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Steiner's Formula for the Volume of a Parallel Hypersurface in a Riemannian Manifold.

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1. – Introduction.

Let P be a connected orientable embedded hypersurface of compact closure in an oriented Riemannian manifold M. We suppose that P, M and the embedding of P in M are all of class C^ω (unless stated otherwise), and that the manifold topology of P coincides with the subspace topology. For small $r > 0$ we put

$$T(P, r) = \{p \in M \mid \text{there exists a geodesic of length} \leqslant r \text{ from } p \text{ to } P \text{ meeting } P \text{ orthogonally}\} .$$

This is the tube of radius r about P. Also let

$$P_r = \{p \in T(P, r) | d(p, P) = r\} .$$

Because both P and M are orientable, P_r will have two components, P_r^+ and P_r^-. These are the hypersurfaces parallel to P. Let

$S_P^\pm(r) = (n-1)$-dimensional volume of $P_r^\pm$,

$V_P^\pm(r) = n$-dimensional volume of the portion of $T(P, r)$ lying between $P_r^\pm$ and P.

We call the $V_P^\pm(r)$ the volumes of the « half-tubes ». It is not hard to see that $S_P^\pm(r) = (d/dr)\, V_P^\pm(r)$.

The purpose of this article is to determine the terms of order less than or equal 5 in the power series expansion for $V_P^\pm(r)$; in the case that M is

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Euclidean space or a simply connected rank 1 symmetric space we give the complete formulas.

Formulas of this sort were first considered by Steiner in 1840 [STE]. He considered curves in the plane and surfaces in $\boldsymbol{R}^3$, assumed that they were closed and convex, and found the formula for $S_P^+(r)$, where P_r^+ is a parallel hypersurface determined by the outward normal. Steiner also gave a formula for the volume of the convex body whose boundary is P_r^+. In our notation this would be $V + V_P^+(r)$, where V is the volume of the convex body whose boundary is P. Generalizations of Steiner's formulas to higher dimensions and to spaces of constant positive curvature have been given by several authors [AL2], [HA1-5], [HZ], [OH], [VA1-8]. Also Fiala [FI] studied parallel curves on surfaces, and Federer [FD1] generalized Steiner's formula for $V_P^\pm(r)$ for certain subsets of $\boldsymbol{R}^n$ other than submanifolds. In the references at the end of the paper we list other books and papers that treat Steiner's formula and its generalizations.

There is also a relation between the functions $V_P^\pm(r)$ and $S_P^\pm(r)$ and the volumes of tubes as studied in [WY], [BG, pp. 235-256], [FL1], [GR5], [GR6], [GV1-4], [GS, pp. 432-472], [HO], [WO]. Clearly

$$V_P(r) = V_P^+(r) + V_P^-(r) \quad \text{and} \quad S_P(r) = S_P^+(r) + S_P^-(r)$$

are the volumes of $T(P, r)$ and its boundary $P_r^+ \cup P_r^-$. In the papers just mentioned the power series expansions for $V_P(r)$ and $S_P(r)$ are given not only for hypersurfaces, but also for submanifolds of arbitrary codimension, which need not be orientable. Thus for orientable hypersurfaces Steiner's formula amounts to a refinement of Weyl's formula for the volume of a tube. In [WY] Weyl showed that when $P \subset \boldsymbol{R}^n$ the function $V_P(r)$ is intrinsic to P, and in fact can be expressed in terms of the curvature of P. This is not true for $V_P^+(r)$ and $V_P^-(r)$, however.

In section 2 we review the (generalized) Fermi coordinates and Fermi vector fields introduced in [GV3]. The Fermi coordinates for a hypersurface are considerably simpler than those of a general submanifold. In section 3 we use the formulas of section 2 to give our expansions for $S_P^\pm(r)$ and $V_P^\pm(r)$. Using this expansion (formula (3.25)) one obtains at once the following comparison theorem.

THEOREM 1.1. *Let M be an analytic oriented Riemannian manifold and P a connected, orientable, analytically embedded minimal hypersurface with compact closure. Suppose that M has positive (negative) Ricci curvature at all points of P. Then for sufficiently small $r > 0$*

$$V_P^\pm(r) < V_P^{n\pm}(r) \qquad \big(V_P^\pm(r) > V_P^{n\pm}(r)\big)$$

where $V_P^{n\pm}(r)$ *denotes the volume of the half-tubes of radius* r *about* P *if* P *were in Euclidean space* (*that is, formula* (4.6)).

Finally in section 4 we write down formulas for $S_P^{\pm}(r)$ when P is a hypersurface in Euclidean space or in a rank one symmetric space.

2. – Fermi coordinates and Fermi fields.

Let M be a C^∞ Riemannian manifold of dimension n with metric tensor field $\langle\, ,\rangle$. Denote by $\mathfrak{X}(M)$ the C^∞ vector fields on M, and let ∇ and R be the Riemannian connection and curvature of M. Here R is given by $R_{XY} = \nabla_{([X,Y])} - [\nabla_X, \nabla_Y]$ for $X, Y \in \mathfrak{X}(M)$. Sometimes we write $R(X, Y)Z$ instead of $R_{XY}Z$. Denote by ϱ and τ the Ricci and scalar curvatures of M.

Assume that M is orientable and that P is an orientable hypersurface. Then on P there is a globally defined vector field N with $\|N\| = 1$ that is everywhere normal to P. One orientation of P is determined by N, the other by $-N$.

Let $(x_1, \ldots, x_n)$ be a system of (generalized) Fermi coordinates as introduced in [GV3]. They are defined in a neighborhood U of P. Here $x_1, \ldots, x_{n-1}$ when restricted to P form a coordinate system on P, while x_n measures the distance normal to P. We also recall the notion of *tangential Fermi field* which in [GV3] was defined to be a vector field of the form $A = \sum_{a=1}^{n-1} g_a(\partial/\partial x_a)$, where the g_a's are constants. The vector field N is a *normal Fermi field* in the sense of [GV3] because $N = \partial/\partial x_n$. We denote by $\mathfrak{X}(P, m)^T$ the finite dimensional Abelian subalgebra of $\mathfrak{X}(U)$ formed by the tangential Fermi fields.

In order to derive our generalization of Steiner's formula it will be necessary to calculate various covariant derivatives of the Fermi fields. We do this in two elementary lemmas following the scheme of [GR5]. The calculations for a hypersurface are simpler than those for a general submanifold. We write $\nabla^p_{X_1 \ldots X_p} Y = \nabla_{X_1}\nabla_{X_2} \ldots \nabla_{X_p} Y$.

LEMMA 2.1. *For* $A, B \in \mathfrak{X}(P, m)^T$ *we have*

(2.1) $\quad [N, A] = [A, B] = 0\,;$

(2.2) $\quad (\nabla^p_{N \ldots N} N)_{\gamma(t)} = 0\,,\quad$ *where* γ *is a geodesic normal to* P *at some point* $m \in P$;

(2.3) $\quad (\nabla^p_{AN \ldots N} N)_m = \ldots = (\nabla^p_{N \ldots NAN} N)_m = 0\,;$

(2.4) $\quad (\nabla^p_{N \ldots N} A)_m = -(\nabla^{p-2}_{N \ldots N} R_{NA} N)_m \quad$ *for* $p \geqslant 2$.

PROOF. (2.1) is obvious from the definition of Fermi field. Furthermore $N_{\gamma(t)} = \pm\gamma'(t)/\|\gamma'(t)\|$, so that $(\nabla_N N)_{\gamma(t)} = 0$, because γ is a geodesic. Taking successive covariant derivatives we get (2.2). To prove (2.3) we put

$$A_s = (\nabla^p_{N\ldots A\ldots N} N)_m$$

where A occurs in the s-th place. From (2.2) and the fact that $[N, A] = 0$ it follows that

$$A_s - A_{s-1} = -(\nabla^{s-2}_{N\ldots N} R_{NA} \nabla^{p-s}_{N\ldots N} N)_m\,. \tag{2.5}$$

The right hand side of (2.5) can be expressed in terms of the covariant derivatives of R, A and N at m. However, if $2 \leqslant s < p$ each term contains a factor of the form $(\nabla^u_{N\ldots N} N)_m$ with $u > 0$. Hence $A_s = A_{s-1}$ for $2 \leqslant s < p$. On the other hand from (2.2) it follows that $A_1 = 0$. Hence we get (2.3).

Finally for (2.4) we use the definition of the curvature operator, (2.1), (2.2) and (2.3) to obtain:

$$(\nabla^p_{N\ldots N} A)_m = (\nabla^p_{N\ldots NA} N)_m = (\nabla^p_{N\ldots NAN} N - \nabla^{p-2}_{N\ldots N} R_{NA} N)_m = -(\nabla^{p-2}_{N\ldots N} R_{NA} N)_m\,.$$

In order to describe the embedding of P in M we shall make use of that version of the second fundamental form known as the shape operator S (see [BC, pp. 195-212], [ON, pp. 189-193]). We regard S as a linear transformation $\mathfrak{X}(P) \to \mathfrak{X}(P)$ given by

$$SA = -\nabla_A N\,.$$

It is well known (and easy to prove) that $\langle SA, B\rangle = \langle A, SB\rangle$. When the opposite orientation of P is used, we must change S to $-S$. Next we use lemma 2.1 to compute the first four covariant derivatives of $A \in \mathfrak{X}(P, m)^T$ with respect to N at m. Let $\nabla^p_{N\ldots N}(R)$ denote the p-th covariant derivative of the curvature operator.

LEMMA 2.2. *We have*

$$(\nabla_N A)_m = -(SA)_m\,, \tag{2.6}$$

$$(\nabla^2_{NN} A)_m = -(R_{NA} N)_m\,, \tag{2.7}$$

$$(\nabla^3_{NNN} A)_m = \big(-\nabla_N(R)_{NA} N + R_{NSA} N\big)_m\,, \tag{2.8}$$

$$(\nabla^4_{NNNN} A)_m = \big(-\nabla^2_{NN}(R)_{NA} N + 2\nabla_N(R)_{NSA} N + R(N, R_{NA} N) N\big)_m\,. \tag{2.9}$$

PROOF. (2.6) follows from (2.1) and the definition of S. Also (2.7) is a special case of (2.4). Equations (2.8)-(2.9) are found by expanding (2.4).

We have for (2.8) that

$$(\nabla^3_{NNN}A)_m = -(\nabla_N R_{NA}N)_m = \big(-\nabla_N(R)_{NA}N + R_{NSA}N\big)_m .$$

Next for (2.9) we compute

$$\begin{aligned}(\nabla^4_{NNNN}A)_m &= -(\nabla^2_{NN}R_{NA}N)_m\\ &= -\big(\nabla^2_{NN}(R)_{NA}N + 2\nabla_N(R)(N,\nabla_N A)N + R(N,\nabla^2_{NN}A)N\big)_m\\ &= \big(-\nabla^2_{NN}(R)_{NA}N + 2\nabla_N(R)_{NSA}N + R(N, R_{NA}N)N\big)_m .\end{aligned}$$

3. – The power series for the volume of a half-tube about a hypersurface.

Let ω denote the volume element of M. We first compute the terms of order at most 4 in the power series expansion of $\omega_{1\ldots n} = \omega(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, where $(x_1, \ldots, x_n)$ is a system of Fermi coordinates at m for the hypersurface P. Here, we shall use the index conventions $1 \leqslant \alpha, \beta, \gamma, \delta \leqslant n$ and $1 \leqslant a, b, c, d \leqslant n-1$. Denote by R^T the curvature operator of the submanifold P, and let ϱ^T, τ^T be the corresponding Ricci and scalar curvatures. The following notation will be used:

$$r = x_n, \qquad X_\alpha = \frac{\partial}{\partial x_\alpha}, \qquad dx_\alpha = \theta_\alpha, \qquad \langle R(X_\alpha, X_\beta)X_\gamma, X_\delta\rangle = R_{\alpha\beta\gamma\delta},$$

$$\langle R^T(X_a, X_b)X_c, X_d\rangle = R^T_{abcd}, \qquad \varrho(X_\alpha, X_\beta) = \varrho_{\alpha\beta}, \qquad \varrho^T(X_a, X_b) = \varrho^T_{ab},$$

$$\|R\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{n} R^2_{\alpha\beta\gamma\delta}, \qquad \|R^T\|^2 = \sum_{a,b,c,d=1}^{n-1} R^{T2}_{abcd}, \qquad \langle R, R^T\rangle = \sum_{a,b,c,d=1}^{n-1} R_{abcd}R^T_{abcd},$$

$$\|\varrho\|^2 = \sum_{\alpha,\beta=1}^{n} \varrho^2_{\alpha\beta}, \qquad S_{ab} = \langle SX_a, X_b\rangle, \qquad \langle H, N\rangle = \sum_{a=1}^{n-1} S_{aa} .$$

(Here H is the mean curvature vector field.)

If ξ is a tensor field of type (2, 0) on M and η is a tensor field of type (2, 0) on P, we write $\langle \xi, \eta\rangle = \sum_{a,b=1}^{n-1} \xi_{ab}\eta_{ab}$. Finally we write $\langle R(N, X_a)N, X_b\rangle = R_{NaNb}$, etc. In all of the above it is assumed that all quantities are evaluated at the center m of the system of Fermi coordinates.

The curvature operators R and R^T and the shape operator S are related by the Gauss equation:

$$R^T_{abcd} - R_{abcd} = S_{ac}S_{bd} - S_{ad}S_{bc} . \tag{3.1}$$

Let Ω, Ω^T be the curvature forms corresponding to R and R^T. Then $\Omega_{\alpha\beta}(X_\gamma, X_\delta) = R_{\alpha\beta\gamma\delta}$ and $\Omega^T_{ab}(X_c, X_d) = R^T_{abcd}$. Also let $S\theta_a$ be the 1-form given by $S\theta_a(X_b) = \theta_a(SX_b) = \langle X_a, SX_b\rangle = S_{ab}$ at m. We can rewrite (3.1) as

$$\Omega^T_{ab} - \Omega_{ab} = S\theta_a \wedge S\theta_b = S(\theta_a \wedge \theta_b)\,. \tag{3.2}$$

Next let A be a $2k \times 2k$ skew-symmetric matrix with entries A_{ab} from a commutative algebra over $\boldsymbol{R}$. We recall that the Pfaffian $A_{1\ldots 2k}$ of A is given by

$$A_{1\ldots 2k} = \sum_{\sigma\in S_k} \varepsilon_\sigma A_{\sigma(1)\sigma(2)} \wedge \ldots \wedge A_{\sigma(2k-1)\sigma(2k)}, \tag{3.3}$$

where $\wedge$ denotes the multiplication of the algebra,

$$S_k = \{\sigma \in \mathfrak{S}_{2k} | \sigma(1) < \sigma(3) < \ldots < \sigma(2k-1) \text{ and } \sigma(2i-1) < \sigma(2i) \quad \text{for } i = 1, \ldots, k\}\,,$$

and ε_σ is the sign of σ. Here $\mathfrak{S}_{2k}$ denotes the symmetric group of degree $2k$. We denote by $A_{a_1\ldots a_{2c}}$ the Pfaffian of the skew-symmetric submatrix of A in which the indices range over the values $a_1, \ldots, a_{2c}$.

We shall be concerned with the case when A is the matrix $\Omega^T - \Omega$ whose entries are $\Omega^T_{ab} - \Omega_{ab}$. The following is a generalization of the Gauss equation that we shall need for our generalization of Steiner's formula.

LEMMA 3.1. *We have*

$$S(\theta_{a_1} \wedge \ldots \wedge \theta_{a_{2c}}) = \frac{1}{1.3\ldots(2c-1)} (\Omega^T - \Omega)_{a_1\ldots a_{2c}}, \tag{3.4}$$

$$S(\theta_{a_1} \wedge \ldots \wedge \theta_{a_{2c+1}}) = \frac{1}{1.3\ldots(2c+1)} \sum_{b=1}^{2c+1} (-1)^{b+1} S\theta_b \wedge (\Omega^T - \Omega)_{a_1\ldots \hat{a}_b \ldots a_{2c+1}}\,. \tag{3.5}$$

PROOF. We have from (3.2) that

$$(\Omega^T - \Omega)_{a_1\ldots a_{2c}} = A_{a_1\ldots a_{2c}} \tag{3.6}$$

where A is the matrix whose entries are $S\theta_a \wedge S\theta_b$. Using the definition (3.3) of the Pfaffian $A_{a_1\ldots a_{2c}}$ we see that all the terms on the right hand side of (3.3) are the same. It follows then from (3.6) that

$$(3.7)\qquad (\Omega^T-\Omega)_{a_1\dots a_{2c}}=1.3\dots(2c-1)S\theta_{a_1}\wedge\dots\wedge S\theta_{a_{2c}}.$$

We also have

$$(3.8)\qquad S(\theta_{a_1}\wedge\dots\wedge\theta_{a_{2c}})=S\theta_{a_1}\wedge\dots\wedge S\theta_{a_{2c}},$$

and from (3.7) and (3.8) we get (3.4).

To prove (3.5) we compute as follows:

$$\begin{aligned} S(\theta_{a_1}\wedge\dots\wedge\theta_{a_{2c+1}}) &= \frac{1}{2c+1}\sum_{b=1}^{2c+1}(-1)^{b+1}S\theta_{a_b}\wedge S(\theta_{a_1}\wedge\dots\wedge\hat{\theta}_{a_b}\wedge\dots\wedge\theta_{a_{2c+1}}) \\ &= \frac{1}{1.3\dots(2c+1)}\sum_{b=1}^{2c+1}(-1)^{b+1}S\theta_{a_b}\wedge(\Omega^T-\Omega)_{a_1\dots\hat{a}_b\dots a_{2c+1}}. \end{aligned}$$

We shall also need some of the formalism of [GR2], [GR3]. Let R_0 be a tensor field on P of the same type as the curvature operator of P. We write $R_0(W\wedge X)(Y\wedge Z)$ for the value of R_0 on $W, X, Y, Z \in \mathfrak{X}(P)$, and we assume that R_0 has all the symmetry properties of the curvature tensor field of P. In [GR3] the s-th power of R_0 is defined as a special case of multiplication of double forms. Here R_0^s is given by

$$\begin{aligned} (3.9)\qquad & R_0^s(X_1\wedge\dots\wedge X_{2s})(Y_1\wedge\dots\wedge Y_{2s}) \\ &= \sum_{\pi,\sigma\in\tilde{S}_s}\varepsilon_\pi\varepsilon_\sigma R_0(X_{\pi(1)}\wedge X_{\pi(2)})(Y_{\sigma(1)}\wedge Y_{\sigma(2)})\dots R_0(X_{\pi(2s-1)}\wedge X_{\pi(2s)})(Y_{\sigma(2s-1)}\wedge Y_{\sigma(2s)}), \end{aligned}$$

where $\tilde{S}_s=\{\sigma\in\mathfrak{S}_{2s}|\sigma(2t-1)<\sigma(2t)\text{ for }t=1,\dots,s\}$. Also following [GR2] it will be useful to consider the contraction operators C^t. These are defined inductively by $C^0(R_0^s)=R_0^s$ and

$$\begin{aligned} (3.10)\qquad & C^t(R_0^s)(X_1\wedge\dots\wedge X_{2s-t})(Y_1\wedge\dots\wedge Y_{2s-t}) \\ &= \sum_{a=1}^{n-1}C^{t-1}(R_0^s)(X_1\wedge\dots\wedge X_{2s-t}\wedge E_a)(Y_1\wedge\dots\wedge Y_{2s-t}\wedge E_a) \end{aligned}$$

for $X_1,\dots,X_{2s-t}$, $Y_1,\dots,Y_{2s-t}\in\mathfrak{X}(P)$, where $\{E_1,\dots,E_{n-1}\}$ is an arbitrary local frame. We shall write $(R_0)_{a_1\dots a_{2s}b_1\dots b_{2s}}$ for $R_0^s(X_{a_1}\wedge\dots\wedge X_{a_{2s}})(Y_{b_1}\wedge\dots\wedge Y_{b_{2s}})$, etc. Further let $S(a_1,\dots,a_d)$ denote the determinant of the submatrix of S in which the indices range over the values $a_1,\dots,a_d$. Then

$$S(\theta_{a_1}\wedge\dots\wedge\theta_{a_d})(X_{a_1}\wedge\dots\wedge X_{a_d})=S(a_1\dots a_d).$$

COROLLARY 3.2. (Generalized Gauss equations). *We have*

$$S(a_1 \dots a_{2c}) = \frac{2^c}{(2c)!} (R^T - R)^c_{a_1 \dots a_{2c}\, a_1 \dots a_{2c}}, \tag{3.11}$$

$$S(a_1 \dots a_{2c+1}) = \tag{3.12}$$
$$= \frac{2^c}{(2c+1)!} \sum_{b,d=1}^{2c+1} (-1)^{b+d} S_{a_b a_d} (R^T - R)^c_{a_1 \dots \hat{a}_b \dots a_{2c+1}\, a_1 \dots \hat{a}_d \dots a_{2c+1}} .$$

PROOF. (3.11) is immediate from (3.4) and the definition of the Pfaffian. Furthermore (3.12) follows from (3.5) and (3.11).

Recall that (using our notational conventions given at the beginning of the section)

$$\langle S, C^{2c-1}(R^T - R)^c \rangle = \sum_{a,b=1}^{n-1} S_{ab} C^{2c-1}(R^T - R)^c_{ab} .$$

LEMMA 3.3. *We have*

$$\sum_{a_1, \dots, a_{2c}=1}^{n-1} S(a_1 \dots a_{2c}) = \frac{2^c}{(2c)!} C^{2c}(R^T - R)^c, \tag{3.13}$$

$$\sum_{a_1, \dots, a_{2c+1}=1}^{n-1} S(a_1 \dots a_{2c+1}) = \tag{3.14}$$
$$= \frac{2^c}{(2c)!} \{ \langle H, N \rangle C^{2c}(R^T - R)^c - 2c \langle S, C^{2c-1}(R^T - R)^c \rangle \} .$$

PROOF. (3.13) is obvious from (3.11). For (3.14) we compute from (3.12) as follows:

$$\begin{aligned}
\sum_{a_1, \dots, a_{2c+1}=1}^{n-1} S(a_1 \dots a_{2c+1}) &= \frac{2^c}{(2c+1)!} \sum_{a_1, \dots, a_{2c+1}=1}^{n-1} \\
&\quad \cdot \left\{ \left\{ \sum_{d<b} + \sum_{d=b} + \sum_{d>b} \right\} (-1)^{b+d} S_{a_b a_d} (R^T - R)^c_{a_1 \dots \hat{a}_b \dots a_{2c+1}\, a_1 \dots \hat{a}_d \dots a_{2c+1}} \right\} \\
&= \frac{2^c}{(2c+1)!} \left\{ (2c+1) \langle H, N \rangle C^{2c}(R^T - R)^c \right. \\
&\quad \left. - \sum_{\substack{b,d=1 \\ b \neq d}}^{2c+1} \sum_{a_b, a_d=1}^{n-1} S_{a_b a_d} C^{2c-1}(R^T - R)^c_{a_b a_d} \right\} \\
&= \frac{2^c}{(2c)!} \{ \langle H, N \rangle C^{2c}(R^T - R)^c - 2c \langle S, C^{2c-1}(R^T - R)^c \rangle \} .
\end{aligned}$$

We note the following special cases of (3.13) and (3.14):

$$\sum_{a=1}^{n-1} S(a) = \sum_{a=1}^{n-1} S_{aa} = \langle H, N\rangle , \tag{3.15}$$

$$\sum_{a,b=1}^{n-1} S(ab) = \sum_{a,b=1}^{n-1} (S_{aa}S_{bb} - S_{ab}^2) = \sum_{a,b=1}^{n-1} (R^T_{abab} - R_{abab}) = \tau^T - \tau + 2\varrho_{NN} , \tag{3.16}$$

$$\sum_{a,b,c=1}^{n-1} S(abc) = \langle H, N\rangle(\tau^T - \tau + 2\varrho_{NN}) - 2\langle S, \varrho^T - \varrho\rangle - 2\sum_{a=1}^{n-1} R_{NSa\,Na} , \tag{3.17}$$

$$\begin{aligned}\sum_{a,b,c,d=1}^{n-1} S(abcd) = &\sum_{a,b,c,d=1}^{n-1} \{(R^T_{abcd} - R_{abcd})^2 - 4(R^T_{abac} - R_{abac})(R^T_{dbdc} - R_{dbdc})\} \\ &+ \Big\{\sum_{a,b=1}^{n-1} (R^T_{abab} - R_{abab})\Big\}^2 \\ = &\|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4\sum_{a,b,c=1}^{n-1} R^2_{abcN} - 8\sum_{a,b=1}^{n-1} R^2_{aNbN} - 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle \\ &- 4\|\varrho\|^2 + 8\sum_{a=1}^{n-1}\varrho^2_{aN} + 8\varrho^2_{NN} - 8\sum_{a,b=1}^{n-1}(\varrho^T_{ab} - \varrho_{ab})R_{aNbN} + (\tau^T - \tau)^2 + 4\varrho_{NN}(\tau^T - \tau) .\end{aligned} \tag{3.18}$$

We are now able to prove

THEOREM 3.4. *The power series expansion for $\omega_{1\dots n}$ in $x_n = r$ is given by*

$$\begin{aligned}\omega_{1\dots n}(\exp_m rN_m) = \Big\{&1 - \langle H, N\rangle r + \frac{1}{2}(\varrho_{NN} + \tau^T - \tau)r^2 \\ &+ \frac{1}{6}\big(-\nabla_N(\varrho)_{NN} + \langle H, N\rangle(\varrho_{NN} - \tau^T + \tau) + 2\langle S, \varrho^T - \varrho\rangle\big) r^3 \\ &+ \frac{1}{24}\Big(-\nabla^2_{NN}(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2\sum_{a=1}^{n-1}\nabla_N(R)_{Na\,NSa} + \|R^T\|^2 \\ &- 2\langle R, R^T\rangle + \|R\|^2 - 4\sum_{a,b,c=1}^{n=1} R^2_{Nabc} - 2\sum_{a,b=1}^{n-1} R^2_{NaNb} - 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle \\ &- 4\|\varrho\|^2 + 8\sum_{a=1}^{n-1}\varrho^2_{aN} - \varrho^2_{NN} + (\tau^T - \tau)^2 - 2\varrho_{NN}(\tau^T - \tau)\Big) r^4 + O(r^5)\Big\}_m .\end{aligned} \tag{3.19}$$

PROOF. We have

$$\left(\frac{\partial^k}{\partial x_n^k}\omega_{1\dots n}\right)_m = (N^k\omega_{1\dots n})_m .$$

Using the formulas of section 2 we obtain

$$(3.20)\qquad (N\omega_{1\dots n})_m = \sum_{\alpha=1}^{n} \omega(X_1\wedge\dots\wedge\nabla_N X_\alpha\wedge\dots\wedge X_n)_m = \sum_{a=1}^{n-1} \langle\nabla_N X_a, X_a\rangle_m$$

$$= -\sum_{a=1}^{n-1} S_{aa}(m) = -\langle H, N\rangle_m .$$

Similarly

$$(3.21)\qquad (N^2\omega_{1\dots n})_m = \Big\{\sum_{\alpha=1}^{n} \omega(X_1\wedge\dots\wedge\nabla^2_{NN}X_\alpha\wedge\dots\wedge X_n)$$

$$+ 2\sum_{1\leqslant\alpha<\beta\leqslant n} \omega(X_1\wedge\dots\wedge\nabla_N X_\alpha\wedge\dots\wedge\nabla_N X_\beta\wedge\dots\wedge X_n)\Big\}_m$$

$$= \Big\{\sum_{a=1}^{n-1}\langle\nabla^2_{NN}X_a, X_a\rangle + \sum_{a,b=1}^{n-1}(S_{aa}S_{bb} - S^2_{ab})\Big\}_m$$

$$= \Big\{-\sum_{a=1}^{n-1} R_{NaNa} + \sum_{a,b=1}^{n-1}(R^T_{abab} - R_{abab})\Big\}_m$$

$$= \{\varrho_{NN} + \tau^T - \tau\}_m .$$

Next we have

$$(3.22)\qquad (N^3\omega_{1\dots n})_m$$

$$= \Bigg\{\sum_{a=1}^{n-1}\langle\nabla^3_{NNN}X_a, X_a\rangle + 3\sum_{a,b=1}^{n-1}\det\begin{pmatrix}\langle\nabla^2_{NN}X_a, X_a\rangle & \langle\nabla^2_{NN}X_a, X_b\rangle\\ \langle\nabla_N X_b, X_a\rangle & \langle\nabla_N X_b, X_b\rangle\end{pmatrix}$$

$$+ \sum_{a,b,c=1}^{n-1}\det\begin{pmatrix}\langle\nabla_N X_a, X_a\rangle & \langle\nabla_N X_a, X_b\rangle & \langle\nabla_N X_a, X_c\rangle\\ \langle\nabla_N X_b, X_a\rangle & \langle\nabla_N X_b, X_b\rangle & \langle\nabla_N X_b, X_c\rangle\\ \langle\nabla_N X_c, X_a\rangle & \langle\nabla_N X_c, X_b\rangle & \langle\nabla_N X_c, X_c\rangle\end{pmatrix}\Bigg\}_m$$

$$= \Big\{\sum_{a=1}^{n-1}(-\nabla_N(R)_{NaNa} + R_{NSaNa}) + 3\sum_{a,b=1}^{n-1}(R_{NaNa}S_{bb} - R_{NaNb}S_{ab}) - \sum_{a,b,c=1}^{n-1} S(abc)\Big\}_m$$

$$= \{-\nabla_N(\varrho)_{NN} + 3\varrho_{NN}\langle H, N\rangle - \langle H, N\rangle(\tau^T - \tau + 2\varrho_{NN}) + 2\langle S, \varrho^T - \varrho\rangle\}_m$$

$$= \{-\nabla_N(\varrho)_{NN} + \langle H, N\rangle(\varrho_{NN} - \tau^T + \tau) + 2\langle S, \varrho^T - \varrho\rangle\}_m .$$

For the coefficient of r^4 we compute

$$(3.23)\quad (N^4\omega_{1\ldots n})_m = \Bigg\{\sum_{a=1}^{n-1}\langle\nabla^4_{NNNN}X_a, X_a\rangle$$

$$+ 4\sum_{a,b=1}^{n-1}\det\begin{pmatrix}\langle\nabla^3_{NNN}X_a, X_a\rangle & \langle X^3_{NNN}X_a, X_b\rangle\\ \langle\nabla_N X_b, X_a\rangle & \langle\nabla_N X_b, X_b\rangle\end{pmatrix}$$

$$+ 3\sum_{a,b=1}^{n-1}\det\begin{pmatrix}\langle\nabla^2_{NN}X_a, X_a\rangle & \langle\nabla^2_{NN}X_a, X_b\rangle\\ \langle\nabla^2_{NN}X_b, X_a\rangle & \langle\nabla^2_{NN}X_b, X_b\rangle\end{pmatrix}$$

$$+ 6\sum_{a,b,c=1}^{n-1}\det\begin{pmatrix}\langle\nabla^2_{NN}X_a, X_a\rangle & \langle\nabla^2_{NN}X_a, X_b\rangle & \langle\nabla^2_{NN}X_a, X_c\rangle\\ \langle\nabla_N X_b, X_a\rangle & \langle\nabla_N X_b, X_b\rangle & \langle\nabla_N X_b, X_c\rangle\\ \langle\nabla_N X_c, X_a\rangle & \langle\nabla_N X_c, X_b\rangle & \langle\nabla_N X_c, X_c\rangle\end{pmatrix} + \sum_{a,b,c,d=1}^{n-1}S(abcd)\Bigg\}_m$$

$$= \Bigg\{\sum_{a=1}^{n-1}\left(-\nabla^2_{NN}(R)_{NaNa} + 2\nabla_N(R)_{NSaNa} + \langle R(N, R_{Na}N)N, a\rangle\right)$$

$$+ 4\sum_{a,b=1}^{n-1}\Big(\big(\nabla_N(R)_{NaNa} - R_{NSaNa}\big)S_{bb} - \big(\nabla_N(R)_{NaNb} - R_{NSaNb}\big)S_{ab}\Big)$$

$$+ 3\sum_{a,b=1}^{n-1}(R_{NaNa}R_{NbNb} - R^2_{NaNb}) + \frac{1}{6}C^4(R^T - R)^2$$

$$- 6\sum_{a,b,c=1}^{n-1}\big(R_{NaNa}(S_{bb}S_{cc} - S^2_{bc}) - 2R_{NaNb}(S_{ab}S_{cc} - S_{bc}S_{ca})\big)\Bigg\}_m$$

$$= \Bigg\{-\nabla^2_{NN}(\varrho)_{NN} - 2\sum_{a=1}^{n-1}\nabla_N(R)_{NSaNa} - 2\sum_{a,b=1}^{n-1}R^2_{NaNb} + 4\nabla_N(\varrho)_{NN}\langle H, N\rangle$$

$$+ 3\varrho^2_{NN} + \sum_{a,b,c,d=1}^{n-1}\left((R^T_{abcd} - R_{abcd})^2 - 4(R^T_{acbc} - R_{acbc})(R^T_{adbd} - R_{adbd})\right)$$

$$+ \left(\sum_{a,b=1}^{n-1}(R^T_{abab} - R_{abab})\right)^2 - 6\varrho_{NN}\sum_{a,b=1}^{n-1}(R^T_{abab} - R_{abab})$$

$$+ 8\sum_{a,b,c=1}^{n-1}R_{NaNb}(R^T_{acbc} - R_{acbc})\Bigg\}_m$$

$$= \Bigg\{-\nabla^2_{NN}(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2\sum_{a=1}^{n-1}\nabla_N(R)_{NSaNa}$$

$$- 6\sum_{a,b=1}^{n-1}R^2_{NaNb} + 3\varrho^2_{NN} + \|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4\sum_{a,b,c=1}^{n-1}R^2_{Nabc}$$

$$- 4\sum_{a,b=1}^{n-1}(\varrho^T_{ab} - \varrho_{ab} + R_{aNbN})^2 + (\tau^T - \tau + 2\varrho_{NN})^2$$

$$- 6\varrho_{NN}(\tau^T - \tau + 2\varrho_{NN}) + 8\sum_{a,b=1}^{n-1}R_{NaNb}(\varrho^T_{ab} - \varrho_{ab} + R_{aNbN})\Bigg\}_m$$

$$= \Bigg\{-\nabla^2_{NN}(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2\sum_{a=1}^{n-1}\nabla_N(R)_{NSaNa} - 5\varrho^2_{NN}$$

$$- 2 \sum_{a,b=1}^{n-1} R_{NaNb}^2 + \|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4 \sum_{a,b,c=1}^{n-1} R_{Nabc}^2$$

$$\left. - 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle - 4\|\varrho\|^2 + 8 \sum_{a=1}^{n-1} \varrho_{aN}^2 + 4\varrho_{NN}^2 + (\tau^T - \tau)^2 - 2\varrho_{NN}(\tau^T - \tau)\right\}_m$$

$$= \left\{- \nabla_{NN}^2(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2 \sum_{a=1}^{n-1} \nabla_N(R)_{NSaNa}\right.$$

$$+ \|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4 \sum_{a,b,c=1}^{n-1} R_{Nabc}^2 - 2 \sum_{a,b=1}^{n-1} R_{NaNb}^2$$

$$\left. - 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle - 4\|\varrho\|^2 + 8 \sum_{a=1}^{n-1} \varrho_{aN}^2 - \varrho_{NN}^2 + (\tau^T - \tau)^2 - 2\varrho_{NN}(\tau^T - \tau)\right\}_m .$$

From (3.20)-(3.23) we get (3.19).

Now we compute the power series for $S_P^{\pm}(r)$ and $V_P^{\pm}(r)$.

Theorem 3.5. *We have*

$$(3.24)\qquad S_P^{\pm}(r) = \int_P \left\{1 \mp \langle H, N\rangle r + \frac{1}{2}(\varrho_{NN} + \tau^T - \tau) r^2\right.$$
$$\pm \frac{1}{6}(- \nabla_N(\varrho)_{NN} + \langle H, N\rangle(\varrho_{NN} - \tau^T + \tau) + 2\langle S, \varrho^T - \varrho\rangle) r^3$$
$$+ \frac{1}{24}\Big(- \nabla_{NN}^2(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2 \sum_{a=1}^{n-1} \nabla_N(R)_{NaNSa}$$
$$+ \|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4 \sum_{a,b,c=1}^{n-1} R_{Nabc}^2 - 2 \sum_{a,b=1}^{n-1} R_{NaNb}^2$$
$$- 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle - 4\|\varrho\|^2 + 8 \sum_{a=1}^{n-1} \varrho_{aN}^2 - \varrho_{NN}^2$$
$$\left. + (\tau^T - \tau)^2 - 2\varrho_{NN}(\tau^T - \tau)\Big) r^4\right\}_m dm + O(r^5),$$

$$(3.25)\qquad V_P^{\pm}(r) = r\int_P \left\{1 \mp \frac{1}{2}\langle H, N\rangle r + \frac{1}{6}(\varrho_{NN} + \tau^T - \tau) r^2\right.$$
$$\pm \frac{1}{24}(- \nabla_N(\varrho)_{NN} + \langle H, N\rangle(\varrho_{NN} - \tau^T + \tau) + 2\langle S, \varrho^T - \varrho\rangle) r^3$$
$$+ \frac{1}{120}\Big(- \nabla_{NN}^2(\varrho)_{NN} + 4\langle H, N\rangle\nabla_N(\varrho)_{NN} - 2 \sum_{a=1}^{n-1} \nabla_N(R)_{NaNSa}$$
$$+ \|R^T\|^2 - 2\langle R, R^T\rangle + \|R\|^2 - 4 \sum_{a,b,c=1}^{n-1} R_{Nabc}^2 - 2 \sum_{a,b=1}^{n-1} R_{NaNb}^2$$
$$- 4\|\varrho^T\|^2 + 8\langle\varrho, \varrho^T\rangle - 4\|\varrho\|^2 + 8 \sum_{a=1}^{n-1} \varrho_{aN}^2 - \varrho_{NN}^2$$
$$\left. + (\tau^T - \tau)^2 - 2\varrho_{NN}(\tau^T - \tau)\Big) r^4\right\}_m dm + O(r^6).$$

PROOF. We have

$$S_P^{\pm}(r) = \int_P \omega_{1\dots n}(\exp_m(\pm rN_m))\,dm\,. \tag{3.26}$$

(This is an obvious generalization of lemma 7.1 of [GR6].) Then (3.24) follows immediately from (3.19) and (3.26). Also (3.25) is a consequence of (3.24) and the relation $S_P^{\pm}(r) = (d/dr)\,V_P^{\pm}(r)$.

4. – The volumes of half-tubes in Euclidean space and rank 1 symmetric spaces.

In this section we give the complete formula for $S_P^{\pm}(r)$ where P is an orientable hypersurface in Euclidean space or in an orientable rank 1 symmetric space. Then $V_P^{\pm}(r)$ can be determined from the relation $(d/dr)V_P^{\pm}(r) = S_P^{\pm}(r)$. This sometimes involves a messy integration, and so we calculate $V_P^{\pm}(r)$ explicitly only when it is convenient to do so.

We shall need certain mean curvatures to understand the geometrical significance of the coefficients in the expansion of $S_P^{\pm}(r)$.

DEFINITION. Let P be an orientable hypersurface of an orientable Riemannian manifold M. The *c-th integrated mean curvature* k_c of P in M is given by

$$k_{2c} = k_{2c}(M, P) = \frac{1}{2^c c!} \sum_{a_1\dots a_{2c}=1}^{n-1} \int_P S(a_1 \dots a_{2c})_m\,dm\,, \tag{4.1}$$

$$k_{2c+1} = k_{2c+1}(M, P) = \frac{1}{2^c c!} \sum_{a_1\dots a_{2c+1}=1}^{n-1} \int_P S(a_1 \dots a_{2c+1})_m\,dm\,. \tag{4.2}$$

Although the integrated mean curvatures are defined in terms of the shape operator S, it is almost possible to eliminate the dependence on S.

LEMMA 4.1. *We have*

$$k_{2c} = \frac{1}{c!\,(2c)!} \int_P C^{2c}(R^T - R)^c_m\,dm\,, \tag{4.3}$$

$$k_{2c+1} = \frac{1}{c!\,(2c)!} \int_P \{\langle H, N\rangle\, C^{2c}(R^T - R)^c - 2c\,\langle S, C^{2c-1}(R^T - R)^c\rangle\}_m\,dm\,. \tag{4.4}$$

PROOF. This is immediate from (4.1), (4.2) and lemma 3.3.

In [WY] Weyl introduced the quantities k_c (for c even) in the more general situation when P is a submanifold of arbitrary codimension and possibly nonorientable. These are the coefficients in his formula for the volume $V_P(r)$ of a tube about P, when P is a submanifold of a sphere or Euclidean space. His main point was that the k_{2c} are completely expressible in terms of the curvature operator R^T of P. (Actually when P is a submanifold of a sphere, the k_{2c} also depend on the value of the constant curvature of the sphere.) We shall generalize these results to the case of a hypersurface of a rank 1 symmetric space, and at the same time obtain formulas for $S_P^{\pm}(r)$ and $V_P^{\pm}(r)$.

In contrast to the situation with the k_{2c} it is clear that the k_{2c+1} depend on the particular embedding of P in M, even when M is a Euclidean space or a rank 1 symmetric space. Nonetheless we see from (4.4) that this dependence is linear instead of something more complicated.

The integrated mean curvatures have also been used in [AL2], [BF], [FD1], [FL2], [HA1], [SA], [SW], [VA1-8], but in these papers no use was made of the relations between curvature and the second fundamental form given by lemma 4.1. In the notation of Allendoerfer [AL2] (writing n for Allendoerfer's $n+1$) we have

$$k_{2c} = 1.3 \ldots (2c-1) M_{n-2c-1} \quad \text{and} \quad k_{2c-1} = 1.3 \ldots (2c-1) M_{n-2c}\,.$$

We shall use Jacobi fields to compute $S_P^{\pm}(r)$ for a hypersurface in Euclidean space or a rank 1 symmetric space. Our method is to exploit the relation between Jacobi fields and Fermi fields given in [GR6]. Let γ be a unit speed geodesic in M meeting P orthogonally at a point m. We may assume that $\gamma(0) = m$; then $\gamma'(0) \in P_m^{\perp}$. Let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates such that $\gamma'(t) = N_{\gamma(t)} = X_{n|\gamma(t)}$.

LEMMA 4.2. *When restricted to γ the following are Jacobi fields:*

$$X_{1|\gamma(t)}, \ldots, X_{n-1|\gamma(t)}, N_{\gamma(t)}\,.$$

For a proof see [GR6] or the appendix of [GV2].

There are standard formulas (see for example [BS, p. 87]) that relate Jacobi fields and parallel fields for Euclidean space and the rank 1 symmetric spaces. Because of this we shall be able to express Fermi fields in terms of parallel fields. This will allow us to compute $\omega_{1\ldots n}$, and hence also $S_P^{\pm}(r)$.

In terms of the Fermi coordinates $(x_1, \ldots, x_n)$ for P at m we define a parallel frame field $\{E_1, \ldots, E_n\}$ along the geodesic γ. Let $E_{\alpha|\gamma(t)}$ be the parallel translate of $X_{\alpha|m}$ along γ for $\alpha = 1, \ldots, n$. Then $E_{n|\gamma(t)} = X_{n|\gamma(t)} = N_{\gamma(t)}$. This holds for any Riemannian manifold. However the expressions for the X_a in terms of the E_a for $1 \leqslant a \leqslant n-1$ depend on the particular Riemannian manifold. Also let $S_m E_a$ denote the vector field along γ such that $S_m E_{a|\gamma(t)}$ is the parallel translate of $S_m E_{a|m}$.

We are now ready to give our formulas for $S_P^{\pm}(r)$ and $V_P^{\pm}(r)$. For Euclidean space we have

THEOREM 4.3. *Let P be an orientable hypersurface of $\boldsymbol{R}^n$. Then*

$$S_P^{\pm}(r) = \int_P \det(I \mp rS_m)\, dm = \tag{4.5}$$
$$= \sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}}{1.3 \ldots (2c-1)} r^{2c} \mp \sum_{c=0}^{[n/2-1]} \frac{k_{2c+1}}{1.3 \ldots (2c+1)} r^{2c+1},$$

$$V_P^{\pm}(r) = \int_P \int_0^r \det(I \mp tS_m)\, dt\, dm = \tag{4.6}$$
$$= \sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}}{1.3 \ldots (2c+1)} r^{2c+1} \mp \sum_{c=0}^{[n/2-1]} \frac{k_{2c+1}}{1.3 \ldots (2c+1)(2c+2)} r^{2c+2}.$$

PROOF. For $\boldsymbol{R}^n$ each Jacobi field is of the form $\sum (a_\alpha + tb_\alpha) E_\alpha$, where each E_α is parallel. From this fact, lemma 4.2 and the initial conditions given by $(\nabla_A N)_m = -(SA)_m$ we find that

$$X_{a|\gamma(t)} = (I - tS_m) E_{a|\gamma(t)}, \qquad a = 1, \ldots, n-1. \tag{4.7}$$

From (4.7) it follows easily that

$$\omega_{1\ldots n}(\exp_m(\pm rN_m)) = \det(I \mp rS_m) = \sum_{c=0}^{n-1} (\mp r)^c \frac{1}{c!} \sum_{a_1, \ldots, a_c = 1}^{n-1} S(a_1 \ldots a_c)_m. \tag{4.8}$$

Integrating (4.8) and using the integrated mean curvatures we find (4.5). A further integration yields (4.6).

It is not much more difficult to find the formulas for $S_P^{\pm}(r)$ when P is a hypersurface of a space of constant curvature λ. For convenience we assume $\lambda > 0$. The formulas for the case $\lambda < 0$ can be found by changing all of the trigonometric functions to hyperbolic functions. Similar remarks will apply to the other rank 1 symmetric spaces.

THEOREM 4.4. *Let P be an orientable hypersurface in a sphere $S^n(\lambda)$ with constant curvature $\lambda > 0$. Let $k_c(\lambda) = k_c(S^n(\lambda), P)$. Then*

$$(4.9)\qquad S_P^{\pm}(r) = \int_P \det\left(\cos\sqrt{\lambda}\, r I \mp \frac{\sin\sqrt{\lambda}\, r}{\sqrt{\lambda}} S_m\right) dm$$
$$= (\cos\sqrt{\lambda}\, r)^{n-1}\left\{\sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}(\lambda)}{1.3\ldots(2c-1)}\left(\frac{\tan\sqrt{\lambda}\, r}{\sqrt{\lambda}}\right)^{2c}\right.$$
$$\left.\mp \sum_{c=0}^{[n/2-1]} \frac{k_{2c+1}(\lambda)}{1.3\ldots(2c+1)}\left(\frac{\tan\sqrt{\lambda}\, r}{\sqrt{\lambda}}\right)^{2c+1}\right\}.$$

PROOF. For $S^n(\lambda)$ each Jacobi field is of the form $\sum(a_\alpha\cos\sqrt{\lambda}t + b_\alpha \sin\sqrt{\lambda}t)E_\alpha$ where each E_α is parallel. Using this fact and proceeding as in theorem 4.3 we find that

$$(4.10)\qquad X_{a|\gamma(t)} = \left(\cos\sqrt{\lambda}\, t I - \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} S_m\right) E_{a|\gamma(t)}, \qquad a = 1, \ldots, n-1.$$

From (4.10) it follows that

$$(4.11)\qquad \omega_{1\ldots n}(\exp_m(\mp rN_m)) = (\cos\sqrt{\lambda}\, r)^{n-1}\det\left(I \mp \frac{\tan\sqrt{\lambda}\, r}{\sqrt{\lambda}} S_m\right)$$
$$= (\cos\sqrt{\lambda}\, r)^{n-1}\sum_{c=0}^{n-1}\left(\mp\frac{\tan\sqrt{\lambda}\, r}{\sqrt{\lambda}}\right)^c \frac{1}{c!}\sum_{a_1,\ldots,a_c=1}^{n-1} S(a_1\ldots a_c)_m.$$

The rest of the proof is the same as that of theorem 4.3.

The computations for the other rank one symmetric spaces are more complicated.

THEOREM 4.5. *Let P be an orientable hypersurface in the complex projective space $CP^n(\mu)$ with constant holomorphic sectional curvature $\mu > 0$. Then*

$$(4.12)\qquad S_P^{\pm}(r) = \left(\cos\frac{1}{2}\sqrt{\mu}\, r\right)^{2n}\int_P \det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\mu}\, r\right)^2 \delta_{a2n-1}\delta_{b2n-1}\right.$$
$$\left.\mp \frac{2}{\sqrt{\mu}}\left(\tan\frac{1}{2}\sqrt{\mu}\, r\right) S_{ab}(m)\right) dm,$$

where $a, b = 1, \ldots, 2n-1$ and $E_{2n-1} = JN$.

PROOF. Let $\{E_1, \ldots, E_{2n-2}, E_{2n-1} = JN\}$ be a parallel frame field along γ. Here J denotes the almost complex structure. Proceeding as in theorem 4.3

we find that

$$
(4.13)\quad \begin{cases} X_{c|\gamma(t)} = \left\{\cos\frac{1}{2}\sqrt{\mu}\,tI - \frac{2}{\sqrt{\mu}}\sin\frac{1}{2}\sqrt{\mu}\,tS_m\right\}E_{c|\gamma(t)}, & c=1, \ldots, 2n-2, \\ X_{2n-1|\gamma(t)} = \left\{\cos\sqrt{\mu}\,tI - \frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}\,tS_m\right\}E_{2n-1|\gamma(t)}. \end{cases}
$$

From (4.13) it follows that

$$
\omega_{1\ldots n}(\exp_m \pm rN_m) = \left(\cos\frac{1}{2}\sqrt{\mu}\,r\right)^{2n}\det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\mu}\,r\right)^2\delta_{a2n-1}\delta_{b2n-1} \mp \frac{2}{\sqrt{\mu}}\tan\frac{1}{2}\sqrt{\mu}r\,S_{ab}(m)\right)
$$

where $a, b = 1, \ldots, 2n-1$. This implies (4.12).

THEOREM 4.6. *Let P be an orientable hypersurface in the quaternionic projective space $QP^n(\nu)$ with maximum sectional curvature $\nu > 0$. Then*

$$
(4.14)\quad S_P^{\pm}(r) = \left(\cos\frac{1}{2}\sqrt{\nu}\,r\right)^{4n+2}\int_P \det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\nu}\,r\right)^2\sum_{\alpha=4n-3}^{4n-1}\delta_{\alpha a}\delta_{\alpha b} \mp \frac{2}{\sqrt{\nu}}\left(\tan\frac{1}{2}\sqrt{\nu}\,r\right)S_{ab}(m)\right)dm\,,
$$

where $a, b = 1, \ldots, 4n-1$ and $E_{4n-1} = IN$, $E_{4n-2} = JN$, $E_{4n-3} = KN$.

PROOF. In a neighborhood of $m = \gamma(0)$ there are locally defined almost complex structures I, J and K such that $IJ = -JI = K$, etc. Let $\{E_1, \ldots, E_{4n-4}, E_{4n-3} = KN, E_{4n-2} = JN, E_{4n-1} = IN\}$ be a parallel frame field along γ. In order to achieve this it is necessary to make the proper choice of the locally defined almost complex structures I, J and K. Proceeding as in theorem 4.3 we find

$$
(4.15)\quad \begin{cases} X_{c|\gamma(t)} = \left(\cos\frac{1}{2}\sqrt{\nu}\,t\right)\left\{I - \frac{2}{\sqrt{\nu}}\tan\frac{1}{2}\sqrt{\nu}\,tS_m\right\}E_{c|\gamma(t)}, \quad c=1, \ldots, 4n-4, \\ X_{\alpha|\gamma(t)} = \left(\cos\frac{1}{2}\sqrt{\nu}\,t\right)^2\left\{\left(1-\left(\tan\frac{1}{2}\sqrt{\nu}\,t\right)^2\right)I - \frac{2}{\sqrt{\nu}}\tan\frac{1}{2}\sqrt{\nu}\,tS_m\right\}E_{\alpha|\gamma(t)}, \\ \qquad \alpha = 4n-3,\ 4n-2,\ 4n-1. \end{cases}
$$

From (4.15) it follows that

$$\omega_{1\dots n}(\exp_m \pm rN_m) = \left(\cos\frac{1}{2}\sqrt{\nu}r\right)^{4n+2}\det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\nu}r\right)^2\sum_{\alpha=4n-3}^{4n-1}\delta_{\alpha a}\delta_{\alpha b} \mp \frac{2}{\sqrt{\nu}}\left(\tan\frac{1}{2}\sqrt{\nu}r\right)S_{ab}(m)\right),$$

where $a, b = 1, \dots, 4n-1$. This gives the required formula (4.14).

THEOREM 4.7. *Let P be an orientable hypersurface in the Cayley plane* $\mathrm{Cay}\,P^2(\zeta)$ *with maximum sectional curvature $\zeta > 0$. Then*

$$S_P^{\pm}(r) = \left(\cos\frac{1}{2}\sqrt{\zeta}r\right)^{22}\int_P \det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\zeta}r\right)^2\sum_{\alpha=9}^{15}\delta_{\alpha a}\delta_{\alpha b} \mp \frac{2}{\sqrt{\zeta}}\left(\tan\frac{1}{2}\sqrt{\zeta}r\right)S_{ab}(m)\right)dm, \tag{4.16}$$

where $a, b = 1, \dots, 15$.

PROOF. We proceed as in theorem 4.6. Let $\{E_1, \dots, E_8, E_9 = I_1N, \dots$ $\dots, E_{15} = I_7N\}$ be a parallel frame field along γ with respect to a proper choice of the almost complex structures I_α. Then we find

$$\begin{cases} X_{c|\gamma(t)} = \left(\cos\frac{1}{2}\sqrt{\zeta}t\right)\left\{I - \frac{2}{\sqrt{\zeta}}\tan\frac{1}{2}\sqrt{\zeta}tS_m\right\}E_{c|\gamma(t)}, & c = 1, \dots, 8, \\ X_{\alpha|\gamma(t)} = \left(\cos\frac{1}{2}\sqrt{\zeta}t\right)^2\left\{\left(1 - \left(\tan\frac{1}{2}\sqrt{\zeta}t\right)^2\right)I - \frac{2}{\sqrt{\zeta}}\tan\frac{1}{2}\sqrt{\zeta}tS_m\right\}E_{\alpha|\gamma(t)}, & \alpha = 9, \dots, 15. \end{cases} \tag{4.17}$$

From (4.17) we find

$$\omega_{1\dots n}(\exp_m \pm rN_m) = \left(\cos\frac{1}{2}\sqrt{\zeta}r\right)^{22}\det\left(\delta_{ab} - \left(\tan\frac{1}{2}\sqrt{\zeta}r\right)^2\sum_{\alpha=9}^{15}\delta_{\alpha a}\delta_{\alpha b} \mp \frac{2}{\sqrt{\zeta}}\left(\tan\frac{1}{2}\sqrt{\zeta}r\right)S_{ab}(m)\right),$$

where $a, b = 1, \dots, 15$. This implies (4.16).

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