

Then, there exists a solution $\overline{u} \in Q$ of the inequalities

(1) $$g(\overline{u}, w) \leq 0 \quad \textit{for every } w \in Q\,.$$

Moreover, the solution is unique, provided the sign $>$ holds in (ii) *whenever $u \neq v$.*

REMARK. When dealing with problem (5.19) in the proof of Theorem 5.1, we apply the above result with $X = H_0^1(\mathcal{O})$, the set Q given by (5.20) and the function g, for fixed m, given by

$$g(u, v) = \sum_{i=1}^{n} (a_i(x, u, \nabla u), D_i u - D_i v) + (a_{0m}(x, u, \nabla u), u - v) + \\ + \lambda_m(u, u - v) - (f_m \wedge T_m, u - v)$$

for every $u, v \in Q$, λ_m being chosen conveniently large. Note, in particular, that (ii) and (v) follow from the monotonicity and growth assumptions (5.5) and (5.13).

Similarly, for problem (5.37) in the proof of Theorem 5.2 we take X as above, Q given by (5.38) and

$$g(u, v) = \sum_{i=1}^{n} (a_i(x, u, \nabla u), D_i u - D_i v) - (\wedge A(v_\alpha), u - v)$$

for every $u, v \in Q$.

Proof of Theorem. The uniqueness of the solution is an immediate consequence of (ii), where $>$ holds if $u \neq v$.

To prove the existence we set for every $w \in Q$

$$G(w) := \{v \in Q | v(v, w) \leq 0\}$$

and we prove that

$$\bigcap G(w) \neq \emptyset \tag{2}$$

where $\bigcap$, as in the following, carries over all $w \in Q$.

The proof of (2) is achieved in two steps. We first prove that

$$\bigcap G(w) = \bigcap \overline{G(w)} \tag{3}$$

the closure being in the weak topology of X. We then prove

$$\bigcap \overline{G(w)} \neq \emptyset\,. \tag{4}$$

In order to prove (3) we remark that, by (iv), the set

$$H(w) := \{v \in Q | 0 \leq g(w, v)\}$$

for each fixed w is convex and closed (hence also weakly closed) and, moreover, by (ii), $G(w) \subset H(w)$. Therefore, (3) follows from

$$\bigcap H(w) \subset \bigcap G(w) \tag{5}$$

which in turn is immediately seen to be a consequence of (iii) and (iv). [If (5) is false, then $0 \leq g(w, v)$ for some $v \in Q$ and all $w \in Q$, whereas $g(v, \overline{w}) > 0$ for some $\overline{w} \in Q$. Thus $g(v + t(\overline{w} - v), v) \geq 0$ for all $t \in [0, 1]$ and, by (iii), $g(v + \bar{t}(\overline{w} - v), \overline{w}) > 0$ for some $\bar{t} \in]0, 1[$, which, by (iv), together imply

$$g(v + \bar{t}(\overline{w} - v), v + \bar{t}(\overline{w} - v)) > 0 ,$$

hence a contradiction with (i).]

The proof of (4) follows from a well known Lemma of Ky Fan [20], once we notice that the following properties are satisfied in consequence of (v) and (i), (iv), respectively:

(j) every $\overline{G(w)}$ is a weakly closed subset of X, moreover, $\overline{G(w_0)}$ is weakly compact in X;

(jj) the convex hull of every finite set $w_1, \ldots, w_s$ in Q is contained in the corresponding union

$$\overline{G(w_1)} \cup \ldots \cup \overline{G(w_s)} .$$

The proof is thus complete.

For more general existence results of this kind, see [18] and [27].

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